Prob. 11

1. $f(x,y) = \sin(x^2+y^2) + \arcsin(y^2)$

$$\frac{\partial f}{\partial y} = 2y\cos(x^2+y^2) + \frac{2y}{\sqrt{1-y^4}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -4xy\sin(x^2+y^2)$$

(it'll be easier to calculate $\frac{\partial f}{\partial x}$ first and use the property $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$)

2. it should be $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$.

it's called the Laplacian of z

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 z}{\partial x \partial y}\cos\theta\sin\theta$$

$$\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial^2 z}{\partial x^2} r^2\sin^2\theta + \frac{\partial^2 z}{\partial y^2} r^2\cos^2\theta - 2\frac{\partial^2 z}{\partial x \partial y} r^2\sin\theta\cos\theta$$
$$-\left(\frac{\partial z}{\partial x} r\cos\theta + \frac{\partial z}{\partial y} r\sin\theta\right)$$

$$\Rightarrow \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} = \frac{\partial^2 z}{\partial x^2}(\cos^2\theta + \sin^2\theta) + \frac{\partial^2 z}{\partial y^2}(\cos^2\theta +$$

$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

Prob. 12. 1. $\frac{\partial(xy^2)}{\partial y} = 2xy = \frac{\partial(yx^2)}{\partial x} \Rightarrow \vec{F}$ is conservative.

2. $\mathrm{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial(xy^2)}{\partial x} + \frac{\partial(yx^2)}{\partial y} = y^2 + x^2$

Prob. 13. $\vec{r}(u,v) = (u, \; u\cos v, \; u\sin v)$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x & y & z \end{array}$$

which satisfies $\quad x^2 = y^2 + z^2$

$$\iff \quad -x^2 + y^2 + z^2 = 0$$

which is an elliptic cone. cf. p815.

Prob. 14. Solution 1: $\quad \vec{r}(u,v) = (2,2,3) \iff u = 2, \; v = -1$

$$\frac{\partial \vec{r}}{\partial u}(2,2,3) = (1, \; 0, \; 2u)\Big|_{\substack{u=2 \\ v=-1}} = (1, 0, 4)$$

$$\frac{\partial \vec{r}}{\partial v}(2,2,3) = (0, \; 4v, \; 1)\Big|_{\substack{u=2 \\ v=-1}} = (0, -4, 1)$$

$$\vec{N}(2,2,3) = \frac{\partial \vec{r}}{\partial u}(2,2,3) \times \frac{\partial \vec{r}}{\partial v}(2,2,3) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 4 \\ 0 & -4 & 1 \end{vmatrix} = (16, -1, -$$

Eqn of the tangent plane:

$$\vec{N} \cdot (x-2, y-2, z-3) = 0$$

$$\iff 16(x-2) - (y-2) - 4(z-3) = 0$$

Solution 2: the surface satisfies

$$f(x,y,z) = y - 2(z - x^2)^2 = 0$$

$$\vec{N} = \nabla f(2,2,3) = (-4(z-x^2)(-2x), \; 1, \; -4(z-x^2))\Big|_{(2}$$

$$= (8x(z-x^2), \; 1, \; -4(z-x^2))\Big|_{(2,2,3)}$$

$$= (-16, 1, 4)$$

(which is the opposite of the normal vector obtained in solution 1, but will give the same equation of the tangent plane.)

Prob. 15.

$$\text{Arc length} = \int_0^{\pi} \| \vec{r}'(t) \| \, dt$$

$$= \int_0^{\pi} \Big((2t)^2 + (\cos t - \cos t + t\sin t)^2 + (-\sin t + \sin t + t\cos t)^2 \Big)^{1/2} dt$$

$$= \int_0^{\pi} (4t^2 + t^2)^{1/2} dt$$

$$= \sqrt{5} \int_0^{\pi} t \, dt$$

$$= \frac{\sqrt{5}}{2} t^2 \Big|_0^{\pi} = \frac{\sqrt{5}}{2} \pi^2$$

Prob. 16. $\vec{F} = (y^2 + x)\vec{i} - (x^2 - y)\vec{j} + z\vec{k}$

M N P

1. $$\text{curl } \vec{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right)\vec{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right.$$

$$= (0 - 0)\vec{i} - (0 - 0)\vec{j} + (-2x - 2y)\vec{k}$$

$$= -2(x - y)\vec{k}$$

2. $$\text{div } \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= 1 + 1 + 1$$

$$= 3$$

Prob. 17

1. $x = \cos t$, $y = \sin t$, $t \in [0, \frac{\pi}{2}]$.

$$\int_C xy\, ds$$

$$= \int_0^{\frac{\pi}{2}} x(t)\, y(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= \int_0^{\pi/2} \cos t \sin t \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

$$= \int_0^{\pi/2} \cos t \sin t\, dt$$

$$= \frac{1}{2} \sin^2 t \Big|_0^{\pi/2}$$

$$= \frac{1}{2}$$

2.

$$\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = -\frac{1(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1\cdot(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2}$$

Therefore, if denote $\vec{F} = M\vec{i} + N\vec{j}$, then $\vec{F}$, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are all defined in $R = \{x^2+y^2 > 1\}$ and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\Rightarrow \vec{F}$ conserv

3. By 2, $\vec{F}$ is conservative.

By integration, find $\Phi = -\arctan\left(\frac{x}{y}\right)$ is a potential of $\vec{F}$,

i.e. $\nabla \Phi = \vec{F}$.

$$\Rightarrow \int_{C(t)} \vec{F} \cdot d\vec{r} = \Phi(c(2)) - \Phi(c(1))$$

$$= \Phi(e^2, 2) - \Phi(e, 1)$$

$$= -\arctan \frac{e^2}{2} + \arctan e$$

4. $x = 3\cos t, \quad y = 3\sin t, \quad t \in [0, 2\pi]$

$$\int_C \frac{-y\,dx + x\,dy}{x^2+y^2}$$

$$= \int_0^{2\pi} \frac{-3\sin t\,(-3\sin t\,dt) + 3\cos t\,(3\cos t\,dt)}{(3\cos t)^2 + (3\sin t)^2}$$

$$= \int_0^{2\pi} \frac{9\,dt}{9} = \int_0^{2\pi} dt$$

$$= 2\pi$$

5. By Green's Thm.

$$\int_C xy^2\,dx + (x^2y + 2x)\,dy$$

$$= \iint_D \left(\frac{\partial(x^2y + 2x)}{\partial x} - \frac{\partial(xy^2)}{\partial y} \right) dx\,dy$$

$$= \iint_D 2\,dx\,dy = 2\iint_D dx\,dy = 2\,\text{Area}(D)$$

6. Let D_+ denote the half-disc.

Then by the divergence theorem,

$$\int_L \vec{F}_1 \cdot \vec{N} \, ds$$

$$= \iint_{D_+} \operatorname{div} \vec{F}_1 \, dx\,dy$$

$$= \iint_{D_+} \left(\frac{\partial (x-y)}{\partial x} + \frac{\partial (-1)}{\partial y} \right) dx\,dy$$

$$= \iint_{D_+} dx\,dy$$

$$= \text{Area}(D_+)$$

$$= \frac{1}{2}\left(\pi \cdot 1^2\right)$$

$$= \frac{\pi}{2}$$

7. Similarly, by the divergence theorem,

$$\int_L \vec{F}_2 \cdot \vec{N} \, ds$$

$$= \iint_{D_+} \operatorname{div} \vec{F} \, dx\,dy$$

$$= \iint_{D_+} \left(\frac{\partial (x^2+y^2)}{\partial x} + \frac{\partial (x^2-y^2)}{\partial y} \right) dx\,dy$$

$$= \iint_{D_+} (2x - 2y) \, dx\,dy$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^1 2(r\cos\theta - r\sin\theta) \, r \, dr\,d\theta$$

$$= \frac{4}{3}$$

Prob. 18.

1. $$\iint_S \vec{F}\cdot\vec{N}\,dS$$

$$= \int_0^2\int_0^1 (3x, 2xz, 3)\cdot(0,0,1)\Big|_{z=0}\,dx\,dy$$

$$= \int_0^2\int_0^1 3\,dx\,dy$$

$$= 6$$

2. Use the formula on the bottom on page P1119 and the example on page P1120,

$$\iint_S \vec{F}\cdot\vec{N}\,dS$$

Remark the 1st octant should be that x, y, z are all ≥ 0

$$= \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

where $r(u,v) = a\sin u\cos v\,\vec{i} + a\sin u\sin v\,\vec{j} + a\cos u\,\vec{k}$

$$D = \{(u,v) \mid u\in[0,\tfrac{\pi}{2}],\ v=[0,\tfrac{\pi}{2}]\},$$

and $\vec{F}\Big|_S = z\vec{k}\Big|_S = a\cos u\,\vec{k}$

$$\Rightarrow \text{flux} = \int_0^{\pi/2}\int_0^{\pi/2}\begin{vmatrix} 0 & 0 & a\cos u \\ a\cos u\cos v & a\cos u\sin v & -a\sin u \\ -a\sin u\sin v & a\sin u\cos v & 0 \end{vmatrix}\,du\,dv$$

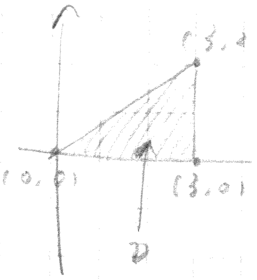
$$= \int_0^{\pi/2}\int_0^{\pi/2} a^2\cos u\sin u\,du\,dv$$

$$= a^2\frac{\pi}{2}\cdot\frac{1}{2}\sin^2 u\Big|_0^{\pi/2}$$

$$= \frac{a^2\pi}{4}$$

Prob. 19

1. $$\text{Area} = \iint_D \sqrt{1+z_x^2+z_y^2}\,dx\,dy$$



$$= \iint_D \sqrt{1+(2x)^2+1^2}\,dx\,dy$$

$$= \iint_D \sqrt{2+4x^2}\,dx\,dy$$

$$= \int_0^3 \int_0^{2/3\,x} \sqrt{2+4x^2}\,dy\,dx$$

$$= \frac{2}{3}\int_0^3 x\sqrt{2+4x^2}\,dx$$

$$\overset{u=x^2}{=} \frac{1}{3}\int_0^9 \sqrt{2+4u}\,du$$

$$= \frac{1}{18}(2+4u)^{3/2}\Big|_0^9$$

$$= \frac{1}{18}\left(38^{3/2} - 2^{3/2}\right)$$

2. The integral should be $\iint_S \vec{A}\cdot\vec{N}\,dS$.

By Gauss' theorem (Divergence theorem)

$$\iint_S \vec{A}\cdot\vec{N}\,dS$$

$$= \iiint_V \operatorname{div}\vec{A}\,dx\,dy\,dz$$

$$= \iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial 0}{\partial z}\right) dx\,dy\,dz$$

$$= 2\,\mathrm{Vol}(B(a))$$ ← Ball with radius a

$$= 2\cdot\frac{4}{3}\pi a^3$$ ⟶ can be found by integration using spherical coordinates

Prob. 20. 1. Using spherical coordinates,

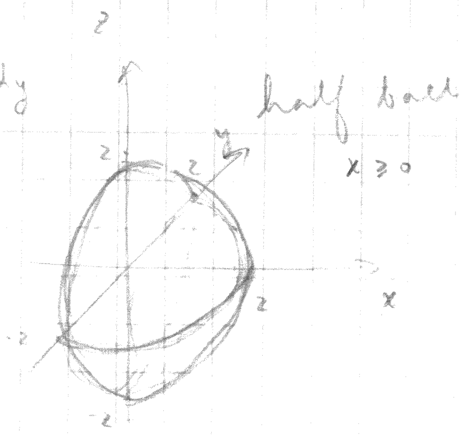
$$\iiint_B (x^2+y^2+z^2)^2 \, dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{3} r^4 \cdot r^2 \sin\varphi \, dr\,d\varphi\,d\theta$$

$$= \frac{1}{6} 3^6 \cdot 2 \cdot 2\pi$$

$$= 3^5 \cdot 2\pi$$

2. $$\int_{-2}^{2}\int_0^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2+y^2+z^2} \, dz\,dx\,dy$$



$$= \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{2} (r\sin\varphi\sin\theta)^2 \, r \cdot r^2 \sin\varphi \, dr\,d\varphi\,d\theta$$

$$= \left(\int_{-\pi/2}^{\pi/2} \sin^2\theta \, d\theta\right)\left(\int_0^{\pi} \sin^3\varphi \, d\varphi\right)\left(\int_0^{2} r^4 \, dr\right)$$

$$= \frac{\pi}{2} \cdot \frac{4}{3} \cdot \frac{32}{5}$$

$$= \frac{64\pi}{15}$$

3. $$\int_0^2\int_0^{y^3}\int_0^{y^2} f(x,y,z)\,dz\,dx\,dy = \int_0^2\int_0^{y^2}\int_0^{y^3} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^4\int_0^{\sqrt{z}}\int_0^{\sqrt{z^3}} f(x,y,z)\,dx\,dy\,dz = \int_0^4\int_0^{\sqrt{z^3}}\int_0^{\sqrt{z}} f(x,y,z)\,dy\,dx\,dz$$

$$= \int_0^8\int_0^{x^{1/3}}\int_0^{x^{2/3}} f(x,y,z)\,dz\,dy\,dx = \int_0^8\int_0^{x^{2/3}}\int_0^{x^{1/3}} f(x,y,z)\,dy\,dz\,dx$$

4. refer to the solution for practice midterm 2.
(problem 20)

Prob. 21.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u - \cos(u+v) & -\sin(u+v) \\ e^u \cos v & -e^u \sin v \end{vmatrix}$$

$$= -e^u \left(2u\sin v - \cos(u+v)\sin v - \sin(u+v)\cos v \right)$$

$$= -e^u \left(2u\sin v - \sin(u+2v) \right)$$