Math 203 Homework #8

13.10 #39. Use Lagrange multipliers to do 13.9 #11

Maximize $V(a,b,c) = \frac{4\pi}{3}abc$ subject to $a+b+c = K > 0$

V is max if and only if $\tilde{V}(a,b,c) = abc$ is max, so can drop the $\frac{4\pi}{3}$ ...

$a, b, c > 0$

$\nabla V = \lambda \nabla g$ $\qquad g(a,b,c) = a+b+c$

$(bc, ac, ab) = (\lambda, \lambda, \lambda)$

$$\left.\begin{array}{l} bc = \lambda \\ ac = \lambda \\ ab = \lambda \\ a+b+c = K \end{array}\right\} \qquad \left.\begin{array}{l} abc = a\lambda \\ abc = b\lambda \\ abc = c\lambda \end{array}\right\}$$

So $a\lambda = b\lambda = c\lambda$

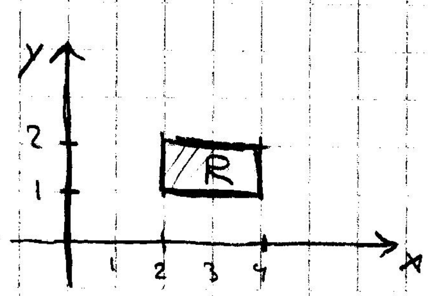
since $\lambda \neq 0$ $\quad a = b = c$.

So the ellipsoid of max volume, subject to $a+b+c = K$ is a sphere.

14.1 #56. $$\int_1^2 \int_2^4 dx\,dy = \int_1^2 2\,dy = 2$$

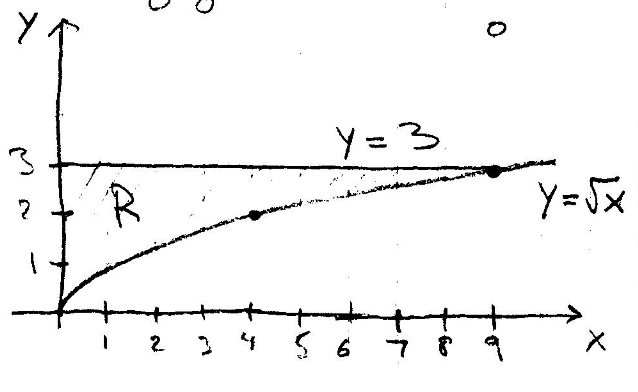
$$\int_2^4 \int_1^2 dy\,dx = \int_2^4 dx = 2$$

$= \checkmark$

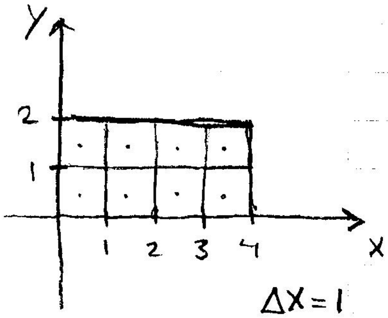


14.1 #62. $$\int_0^9 \int_{\sqrt{x}}^3 dy\,dx = \int_0^9 (3-\sqrt{x})\,dx = 3x - \frac{2}{3}x^{3/2}\Big|_0^9 = 27 - 18 = 9$$

$$\int_0^3 \int_0^{y^2} dx\,dy = \int_0^3 y^2\,dy = y^3/3\Big|_0^3 = 27/3 = 9 \quad = \checkmark$$



14.2 # 2



$$\frac{1}{2}\int_0^4\int_0^2 x^2 y\,dy\,dx = \frac{1}{4}\int_0^4 x^2y^2\Big|_{y=0}^{2} dx = \int_0^4 x^2\,dx$$

$$= \frac{1}{3}x^3\Big|_0^4 = \boxed{64/3} = \boxed{21.\overline{3}}$$

$$\sum_{i=1}^{4}\sum_{j=1}^{2} f(x_{i,j}, y_{i,j})\,\Delta A = \cdots$$

$$x_{i,j} = \tfrac{1}{2} + i - 1 = i - 1/2,\quad y_{i,j} = \tfrac{1}{2} + j - 1 = j - 1/2$$

$\Delta x = 1$ $\quad \Delta A = \Delta x \cdot \Delta y = 1\cdot 1 = 1$

$\Delta y = 1$

$$\cdots = \sum_{i=1}^{4}\sum_{j=1}^{2} \tfrac{1}{2}(i-1/2)^2(j-1/2) = \tfrac{1}{2}\Big(\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right) + \left(\tfrac{3}{2}\right)^2\left(\tfrac{1}{2}\right) + \left(\tfrac{5}{2}\right)^2\left(\tfrac{1}{2}\right) + \left(\tfrac{7}{2}\right)\left(\tfrac{1}{2}\right)$$

$$+ \left(\tfrac{1}{2}\right)^2\left(\tfrac{3}{2}\right) + \left(\tfrac{3}{2}\right)^2\left(\tfrac{3}{2}\right) + \left(\tfrac{5}{2}\right)^2\left(\tfrac{3}{2}\right) + \left(\tfrac{7}{2}\right)^2\left(\tfrac{3}{2}\right)\Big)$$

$\boxed{= 21}$

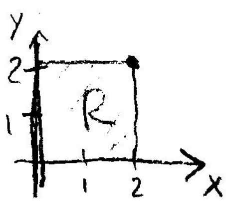
Homework #9

14.2#28. $V = \int_0^2\int_x^2 (4-y^2)\,dy\,dx = \int_0^2 \left(4y - y^3/3\right)\Big|_{y=x}^{2} dx$

$$= \int_0^2 \left(8 - 8/3 - \left(4x - \tfrac{x^3}{3}\right)\right)dx = \frac{32}{3} - 2x^2\Big|_0^2 + \frac{x^4}{12}\Big|_0^2$$

$$= 32/3 - 8 + \frac{16}{12} = \frac{32}{3} + \frac{4}{3} - 8 = 12 - 8 = \boxed{4}$$

#61.



$$\text{Average} = \frac{1}{4}\int_0^2\int_0^2 (x^2+y^2)\,dx\,dy$$

$$= \frac{1}{4}\int_0^2 \left(\frac{x^3}{3} + y^2x\right)\Big|_{x=0}^{2} dy = \frac{1}{4}\int_0^2 \left(8/3 + 2y^2\right)dy$$

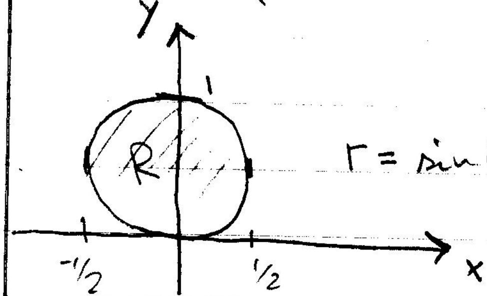
$$= \frac{1}{4}\left(\frac{16}{3} + \frac{16}{3}\right) = \frac{32}{4\cdot 3} = \boxed{8/3}$$

14.3 #10. $\int_0^{\pi} \int_0^{\sin\theta} r^2\,dr\,d\theta = \frac{1}{3} \int_0^{\pi} \sin^3\theta\,d\theta$

$= \frac{1}{3} \int_0^{\pi} (1-\cos^2\theta)\sin\theta\,d\theta \qquad = +\frac{1}{3} \int_{-1}^{1} (1-u^2)\,du$

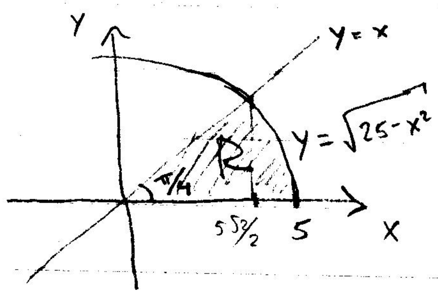
$u = \cos\theta \quad du = -\sin\theta\,d\theta$

$= \frac{1}{3}\left(2 - \frac{2}{3}\right) = \frac{2}{3}\cdot\frac{2}{3} = \boxed{4/3}$



14.3 #28. $\int_0^{5\sqrt{2}/2} \int_0^{x} xy\,dy\,dx + \int_{5\sqrt{2}/2}^{5} \int_0^{\sqrt{25-x^2}} xy\,dy\,dx = \iint_R xy\,dA$

=



$= \int_0^{\pi/4} \int_0^{5} \underbrace{r\cos\theta}_{x}\, \underbrace{r\sin\theta}_{y}\, \underbrace{r\,dr\,d\theta}_{dA}$

$= \int_0^{\pi/4} \int_0^{5} r^3 \cos\theta \sin\theta\,dr\,d\theta$

$\sin(2\theta) = 2\cos\theta\sin\theta$

$= \int_0^{\pi/4} \frac{1}{2}\sin(2\theta)\,d\theta \cdot \frac{625}{4} = \frac{625}{4}\cdot\left(-\frac{1}{4}\cos(2\theta)\right)\Big|_0^{\pi/4}$

$= \frac{625}{16}(1-0) = \boxed{\frac{625}{16}}$

14.3 #41. $V_{\text{no hole}} = \int_0^{2\pi} \int_0^{4} 25\, e^{-r^2/4}\, r\,dr\,d\theta = 100\pi\left(1 - 1/e^4\right)$

$9/10\, V_{\text{no hole}} = V_{\text{hole drilled out of radius } a} = \int_0^{2\pi} \int_a^{4} 25\, e^{-r^2/4}\, r\,dr\,d\theta$

$\parallel \qquad\qquad\qquad\qquad \parallel$

$90\pi\left(1 - 1/e^4\right) \qquad = 100\pi\left(e^{-a^2/4} - 1/e^4\right)$

$$9 - \frac{9}{e^4} = 10e^{-a^2/4} - \frac{10}{e^4}$$

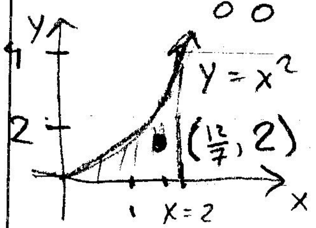
$$9 + \frac{1}{e^4} = 10e^{-a^2/4} \qquad e^{-a^2/4} = \frac{1}{10}\left(9 + \frac{1}{e^4}\right)$$

$$-\frac{a^2}{4} = -\ln 10 + \ln\left(9 + \frac{1}{e^4}\right)$$

$$\boxed{a = 2\sqrt{\ln(10) - \ln\left(9 + \frac{1}{e^4}\right)} \approx 0.64289\ldots}$$

14.4#12.

$$M = \int_0^2\int_0^{x^2} kxy\,dy\,dx = \int_0^2 kx\,\frac{y^2}{2}\Big|_{y=0}^{x^2} dx$$



$$= \frac{k}{2}\int_0^2 x^5\,dx = \frac{k}{2}\,\frac{2^6}{6} = \frac{2^4\cdot k}{3}$$

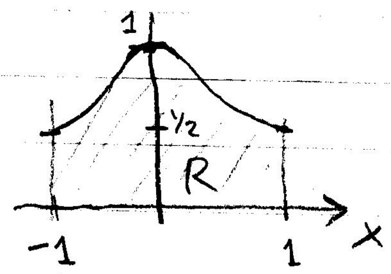
$$= \boxed{\frac{16}{3}k}$$

$$x_{CM} = \frac{1}{M}\int_0^2\int_0^{x^2} x\cdot kxy\,dy\,dx = \frac{3}{16k}\,\frac{k}{2}\int_0^2 x^6\,dx = \frac{3}{32}\,\frac{2^7}{7} = \frac{3\cdot 4}{7} = \boxed{\frac{12}{7}}$$

$$y_{CM} = \frac{1}{M}\int_0^2\int_0^{x^2} y\cdot kxy\,dy\,dx = \frac{3}{16k}\,\frac{k}{3}\int_0^2 x^7\,dx = \frac{1}{16}\,\frac{2^8}{8} = \boxed{2}$$

C. of M. $= \boxed{\left(\frac{12}{7}, 2\right).}$

14.4#14



$$M = k\int_{-1}^1\int_0^{1/(1+x^2)} dy\,dx = k\int_{-1}^1 \frac{1}{1+x^2}\,dx$$

$$= 2k\int_0^1 \frac{1}{1+x^2}\,dx = 2k\,\mathrm{Arctan}(x)\Big|_{x=0}^1$$

$x_{CM} = 0$ by symmetry $\qquad = 2\cdot\frac{\pi}{4}k = \boxed{\frac{\pi}{2}k}$

$$y_{CM} = \frac{1}{M}\iint_R y\,dA = \frac{2}{\pi k}\,k\int_{-1}^1\int_0^{1/(1+x^2)} y\,dy\,dx = \frac{2}{\pi}\,\frac{1}{2}\int_{-1}^1 \frac{1}{(1+x^2)^2}\,dx$$

$$= \frac{2}{\pi} \int_0^1 \frac{1}{(1+x^2)} \cdot \underbrace{\frac{1}{(1+x^2)} dx}_{du} = \frac{2}{\pi} \int_0^{\pi/4} \frac{1}{1+\tan^2 u}\, du$$

$u = \arctan x \quad du = \frac{1}{1+x^2}\, dx \qquad = \frac{2}{\pi} \int_0^{\pi/4} \cos^2 u \, du$

so $x = \tan u$

$\cos^2 u + \sin^2 u = 1 \qquad 1 + \tan^2 u = \sec^2 u$

$\cos(2u) = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 \quad \cos^2 u = \frac{1}{2}(1 + \cos(2u))$

$$= \frac{2}{\pi} \frac{1}{2} \int_0^{\pi/4} (1 + \cos(2u))\, du = \frac{1}{\pi}\left(\frac{\pi}{4} + \frac{1}{2} \sin(2u) \Big|_0^{\pi/4}\right)$$

$$= \boxed{\frac{1}{4} + \frac{1}{2\pi}}$$