Solution to practice ~~midterm~~ final:

prob. 6 i) $y'' + y = \sin x$

char. eq.: $r^2 + 1 = 0$

$\longrightarrow \Rightarrow r = \pm i$

$\sin x$ corresponds to $r = \pm i$, which is a ~~characteristic~~ pair of complex conjugate ~~appearing~~ characteristic roots, (with multiplicity 1). So ~~we~~ try:

$y_p = x(A\cos x + B\sin x)$

$y_p' = A\cos x + B\sin x + x(-A\sin x + B\cos x)$ (prod. rule)

$\Rightarrow y_p'' = -A\sin x + B\cos x - A\sin x + B\cos x + x(-A\cos x - B\sin x)$

$\Rightarrow y_p'' + y_p = -2A\sin x + 2B\cos x \overset{\text{want}}{=} \sin x$

$\Rightarrow \boxed{A = -\frac{1}{2}}, \boxed{B = 0}$

$\Rightarrow y_p = -\frac{1}{2}x\cos x$

$\Rightarrow$ gen. sol: $y = y_c + y_p$

$$\boxed{y = C\cos(x) + D\sin(x) - \frac{1}{2}x\cos x}$$

Solution to prac. final :

prob. 6 (continued)

ii) $y'' - y' - 2y = 2xe^x + x^2$

char. eq.: $r^2 - r - 2 = 0$

$$r = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1}{2} \pm \frac{3}{2}$$

$r = 2$ or $r = -1$

We try $y_p = Axe^x + Be^x + Cx^2 + Dx + E$ $\quad (\times -2)$

$y_p' = Ae^x + Axe^x + Be^x + 2Cx + D$ $\quad (\times -1)$

$y_p'' = Ae^x + Ae^x + Axe^x + Be^x + 2C$

$= (2A + B)e^x + Axe^x + 2C$

$y_p'' - y_p' - 2y_p = -2Axe^x + 2(A-B)e^x - 2Cx^2 - 2(C+D)x$
$+ 2C - D - 2E$

$\overset{\text{want}}{=} 2xe^x + x^2$

$\Rightarrow -2A = 2 \Rightarrow \boxed{A = -1}$

$A - B = 0 \Rightarrow \boxed{B = -1}$

$-2C = 1 \Rightarrow \boxed{C = -\frac{1}{2}}$

$C + D = 0 \Rightarrow \boxed{D = \frac{1}{2}}$

$2C - D - 2E = 0$

$\Rightarrow -1 - \frac{1}{2} = 2E$

$\boxed{E = -\frac{3}{4}}$

<u>Solution to practice final:</u>

<u>prob. 6 ii) (continued):</u>

$\Rightarrow y_p = -xe^x - e^x - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}$

$\Rightarrow y = y_c + y_p$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-x} + y_p}$$

iii) $y'' - 5y' + 4y = e^{2x}\cos x + e^{2x}\sin x$

char. eq.: $r^2 - 5r + 4 = 0$

$r = 4 \quad , r = 1$

we try $y_p = Ae^{2x}\cos x + Be^{2x}\sin x \qquad (\times 4)$

$y_p' = 2Ae^{2x}\cos x + 2Be^{2x}\sin x - Ae^{2x}\sin x + Be^{2x}\cos x$

$= (2A+B)e^{2x}\cos x + (2B-A)e^{2x}\sin x \qquad (\times -5)$

$y_p'' = (4A+2B)e^{2x}\cos x + (4B-2A)e^{2x}\sin x$

$- (2A+B)e^{2x}\sin x + (2B-A)e^{2x}\cos x$

$= (3A+4B)e^{2x}\cos x + (3B-4A)e^{2x}\sin x \qquad (\times 1)$

$y_p'' - 5y_p' + 4y_p = (\cancel{4A} - \cancel{10A} - \cancel{5B} + \cancel{3A} + \cancel{4B})e^{2x}\cos x$

$+ (\cancel{4B} - \cancel{10B} + \cancel{5A} + \cancel{3B} - \cancel{4A})e^{2x}\sin x$

$= (-3A - B)e^{2x}\cos x + (A - 3B)e^{2x}\sin x$

$\overset{\text{want}}{=} e^{2x}\cos x + e^{2x}\sin x$

Solution to practice final:

prob. 6 iii) (continued)

So we get the system

$$\begin{pmatrix} -3 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -3 & 1 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ -\frac{2}{5} \end{pmatrix}$$

$$\Rightarrow \boxed{y_p = -\frac{1}{5} e^{2x}\cos x - \frac{2}{5} e^{2x}\sin x}$$

$$\Rightarrow y = y_c + y_p$$

$$\boxed{y = c_1 e^{4x} + c_2 e^{x} + y_p}$$