Midterm II

MAT127C Fall 2023

Wed Nov 8, 8:30-9:50 pm, Frey 100

Name: (please print)	**ID #**:
Your lecture: (see list below)	

Lecture 01	MW 4:00- 5:20pm Harriman Hall 108	Insung Park
Lecture 02	TuTh 10:00-11:20am Earth&Space 131	Mikhail Movshev
Lecture 03	TuTh 5:30- 6:50pm Earth&Space 131	Charles Cifarelli

You are allowed to used calculators on this test

No notes, books. You must show your reasoning, not just the answer. Answers without justification will get only partial credit.

Please cross out anything that is not part of your solution — e.g., some preliminary computations that you didn't need. All answers should be simplifies if possible — e.g., sin(0) should be replaced by 0. However, unless instructed, do not replace exact answers by approximate ones — e.g. do not replace $\sqrt{2}$ by 1.41

	1	2	3	4	5	**Total**
	20pt	20pt	20pt	20pt	20pt	100pts
Grade						

Problem 1. Consider the following differential equations

a) $y' = -x/y$ b) $y^2 y' = 1/3$ c) $y' = -y - 1$ d) $x^{-1} y' = 1$

1. Each of the following functions is a solution to one of the differential equations listed above. Indicate which differential equation with the corresponding letter (a,b,c or d) on the given line.

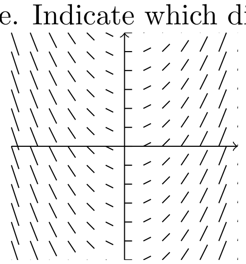
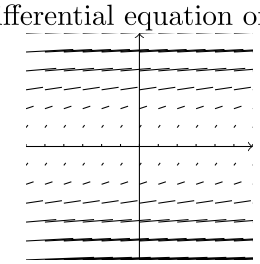
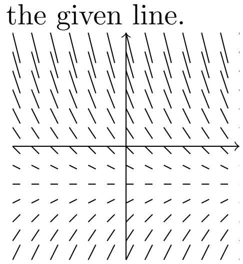
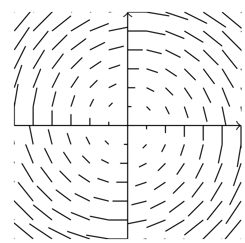
$y(x) = ce^{-x} - 1$ ________________

$y(x) = \sqrt{c - x^2}$ ________________

$y(x) = x^2/2 + c$ ________________

$y(x) = \sqrt[3]{c + x}$ ________________

2. Each of the following slope fields belongs to one of the differential equations listed above. Indicate which differential equation on the given line.

________________ ________________ ________________ ________________

3. Find the equilibrium solutions of the differential equations given above (if any). Write the equation of the equilibrium solutions in the space provided below. If the equation does not have equilibrium solutions, write none.

a ________________ b ________________ c ________________ d ________________

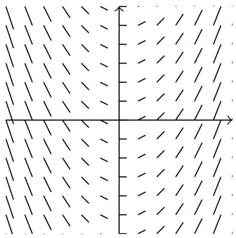
Solution. **1.** $y(x) = ce^{-x} - 1$ (c)

$y(x) = \sqrt{c - x^2}$ (a)

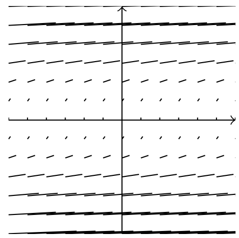
$y(x) = x^2/2 + c$ (d)

$y(x) = \sqrt[3]{c + x}$ (b)

2.



(d)



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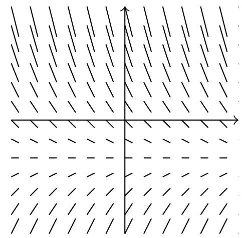
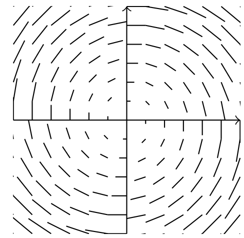


(c)



(a)

3. None b None c $y(x) = -1$ d None

□

Problem 2.

1. Use the binomial series to find the fourth derivative $f^{(4)}(0)$ and the twenty-first derivative $f^{(21)}(0)$ for the function $f(x) = \dfrac{1}{(4+x^2)^2}$
2. Find the degree 3 Taylor polynomial with $a = 0$ of $g(x) = 2 + x^2 + \dfrac{x}{(4+x^2)^2}$

Solution. **1.** To find the fourth and twenty-first derivatives of the function $f(x) = \frac{1}{(4+x^2)^2} = \frac{1}{(4)^2}\frac{1}{(1+x^2/4)^2}$, we can utilize the binomial series expansion. The binomial series allows us to express the function as a power series and find the desired derivatives at $x = 0$.

The binomial series for $(1+z)^n$ is given by:

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!} z^k$$

In our case, we have $\alpha = -2$

$$\frac{1}{(1+z)^2} = \sum_{k=0}^{\infty} \frac{-2(-2-1)(-2-2)\ldots(-2-k+1)}{k!} z^k = \sum_{k=0}^{\infty} (-1)^k \frac{2(2+1)(2+2)\ldots(2+k-1)}{k!} z^k$$

and $z = \frac{x^2}{4}$ So

$$f(x) = \frac{1}{16}\frac{1}{(1+x^2/4)^2} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{16} \frac{2(2+1)(2+2)\ldots(2+k-1)}{k!} \frac{x^{2k}}{4^k} = \tag{1}$$
$$= \frac{1}{16} - \frac{x^2}{32} + \frac{3}{256}x^4 + \ldots$$

We identify it with the Taylor expansion formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \tag{2}$$

in our case $a = 0$. In the formula (1) we do not see x^n with odd n. This means that $\frac{f^{(n)}(0)}{n!} = 0$ for odd n. In particular $f^{(21)}(0) = 0$. If $n = 4 = 2 \times 2$ we read off the coefficient $\frac{f^{(4)}(0)}{4!}$ from (1) to be equal to

$$(-1)^2 \frac{1}{16} \frac{2\ldots(2+2-1)}{2!} \frac{1}{4^2} = \frac{1}{16}\frac{2\cdot 3}{2!}\frac{1}{4^2} = \frac{3}{256} = \frac{f^{(4)}(0)}{24}$$

and $f^{(4)}(0) = \frac{3}{256} \times 24 = \frac{9}{32}$

2. We will keep terms of degree ≤ 3

$$g(x) = 2 + x^2 + \frac{x}{(4+x^2)^2} = 2 + x^2 + x\frac{1}{16}\left(1 - \frac{x^2}{2} + \frac{3}{32}x^4 + \dots\right)$$

$$= 2 + x^2 + \frac{x}{16} - \frac{x^3}{32} + \cdots$$

The Taylor polynomial of degree tree is $T_3(x) = 2 + x^2 + \frac{x}{16} - \frac{x^3}{32}$.

$\square$

Problem 3. Consider the function $f(x) = \sin(\frac{x}{2})$.

1. Calculate the degree three Taylor polynomial $T_3(x)$ with $a = 0$.

2. Use Taylor's inequality to estimate the error of the approximation $f(x) \approx T_3(x)$ on the interval $|x| \leq 2$.

3. Determine the number of terms of the Maclaurin series for $f(x)$ are required to estimate the number $\sin(1)$ with an error less than .01.

Solution. The Taylor polynomial of a function $f(x)$ of degree n centered at a is given by:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where $T_n(x)$ is the Taylor polynomial, $f(a)$ is the value of the function at $x = a$, and $f^{(k)}(a)$ represents the k-th derivative of the function $f(x)$ evaluated at $x = a$.

1. We want to find the Taylor polynomial of degree 3 for the function $f(x) = \sin(\frac{x}{2})$ centered at $a = 0$.

1. Calculate the function and its derivatives:

$$f(x) = \sin\left(\frac{x}{2}\right)$$
$$f(0) = \sin(0) = 0$$

2. Calculate the first derivative:

$$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$
$$f'(0) = \frac{1}{2}\cos(0) = \frac{1}{2}$$

3. Calculate the second derivative:

$$f''(x) = -\frac{1}{4}\sin\left(\frac{x}{2}\right)$$
$$f''(0) = -\frac{1}{4}\sin(0) = 0$$

4. Calculate the third derivative:

$$f'''(x) = -\frac{1}{8}\cos\left(\frac{x}{2}\right)$$
$$f'''(0) = -\frac{1}{8}\cos(0) = -\frac{1}{8}$$

5. Assemble the Taylor polynomial of degree 3:

$$\begin{aligned}T_3(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 \\ &= 0 + \frac{1}{2}x + 0 - \frac{1}{8}\cdot\frac{x^3}{3!} \\ &= \frac{1}{2}x - \frac{1}{48}x^3\end{aligned}$$

The Taylor polynomial of degree 3 for the function $f(x) = \sin\left(\frac{x}{2}\right)$ centered at $a = 0$ is given by $T_3(x) = \frac{1}{2}x - \frac{1}{48}x^3$.

2. We have the approximation $f(x) \approx T_3(x)$ on the interval $|x| \leq 2$.

According to Taylor's inequality, we have:

$$|R_3(x)| \leq \frac{M}{4!}|x|^4$$

Now, let's find the value of M, which is the maximum of $|f^{(4)}(x)|$ on the interval $[-2, 2]$. By utilizing our previous computations we see that $f^{(4)}(x) = \frac{1}{16}\sin\left(\frac{x}{2}\right)$ and $|f^{(4)}(x)| = |\frac{1}{16}\sin\left(\frac{x}{2}\right)| \leq \frac{1}{16}$.

$|R_3(x)| \leq \frac{1}{16\times 24}|x|^4 \leq \max_{-2\leq x\leq 2}\frac{1}{16\times 24}|x|^4 = \frac{1}{16\times 24}|2|^4 = \frac{1}{24}$.

3. Our previous computation of the derivatives generalizes to the following expressions for $f^{4n}(x)$, $f^{4n+1}(x)$, $f^{4n+2}(x)$, and $f^{4n+3}(x)$:

$$f^{4n}(x) = \frac{1}{2^{4n}}\sin\left(\frac{x}{2}\right), \quad f^{4n+1}(x) = \frac{1}{2^{4n+1}}\cos\left(\frac{x}{2}\right),$$
$$f^{4n+2}(x) = -\frac{1}{2^{4n+2}}\sin\left(\frac{x}{2}\right), \quad f^{4n+3}(x) = -\frac{1}{2^{4n+3}}\cos\left(\frac{x}{2}\right).$$

In all cases, it holds that $|f^n(x)| \leq \frac{1}{2^n}$, since $|\pm\cos(x/2)|$ and $|\pm\sin(x/2)|$ are both less than or equal to 1.

Now, we can use Taylor's inequality to estimate the error of the approximation:

$$|R_n(x)| \leq \frac{1}{2^{n+1}(n+1)!}x^{n+1}$$

We want to find n such that $|R_n(1)| < 0.01$. From the above inequality, we have:

$$|R_n(1)| \leq \frac{1}{2^{n+1}(n+1)!} < 0.01$$

By computing the values of $\frac{1}{2^{n+1}(n+1)!}$ for $n = 1$ (resulting in $\frac{1}{8}$), $n = 2$ (resulting in $\frac{1}{48}$), and $n = 3$ (resulting in $\frac{1}{384}$), it's clear that $\frac{1}{384} < 0.01$.

Therefore, we conclude that it suffices to use 3 terms in the Taylor series to ensure that the error is less than 0.01.

□

Problem 4. A stone is dropped from the deck of a ship into the ocean. The downward velocity $v(t)$ of the stone satisfies the following differential equation:

$$\frac{dv}{dt} = 10 - 3v^2$$

Suppose the stone enters the water with an initial velocity of 1 m/s. You should use Euler's method with three time steps to approximate the velocity of the stone one second after it is dropped into the water.

Fill in the table below with the approximations at each step. Be sure to include all your work to receive full credit.

Time			
Velocity			

Solution. We are using Euler's method to approximate the velocity of the stone one second after it is dropped into the water. The given differential equation is:

$$\frac{dv}{dt} = 10 - 3v^2$$

The initial conditions are:

$$\begin{aligned} v_0 &= 1\,\text{m/s} \quad \text{(initial velocity)} \\ \Delta t &= \frac{1}{3}\,\text{second} \quad \text{(time step)} \\ t_1 &= \frac{1}{3}\,\text{second} \\ t_2 &= \frac{2}{3}\,\text{second} \\ t_3 &= 1\,\text{second} \end{aligned}$$

Now, let's calculate the values at each time step:

1. At $t_1 = \frac{1}{3}$ seconds:

$$v_1 = v_0 + \frac{dv}{dt}\Delta t$$

Substituting the values:

$$v_1 = 1 + \left(10 - 3 \cdot (1^2)\right) \cdot \frac{1}{3} = \frac{10}{3}\,\text{m/s}$$

2. At $t_2 = \frac{2}{3}$ seconds:

$$v_2 = v_1 + \frac{dv}{dt}\Delta t$$

Substituting the values:

$$v_2 = \frac{10}{3} + \left(10 - 3 \cdot \left(\frac{10}{3}\right)^2\right) \cdot \frac{1}{3} = -\frac{40}{9} \text{ m/s}$$

3. At $t_3 = 1$ second:

$$v_3 = v_2 + \frac{dv}{dt}\Delta t$$

Substituting the values:

$$v_3 = -\frac{40}{9} + \left(10 - 3 \cdot \left(-\frac{40}{9}\right)^2\right) \cdot \frac{1}{3} = -\frac{1690}{81}$$

Time	$\frac{1}{3}$ second	$\frac{2}{3}$ second	1 second
Velocity	$\frac{10}{3}$ m/s	$-\frac{40}{9}$ m/s	$-\frac{1690}{81}$ m/s

□

Problem 5.

1. Find the general solution of

$$x^{-2}y' = y^2$$

2. Find the solution of

$$y' + x^3 y = x^3$$

which satisfies $y(1) = 2$

Solution. **1.** To find the general solution to the differential equation $\frac{dy}{dx} = x^2y^2$, we can use separation of variables.

Separating the variables:

$$\frac{dy}{y^2} = x^2 dx$$

Now, integrate both sides:

$$-\frac{1}{y} = \int \frac{1}{y^2}\, dy = \int x^2\, dx = \frac{1}{3}x^3 + C$$

Where C is the constant of integration. To solve for y, we can take the reciprocal of both sides:

$$y = -\frac{1}{\left(\frac{1}{3}x^3 + C\right)}$$

Simplifying and absorbing 3 in the constant:

$$y = -\frac{3}{x^3 + C}$$

2. We will solve the differential equation $y' = x^3(1 - y)$ with the initial condition $y(1) = 2$ using separation of variables.

The differential equation is in the form $y' = f(x)g(y)$, where $f(x) = x^3$ and $g(y) = 1 - y$. We can separate variables and integrate:

$$\frac{dy}{1-y} = x^3\,dx$$

Now, integrate both sides:

$$\int \frac{1}{1-y}\,dy = \int x^3\,dx$$

To integrate the left side, we can use a substitution. Let $u = 1-y$, so $-du = dy$:

$$-\int \frac{1}{u}\,du = \int x^3\,dx$$

This simplifies to:

$$-\ln|u| = \frac{x^4}{4} + C_1$$

Now, substitute back for u:

$$-\ln|1-y| = \frac{x^4}{4} + C_1$$

Where C_1 is the constant of integration. Exponentiate both sides to solve for $1-y$:

$$1 - y = \pm e^{-C_1} e^{-\frac{x^4}{4}}$$

Since $\pm e^{-C_1}$ is a constant, we can write it as a single constant, $C_2 = \pm e^{-C_1}$, so:

$$1 - y = C_2 e^{-\frac{x^4}{4}}$$

Solve for y:

$$y = 1 - C_2 e^{-\frac{x^4}{4}}$$

Now, we are given the initial condition $y(1) = 2$. Substitute this into the equation to find C_2:

$$2 = 1 - C_2 e^{-\frac{1^4}{4}}$$

Solving for C_2

$$C_2 = -e^{\frac{1}{4}}$$

So, the final solution is:

$$y = 1 + e^{\frac{1}{4} - \frac{x^4}{4}}$$

This is the solution to the initial value problem $y' = x^3(1 - y)$ with the initial condition $y(1) = 2$.

□