MAT127 Fall 2023

Practice Midterm I

Time and location of the test:

Tue Oct 3 8:30-9:50 pm, Frey 100

Exam will cover sections 8.1-8.6 inclusive.

The actual test will contain 5 problems (some multipart)

Problem 1. Compute limits

(1) $\lim_{n\to\infty} n \ln \frac{1}{n}$
(2) $\lim_{n\to\infty} \sqrt[n^2]{n^5}$
(3) $\lim_{n\to\infty} \frac{1}{1+(-2)^n}$
(4) $\lim_{n\to\infty} \frac{(2n)!}{(3n)!}$
(5) $\lim_{n\to\infty} \sin(\pi + \frac{1}{n})$
(6) $\lim_{n\to\infty} \sin(\lfloor \pi + \frac{1}{n} \rfloor)$, where $x \to \lfloor x \rfloor$ is the integral part of x.
(7) $\lim_{n\to\infty} a_n$ where $a_1 = \pi$, $a_{n+1} = \pi^{-1}(a_n + \pi^2), n \geq 1$

Solution. (1) To find the limit

$$\lim_{n\to\infty} n \ln \frac{1}{n} = -\lim_{n\to\infty} n \ln(n) = -\lim_{x\to\infty} x \ln(x)$$

$x \leq x \ln(x)$ so $\infty = \lim_{x\to\infty} x \leq \lim_{x\to\infty} x \ln(x)$ and $\lim_{n\to\infty} n \ln \frac{1}{n} = -\infty$

(2) To find the limit $\lim_{n\to\infty} \sqrt[n^2]{n^5}$, we can use the properties of limits and exponentials.

First, we rewrite the expression as:

$$\lim_{n\to\infty} e^{\ln\left(\sqrt[n^2]{n^5}\right)}$$

Next, we simplify by moving the exponent $\frac{1}{n^2}$ inside the logarithm:

$$\lim_{n\to\infty} e^{\frac{1}{n^2}\ln(n^5)}$$

Now, we have an indeterminate form $\frac{1}{\infty} \cdot \infty$. To evaluate this, we use L'Hpital's Rule:

$$\lim_{n\to\infty} e^{\frac{1}{n^2}\ln(n^5)} = \lim_{x\to\infty} e^{\frac{1}{x^2}\ln(x^5)} = e^{5\lim_{x\to\infty}\frac{\ln(x)}{x^2}} = e^{5\lim_{x\to\infty}\frac{1/x}{2x}} = e^0 = 1$$

So, $\lim_{n\to\infty} \sqrt[n^2]{n^5}$ is equal to 1.

(3) We compute $\lim_{n\to\infty} \frac{1}{1+(-2)^n}$ n even an odd and see that the limits agree: Even case:

$0 \leq \lim_{n\to\infty} \frac{1}{1+(-2)^{2n}} = \lim_{n\to\infty} \frac{1}{1+4^n} \leq \lim_{n\to\infty} \frac{1}{4^n} = 0$

Odd case: $\lim_{n\to\infty} \frac{1}{1+(-2)^{2n+1}} = \lim_{n\to\infty} \frac{1}{1-2\cdot 4^n} = -\lim_{n\to\infty} \frac{1}{2\cdot 4^n-1}$ We know $\lim_{n\to\infty} 2\cdot 4^n - 1 = \infty \Rightarrow \lim_{n\to\infty} \frac{1}{2\cdot 4^n-1} = 0$. Both limits agree and equal to zero.

(4) To find the limit $\lim_{n\to\infty} \frac{(2n)!}{(3n)!}$, we can use the concept of factorials and properties of limits.

First, we can express the factorials as products of integers:

$$\frac{(2n)!}{(3n)!} = \frac{(2n)(2n-1)(2n-2)\dots(2)(1)}{(3n)(3n-1)(3n-2)\dots(2)(1)}$$

Now, we simplify the expression by canceling out common factors in the numerator and denominator:

$$\frac{(2n)!}{(3n)!} = \frac{1}{3n} \cdot \frac{1}{3n-1} \cdot \frac{1}{3n-2n-1} \leq \frac{1}{3n}$$

By squeeze theorem since $0 \leq \frac{(2n)!}{(3n)!} \leq \frac{1}{3n}$, $0 \leq \lim_{n\to\infty} \frac{(2n)!}{(3n)!} \leq \lim_{n\to\infty} \frac{1}{3n} = 0$. So $\lim_{n\to\infty} \frac{(2n)!}{(3n)!} = 0$.

(5) sin() is a continuous function. $\lim_{n\to\infty} \pi + \frac{1}{n} = \pi$. So $\lim_{n\to\infty} \sin(\pi + \frac{1}{n}) = \sin(\pi) = 0$.

(6) $\lfloor x \rfloor$ has discontinuity only at integers. Thus $\lim_{n\to\infty} \lfloor \pi + \frac{1}{n} \rfloor = \lfloor \pi \rfloor$. By continuity of $\sin(x)$ $\lim_{n\to\infty} \sin(\lfloor \pi + \frac{1}{n} \rfloor) = \sin(\lfloor \pi \rfloor)$

(7) Denote $L = \lim_{n\to\infty} a_n$. Equation $a_{n+1} = \pi^{-1}(a_n + \pi^2)$ implies that

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (\pi^{-1}(a_n + \pi^2)) =$$

$$\pi^{-1} \lim_{n\to\infty} ((a_n + \pi^2)) = \pi^{-1}(\lim_{n\to\infty} a_n) + \pi = \pi^{-1} L + \pi$$

We conclude that $L = \frac{\pi}{1-\pi^{-1}}$ if we assume that the limit L exists. The sequence a_n is bounded: evidently $a_1 = \pi \leq \frac{\pi}{1-\pi^{-1}} = \pi(1+\pi^{-1}+\pi^{-2}+\cdots)$. If $a_n \leq \pi(1+\pi^{-1}+\pi^{-2}+\cdots) \Rightarrow a_{n+1} = \pi^{-1}(a_n+\pi^2) = \pi^{-1}a_n+\pi \leq \pi^{-1}\pi(1+\pi^{-1}+\pi^{-2}+\cdots)+\pi = \pi(1+\pi^{-1}+\pi^{-2}+\cdots) = L$. The sequence a_n is increasing: $a_1 = \pi \leq \pi^{-1}a_1+\pi = 1+\pi$, $a_n \leq a_{n+1} \Rightarrow \pi^{-1}a_n+\pi \leq \pi^{-1}a_{n+1}+\pi \Rightarrow a_{n+1} \leq a_{n+2}$. Thus the sequence is increasing and bounded. From this we conclude that $\lim a_n = L$

□

Problem 2.

(1) Represent $-4.555555\ldots$ as a rational
(2) $s_n = \sum_{k=0}^{n} a_k = \frac{2+|n|}{3+|n|}$. Find a_k and $\sum_{k=0}^{\infty} a_k$
(3) Compute if exists $\sum_{n=0}^{\infty} \frac{2^n-3^n}{7^n}$, $\sum_{n=0}^{\infty} \frac{7^n-2^n}{3^n}$, $\sum_{n=0}^{\infty} \frac{3^n-7^n}{2^n}$
(4) Compute if exists $\sum_{n=0}^{\infty} \frac{1}{7^{(-1)^n n}}$,
(5) Of the following series listed below, select ALL which are geometric series.
   (a) $\sum_{n=1}^{\infty} \frac{\pi^{n-1}}{n}$
   (b) $\sum_{n=1}^{\infty} 2^{n^{-1}}$
   (c) $\sum_{n=0}^{\infty} \frac{5^{2n+7}}{3^{10n-4}}$
   (d) $\sum_{n=0}^{\infty} n^{-\frac{1}{2}}$
   (e) $\sum_{n=0}^{\infty} e^{-2n+4}$

Solution. (1) $-4.555555\cdots = -4 - \sum_{n=1}^{\infty} \frac{5}{10^k} = -4 - \frac{5}{10}\frac{1}{1-1/10} = -41/9$.

(2) The absolute value sign is irrelevant since $n > 0$. a_k is equal to $s_k - s_{k-1} = \frac{2+k}{3+k} - \frac{2+k-1}{3+k-1} = \frac{2+k}{(3+k)(2+k)}$. $\sum_{k=0}^{\infty} \frac{2+k}{(3+k)(2+k)} = \lim_{n\to\infty} \frac{2+n}{3+n} = \lim_{n\to\infty} \frac{2/n+1}{3/n+1} = 1$.

(3) $\sum_{n=0}^{\infty} \frac{2^n-3^n}{7^n} = \sum_{n=0}^{\infty} \frac{2^n}{7^n} - \sum_{n=0}^{\infty} \frac{3^n}{7^n} = \frac{1}{1-2/7} - \frac{1}{1-3/7}$ we use that $2/7, 3/7 < 1$.

$\sum_{n=0}^{\infty} \frac{7^n-2^n}{3^n} = \sum_{n=0}^{\infty} \frac{7^n}{3^n} - \sum_{n=0}^{\infty} \frac{2^n}{3^n}$. The first series is divergent because $r = 7/3 > 1$. The second series is convergent $r = 2/3$. Overall the difference is diverging.

By the same reason $\sum_{n=0}^{\infty} \frac{7^n-2^n}{3^n}$ is diverging.

(4) We break $\sum_{n=0}^{\infty} \frac{1}{7^{(-1)^n n}}$ into a pair sums according n is even or odd:

$$\sum_{n=0}^{\infty} \frac{1}{7^{(-1)^n n}} = \sum_{n=0}^{\infty} \frac{1}{7^{(-1)^{2n} 2n}} + \sum_{n=0}^{\infty} \frac{1}{7^{(-1)^{2n+1}(2n+1)}}$$

$$\sum_{n=0}^{\infty} \frac{1}{7^{2n}} + \sum_{n=0}^{\infty} 7^{2n+1}$$

The first is converging geometric series with $r = 1/7^2 < 1$. The second is diverging geometric series with $r = 7^2 > 1$. Overall the series is diverging.

(5) (c) and (e) are the only geometric series in the list with $r = \frac{5^2}{3^1 0}$ and $r = e^{-2}$ respectively.

□

Problem 3. (1) $\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = A$. Estimate the error of approximation $R_N = A - \sum_{n=-N}^{N} \frac{1}{1+n^2}$ for $N = 10$.

(2) $\sum_{n=0}^{\infty} \frac{n^2+n}{n^4+3n^2+1}$. Determine whether the sum is convergent.

(3) Give an example of a divergent series $\sum_{n=0}^{\infty} b_n$ an such that $\sum_{n=0}^{\infty} b_n^2$ is convergent. Explain briefly why your series satisfies these two conditions.

(4) Consider the sum $\sum_{n=0}^{\infty} \frac{(-1)^n}{1+n^2}$. We use the sum of the first 10 terms to approximate the sum of this series. Estimate the error involved in this approximation.

Solution. (1)

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = A \Rightarrow R_N = A - \sum_{n=-N}^{N} \frac{1}{1+n^2} =$$

$$\sum_{n=-\infty}^{-N-1} \frac{1}{1+n^2} + \sum_{n=N+1}^{\infty} \frac{1}{1+n^2} = 2 \sum_{n=N+1}^{\infty} \frac{1}{1+n^2} \leq \int_{N+1}^{\infty} \frac{dx}{1+x^2} \leq \int_{N+1}^{\infty} \frac{dx}{x^2} = \frac{2}{N+1}$$

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(2) We want to compare the sequence $a_n = \frac{n^2+n}{n^4+3n^2+1}$ with $b_n = 1/n^2$. Both sequences are positive $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \frac{(n^2+n)n^2}{n^4+3n^2+1} = \lim_{n\to\infty} \frac{(1+1/n)}{1+3/n^2+1/n^4} = 1$. The sum $\sum_{n=1}^{\infty} 1/n^2$ is $p = 2$ series. It is convergent. So is $\sum_{n=1}^{\infty} a_n$.

(3) $b_n = 1/n$ leads to diverging harmonic series $\sum_{n=1}^{\infty} 1/n$. It is diverging because $\sum_{n=1}^{N} 1/n \geq \int_1^N \frac{dx}{x} = \ln(N)$ On the other hand $b_n^2 = 1/n^2$ leads to converging $p = 2$ series $\sum_{n=1}^{\infty} 1/n^2$.

(4) The sequence $b_n = \frac{1}{1+n^2}$ is decreasing. We can use alternating series test to show convergence $\sum_{n=0}^{\infty} \frac{(-1)^n}{1+n^2}$. By the same test the error of approximation $R_N \leq b_{N+1}$. We conclude that $R_{10} \leq \frac{1}{1+(101)^2}$

□

Theorem 0.1. *(Root Test) Let* $\lim_{n\to\infty} |a_n|^{1/n} = L$

(1) *If* $L < 1$ *then* $\sum_{k=1}^{\infty} a_k$ *is convergent*
(2) *If* $L > 1$ *then* $\sum_{k=1}^{\infty} a_k$ *is divergent.*
(3) *If* $L = 1$ *then the Root Test is inconclusive.*

Problem 4. Determine convergence:

(1) $\sum_{n=1}^{\infty} \frac{\cos(\ln(n))}{n!}$
(2) $\sum_{n=1}^{\infty} \frac{x^{n^2}}{(2n)^{2n}}$

Solution. (1) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\cos(\ln(n))}{n!}$. a_n satisfies $|a_n| \leq \frac{1}{n!} = b_n$. By Ratio Test $\frac{|b_{n+1}|}{|b_n|} = \frac{1/(n+1)!}{1/n!} = \frac{1}{n+1} \Rightarrow \lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n\to\infty} 1/(n+1) = 0 \Rightarrow \sum_{n=1}^{\infty} 1/n!$ is convergent $\Rightarrow \sum_{n=1}^{\infty} \frac{|\cos(\ln(n))|}{n!}$ is convergent $\Rightarrow \sum_{n=1}^{\infty} \frac{\cos(\ln(n))}{n!}$ is convergent absolutely $\Rightarrow \sum_{n=1}^{\infty} \frac{\cos(\ln(n))}{n!}$ is convergent.

(2) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x^{n^2}}{(2n)^{2n}}$, $|a_n|^{1/n} = \frac{|x|^{n^2/n}}{(2n)^{2n/n}} = \frac{|x|^n}{(2n)^2}$ Fix x, set $n = y$. We ca use l'Hpital's Rule two times to compute the limit

$$\frac{1}{4} \lim_{y\to\infty} \frac{|x|^y}{y^2} = \frac{1}{4} \lim_{y\to\infty} \frac{\ln(x)|x|^y}{2y} =$$

$$\frac{1}{4} \lim_{y\to\infty} \frac{\ln(x)^2|x|^y}{2}$$

The limit is infinite if $|x| > 1$ and zero if $|x| \leq 1$. Thus the interval of convergence of the series is $|x| \leq 1$.

□

Definition: Series $\sum_{k=1}^{\infty} a_k$ are conditionally convergent if it is convergent but

$$\sum_{k=1}^{\infty} |a_k| = \infty.$$

Problem 5. Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

(1) $\sum_{n=1}^{\infty} \frac{(n+2)!}{n!5^n}$

(2) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{1+n^2}$

Solution. (1) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(n+2)!}{n!5^n}$,

$$a_n = \frac{(n)!(n+1)(n+2)}{n!5^n} = \frac{(n+1)(n+2)}{5^n}$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1+1)(n+1+2)5^{-(n+1)}}{(n+1)(n+2)5^{-n}} = \frac{1}{5}\frac{(n+2)(n+3)}{(n+1)(n+2)} = \frac{1}{5}\frac{n+3}{n+1}$$

Since $\lim \frac{|a_{n+1}|}{|a_n|} = \lim \frac{1}{5}\frac{n+3}{n+1} = 1/5 < 1$ by Ratio Test the series is absolutely and unconditionally convergent.

(2) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{n}{1+n^2}$ $f(n) = a_n$ where $f(x) = \frac{x}{1+x^2}$. $f'(x) = \frac{1-x^2}{(x^2+1)^2}$ is negative for $x > 1 \Rightarrow f(x)$ is decreasing for $x \geq 1$. By Alternating Series test $\sum_{n=1}^{\infty} a_n$ is convergent. However, the sum $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{1+n^2}$ is diverging. To see this we compare it with harmonic series $b_n = 1/n$. $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \frac{n^2}{1+n^2} = 1$ By the limit comparison test $\sum_{n=1}^{\infty} |a_n|$ is diverging because harmonic series $\sum_{n=1}^{\infty} b_n$. Thus $\sum_{n=1}^{\infty} a_n$ is converging only conditionally.

□

Problem 6. Find the radius R and interval of convergence for the power series

(1) $\sum_{n=0}^{\infty} \frac{n(x-2)^{3n}}{(n+1)(n+2)}$

(2) $\sum_{n=0}^{\infty} \frac{n!x^n}{(n+3)!(n+6)!}$

Solution. (1) $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{n(x-2)^{3n}}{(n+1)(n+2)}$

$$|a_{n+1}|/|a_n| = \frac{(n+1)|x-2|^{3(n+1)}(n+1)(n+2)}{(n+1+1)(n+1+2)(n)|x-2|^{3n}}$$

$$|x-2|^3 \frac{(n+1)^2(n+2)}{(n+2)(n+3)n}$$

$\lim |a_{n+1}|/|a_n| = |x-2|^3$ The series is convergent if $|x-2| < 1$. In other words if x satisfies $1 < x < 3 \Rightarrow$ the series is convergent.

If $x = 3$ the series become $\sum_{n=0}^{\infty} \frac{n(3-2)^{3n}}{(n+1)(n+2)} = \sum_{n=0}^{\infty} \frac{n}{(n+1)(n+2)}$ We can use $b_n = 1/n$ to show that $\lim a_n/b_n = 1$. Since the harmonic series $\sum 1/n$ is diverging $\Rightarrow \sum_{n=0}^{\infty} \frac{n}{(n+1)(n+2)}$ is diverging.

If $x = 1$ the series become $\sum_{n=0}^{\infty} \frac{n(1-2)^{3n}}{(n+1)(n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n n}{(n+1)(n+2)}$. $f(x) = \frac{x}{(x+1)(x+2)}$ is decreasing for $x > 2^{1/2}$ ($f'(x) = \frac{2-x^2}{(x+1)^2(x+2)^2}$). We see that $a_n = \frac{n}{(n+1)(n+2)} = f(n)$ is a decreasing sequence with $\lim a_n = 0$. By the Alternating Series test $\sum_{n=0}^{\infty} \frac{(-1)^n n}{(n+1)(n+2)}$ is convergent. The interval of convergence for our series is $1 \leq x < 3$.

(2) $\sum_{n=0}^{\infty} \frac{n!x^n}{(n+3)!(n+6)!}$.The solution to this problem closely mirrors the the previous problem.

□

Recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Problem 7. Find a power series representation (at 0) of the function

(1) $\int_0^x \frac{\sin(t)}{t} dt$

(2) $\log \frac{1+x}{1-x}$

(3) $\frac{1+x}{1-x^3}$

and compute its radius of convergence.