MAT203 Fall 2011

Practice Final

The actual Final exam will consist of twelve problems that cover sections 11.1-15.7 (inclusive)

Problem 1 Show that the line $x = 3 + t, y = 1 + 2t, z = 1 - 2t$ is parallel to the plane $2x + 3y + 4z = 5$.

Problem 2

1. Write the equation of the tangent line to the curve with parametric equation $\mathbf{r}(t) = \sqrt{t}\boldsymbol{i} + \boldsymbol{j} + t^4\boldsymbol{k}$ at a point at the point $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.
2. Find the distance from the tangent line to $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

Problem 3 Which one of the following is the same as $\phi = \pi/6$ in spherical coordinates?

1. $z = \sqrt{x^2 + y^2}$ in Cartesian coordinates.
2. $z = 3r$ in cylindrical coordinates.
3. $z = \sqrt{r}$ in cylindrical coordinates.
4. $z^2 = 3(x^2 + y^2)$ in Cartesian coordinates.
5. None of the above.

Problem 4 Consider the curve $\mathbf{r}(t) = \sqrt{2}\cos t\boldsymbol{i} + \sin t\boldsymbol{j} + \sin t\boldsymbol{k}$.

1. Find the unit tangent vector $\mathbf{T}(t)$ and the principal normal unit vector $\mathbf{N}(t)$.
2. Compute the curvature κ

Problem 5 Evaluate the integrals

1. $$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dxdy$$
2. $$\int_0^1 \int_{\sin^{-1} y}^{\frac{\pi}{2}} \cos x \sqrt{1 + \cos x^2}\, dxdy.$$

Problem 6 A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ and outside the circle $x^2 + y^2 = 1$. Find the mass if the density at any point is the inverse to its distance from the origin.

Recall that directional derivative of a function $f(x, y)$ at a point x_0, y_0 along a unit vector $\boldsymbol{u} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$ is

$$D_{\boldsymbol{u}}(x_0, y_0) = \lim_{t\to 0} \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}$$

If f is a differentiable function then $D_{\boldsymbol{u}} f(x, y) = f_x \cos\theta + f_y \sin\theta$. Note that $a\cos\theta + b\sin\theta$ has the greatest value (as a function of θ) when $a\boldsymbol{i} + b\boldsymbol{j}$ and $\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$ are pointing in the same direction.

Problem 7

1. Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin(xy)$ at the point $(1, 0)$ has the value 1.

2. Find all points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\boldsymbol{i} + \boldsymbol{j}$.

3. Find the differential of $z = e^{x+y} \ln(y^2)$ and the linear approximation at $(2, 2)$

Problem 8 Show that the following limits do not exist:

1. $$\lim_{(x,y)\to(0,0)} \frac{x + \sin y}{2x + y}$$

2. $$\lim_{(x,y)\to(0,0)} \frac{7x^2 y(x - y)}{x^4 + y^4}$$

Problem 9 Show that the surfaces $z = 7x^2 - 12x - 5y^2$ and $xyz^2 = 2$ intersect orthogonally at the point $(2, 1, -1)$.

Problem 10 Determine the global max and min of the function $f(x, y) = x^2 - 2x + 2y^2 - 2y + 2xy$ over the region $-1 \le x \le 1, 0 \le y \le 2$.

Problem 11

1. Let $f(x, y) = \sin(x^2 + y^2) + \arcsin(y^2)$. Calculate:

$$\frac{\partial^2 f}{\partial x \partial y}$$

2. If $z = f(x, y)$, where $x = r\cos\theta, y = r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial^2 z}{\partial r}$$

Problem 12 Let $\mathbf{F}$ be the plane vector field $xy^2\boldsymbol{i} + yx^2\boldsymbol{j}$.

1. Is $\mathbf{F}$ a conservative vector field? Why?

2. Calculate the divergence of $\mathbf{F}$.

Problem 13 Determine the surface given by the parametric representation

$$r(u, v) = u\boldsymbol{i} + u\cos(v)\boldsymbol{j} + u\sin(v)\boldsymbol{k}$$

Problem 14 Find the equation of the tangent plane to the surface given by

$$r(u, v) = u\boldsymbol{i} + 2v^2\boldsymbol{j} + (u^2 + v)\boldsymbol{k}$$

at the point $(2, 2, 3)$.

Problem 15 Find the arc length of the curve

$$\mathbf{r}(t) = t^2\boldsymbol{i} + (\sin t - t\cos t)\boldsymbol{j} + (\cos t + t\sin t)\boldsymbol{k}, 0 \leq t \leq \pi.$$

Problem 16 Let $\mathbf{F} = (y^2 + x)\boldsymbol{i} - (x^2 - y)\boldsymbol{j} + z\boldsymbol{k}$.

1. Find curl$\mathbf{F}$.

2. Find div$\mathbf{F}$.

Problem 17

1. Calculate the line integral $\int_C xyds$ if C is the portion of the unit circle in the first quadrant (i.e. $x^2 + y^2 = 1$ with $x \geq 0, y \geq 0$).

2. Let R be the region in xy-plane defined by $x^2 + y^2 > 1$. Show that $\mathbf{F} = \frac{-y\boldsymbol{i}+x\boldsymbol{j}}{x^2+y^2}$ is a conservative vector field on R.

3. Let C be the path (e^t, t), $1 \leq t \leq 2$. Evaluate the integral $\int_C \frac{-ydx+xdy}{x^2+y^2}$

4. Evaluate $\int_C \frac{-ydx+xdy}{x^2+y^2}$, where $C := \{(x, y) : x^2 + y^2 = 9\}$.

5. Let C be boundary of a square D. Compute $\int_C xy^2dx + (x^2y + 2x)dy$ as a function of Area(D).

6. Let L be the boundary of the half-disk $\{(x, y)|x^2 + y^2 \leq 1, x \geq 0\}$. Let $\mathbf{F}_1 = (x - y)\boldsymbol{i} - \boldsymbol{j}$. Find by evaluating a line integral the outward flux of $\mathbf{F}_1$ across L

7. Let $\mathbf{F}_2 = (x^2 + y^2)\boldsymbol{i} + (x^2 - y^2)\boldsymbol{j}$. Use Greens theorem to find the outward flux of $\mathbf{F}_2$ across L.

Problem 18 1. If $\mathbf{F} = 3x\boldsymbol{i} + 2xz\boldsymbol{j} + 3\boldsymbol{k}$, evaluate the flux of $\mathbf{F}$ across the surface $S : z = 0, 0 \leq x \leq 1, 0 \leq y \leq 2$ (where the normal is to be in the positive z direction).

2. Find the flux of the field $\mathbf{F} = z\boldsymbol{k}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant (this is the 1/8-th of space in which x, y and z are all ≤ 0) with normal taken in the direction away from the origin.

Problem 19 1. Compute the surface area of the graph $z = 1 + x^2 + y$ over the triangular region formed by the points $(0, 0), (3, 0)$, and $(3, 2)$.

2. Find the integral $\int\int_S \mathbf{A} \cdot dS$ for $\mathbf{A} = x\boldsymbol{i} + z\boldsymbol{j}$ and the surface S of a sphere of radius a.

Problem 20 1. Evaluate $\int\int\int_B (x^2 + y^2 + z^2)^2 dxdydz$, where B the ball with center the origin and radius 3.

2.

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2} dzdxdy$$

3. Give five other iterated integrals that are equal to

$$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x, y, z) dzdxdy$$

4. Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 9$ and outside the cylinder $x^2 + y^2 = 1$.

Problem 21 Find Jacobian of

$$x = u^2 - \sin(u + v), y = e^u \cos v$$