MAT303 Spring 2009

Practice Midterm I

The actual final will consist of five problems with no more then two subproblems

You will be allowed to use calculators

Problem 1.

In a small town of 100 persons, there are $P(t)$ persons having the flu after t days. Assume that the rate of increase of $P(t)$ satisfies the differential equation:

$$\frac{dP}{dt} = \frac{kP(100 - P)}{100},$$

where k is some constant.

1. Should k be positive or negative? Explain your answer.
2. Find the solution if at time $t = 0$ only one person has the flu.
3. When t gets large, does $P(t)$ approach some fixed value? If yes, what is this value?

Problem 2. Find the solutions of the initial value problems:

- $$\frac{dv}{dx} - \frac{1}{x}v = 2x, \quad v(1) = 0$$

- $$(1+x)\frac{dy}{dx} = 4y, \quad y(0) = 1.$$

- $$(xy')^2 + y^2 = 1, \quad y(1) = 0 \text{ Hint: take the square root first}$$

- $$\frac{dv}{dx} - \frac{1}{x}v = xv^6, \quad v(1) = 1$$

Problem 3.

1. Check that the following differential equation is exact and find its general solution:

$$(4x - y)dx + (6y - x)dy = 0.$$

   Will be extensively discussed on the review

2. Show that the following equation is not exact:

$$(xy + y^2)dx + x^2 dy = 0. \tag{1}$$

3. Find the general solution of equation (1).

Problem 4. Suppose that the air resistance of a drop of water (of mass m) in free fall is proportional to the cube of its velocity, so that its equation of motion is

$$m\frac{dv}{dt} = -kv^3 - mg$$

where h is the height of the drop of water, k is some positive constant and $g = 9.8m/s^2$ is the gravitational acceleration.

What is the terminal velocity of the drop of water (the velocity it ultimately reaches)?

Problem 5.

Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ -0 & \text{if } (x,y) = (0,0) \end{cases}$$

and the associated differential equation

$$\frac{dy}{dt} = \frac{xy}{x^2+y^2}$$

1. Find the points where $f(x,y)$ is discontinuous (remember to justify your answer). Compute $\frac{\partial f}{\partial y}$, and the points where $\frac{\partial f}{\partial y}$ is discontinuous.
2. For what values of a and b the Initial Value Problem (IVP) $y(a) = b$ is guaranteed to have a unique solution? For what values a, b there is no guarantee that the IVP $y(a) = b$ has a solution?
3. Plot the slope field for this differential equation.
4. Considering your answer for part (2) and the slope field you drew for (3), do you think any solutions with initial condition $y(0) = 0$ exist, and if so, do you think there is a unique solution?
5. Find the differential equation's general solution (there's an easy way and a hard way!).

Problem 6.

Evaluate $dF(x,y)$ when $F(x,y)$ is the given 2-variable function. Then solve the stated differential equation. If you know how to do these, there is no work whatsoever

1. $F(x,y) = x^2 + xy + y^3$
2. $F(x,y) = x^2 \cos(y)$