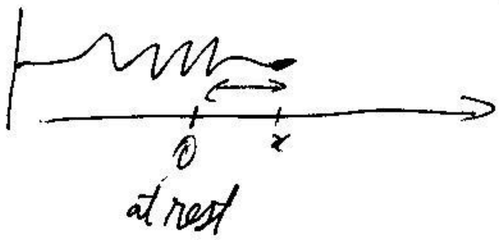
Solutions to 3.4 HW problems



2) $-kx = mx''$

$mx'' + kx = 0$

$x'' + \frac{k}{m}x = 0$

char. equ. $s^2 = -\frac{k}{m}$

$\Rightarrow s = \pm i\sqrt{\frac{k}{m}}$

$\Rightarrow x(t) = C\cos\left(\sqrt{\frac{k}{m}}\,t - \alpha\right)$

Period $= T = 2\pi\sqrt{\frac{m}{k}} = \frac{\pi}{4}$ s.

$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{8}{2\pi}$ Hz $= \frac{4}{\pi}$ Hz

$\omega = 2\pi f = \sqrt{\frac{k}{m}} = \sqrt{\frac{48}{0.75}} = \sqrt{\frac{48}{3}\times 4}$

$= 8$ rad/s

4) $m = 0.25$ kg

$k = \frac{9}{0.25} = 36$ N/m

$x(0) = 1$ m

$x'(0) = -5$ m/s

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{36}{0.25}} = \sqrt{36\times 4} = 12$ rad/s

$\Rightarrow x(t) = A\cos(12t) + B\sin(12t)$

3.4 problems (continued)

4) (continued)

$$x(0) = 1 \Rightarrow \boxed{A = 1}$$

$$\Rightarrow x(t) = \cos(12t) + B\sin(12t)$$

$$x'(0) = -5$$

$$\Rightarrow 12B = -5$$

$$\Rightarrow \boxed{B = -\frac{5}{12}}$$

$$\Rightarrow x(t) = \cos(12t) - \frac{5}{12}\sin(12t)$$

$$= C\cos(12t - \alpha),$$

with $C = \sqrt{1^2 + \left(\frac{5}{12}\right)^2}$

$$= \frac{1}{12}\sqrt{12^2 + 5^2}$$

$$\boxed{C = \frac{13}{12} \quad m}$$

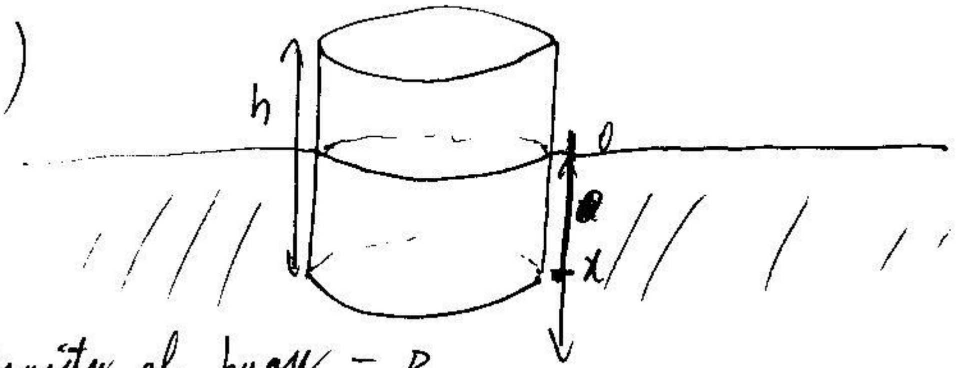
$$\alpha = \tan^{-1}\left(\frac{-\frac{5}{12}}{1}\right)$$

$$= -\tan^{-1}\left(\frac{5}{12}\right) \simeq -0.39 \text{ rad}.$$

This is consistent with the book's answer because angles which differ by a multiple of 2π rad are equivalent.

$$\Rightarrow \boxed{x(t) \simeq \frac{13}{12}\cos(12t + 0.39) \quad m}$$

10)



density of buoy $= \rho$

density of water $= 1\ g/cm^3$

$$\left(\begin{aligned} &= \frac{10^{-3}}{(10^{-2})^3}\ kg/m^3 \\ &= \frac{10^{-3}}{10^{-6}}\ kg/m^3 \\ &= 1000\ kg/m^3 \end{aligned} \right)$$ ← Ignore this

⚠ Typo in the book: Weight $= mg = \pi r^2 h \underline{\underline{\rho}} g$

$$\Sigma F = m x''$$

$$\Rightarrow \pi r^2 h \rho g - \pi r^2 x g = m x''$$ (note that x-axis is directed downwards)

$$\Rightarrow \boxed{m x'' + (\pi r^2 g) x = \pi r^2 h \rho g}$$

But m = mass of buoy $= \pi r^2 h \rho$ grams

$$\Rightarrow \pi r^2 h \rho\, x'' + (\pi r^2 g) x = \pi r^2 h \rho g$$

We divide by $\pi r^2 h \rho$, to get:

$$\boxed{x'' + \frac{g}{h\rho} x = g} \qquad (*)$$

3.4 problems (continued)

char. eq.: $s^2 + \frac{g}{hp} = 0$

$\Rightarrow s = \pm i \sqrt{\frac{g}{hp}}$

$\Rightarrow \boxed{x_c(t) = A\cos\left(\sqrt{\frac{g}{hp}}\, t\right) + B\sin\left(\sqrt{\frac{g}{hp}}\, t\right)}$

We need to find x_p.

Assume $x_p = D$ (D constant)

$\Rightarrow \frac{g}{hp} D = g$ (by substituting in (*))

$\Rightarrow \boxed{D = hp}$

$\Rightarrow \boxed{x_p = hp}$

$\Rightarrow x(t) = x_c + x_p$

$\boxed{x(t) = A\cos\left(\sqrt{\frac{g}{hp}}\, t\right) + B\sin\left(\sqrt{\frac{g}{hp}}\, t\right) + hp}$

We have initial conditions $\begin{cases} x(0) = 0 \\ x'(0) = 0 \end{cases}$

$x(0)=0 \Rightarrow A + hp = 0 \Rightarrow \boxed{A = -hp}$

$x'(0) = 0 \Rightarrow \sqrt{\frac{g}{hp}}\, B = 0 \Rightarrow \boxed{B = 0}$

3.4 problems (continued)

So we have found that:

$$\boxed{x(t) = hp\left(1 - \cos\left(\sqrt{\frac{g}{hp}}\, t\right)\right)} \qquad (cm)$$

Hence, we see that the equilibrium position (which is $= x_p$ in this case) is:

$$x_e = hp$$

and the period is:

$$T = 2\pi\sqrt{\frac{hp}{g}}$$

You then use a calculator to compute numerically:

$$x_e = 200 \times 0.5 = 100 \text{ cm}$$

$$T = 2\pi\sqrt{\frac{200 \times 0.5}{980}}$$

$$= \frac{20\pi}{\sqrt{980}} \qquad (s)$$

$$T = \frac{2}{7}\pi\sqrt{5} \qquad (s)$$

$$\boxed{T \simeq 2.01 s}$$

20) With damping:

$$m x'' + c x' + k x = 0$$

$$2 x'' + 16 x' + 40 x = 0$$

$$\Rightarrow x'' + 8 x' + 20 x = 0$$

char. eq.: $s^2 + 8s + 20 = 0$

$$\Rightarrow s = \frac{-8 \pm \sqrt{64 - 80}}{2}$$

$$= -4 \pm i2$$

$$\Rightarrow \boxed{x(t) = A e^{-4t} \cos(2t) + B e^{-4t} \sin(2t)}$$

$$x(0) = 5 \Rightarrow \boxed{A = 5}$$

$$\Rightarrow x(t) = 5 e^{-4t} \cos(2t) + B e^{-4t} \sin(2t)$$

$$x'(t) = -20 e^{-4t} \cos(2t) - 10 e^{-4t} \sin(2t)$$

$$- 4 B e^{-4t} \sin(2t) + 2 B e^{-4t} \cos(2t)$$

$$x'(0) = 4 \Rightarrow -20 + 2B = 4$$

$$\Rightarrow \boxed{B = 12}$$

$$\Rightarrow x(t) = e^{-4t} (5 \cos(2t) + 12 \sin(2t))$$

$$\boxed{\begin{aligned} x(t) &= 13 e^{-4t} \cos(2t - \alpha) \\ \alpha &= \tan^{-1}\left(\frac{12}{5}\right) \simeq 1.18 \text{ (rad)} \end{aligned}}$$

3.4 problems (continued)

20) (continued) <u>without damping ($c = 0$)</u>

$$2x'' + 40x = 0$$

$$\Rightarrow x'' + 20x = 0$$

$$\Rightarrow s^2 = -20$$

$$\Rightarrow s = \pm 2i\sqrt{5}$$

$$\Rightarrow u(t) = a\cos(2\sqrt{5}\,t) + b\sin(2\sqrt{5}\,t)$$

$u(0) = 5 \Rightarrow a = 5$

$$\Rightarrow u(t) = 5\cos(2\sqrt{5}\,t) + b\sin(2\sqrt{5}\,t)$$

$u'(0) = 4 \Rightarrow 2\sqrt{5}\,b = 4$

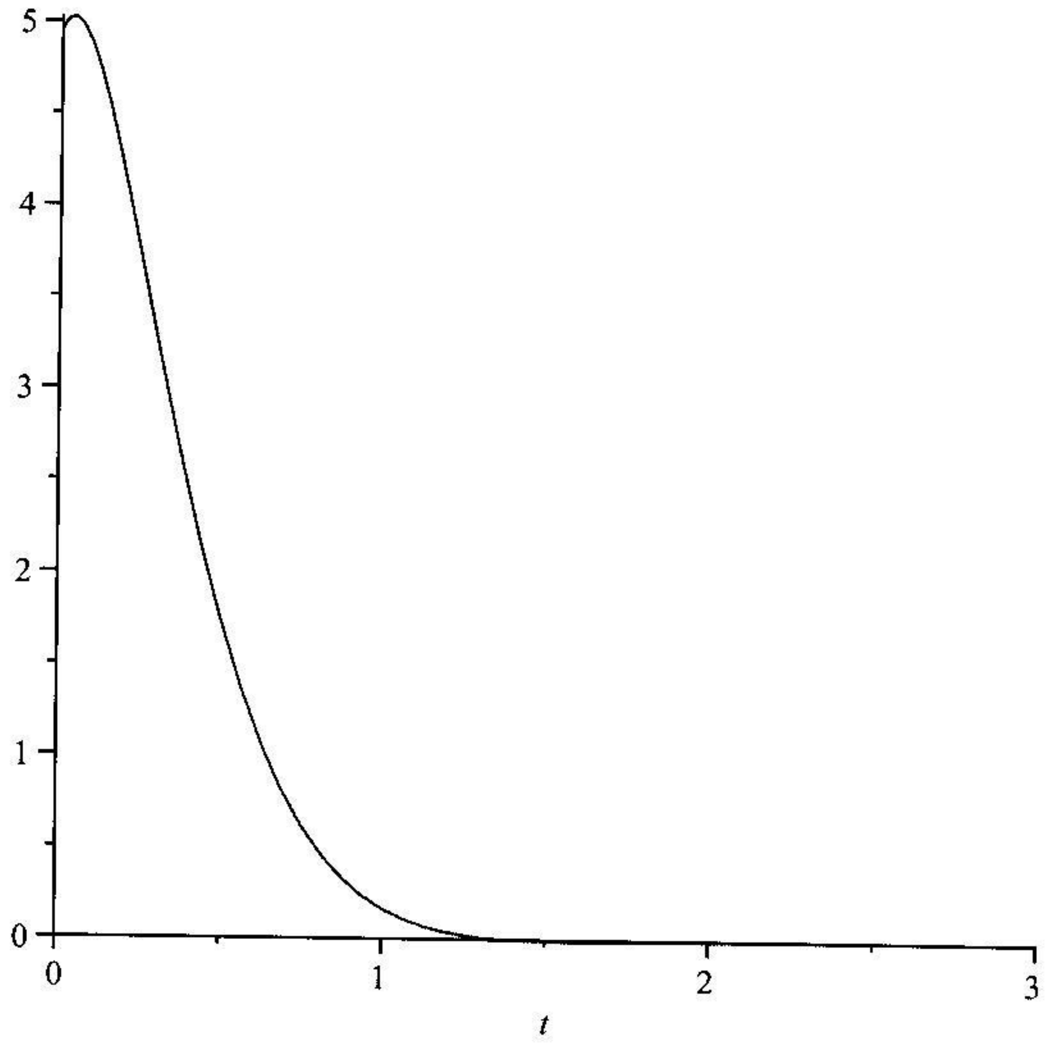
$$\Rightarrow b = \frac{2}{\sqrt{5}}$$

$$b = \frac{2\sqrt{5}}{5}$$

$$\Rightarrow u(t) = 5\cos(2\sqrt{5}\,t) + \frac{2\sqrt{5}}{5}\sin(2\sqrt{5}\,t)$$

$$\Rightarrow \boxed{u(t) \simeq 5.08\cos(2\sqrt{5}\,t - 0.18)}$$

$plot(13 \cdot \exp(-4 \cdot t) \cdot \cos(2 \cdot t - 1.18), t = 0 ..3)$



$plot(5.08 \cdot \cos(2 \cdot \text{sqrt}(5) \cdot t - 0.18), t = 0..3)$