Practice Exam Solutions 1–10

1. line $x = 3 + t$ $y = 1 + 2t$ $z = 1 - 2t$

has direction vector $\vec{u} = \langle 1, 2, -2 \rangle$

plane $2x + 3y + 4z = 5$ has normal vector

$\vec{n} = \langle 2, 3, 4 \rangle$

compute $\vec{u} \cdot \vec{n} = 1 \cdot 2 + 2 \cdot 3 + (-2) \cdot 4 = 0$

thus the line is orthogonal to the normal vector of the plane, and hence is parallel to the plane.

2. $r(t) = \sqrt{t}\,\hat{i} + \hat{j} + t^4 \hat{k}$, find tangent line at $\hat{i} + \hat{j} + \hat{k}$:

$r'(t) = \frac{1}{2\sqrt{t}}\hat{i} + 4t^3 \hat{k}$

what is t when $r(t) = \hat{i} + \hat{j} + \hat{k}$?

solve $\sqrt{t}\,\hat{i} + \hat{j} + t^4 \hat{k} = \hat{i} + \hat{j} + \hat{k}$

$\Rightarrow t = 1$

so direction vector for tangent line is

$r'(1) = \frac{1}{2}\hat{i} + 4\hat{k}$

a) thus the tangent line through $\hat{i} + \hat{j} + \hat{k}$ has equations

$x = 1 + \frac{1}{2}t$, $y = 1$, $z = 1 + 4t$

or $2(x-1) = y - 1 = \frac{z-1}{4}$.

b). point on ~~plane~~ line $P = (1, 1, 1)$

point off line $Q = (1, 2, 3)$

direction vector for line $\vec{u} = \langle \frac{1}{2}, 0, 4 \rangle$

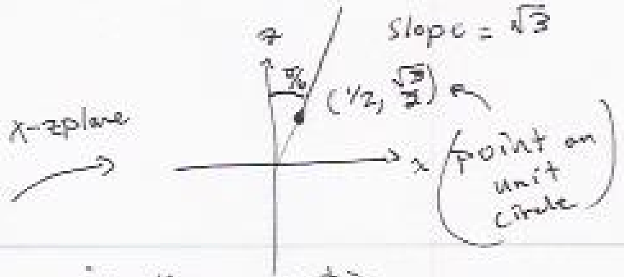
$\vec{PQ} = \langle 1-1, 2-1, 3-1 \rangle = \langle 0, 1, 2 \rangle$

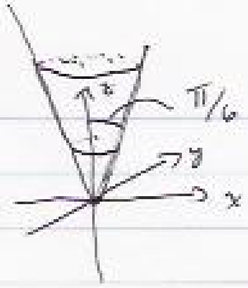
$\|\vec{u}\| = \sqrt{\frac{1}{4} + 16} = \sqrt{\frac{1}{4} + \frac{64}{4}} = \frac{\sqrt{65}}{2}$

distance btwn point and line:

$$D = \frac{\|\vec{PQ} \times \vec{u}\|}{\|\vec{u}\|} = \frac{\left\| \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ \frac{1}{2} & 0 & 4 \end{vmatrix} \right\|}{\sqrt{65}/2} = \frac{\|\langle 4, 1, -\frac{1}{2} \rangle\|}{\frac{\sqrt{65}}{2}}$$

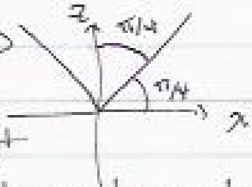
$$= \frac{\sqrt{16 + 1 + \frac{1}{4}}}{\sqrt{65}/2} = \frac{\frac{\sqrt{69}}{2}}{\sqrt{65}/2} = \sqrt{\frac{69}{65}}$$





3. $\phi = \pi/6$ is the equation for a cone

a) $z = \sqrt{x^2+y^2}$ is not the same because if we look at, say, the xz-plane $(y=0)$ we have $z = \sqrt{x^2} \Rightarrow$

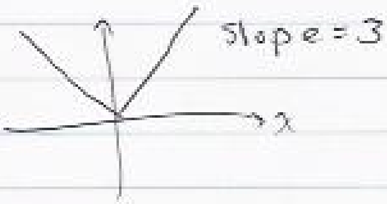


so this is a cone, but it doesn't have the right ~~angle~~ angle.

b) $z = 3r \overset{\text{convert}}{=} 3\sqrt{x^2+y^2}$

in xz plane:

$z = 3\sqrt{x^2}$

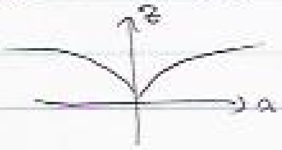


So not the right cone again.

c). $z = \sqrt{r} = \sqrt{\sqrt{x^2+y^2}} = \cancel{x^2+y^2}$

$\Rightarrow \quad z^2 = \sqrt{x^2+y^2}$

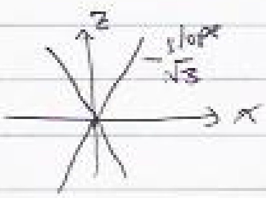
in xz plane: $z^2 = \sqrt{x^2}$



not a cone.

d). $z^2 = 3(x^2+y^2)$

in xz plane:



$z^2 = 3x^2$

$z = \pm(\sqrt{3})x$

has the right slope, but includes points with negative values of z (below xy plane, so this is not the same as $\varphi = \pi/6$.

~~e (or 5)~~

So none of the above are correct

4.1 $\vec{r}(t) = \sqrt{2}\cos t\,\hat{i} + \sin t\,\hat{j} + \sin t\,\hat{k}$

find unit tangent $\hat{T}(t)$ and unit normal $\hat{N}(t)$

tangent vector $\vec{r}\,'(t) = -\sqrt{2}\sin t\,\hat{i} + \cos t\,\hat{j} + \cos t\,\hat{k}$

$$\|\vec{r}\,'(t)\| = \sqrt{2\sin^2 t + \cos^2 t + \cos^2 t} = \sqrt{2(\sin^2 t + \cos^2 t)} = \sqrt{2}$$

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = -\sin t\,\hat{i} + \frac{1}{\sqrt{2}}\cos t\,\hat{j} + \frac{1}{\sqrt{2}}\cos t\,\hat{k}$$

$$\hat{T}'(t) = -\cos t\,\hat{i} - \frac{1}{\sqrt{2}}\sin t\,\hat{j} - \frac{1}{\sqrt{2}}\sin t\,\hat{k}$$

$$\|\hat{T}'(t)\| = \left(\cos^2 t + \frac{1}{2}\sin^2 t + \frac{1}{2}\sin^2 t\right)^{1/2} = 1$$

so $\hat{N}(t) = \dfrac{\hat{T}'(t)}{\|\hat{T}'(t)\|} = -\cos t\,\hat{i} - \dfrac{1}{\sqrt{2}}\sin t\,\hat{j} - \dfrac{1}{\sqrt{2}}\sin t\,\hat{k}$

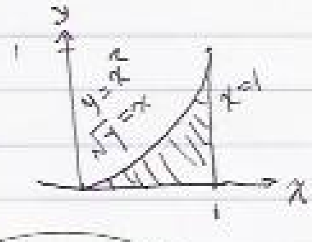
4.2 find curvature $\kappa = \dfrac{\|\hat{T}'(t)\|}{\|\vec{r}\,'(t)\|} = \dfrac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \|\hat{T}'(s)\|$

arc length parameter

use the first formula for κ since we have already compute these values.

$$\kappa = \frac{\|\hat{T}'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{1}{\sqrt{2}}.$$

5.1 $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy$ difficult to integrate e^{x^3} so swith order



$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{x^2} e^{x^3}\,dy\,dx = \int_0^1 y e^{x^3}\Big|_0^{x^2} dx = \int_0^1 x^2 e^{x^3}\,dx$$

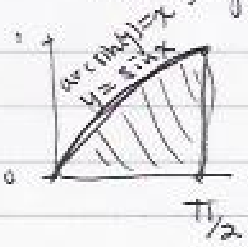
$u = x^3$, $du = 3x^2\,dx$

$$= \int_{x=0}^{1} \frac{1}{3} e^u\,du = \frac{1}{3} e^u\Big|_{x=0}^{1} = \frac{1}{3} e^{x^3}\Big|_{x=0}^{1} = \frac{1}{3}(e-1)$$

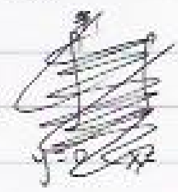
(interpret $\cos x^2$ as $\cos^2(x)$ (otherwise integration is hard))

5.2 $\displaystyle\int_0^1 \int_{\arcsin(y)}^{\pi/2} (\cos x)\sqrt{1+\cos^2(x)}\, dx\, dy$

switch order to make integration easier (try to integrate first, you'll see).



$\Rightarrow$



$$\int_{x=0}^{\pi/2} \int_{y=0}^{\sin x} (\cos x)\sqrt{(1+\cos^2 x)}\, dy\, dx$$

$u = 1+\cos^2 x$

$du = -2\cos x \sin x \, dx$

$$= \int_{x=0}^{\pi/2} \sin x \cos x \sqrt{1+\cos^2 x}\, dx$$

$$= -\frac{1}{2}\int_{x=0}^{\pi/2} \sqrt{u}\, du = -\frac{1}{2}\cdot\frac{2}{3} u^{3/2}\Big|_{x=0}^{\pi/2}$$

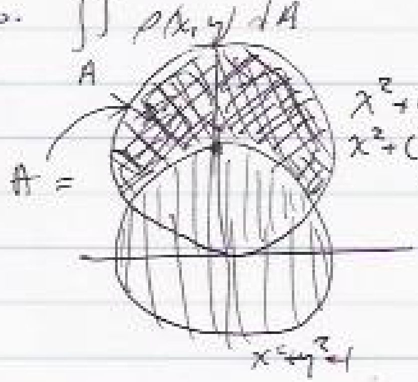
$$= -\frac{1}{3}\left(1+\cos^2 x \Big|_{x=0}^{\pi/2}\right)$$

$$= -\frac{1}{3}\left((1+0) - (1+1)\right)$$

$$= \frac{1}{3}$$

6. $\displaystyle\iint_A \rho(x,y)\, dA$ $\quad \rho(x,y) =$ denisity of mass $= \frac{1}{\sqrt{x^2+y^2}}$ "inverse of distance to origin"

$A =$



$\displaystyle\iint_A \frac{1}{\sqrt{x^2+y^2}}\, dA$ is hard to integrate in Cartesian coordinates

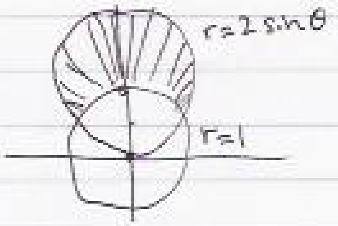
so convert to polar.

$\rho(x,y) = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$

$x^2+y^2=2y \Rightarrow r^2 = 2r\sin\theta$
$r = 2\sin\theta$

$x^2+y^2=1 \Rightarrow r^2=1$
$r=1$

6.2 (cont)



find intersection for bounds on θ:

$2\sin\theta = r = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$\text{mass} = \int_{\theta=\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{r=1}^{2\sin\theta} \underbrace{\frac{1}{r}}_{\rho} \underbrace{r\,dr\,d\theta}_{dA}$$

$$= \int_{\frac{\pi}{6}}^{5\pi/6} (2\sin\theta - 1)\, d\theta$$

$$= -2\cos\theta - \theta \Big|_{\pi/6}^{5\pi/6} = -\left(2\cos\frac{5\pi}{6} + \frac{5\pi}{6}\right) + \left(2\cos\frac{\pi}{6} + \frac{\pi}{6}\right)$$

$$= -\left(2\left(-\frac{\sqrt{3}}{2}\right) + \frac{5\pi}{6}\right) + \left(2\cdot\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right)$$

$$= \sqrt{3} - \frac{5\pi}{6} + \sqrt{3} + \frac{\pi}{6}$$

$$= 2\sqrt{3} - \frac{2\pi}{3}$$

(note this is positive)

7.1 $f(x,y) = x^2 + \sin(xy)$

$D_u f = (2x + y\cos(xy))\cos\theta + x\cos(xy)\sin\theta$

$1 = D_u f(1,0) = (2)\cos\theta + \sin\theta$

~~$1 = 2\cos\theta + \sin\theta$~~

$1 = 2\cos\theta + \sin\theta$

now solve for θ

7.2 $f(x,y) = x^2+y^2-2x-4y$

Maximum change is ~~along~~ in the direction of the gradient:

$$\nabla f(x,y) = (2x-2)\hat{\imath} + (2y-4)\hat{\jmath}$$

if maximum change is $\hat{\imath}+\hat{\jmath}$

then solve

$$\hat{\imath}+\hat{\jmath} = \nabla f(x,y) = (2x-2)\hat{\imath} + (2y-4)\hat{\jmath}$$

$$\Rightarrow \quad \begin{aligned} 1 &= 2x-2 \\ 1 &= 2y-4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \tfrac{3}{2} &= x \\ \tfrac{5}{2} &= y \end{aligned}$$

so $(\frac{3}{2}, \frac{5}{2})$ is the only such point.

7.3 differential of $z = e^{x+y}\ln(y^2) = 2e^{x+y}\ln(y)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$dz = (2e^{x+y}\ln y)\,dx + \left(2e^{x+y}\ln y + \frac{2e^{x+y}}{y}\right)dy$$

linear approximation: $\Delta z \simeq dz$

so

$$z \approx z(2,2) + \left.\frac{\partial z}{\partial x}\right|_{(2,2)}(x-2) + \left.\frac{\partial z}{\partial y}\right|_{(2,2)}(y-2)$$

$$= 2e^4\ln(2) + (2e^4\ln 2)(x-2) + (2e^4\ln(2)+e^4)(y-2)$$

8. 1. $\lim_{(x,y)\to\vec{0}} \frac{x+\sin y}{2x+y}$

if $y=0$ then $\lim_{(x,y)\to\vec{0}} \frac{x}{2x} = \frac{1}{2}$

but if $x=0$ then

$\lim_{(x,y)\to\vec{0}} \frac{\sin(y)}{y} = 1$ (L'Hospitals rule)

since these are not the same, the limit DNE.

8.2 $\lim_{(x,y)\to\vec{0}} \frac{7x^2y(x-y)}{x^4+y^4}$

if $x=0$

then $\lim_{(x,y)\to\vec{0}} \frac{0}{y^4} = 0$

but if $y=-x$

then $\lim_{(x,y)\to\vec{0}} \frac{7x^2(-x)(x-(-(x)))}{x^4+(-x)^4}$

$= \lim_{(x,y)\to\vec{0}} \frac{-7x^2(x)(2x)}{2x^4}$

$= \lim_{(x,y)\to\vec{0}} \frac{-7x^4}{x^4} = -7$

these are not the same, hence the limit DNE.

9. surface are orthogonal if their tangent planes are orthogonal. planes are orthogonal if their normal vectors are orthogonal.

normal vector to surface $z = 7x^2 - 12x - 5y^2$

$0 = 7x^2 - 12x - 5y^2 - z$

is the gradient of $f(x,y,z) = 7x^2 - 12x - 5y^2 - z$

$\nabla f(x,y,z) = \langle 14x - 12, -10y, -1 \rangle$

@ $(2,1,-1)$ $\nabla f(2,1,-1) = \langle 16, -10, -1 \rangle$

normal vector for surface $xyz^2 = 2$

$0 = xyz^2 - 2$

is the gradient of $g(x,y,z) = xyz^2 - 2$

$\nabla g(x,y,z) = \langle yz^2, xz^2, 2xyz \rangle$

@ $(2,1,-1)$ $\nabla g(2,1,-1) = \langle 1, 2, -4 \rangle$

now check that they or orthonal

$\langle 16, -10, -1 \rangle \cdot \langle 1, 2, -4 \rangle$

$= 16 \cdot 1 + (-10) \cdot 2 + (-1)(-4)$

$= 16 - 20 + 4$

$= 0$ ✓

so these surfaces are orthogonal at $(2,1,-1)$.

10. $f(x,y) = x^2 - 2x + 2y^2 - 2y + 2xy$ $\quad$ $-1 \le x \le 1$, $0 \le y \le 2$

find global max/min

compute $\nabla f(x,y) = (2x-2+2y)\hat{i} + (4y-2+2x)\hat{j}$

solve
$0 = 2x - 2 + 2y \rightarrow 4y - 2 + 2x = 0$
$0 = x - 1 + y \qquad 2y - 1 + x = 0$
$y = 1 - x \longrightarrow 2 - 2x - 1 + x = 0$
$y = 0 \longleftarrow 1 - x = 0$
$x = 1$

critical point : $(1,0)$
$f(1,0) = 1 - 2 = -1$

Boundary:

$x = -1$ $\quad f(-1,y) = +1 + 2 + 2y^2 - 2y - 2y$
$= 2y^2 - 4y + 3$
$\frac{d}{dy} f(-1,y) = 4y - 4 = 4(y-1)$ $\qquad \frac{d^2}{dy^2} f(-1,y) = 4$
so $f(-1,y)$ has a min at $y = 1$
$f(-1,1) = 2 - 4 + 3$
$f(-1,1) = 1$

$x = 1$ $\quad f(1,y) = 1 - 2 + 2y^2 - 2y + 2y$
$= 2y^2 - 1$
$f(1,y)$ has a min at $y = 0$
$f(1,0) = -1$ (as before)

$y = 0$ $\quad f(x,0) = x^2 - 2x = x(x-2)$
$\frac{d}{dx} f(x,0) = 2x - 2$
$f(x,0)$ has min at $x = 1$ $\qquad$ which we have already seen.

$y = 2$ $\quad f(x,2) = x^2 - 2x + 8 - 4 + 4x$
$= x^2 + 2x + 4$
$\frac{d}{dx} f(x,2) = 2x + 2$
$f(x,2)$ has min at $x = -1 \Rightarrow f(-1,2) = 3$

10 (contin) check remaining corners for maxima:

$f(1,0) = 1+2 = 3$

$f(1,2) = 1-2+8-4+4$
$= 7$

summary:

global max value is 7 at point (1,2)
global min value is −1 at point (1,0).