Solutions to HW problems

section 2.3

2) The ODE is $v' = -kv$. Solving, we get:

$$v(t) = Ae^{-kt}$$

$$v(0) = v_0 \Rightarrow A = v_0 \Rightarrow \boxed{v(t) = v_0 e^{-kt}}$$

But $x' = v = v_0 e^{-kt}$. We integrate, and get

$$x(t) = -\frac{v_0}{k} e^{-kt} + B$$

$$x(0) = x_0 \Rightarrow -\frac{v_0}{k} + B = x_0 \Rightarrow \underline{B = x_0 + \frac{v_0}{k}}$$

$$\Rightarrow \boxed{x(t) = x_0 + \frac{v_0}{k}\left(1 - e^{-kt}\right)}$$

QED

4) ODE: $v' = -kv^2$

$$\frac{v'}{v^2} = -k \qquad (\text{if } v(t) \neq 0)$$

Integrating, we get

$$-\frac{1}{v(t)} = -kt + C$$

$$\Rightarrow v(t) = \frac{1}{kt - C}$$

$$v(0) = v_0 \Rightarrow v_0 = -\frac{1}{C} \Rightarrow v = \frac{1}{kt + \frac{1}{v_0}}$$

$$\Rightarrow \boxed{v = \frac{v_0}{kv_0 t + 1}}$$

...

4) (continued)

Thus, $x' = \dfrac{v_0}{kv_0 t + 1}$.

We integrate, and get:

$$x(t) = \frac{v_0}{kv_0} \ln|kv_0 t + 1| + C$$

$$= \frac{1}{k} \ln|kv_0 t + 1| + C$$

$x(0) = x_0 \Rightarrow C = x_0$

$$\Rightarrow x(t) = x_0 + \frac{1}{k} \ln|kv_0 t + 1|$$

In problem 2, resistance is $\frac{dv}{dt} = -kv$

In problem 4, resistance is $\frac{dv}{dt} = -kv^2$

Thus, if $|v(t)| < 1$, then $|v^2| < |v|$, so there will be less resistance in the case of problem 4, if the object is moving slowly.

$\lim_{t \to \infty} v(t) = 0$ in both cases,

so if t is very large, then $|v(t)| < 1$, and we see that the object travels an infinite distance in problem 4, but only a finite distance in problem 2. QED

2.3 problems (continued)

10) ODE: $v' = -pv - g$

* $\underline{0 \leq t \leq 20}$:

Eq. 5 gives, using $v_0 = 0$

$$v(t) = \frac{g}{p} e^{-pt} - \frac{g}{p}$$

$g \simeq 32.2 \ ft/s^2$, $p = 0.15$

$$\boxed{v(20) \simeq -203.98 \ ft/s}$$

Eq. 9 gives:

$$y(t) = 10000 + v_T t + \frac{1}{p}(v_0 - v_T)(1 - e^{-pt})$$

$$v_T = -\frac{g}{p} \Rightarrow \boxed{v_T \simeq -214.7 \ ft/s}$$

$$y(20) = 10000 - 214.7 \times 20 + \frac{1}{0.15} \cdot 214.7 \left(1 - e^{-0.15 \times 20}\right)$$

$$\Rightarrow \boxed{y(20) \simeq 7066.1 \ ft/s}$$

10) (continued)

* $\underline{t \geq 20:}$

Let $\tilde{t} = t - 20$

$p = 1.5$ (different from before)

$y_0 \simeq 7066.1$ ft/s

$v_0 \simeq -203.98$ ft/s.

$v_T \simeq -21.5$ ft/s

Using eq. 9, we want to find $\tilde{t}$ s.t. $y(\hat{t}) = 0$, i.e.

$$0 = 7066.1 - 21.5\,\tilde{t} + \frac{1}{1.5}(-204 + 21.5)(1 - e^{-1.5\tilde{t}})$$

solving using command fsolve in Maple:

$\tilde{t} \simeq 323$ s.

$\Rightarrow \boxed{t \simeq 323 + 20 \simeq 343\text{ s.} \simeq 5\text{ min. }43\text{ s.}}$

2.3 problems

22) $p = 0.075$ (in fps units, with $g = 32\ ft/s^2$)

Terminal speed

$$\boxed{|v_\tau| = \sqrt{\frac{g}{p}} = \sqrt{\frac{32}{0.075}} \simeq 20.67\ ft/s}$$

$$v(t) = \sqrt{\frac{g}{p}} \tanh\left(C_2 - t\sqrt{pg}\right)$$

$$C_2 = \tanh^{-1}\left(v_0 \sqrt{\frac{p}{g}}\right) = 0 \quad (\text{since } v_0 = 0)$$

$$\Rightarrow v(t) = -\sqrt{\frac{g}{p}} \tanh\left(t\sqrt{pg}\right)$$

$$y(t) = 10000 - \frac{1}{0.075} \ln\left|\cosh\left(t\sqrt{pg}\right)\right| \overset{\text{want}}{=} 0$$

Then, using Maple's fsolve command:

$$\boxed{t \simeq 485\ s \simeq 8\ \text{min}\ 5\ s.}$$