Prob. 15.

$$F = 3 + \cos(\pi x y) - z = 0$$

$$\nabla F(x,y,z) = (F_x, F_y, F_z) = (-\pi y \sin(\pi x y), -\pi x \sin(\pi x y), -1)$$

$$\nabla F(1,1,\overset{z(1,1)}{2}) = (0, 0, -1)$$

tangent plane: $\nabla F(1,1,2) \cdot (x-1, y-1, z-2) = 0$

$\Rightarrow z - 2 = 0, \quad z = 2.$ (it's a plane !)

Prob. 16. $f(x,y) = x^2 - 3xy + y^2$. $\nabla f(x,y) = (2x - 3y, -3x + 2y)$

- critical pts in the interior: $\nabla f(x,y) = 0$, $(x,y) \in \{x^2+y^2 < 1\}$

$$\Rightarrow (x,y) = (0,0).$$

- The boundary of the region $\{x^2+y^2 \le 1\}$ is $\{x^2+y^2 = 1\}$,

which is a circle and can be parametrized by $x = \cos\theta$, $y = \sin\theta$. $\theta \in [0, 2\pi]$

$$\text{Therefore, } f(x,y)\Big|_{\text{boundary}} = f(\cos\theta, \sin\theta)$$

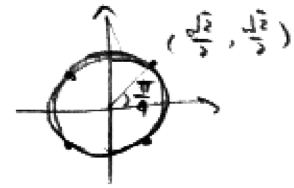
$$= \cos^2\theta - 3\cos\theta\sin\theta + \sin^2\theta$$

$$= 1 - \frac{3}{2}\sin 2\theta. \quad =: f(\theta)$$

If f achieve a global extremum at a point $(x(\theta), y(\theta))$ on the boundary, it must also be a global extremum of the single variable function $f(\theta)$, which is the restriction of $f(x,y)$ on the boundary. Therefore θ must be a critical pt of $f(\theta)$. (We don't need to consider the boundary values of $f(\theta)$ at 0 and 2π because f is periodic.)

$f'(\theta) = -3\cos 2\theta$

$f'(\theta) = 0, \ \theta \in [0, 2\pi] \implies \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$



i.e. $(x, y) = (\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$.

- Compare values at all candidates for global extrema:

(x, y)	$f(x, y)$
$(0, 0)$	0
$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$-\frac{1}{2}$
$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$\frac{5}{2}$
$(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$	$-\frac{1}{2}$
$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$	$\frac{5}{2}$

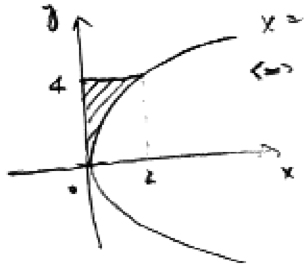
(easier to obtain using $f(\theta) = f|_{(\cos\theta, \sin\theta)} = 1 - \frac{3}{2}\sin 2\theta$ and plug in $\theta = \frac{\pi}{4} + k\frac{\pi}{2}$, $k = 0, 1, 2, 3$)

Conclusion: $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{1}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\frac{1}{2})$ are global minima.

$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{5}{2})$ and $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{5}{2})$ are global maxima.

Prob. 17.

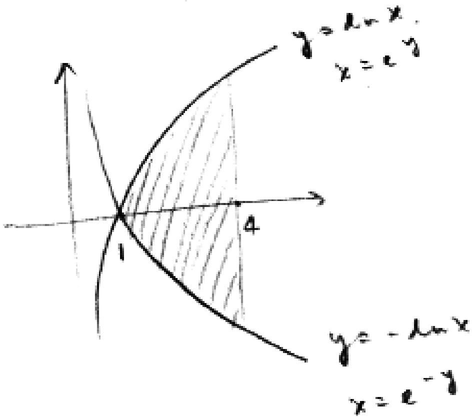
1. $\int_0^4 \int_0^{y^2} f(x,y)\,dx\,dy$



$x = y^2$
$\Leftrightarrow y = \sqrt{x}$ (the upper part, i.e. $y \geq 0$)

$$\Rightarrow \int_0^2 \int_{\sqrt{x}}^4 f(x,y)\,dy\,dx$$

2. $\int_1^4 \int_{-\ln x}^{\ln x} f(x,y)\,dy\,dx$

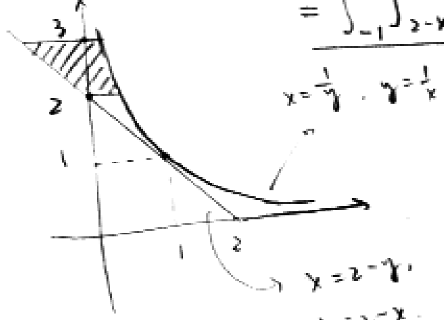


$$\Rightarrow \int_{-\ln 4}^{0} \int_{e^{-y}}^{4} f(x,y)\,dx\,dy + \int_0^{\ln 4} \int_{e^y}^{4} f(x,y)\,dx\,dy.$$

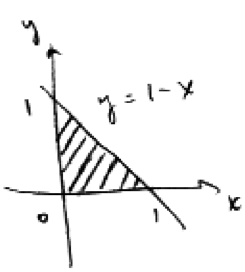
3. $\int_2^3 \int_{2-y}^{\frac{1}{y}} f(x,y)\,dx\,dy$

$$= \underline{\int_{-1}^{0} \int_{2-x}^{3} f(x,y)\,dy\,dx + \int_0^{\frac{1}{3}} \int_2^3 f(x,y)\,dy\,dx + \int_{\frac{1}{3}}^{\frac{1}{2}} \int_2^{\frac{1}{x}} f(x,y)\,dy\,dx}$$

final answer.



Prob. 18.

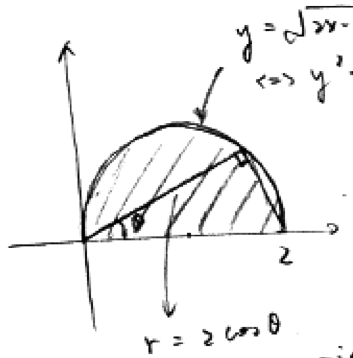


$$\iint_R e^{-x-y}\,dx\,dy = \int_0^1 \int_0^{1-x} e^{-x-y}\,dy\,dx$$

$$= \int_0^1 \left(-e^{-x-y} \Big|_0^{1-x} \right) dx$$

$$= \int_0^1 \left(-e^{-1} + e^{-x} \right) dx = -2e^{-1} + 1$$

Prob. 19.



(use the right triangle !)

$$\int_0^2 \int_0^{\sqrt{2x - x^2}} \sqrt{x^2 + y^2}\, dy\, dx$$

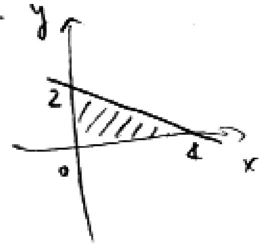
$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r \cdot r\, dr\, d\theta$$

comes from changing to polar coord.

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\, dr\, d\theta.$$

Prob. 20

1.



$$V = \int_0^2 \int_0^{4-2y} (3x + 2y)\, dx\, dy$$

$$= \int_0^2 \left(\frac{3}{2} (4-2y)^2 + 2y(4-2y) \right) dy$$

$$= 2\int_0^2 (y^2 - 8y + 12)\, dy$$

$$= \frac{64}{3}.$$

2. Use the same notation as in Example 3 on page 1007 from the textbook.

In polar coordinates. the bounds for the region is

$2 \le r \le 4$ and $0 \le \theta \le 2\pi$.

and the upper surface of the solid $\sqrt{16 - r^2}$, the lower surface $-\sqrt{16 - r^2}$.

$$\Longrightarrow V = \int_0^{2\pi} \int_2^4 2\sqrt{16 - r^2}\, r\, dr\, d\theta$$

$$= 16\sqrt{3}\,\pi$$

Prob. 21.

$$f(x,y) = y^2 - x^2$$

$$dS = \sqrt{1 + f_x^2 + f_y^2}\ dx\,dy$$

$$= \sqrt{1 + 4x^2 + 4y^2}\ dx\,dy$$

$$\Rightarrow\ A = \iint_{R(x,y)} \sqrt{1+4x^2+4y^2}\ dx\,dy$$

$$= \iint_{R(r,\theta)} \sqrt{1+4r^2}\ r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_1^2 \sqrt{1+4r^2}\ r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{12} (1+4r^2)^{3/2} \Big|_1^2 \right) d\theta$$

$$= \frac{\pi}{6} \left(17\sqrt{17} - 5\sqrt{5} \right)$$

Prob. 22

$$\iiint_V 3xy\ dx\,dy\,dz$$

$$= \int_0^1 \int_0^x \int_0^{xy} 3xy\ dz\,dy\,dx$$

$$= \int_0^1 \int_0^x 3x^2y^2\ dy\,dx$$

$$= \int_0^1 x^5\,dx$$

$$= \frac{1}{6}$$