Problem 1 Find a unit vectors that perpendicular to the graph of functions

1. $\sin(x^2)$ at point with x-coordinate $\sqrt{\frac{\pi}{3}}$
2. $\ln(1+\cos(x))$ at point with x-coordinate $\frac{\pi}{6}$

Present two solutions. One should use the fact that a vector orthogonal to vector (a, b) has coordinates $(-b, a)$. The other solution should use gradients.

a) A tangent vector to the graph $(x, f(x))$ at x_0 is given by $(1, f'(x_0))$. Thus we find tangent vectors

1) $(1, 2x\cos(x^2))\big|_{x=\sqrt{\frac{\pi}{3}}} = \left(1, 2\sqrt{\frac{\pi}{3}}\cos\left(\frac{\pi}{3}\right)\right) = \left(1, \sqrt{\frac{\pi}{3}}\right)$

2) $\left(1, \frac{\sin(x)}{1+\cos(x)}\right)\Big|_{x=\frac{\pi}{6}} = \left(1, \frac{1}{2+\sqrt{3}}\right)$

Making them unitary (divide by the norm $\sqrt{1+(f'(x_0))^2}$)

1) $\frac{1}{\sqrt{1+\frac{\pi}{3}}}\left(1, \sqrt{\frac{\pi}{3}}\right)$ 2) $\frac{1}{\sqrt{1+\frac{1}{(2+\sqrt{3})^2}}}\left(1, \frac{1}{2+\sqrt{3}}\right)$

Finally, interchanging coordinates, we find the normal vectors

1) $\frac{1}{\sqrt{1+\frac{\pi}{3}}}\left(-\sqrt{\frac{\pi}{3}}, 1\right)$ 2) $\frac{1}{\sqrt{1+\frac{1}{(2+\sqrt{3})^2}}}\left(-\frac{1}{2+\sqrt{3}}, 1\right)$

b) We can also use the fact that the graph $(x, f(x))$ is the zero level set of the function $g(x,y) = y - f(x)$. The gradient is then a perpendicular vector to the graph.

1) $g(x,y) = y - \sin(x^2) \Rightarrow \nabla g = (-2x\cos(x^2), 1) = \left(-\sqrt{\frac{\pi}{3}}, 1\right)$

2) $g(x,y) = y - \ln(1+\cos(x)) \Rightarrow \nabla g = \left(-\frac{\sin(x)}{1+\cos(x)}, 1\right) = \left(-\frac{1}{2+\sqrt{3}}, 1\right)$

Finally, making them unitary:

1) $\frac{1}{\sqrt{1+\frac{\pi}{3}}}\left(-\sqrt{\frac{\pi}{3}}, 1\right)$ 2) $\frac{1}{\sqrt{1+\frac{1}{(2+\sqrt{3})^2}}}\left(-\frac{1}{2+\sqrt{3}}, 1\right)$

Problem 2 Give an example of three distinct points collinear to $P = (1, 2, 3)$ and $Q = (3, 2, 1)$.

The equation of the line passing through the points P, Q is given by

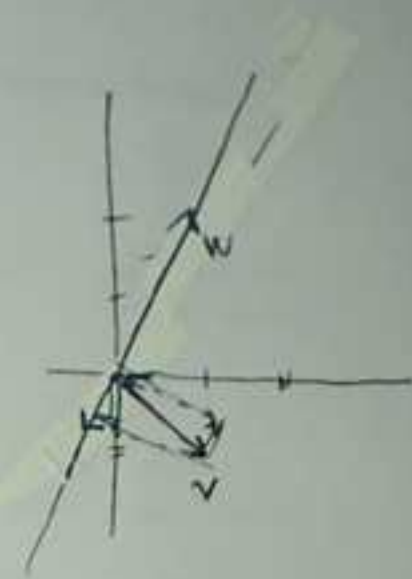
$$\vec{r}(t) = \vec{P} + t(\vec{Q} - \vec{P})$$
$$= (1,2,3) + t(2,0,-2)$$

Now, considering random values of t (for instance, $t = 2,3,4$) we get the points

$$\vec{P_1} = (5, 2, -1)$$
$$\vec{P_2} = (7, 2, -3)$$
$$\vec{P_3} = (9, 2, -5)$$

which by construction are collinear to $\vec{P}, \vec{Q}$.

Problem 3 Find orthogonal projection of vector $v = (1, -1)$ on a line l that contains vector $(1, 2)$. Also find the normal to l component of v.



If we call $\vec{w} = (1,2)$, recall that

$$\text{Proj}_{\vec{w}} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{\|w\|^2} \vec{w}$$

$$= \frac{1-2}{(1+4)} (1,2) = \boxed{\left(-\frac{1}{5}, -\frac{2}{5}\right)}$$

if we call $\vec{N}$ the normal component of $\vec{V}$, we have that

$$\vec{V} = \text{Proj}_{\vec{w}} \vec{V} + \vec{N}$$

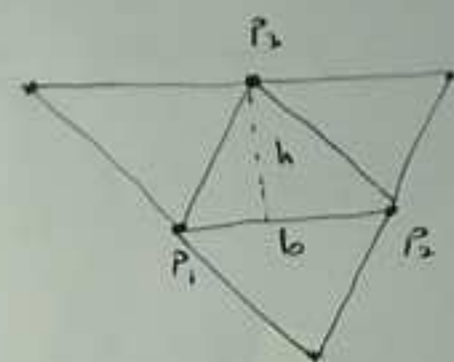
Thus

$$\vec{N} = \vec{V} - \text{Proj}_{\vec{w}} \vec{V} = (1,-1) - \left(-\frac{1}{5}, -\frac{2}{5}\right) = \boxed{\left(\frac{6}{5}, -\frac{3}{5}\right)}$$

Problem 4 Find the area of a parallelogram $ABCD$. The point A, B, C have coordinates $(1,1,1), (1,-2,3), (2,-1,-1)$.

First of all, notice that 3 points do not determine a parallelogram.

Provided that the points are not collinear, we have 3 possible choices (see figure)



In any case, the 3 choices have the same area which is just twice the area of the triangle determined by P_1, P_2, P_3. Now, we can consider this triangle to have the segment $\overline{P_1P_2}$ as a base, with lenght $\|P_2 - P_1\|$, and height h of lenght equal to $(P_3 - P_1) - \mathrm{Proj}_{P_2-P_1}(P_3 - P_1)$. Thus

$$b = \|(0,-3,2)\| = \sqrt{0^2+(-3)^2+2^2} = \sqrt{13}$$

$$\mathrm{Proj}_{P_2-P_1}(P_3-P_1) = \frac{(0,-3,2)\cdot(1,-2,-2)}{(0^2+(-3)^2+(2)^2)}(0,-3,2)$$

$$= \frac{2}{13}(0,-3,2)$$

Then

$$h = \|(1,-2,-2) - \tfrac{2}{13}(0,-3,2)\| = \|(1, -\tfrac{20}{13}, -\tfrac{30}{13})\| = \sqrt{\frac{113}{13}}$$

Then the area of the parallelogram is

$$A = b\cdot h = \sqrt{13}\cdot\sqrt{\frac{113}{13}} = 5\boxed{\sqrt{113}}$$

Problem 5 Determine the $\cos(\theta)$, where θ is the angle enclosed by two intersecting surfaces $x^2 - y^2 + z^2 = 1, x^2 + y^2 + z^2 = 3$ at a point $(1, 1, 1)$.

We do this by determining the cosine of the angle between 2 perpendicular vectors to the surfaces. These are given by the gradients:

$$\nabla(x^2-y^2+z^2) = (2x, -2y, 2z)\Big|_{(1,1,1)} = (2,-2,2) = \vec{n}_1$$

$$\nabla(x^2+y^2+z^2) = (2x, 2y, 2z)\Big|_{(1,1,1)} = (2,2,2) = \vec{n}_2$$

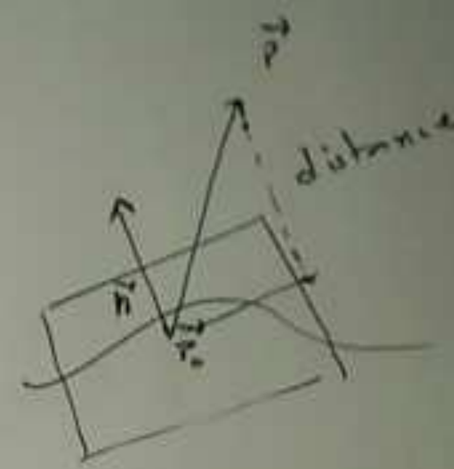
Now,

$$\cos\Theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|n_1\| \cdot \|n_2\|} = \frac{4-4+4}{\sqrt{4+4+4}\sqrt{4+4+4}} = \frac{4}{12} = \frac{1}{3}$$

Problem 6 Determine the distance between point $(1, 0, 1)$ and a tangent plane to the surface

$$xyz = 1$$

at a point $(1, 1, 1)$.



Given a surface, the tangent plane at a point is given in terms of the gradient of the defining function as

$$\frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial z}(z - z_0) = 0$$

as shown in the figure, the distance is determined as the lenght of the projection of $\vec{p} - \vec{P}_0$ to $\vec{n}$.

In this case $\vec{n} = \nabla f|_{(1,1,1)} = (yz, xz, xy)|_{(1,1,1)}$

$= (1, 1, 1)$ and

$\vec{p} - \vec{P}_0 = (1, 0, 1) - (1, 1, 1) = (0, -1, 0)$

so

$$d = \|\mathrm{Proj}_{\vec{n}}(0, -1, 0)\| = \left\| \frac{-1}{3}(1, 1, 1) \right\| = \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9}}$$

$$= \boxed{\frac{1}{\sqrt{3}}}$$

Problem 7 Determine the distance between point (1, 0, 1) and a tangent line to a curve given by parametric equation

$$x(t) = \sin(t)$$

$$y(t) = \cos(2t)$$

$$z(t) = \sin(3t)$$

at a point $t = \pi$.

The tangent line of a curve at a point $\vec{P_0}$ is given by the equation

$$\vec{r}(t) = \vec{P_0} + t\vec{T}$$

where $\vec{T}$ is a tangent vector at $\vec{P_0}$. In this case

$$\vec{P_0} = (\sin(\pi), \cos(2\pi), \sin(3\pi)) = (0, 1, 0)$$

$$\vec{T} = (\cos(\pi), -2\sin(2\pi), 3\cos(3\pi)) = (-1, 0, -3)$$

Now, the distance between a point Q and a line is given by $\dfrac{\|(\vec{Q} - \vec{P_0}) \times \vec{T}\|}{\|\vec{T}\|} = d$

$$(\vec{Q} - \vec{P_0}) \times \vec{T} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ -1 & 0 & -3 \end{vmatrix} = (3, 2, -1)$$

Thus

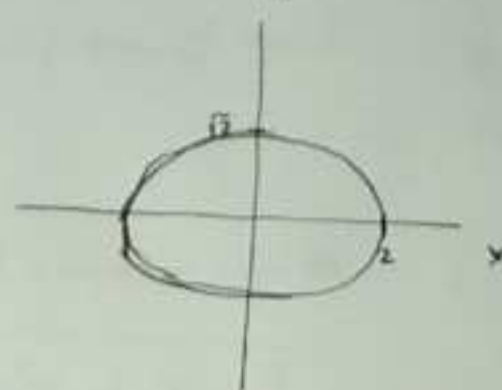
$$d = \frac{\sqrt{9+4+1}}{\sqrt{1+0+9}} = \frac{\sqrt{14}}{\sqrt{10}} = \boxed{\sqrt{\frac{7}{5}}}$$

Problem 8 Determine the type of the quadratic surface and draw the traces at $z = 1$, $y = 0$.

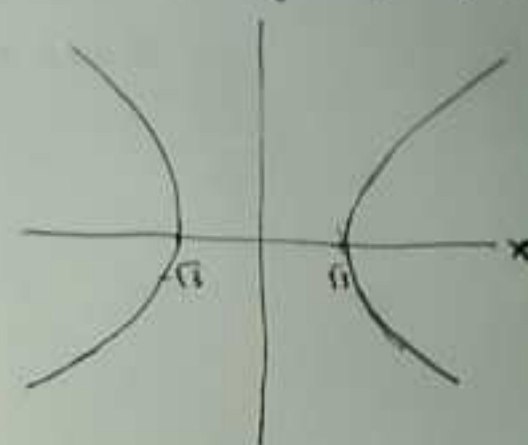
1. $x^2 + 2y^2 - z^2 = 3$
2. $x^2 - 2y^2 - z + 2x = 3$
3. $x^2 - 2y^2 + z^2 = -3$

1.- Hyperboloid of one sheet:

$\boxed{z = 1}$: $\frac{x^2}{4} + \frac{y^2}{2} = 1$



$\boxed{y = 0}$ $\frac{x^2}{3} - \frac{z^2}{3} = 1$



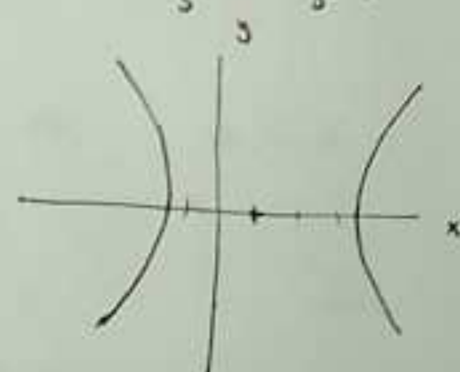
2.- equation equivalent to

$(x+1)^2 - 2y^2 = z + 4$

Hyperbolic paraboloid

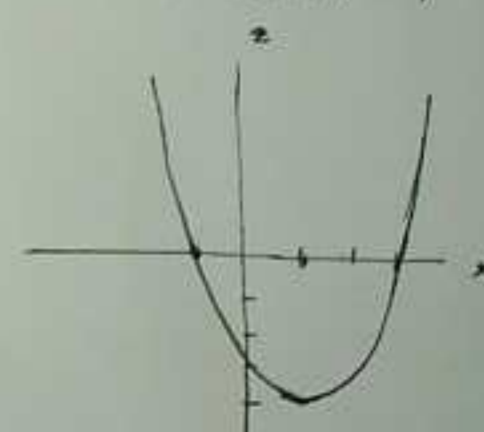
$\boxed{z = 1}$

$\frac{(x+1)^2}{5} - \frac{2}{5} y^2 = 1$



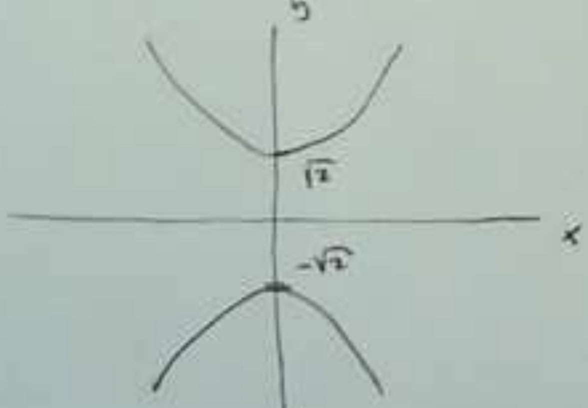
$\boxed{y = 0}$

$z = (x+1)^2 - 4$

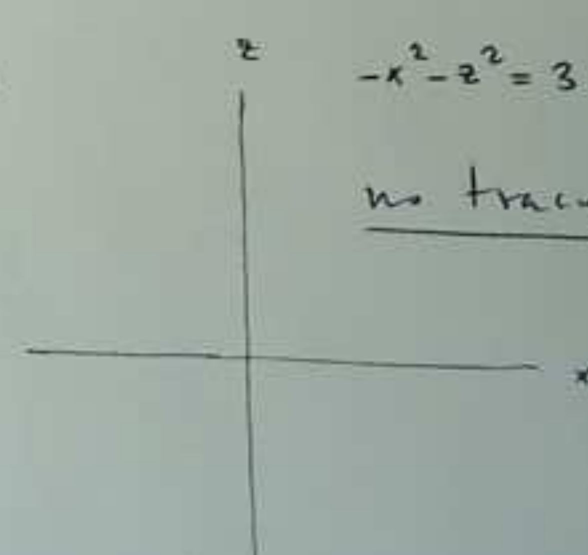


3.- Hyperboloid of two sheets

$\boxed{z = 1}$ $\frac{y^2}{2} - \frac{x^2}{4} = 1$



$\boxed{y = 0}$



$-x^2 - z^2 = 3$

no trace

Problem 9 Write equation of the surface $xyz = 1$ in

1. spherical

2. cylindrical

coordinates

1.- Spherical coord:

$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$

thus

$\rho \sin\phi \cos\theta \cdot \rho \sin\phi \sin\theta \, \rho \cos\theta = 1$ or

$$\boxed{\rho^3 \sin^2\phi \sin\theta \cos^2\theta = 1}$$

2.- Cylindrical coord:

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

thus

$r\cos\theta \, r\sin\theta \, z = 1$ or

$$\boxed{z r^2 \sin\theta \cos\theta = 1}$$

Problem 10 Find

1. the init tangent vector
2. the principal unit normal vector

for the function $r(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$ at $t = 1$

Recall : $T(t) = \dfrac{r'(t)}{\|r'(t)\|}$ $\qquad N(t) = \dfrac{T'(t)}{\|T'(t)\|}$

1) $r'(t) = \hat{i} + 2t\hat{j} + t\hat{k}$; at $t=1$ $\quad r'(1) = \hat{i} + 2\hat{j} + \hat{k}$

$$T(1) = \frac{1}{\sqrt{6}}\left(\hat{i} + 2\hat{j} + \hat{k}\right) = \boxed{\frac{1}{\sqrt{6}}(1, 2, 1)}$$

2) $$T'(t) = \left(\frac{1}{\sqrt{1+5t^2}}(1, 2t, t)\right)' =$$

$$= \left(-\frac{5t}{(1+5t^2)^{3/2}},\ \frac{2}{\sqrt{1+5t^2}} - \frac{10t^2}{(1+5t^2)^{3/2}},\ \frac{1}{\sqrt{1+5t^2}} - \frac{5t^2}{(1+5t^2)^{3/2}}\right)$$

at $t = 1$

$$T'(1) = \left(-\frac{5}{(6)^{3/2}},\ \frac{2}{\sqrt{6}} - \frac{10}{(6)^{3/2}},\ \frac{1}{\sqrt{6}} - \frac{5}{(6)^{3/2}}\right)$$

$$= \frac{1}{\sqrt{6}}\left(-\frac{5}{6},\ 2 - \frac{10}{6},\ 1 - \frac{5}{6}\right) = \frac{1}{\sqrt{6}}\left(-\frac{5}{6},\ \frac{1}{3},\ \frac{1}{6}\right)$$

then

$$N(t) = \frac{6}{\sqrt{5}} \cdot \frac{1}{\sqrt{6}}\left(-\frac{5}{6},\ \frac{1}{3},\ \frac{1}{6}\right) = \boxed{\sqrt{\frac{6}{5}}\left(-\frac{5}{6},\ \frac{1}{3},\ \frac{1}{6}\right)}$$

Problem 11 Find magnitudes of

1. tangential a_T
2. notmal a_N

components of acceleration of the function $r(t) = e^t\mathbf{i} + 2t\mathbf{j} + e^{-t}\mathbf{k}$ at $t = 1$

We have $v(t) = e^t\hat{i} + 2\hat{j} - e^{-t}\hat{k}$ $\quad a(t) = e^t\hat{i} + e^{-t}\hat{k}$

1) $a_T = \dfrac{\vec{v}\cdot\vec{a}}{\|v\|}$, $\quad v(1) = (e, 2, -\frac{1}{e})$ $\quad a(1) = (e, 0, \frac{1}{e})$

thus $\boxed{a_T = \dfrac{e^2 - \frac{1}{e^2}}{\sqrt{e^2 + \frac{1}{e^2} + 4}}}$

2) $a_N = \sqrt{\|a\|^2 - a_T^2} = \sqrt{e^2 + \frac{1}{e^2} - \left(\dfrac{e^2 - \frac{1}{e^2}}{\sqrt{e^2 + \frac{1}{e^2} + 4}}\right)^2}$

$= 2\sqrt{\dfrac{e^2 + \frac{1}{e^2} + 1}{e^2 + \frac{1}{e^2} + 4}}$

Problem 12 Find the curvature of the curve $r(t) = e^t\mathbf{i} + 2t\mathbf{j} + e^{-t}\mathbf{k}$ as a function of t

Recall that $\kappa(t) = \|T'(t)\|$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{e^{2t}+e^{-2t}+4}}\left(e^t\hat{\imath} + 2\hat{\jmath} - e^{-t}\hat{k}\right)$$

$$T'(t) =$$

$$\left(\frac{e^t}{\sqrt{e^{2t}+e^{-2t}+4}} - \frac{(e^{3t}-e^{-t})}{(e^{2t}+e^{-2t}+4)^{3/2}},\ \frac{-2\,(e^{2t}-e^{-2t})}{(e^{2t}+e^{-2t}+4)^{3/2}},\ \frac{e^{-t}}{\sqrt{e^{2t}+e^{-2t}+4}} + \frac{(e^{t}-e^{-3t})}{(e^{2t}+e^{-2t}+4)^{3/2}}\right)$$

Therefore:

$$\|T'(t)\| = \kappa(t) =$$

$$\frac{1}{(e^{2t}+e^{-2t}+4)^{3/2}}\left\|\left(e^t(e^{2t}+e^{-2t}+4)-(e^{3t}-e^{-t}),\ -2\,(e^{2t}-e^{-2t}),\ e^{-t}(e^{2t}+e^{-2t}+4)+(e^{t}-e^{-3t})\right)\right\|$$

$$= \boxed{\frac{2}{(e^{2t}+e^{-2t}+4)^{3/2}}\sqrt{(e^t+e^{-t})^4+(e^t+e^{-t})}}$$

Problem 13 Identify limits that exist and evaluate them

1.

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$$

2.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

3.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$$

Notice that if the limit exists, it has to be independent of the direction we approach $(0,0)$.

1.- The limit doesn't exist. If we make $y=0$, $x\to 0$

$$\lim_{(x,0)\to(0,0)} \frac{x}{x} = 1$$

and if we make $x=0$, $y\to 0$

$$\lim_{(0,y)\to(0,0)} \frac{y}{-y} = -1 .$$

2.- The limit doesn't exists Changing to polar coordinates the function becomes $\frac{r\cos\theta \, r\sin\theta}{r^2} = \cos\theta\sin\theta$

and we want to find, for any θ,

$$\lim_{r\to 0} \cos\theta\sin\theta$$

which is not unique.

3.- Changing to polar coordinates, we have the function

$\frac{\rho^4 \sin^2\theta\cos^2\theta}{\rho^2} = \rho^2\sin^2\theta\cos^2\theta$. Since $\sin\theta, \cos\theta$ are bounded functions, we have

$$\boxed{\lim_{r\to 0} \rho^2\sin^2\theta\cos^2\theta = 0}$$

Problem 14 Identify all removable singularities of the functions

1.

$$f(x,y)=\begin{cases}\frac{x+y}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0)\\ 1 & (x,y)=(0,0)\end{cases}$$

2.

$$f(x,y)=\begin{cases}\frac{x^2-y^2}{x+y} & x+y\neq 0\\ x^2 & x=-y\end{cases}$$

1.- Notice that in polar coordinates,

$f(x,y) = \sin\Theta + \cos\Theta$, . Since $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist,

$f(x,y)$ is not continuous at $(0,0)$. Otherwise $f(x,y)$ is continuous. So $(0,0)$ is not a removable singularity.

2) If $x\neq -y$ $\frac{x^2-y^2}{x+y}$ is continuous. Since

$\frac{x^2-y^2}{x+y} = x-y$ if $x\neq -y$, we have that for any point of the form $(x_0,-x_0)$, $\lim_{(x,y)\to(x_0,-x_0)} f(x,y) = 2x_0$

and then the function has $(0,0)$ and $(2,-2)$ as the only removable singularities (the points $(x,-x)$ where $x^2 = 2x$).

Problem 15 Find the gradients of functions

1.

$$f(x,y)=\sin(\ln(x+y)\cos(xy))$$

2.

$$\cancel{g(x,y)} = \frac{\sqrt{x+y+z}}{1+x^2+y^2+z^2}$$

$g(x,y,z) =$

1.- $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) =$

$$= \left(\cos\left(\ln(x+y)\cdot\cos(xy)\right)\left(\frac{\cos(xy)}{x+y} - y\ln(x+y)\sin(xy)\right), \right.$$

$$\left. \cos\left(\ln(x+y)\cdot\cos(xy)\right)\left(\frac{\cos(xy)}{x+y} - x\ln(x+y)\sin(xy)\right) \right)$$

2.-

$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) =$

$$= \frac{\sqrt{x+y+z}}{(1+x^2+y^2+z^2)} \left(\frac{1+x^2+y^2+z^2}{2\cdot(x+y+z)} - 2x, \frac{1+x^2+y^2+z^2}{2(x+y+z)} - 2y, \frac{1+x^2+y^2+z^2}{2(x+y+z)} - 2z \right)$$

Problem 16 A function $z = g(x, y)$ satisfies equation $F(x, y, g(x, y)) = 0$, where

$$F(x, y, z) = x^2 + zy + y^2 + zx^2 + z^3$$

find partial derivatives g_x, g_y as functions of x, y, z.

Since $F(x, y, g(x, y)) = 0$, using the chain rule we find

$$0 = \frac{\partial F}{\partial x} = 2x + y\frac{\partial g}{\partial x} + 2xg + x^2\frac{\partial g}{\partial x} + 3g^2\frac{\partial g}{\partial x}$$

then $$2x(1+g) + (y + x^2 + 3g^2)\frac{\partial g}{\partial x} = 0$$

$$\therefore \boxed{\frac{\partial g}{\partial x} = -\frac{2x(1+z)}{y + x^2 + 3z^2}}$$

Similarly

$$0 = \frac{\partial F}{\partial y} = g + y\frac{\partial g}{\partial y} + 2y + x^2\frac{\partial g}{\partial y} + 3g^2\frac{\partial g}{\partial y}$$

then $$g + 2y + (y + x^2 + 3g^2)\frac{\partial g}{\partial y} = 0$$

$$\therefore \boxed{\frac{\partial g}{\partial y} = -\frac{2y + z}{y + x^2 + 3z^2}}$$

Problem 17 1. Find formula for a normal vector to level curves of the function $f(x,y) = x^2 + 3x + y - y^3$.

2. Find critical (extreme) points of this function, determine their type.

3. Find directional derivative of f along the vector $(1, -2)$.

1) The normal vector to the level curves is given by the gradient $\boxed{\nabla f = (2x+3)\hat{\imath} + (1-3y^2)\hat{\jmath}.}$

2) The critical points are determined by the vanishing of the gradient:

$$\vec{0} = \nabla f = (2x+3)\hat{\imath} + (1-3y^2)\hat{\jmath} \iff \begin{array}{l} 2x+3=0 \\ 1-3y^2=0 \end{array}$$

$$\Rightarrow \quad x = -\frac{3}{2} \qquad y = \pm\sqrt{\frac{1}{3}}, \qquad \left(-\frac{3}{2}, \frac{1}{\sqrt{3}}\right), \left(-\frac{3}{2}, -\frac{1}{\sqrt{3}}\right)$$

Now, $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -12y$. Since the Hessian is positive $\iff$ critical pt is min, negative $\iff$ it is max, we conclude that $\boxed{\left(-\frac{3}{2}, \frac{1}{\sqrt{3}}\right) \text{ is a maximum,}}$ $\boxed{\left(-\frac{3}{2}, -\frac{1}{\sqrt{3}}\right) \text{ is a minimum.}}$

3) Making $(1,-2)$ unitary we get $\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$. The directional derivative is then

$$\frac{1}{\sqrt{5}}\frac{\partial f}{\partial x} - \frac{2}{\sqrt{5}}\frac{\partial f}{\partial y}$$

$$\boxed{= \frac{2x+3}{\sqrt{5}} - \frac{2(1-3y^2)}{\sqrt{5}}}$$

Problem 18 Find the absolute maximum of the function

$$f(x,y) = x^2 - 3xy + y^2$$

in the region $x^2 + y^2 \leq 1$

First, we can find the relative maximum in the region $\underline{x^2+y^2 < 1}$ using the partial derivatives criterion:

$$\frac{\partial f}{\partial x} = 2x - 3y = 0 \qquad \frac{\partial f}{\partial y} = 2y - 3x = 0 \qquad \Longleftrightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix}$$

Now, $x^2+y^2 \geq 0$ and $-3xy$ could be either positive or negative in any neighborhood of the origin so the point $(0,0)$ cannot be either a min or a max.

If we now restrict ourselves to the boundary $x^2+y^2=1$, changing to polar coordinates we get

$$f(x,y) = r^2 - 3r^2 \sin\theta \cos\theta = 1 - 3\sin\theta\cos\theta \qquad \theta \in [0, 2\pi)$$

and $\frac{df}{d\theta} = -3(\cos^2\theta - \sin^2\theta) = -3\cos(2\theta) = 0$

$$\Longleftrightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

$\frac{d^2 f}{d\theta^2} = 6\sin(2\theta) < 0 \quad \Longleftrightarrow \quad \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$. This corresponds to the points $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

In these points $\boxed{f(x,y) = \frac{5}{2}}$. This is thus the absolute maximum.

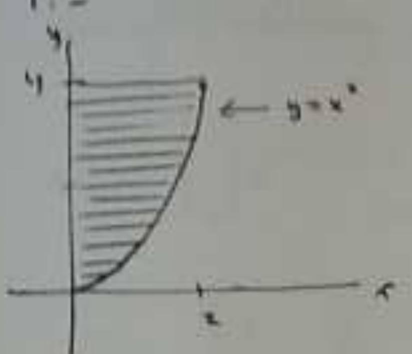
Problem 19 Sketch the region of integration R and switch the order of integration in the following integrals

1.
$$\int_0^4\int_0^{y^2} f(x,y)dxdy$$

2.
$$\int_1^4\int_{-\ln(x)}^{\ln(x)} f(x,y)dydx$$

3.
$$\int_2^3\int_{2-y}^{\frac{1}{y}} f(x,y)dxdy$$

1.-

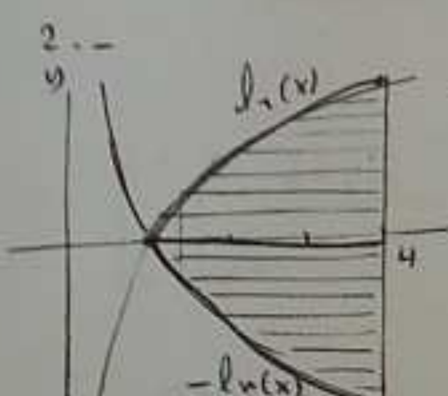


$0 \le y \le 4$

$0 \le x \le y^2$

then $\int_0^4\int_0^{y^2} f(x,y)\,dx\,dy = \int_0^2\int_{\sqrt{x}}^4 f(x,y)\,dy\,dx$

2.-



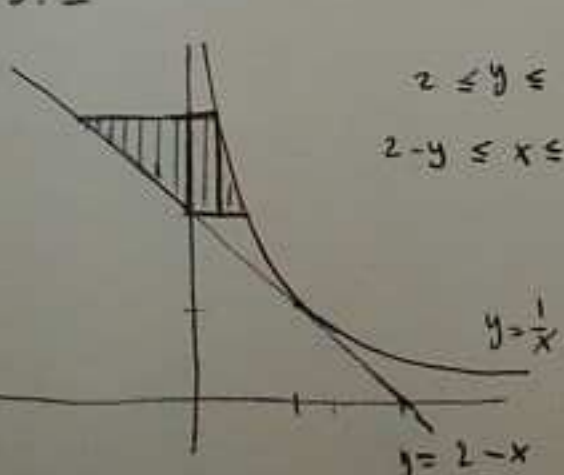
$-\ln(x) \le y \le \ln(x)$

$1 \le x \le 4$

then $\int_1^4\int_{-\ln(x)}^{\ln(x)} f(x,y)\,dy\,dx =$

$$\int_{-\ln(4)}^0\int_{e^x}^4 f(x,y)\,dx\,dy + \int_0^{\ln(4)}\int_{e^{-x}}^4 f(x,y)\,dx\,dy$$

3.-



$2 \le y \le 3$

$2-y \le x \le \frac{1}{y}$

then $\int_2^3\int_{2-y}^{\frac{1}{y}} f(x,y)\,dx\,dy$

$$= \int_{-1}^0\int_{2-x}^3 f(x,y)\,dy\,dx + \int_0^{1/3}\int_2^3 f(x,y)\,dy\,dx$$

$$+ \int_{\frac{1}{3}}^{1/2}\int_2^{1/x} f(x,y)\,dy\,dx$$

Problem 20 1. Evaluate

$$\iint_R e^{-x-y}dxdy$$

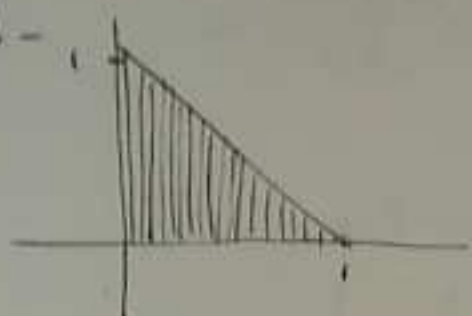
where R is the region in the first quadrant in which $x+y\leq 1$

2. Evaluate

$$\int_0^8 \int_{x^{\frac{1}{3}}}^2 \frac{dydx}{1+y^4}$$

(Hint:change the order of integration first.)

1.-



We can consider the region as

$0 \leq x \leq 1 \quad 0 \leq y \leq 1-x$. Then

$$\iint_R e^{-x-y}dxdy = \int_0^1\int_0^{1-x} e^{-x-y}\,dy\,dx$$

$$= -\int_0^1 \left[e^{-x-y}\right]_0^{1-x} dx = \int_0^1 \left(e^{-x} - \frac{1}{e}\right)dx = -e^{-x}\Big|_0^1 - \frac{1}{e}$$

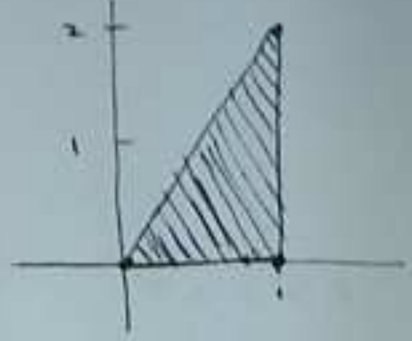
$$= \boxed{1 - \frac{2}{e}}$$

2.-

$$\int_0^8\int_{x^{1/3}}^2 \frac{dxdy}{1+y^4} = \int_0^2\int_0^{y^3} \frac{dx\,dy}{1+y^4} = \int_0^2 \left[\frac{x}{1+y^4}\right]_0^{y^3} dy$$

$$= \frac{1}{4}\int_0^2 \frac{4y^3}{1+y^4}\,dy = \frac{1}{4}\ln\left(1+y^4\right)\Big|_0^2 = \boxed{\frac{1}{4}\ln(17)}$$

Problem 21 Find the mass and center of mass of the triangle with the vertices $(0,0)$, $(1,0)$ and $(1,2)$ whose density is given by $\rho(x,y) = x^2$.



In cartesian coordinates, the triangle is given by the conditions

$0 \le x \le 1 \qquad 0 \le y \le 2x$

Then its mass is

$$m = \int_0^1 \int_0^{2x} x^2 \, dy \, dx = \int_0^1 x^2 y \Big|_0^{2x} dx = 2\int_0^1 x^3 \, dx = \frac{x^4}{2}\Big|_0^1 = \boxed{\frac{1}{2}}$$

The coordinates of the center of mass are given by

$$x_m = \frac{1}{m}\int_0^1 \int_0^{2x} x^3 \, dy \, dx = 2\int_0^1 2x^4 \, dx = \frac{4}{5} x^5 \Big|_0^1 = \frac{4}{5}$$

$$y_m = \frac{1}{m}\int_0^1 \int_0^{2x} y x^3 \, dy \, dx = 2\int_0^1 x^3 \frac{y^2}{2}\Big|_0^{2x} dx = \int_0^1 4x^5 \, dx$$

$$= \frac{4}{6} x^6 \Big|_0^1 = \frac{4}{6}$$

The center of mass is then $\boxed{\left(\frac{4}{5}, \frac{4}{6}\right)}$

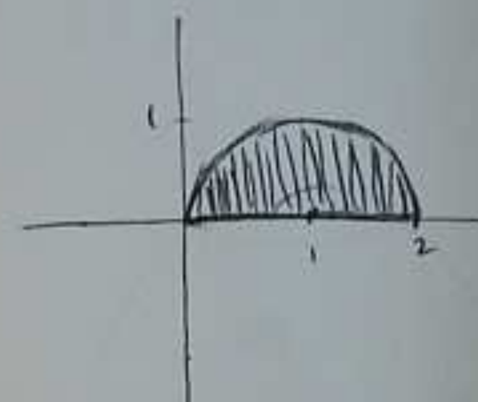
Problem 22 The integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$

is given in orthogonal coordinates. Change it to polar coordinates.

Notice that the function $y=\sqrt{2x-x^2}$ $\quad 0 \le x \le 2$
has a graph corresponding to the upper half of the circle with equation $(x-1)^2+y^2=1$ (see figure)

Then the region of integration corresponds to half (upper) of the disc of radius 1 centered at $(1,0)$. We can describe this region in polar coordinates by

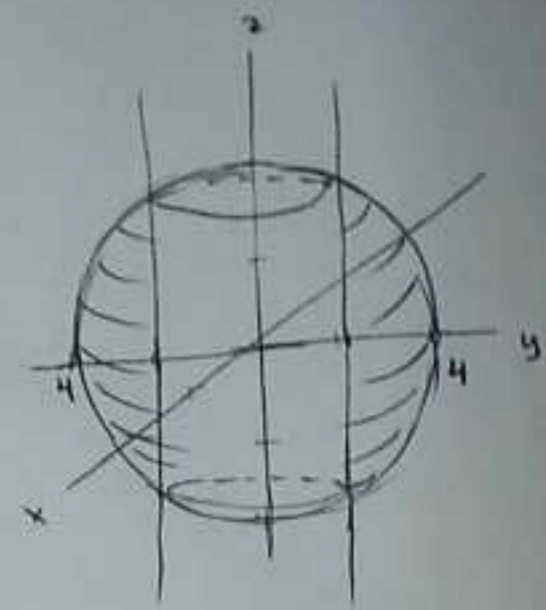


$$0 \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2\cos\theta$$

Then

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \boxed{\int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,dr\,d\theta}$$

Problem 23 Use polar coordinates to set up the integral for the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.



We can use the symmetry of the picture to calculate the volume as twice the volume determined by the northern hemisphere (with boundary determined by $z = \sqrt{16 - x^2 - y^2}$).

Now, notice that in cylindrical (not polar) coordinates, the region is determined by the conditions

$$0 \le \theta \le 2\pi$$
$$0 \le z \le 2\sqrt{3}$$
$$2 \le r \le \sqrt{16 - z^2}$$

The corresponding volume would then be given by

$$\boxed{V = 2 \int_0^{2\pi} \int_0^{2\sqrt{3}} \int_2^{\sqrt{16-z^2}} r \, dr \, dz \, d\theta}$$