Math 203 Practice Midterm Solutions

Problem 1: 1. $\vec{PQ} = (6,6,0)$
$\vec{PS} = (-12,-12,0)$
$\vec{PT} = (18,18,0)$

Since $\vec{PS} = -2\cdot\vec{PQ}$ and $\vec{PT} = 3\vec{PQ}$ all four points are collinear.

2. $\vec{AB} = (0,-4,-2)$ $\quad \vec{BC} = (1,4,2)$
$\vec{AC} = (1,0,0)$ $\quad \vec{BD} = (2,6,1)$
$\vec{AD} = (2,2,-1)$

A, B, C are not collinear
A, B, D are not collinear
A, C, D are not collinear
B, C, D are not collinear

Problem 2.

1. $\vec{v}\cdot(\vec{u}+\vec{w}) = 4$
2. undefined
3. $\vec{v}\times\vec{u} = (-2,-2,4)$

   $\vec{v}\times\vec{u}+\vec{w} = (-1,-3,5)$
4. $(\vec{v}\times\vec{u})\cdot\vec{w} - \vec{u}\cdot\vec{w} = 3$

Problem 3.

$\vec{u}_1\cdot\vec{v}_1 = -22 \quad \|\vec{u}_1\| = 2\sqrt{11} \quad \|\vec{v}_1\| = \sqrt{11} \quad \cos\theta = -1 \quad \theta = \pi$ obtuse

$\vec{u}_2\cdot\vec{v}_2 = -16 \quad \|\vec{u}_2\| = 2\sqrt{10} \quad \|\vec{v}_2\| = 3 \quad \cos\theta = -\frac{4}{15}\sqrt{10} \quad \theta = \arccos(-\frac{4}{15}\sqrt{10})$ obtuse

$\vec{u}_3\cdot\vec{v}_3 = 0 \quad \|\vec{u}_3\| = \sqrt{11} \quad \|\vec{v}_3\| = \sqrt{22} \quad \cos\theta = 0 \quad \theta = \pi/2$ orthogonal

$\vec{u}_4\cdot\vec{v}_4 = 2 \quad \|\vec{u}_4\| = \sqrt{14} \quad \|\vec{v}_4\| = \sqrt{14} \quad \cos\theta = 1/7 \quad \theta = \arccos(1/7)$ acute

$$\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta}$$

Problem 4. $\operatorname{proj}_{\vec{u}}\vec{v} = \dfrac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \dfrac{-2}{30}(1,5,2) = \left(\dfrac{-1}{15}, \dfrac{-1}{3}, \dfrac{-2}{15}\right)$

Component of $\vec{v} \perp \vec{u} = \vec{v} - \operatorname{proj}_{\vec{u}}\vec{v} = (1,5,2) - \left(\frac{-1}{15}, \frac{-1}{3}, \frac{-2}{15}\right)$

$= \left(\dfrac{16}{15}, \dfrac{16}{3}, \dfrac{32}{15}\right)$

Problem 5. 1. $\vec{PQ} = (-3,1,-1)$ $\quad \vec{PQ}\times\vec{PR} = (-4,-5,7)$
$\vec{PR} = (2,-3,-1)$ $\quad \vec{PS} = (-1,-2,-2)$

$(\vec{PQ}\times\vec{PR})\cdot\vec{PS} = 0$ So $\vec{PS}$ is orthogonal to $\vec{PQ}$ and $\vec{PR}$, and so S lies in the plane defined by P, Q and R.

2. $\vec{AB} = (-3,3,0)$ $\quad \vec{AB}\times\vec{AC} = (-3,-3,6)$ $\quad (\vec{AB}\times\vec{AC})\cdot\vec{AD} = -9 \neq 0$
$\vec{AC} = (-1,-1,-1)$ $\quad \vec{AD} = (-2,1,-2)$

so D does NOT lie in the plane determined by A, B, C.

Problem 6. 1. $\overrightarrow{PQ} = (-3, 1, 0)$ $\quad \vec{N}_1 = \overrightarrow{PQ} \times \overrightarrow{PR} = (-2, -6, -6)$

$\overrightarrow{PR} = (0, 2, -2)$ easier to take $\vec{N}_1 = (1, 3, 3)$

$$(x-1) + 3(y+1) + 3(z-1) = 0$$

$$\boxed{x + 3y + 3z = 1}$$

2. $\overrightarrow{AB} = (2, -4, -1)$ $\quad \vec{N}_2 = \overrightarrow{AB} \times \overrightarrow{AC} = (6, 3, 0)$

$\overrightarrow{AC} = (1, -2, -2)$ easier to take $\vec{N}_2 = (2, 1, 0)$

$$2(x+2) + (y-3) = 0 \qquad \boxed{2x + y = -1}$$

3. $$\cos\theta = \frac{\vec{N}_1 \cdot \vec{N}_2}{\|\vec{N}_1\| \|\vec{N}_2\|} = \frac{5}{\sqrt{19}\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{19}} = \frac{\sqrt{95}}{19}$$

4. $\overrightarrow{AQ} = (0, -3, -1)$ $\quad dist(Q, K) = \| proj_{\vec{N}_2/\|\vec{N}_2\|} AQ \|$

$$= \frac{|\overrightarrow{AQ} \cdot \vec{N}_2|}{\|\vec{N}_2\|} = \frac{|-3|}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

Problem 7. $\vec{u} = \overrightarrow{PQ} = (-2, 3, -3)$ $\qquad \overrightarrow{PS} = (3, 5, 4)$

$$\left.\begin{aligned} x &= 1-2t \\ y &= -1+3t \\ z &= -3t \end{aligned}\right\} \qquad \frac{1-x}{2} = \frac{y+1}{3} = \frac{-z}{3}$$

$dist(S, line) = \|$ component of $\overrightarrow{PS}$ which is $\perp$ $\vec{u} \|$

$$= \| \overrightarrow{PS} - proj_{\vec{u}} \overrightarrow{PS} \| = \left\| (3,5,4) - \left(\frac{\overrightarrow{PS} \cdot \vec{u}}{\|\vec{u}\|^2}\right)\vec{u} \right\|$$

$$= \left\| (3,5,4) - \frac{-3}{22}(-2,3,-3) \right\| = \left\| \left(\frac{30}{11}, \frac{119}{22}, \frac{79}{22}\right) \right\|$$

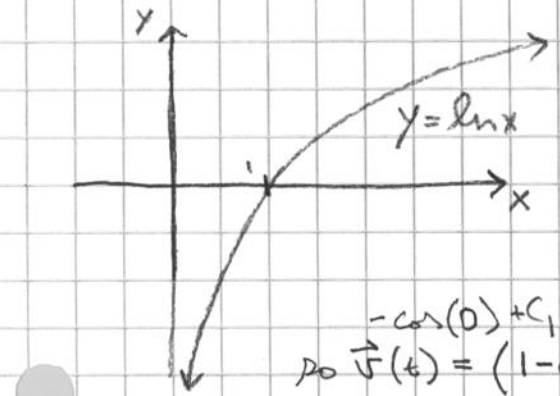
$$= \frac{1}{22}\sqrt{24002}$$

Problem 9. 1. $(x-1)^2 - (y+2)^2 = z+1$

2. $\dfrac{(x-1)^2}{2/3} + \dfrac{(y+2)^2}{2} - \dfrac{(z-1)^2}{2} = 1$

3. $-\dfrac{(x+1)^2}{3} + \dfrac{(y-1)^2}{3/2} + \dfrac{(z-1)^2}{3} = 1$

Problem 10.



1. $y^2+z^2 = (\ln x)^2$

2. $x^2+z^2 = e^{2y}$

Problem 12. $\vec{v}(t) = \int \vec{a}(t)\,dt$

$= \left(-\cos(t)+C_1,\ \frac{1}{2}\sin(2t)+C_2,\ \sin(t+\frac{\pi}{4})+C_3\right)$

$-\cos(0)+C_1 = 1 \Rightarrow C_1=2$ similarly, $C_2=-2$, $C_3=\sqrt{2}/2$

so $\vec{v}(t) = \left(1-\cos(t),\ \frac{1}{2}\sin(2t)-2,\ \sin(t+\frac{\pi}{4})+\frac{\sqrt{2}}{2}\right)$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \left(t-\sin(t)+2,\ -\frac{1}{4}\cos(2t)-2t-\frac{7}{4},\ -\cos(t+\frac{\pi}{4})+\frac{\sqrt{2}}{2}t+3\right)$

Problem 11.

1. cylindrical: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$x^2-y^2=1$

$r^2(\cos^2\theta-\sin^2\theta)=1$

$r^2\cos 2\theta = 1$ or $r^2 = \sec(2\theta)$

spherical: $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$

$x^2-y^2=1$

$\rho^2\sin^2\varphi(\cos^2\theta-\sin^2\theta)=1$

$\rho^2 = \csc^2\varphi\sec(2\theta)$

2. $\underbrace{\rho\sin\varphi\cos\theta}_{x} + \underbrace{\rho\sin\varphi\sin\theta}_{y} - \underbrace{\rho\cos\varphi}_{z} = 1$

$x+y-z=1$

Problem 13. $\vec{r}'(t) = (1,-4t,-2t)$

$\vec{r}''(t) = (0,-4,-2)$

$\hat{T} = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \left(\dfrac{1}{\sqrt{1+20t^2}}, \dfrac{-4t}{\sqrt{1+20t^2}}, \dfrac{-2t}{\sqrt{1+20t^2}}\right)$

$\hat{T}' = \left(\dfrac{-20t}{(1+20t^2)^{3/2}}, \dfrac{-4}{(1+20t^2)^{3/2}}, -\dfrac{2}{(1+20t^2)^{3/2}}\right)$

$\hat{T}(1) = \frac{1}{\sqrt{21}}(1,-4,-2)$

$\hat{N}(1) = \frac{-1}{\sqrt{105}}\cdot(10,2,1)$