Problem 1.

- If the solution curves to a differential equation are all concave up, Euler's method gives an under approximation.

Justification: because the solution curve is concave up \( f(x) = \frac{dy}{dx} \) is increasing.

The approximation is below the real solution curve.

- Use improved Euler method:

  Step-size = 1, \( y' = xy \), \( y(1) = 2 \).

  So \( x_0 = 1, y_0 = 2 \); \( k_{1,0} = 1 \cdot 2 = 2 \), \( u_1 = y_0 + h \cdot k_{1,0} = 2 + 1 \cdot 2 = 4 \)

  \( k_{2,0} = 2 \cdot 4 = 8 \), \( y_2 = y_0 + 1 \cdot \frac{1}{2} (k_{1,0} + k_{2,0}) = 2 + \frac{5}{2} = 4.5 \)

  We get the approximation \( \approx 4.5 \)
\[
\frac{dx}{dy} = x \cdot y
\]

\(x_1 = 1, \quad y_1 = 2\)

\(x_2 = 2\)

\(k_1 = 1 \cdot 2 = 2\)

\(u_2 = 2 + 1 \cdot 2 = 4\)

\(k_2 = 2 \cdot 4 = 8\)

\(y_2 = 2 + 1 \cdot \frac{2+8}{2} = 7\)
Problem 2.

\[
\frac{dy}{dx} = x + y, \quad y(0) = 1, \quad \text{step of size } \ h = 0.1.
\]

\[x_0 = 0, \quad y_0 = 1\]

\[x_1 = 0.1, \quad y_1 = y_0 + h \cdot (x_0 + y_0) = 1 + 0.1 \cdot 1 = 1.1\]

\[x_2 = 0.2, \quad y_2 = y_1 + h \cdot (x_1 + y_1) = 1.1 + 0.1 \cdot (0.1 + 1.1) = 1.22\]
Problem 3.

\[ y'' + cy' + 10y = 0 \]

(see textbook page P176: \( x'' + zpx' + w_0^2 x = 0 \))

In our case: \( p = \frac{1}{2} c, \ w_0 = \sqrt{10} \), consider the signs of \( p^2 - w_0^2 = \frac{1}{4} c^2 - 10 \).

- Overdamped: \( \frac{1}{4} c^2 - 10 > 0 \).
- Critically damped: \( \frac{1}{4} c^2 - 10 = 0 \).
- Underdamped: \( \frac{1}{4} c^2 - 10 < 0 \).

- The solution in the critically case: \( y(t) = (c_1 + c_2 t) e^{-\frac{1}{2}ct} \)

For more accurate graph, see textbook P178.

\[ y \]
\[ 0 \]
\[ t \]

- Choose \( c \) such that underdamped, choose \( c = 0 \) \( \frac{1}{4} c^2 - 10 = -10 < 0 \).

Two conjugate roots: \( \pm i \sqrt{10} \)

General solution: \( y(t) = A \cos (\sqrt{10} + B \sin (\sqrt{10}) \), \( A \) and \( B \) are constants.

\[
\cos (2x) + \sqrt{3} \sin (2x) = 2 \left( \frac{1}{2} \cos (2x) + \frac{\sqrt{3}}{2} \sin (2x) \right) = 2 \left( \cos \left( \frac{2\pi}{3} \right) \cos (2x) - \sin \left( \frac{2\pi}{3} \right) \sin (2x) \right) = 2 \cos \left( 2x - \frac{2\pi}{3} \right).
\]

- General solution in the undamped case:

\[
y(t) = e^{-\frac{1}{2}ct} \left( c_1 \cos \sqrt{10 - \frac{1}{4} c^2} t + c_2 \sin \sqrt{10 - \frac{1}{4} c^2} t \right)
= A \cdot e^{rt} \left( \cos (wt - b) \right)
\]

Here \( A = \sqrt{c_1^2 + c_2^2} \), \( r = -\frac{1}{2} c \), \( w = \sqrt{10 - \frac{1}{4} c^2} \), \( c_1 \), \( c_2 \), \( A \), \( b \)