MAT127 Fall 2023

Practice test III

Location of the Final Exam: Mon. Dec. 18, 11:15-1:45 pm room Javits 100

Final Exam will cover sections 7.1-7.6, 8.1-8.8,

You will be allowed to use calculators. The Final Exam will contain 8 problems (some multipart)

The assessment will draw upon the content discussed in class sessions, homework assignments, quizzes, practice tests, as well as the material previously examined in the midterms. The test aims to comprehensively evaluate the knowledge acquired throughout these learning components. Be prepared to demonstrate understanding and application of the concepts covered in these areas during the test

This practice test covers only Sections 7.4 to 7.6, second-order differential equations and material related to the radius of convergence. For other content, please consult previous practice tests.

Problem 1. Consider the following system of differential equations describing a predator-prey model (x, y are population sizes measured in hundreds):

$$x' = x(2 - x) + 4xy$$

$$y' = y - xy$$

(1) Which of the two variables represents the prey population?

(2) Suppose $y(t_0) = 0$ for some time t_0. Explain why this implies that $y(t) = 0$ for all values of t.

(3) Describe in words the fate of the predator population when the prey population becomes extinct.

Solution. (1) The variable y represents the prey population due to the negative coefficient, specifically equal to -1, associated with the xy term.

(2) In the equation $y' = y(1 - x)$, $y = 0$ implies the condition $y' = 0$, which in turn implies that the value of $y(t)$ remains constant, indicating that the prey population doesn't undergo any changes.

(3) When the prey population becomes extinct ($y = 0$), the equation for x transforms into $x' = x(2 - x) = 2x(1 - x/2)$, which is the logistic equation. As time t tends towards infinity ($t \to \infty$), the solution approaches $x = 2$, representing the carrying capacity of the predator population. This observation indicates that in such a scenario, the predator population persists without extinction, suggesting the presence of omnivorous or mixed-diet predators in this ecological context.

□

Problem 2. Consider the following assumptions concerning the fraction of a piece of bread covered by mold.

- Mold spores fall on the bread at a constant rate.
- When the proportion covered is small, the fraction of the bread covered by mold increases at a rate proportional to the amount of bread covered.
- When the fraction of the bread covered by mold is large, the growth rate decreases.
- In order to survive, mold must be in contact with the bread.

Using these assumptions, write a differential equation that models the proportion of a piece of bread covered by mold. Explain your model in one or two carefully worded sentences. *Note that there is more than one reasonable model that fits these assumptions.*

Solution. One possible model is

$$P' = kP(1 - P)$$

where P is the fraction of the bread covered by mold. When P is small, the growth is nearly exponential. As P approaches 1, the growth rate decreases (but is still positive).

□

Problem 3. Find the general solutions of the following ODEs.

(a) $y'' + 2y' + y = 0$

(b) $y'' + y' - 6y = 0$

(c) $y'' + 2y' + 4y = 0$

Solution. (a) The characteristic equation is $r^2 + 2r + 1 = 0$. This has a repeated root $r = -1$. The general solution of the ODE is $y(t) = c_1 e^{-t} + c_2 t e^{-t}$.

(b) The characteristic equation is $r^2 + r - 6 = 0$. The solutions are $r = 2, -3$. The general solution of the ODE is $y(t) = c_1 e^{2t} + c_2 e^{-3t}$.

(c) The characteristic equation is $r^2 + 2r + 4 = 0$. The solutions are $r = -1 \pm \sqrt{3}i$. The general solution of the ODE is $y(t) = c_1 e^{-t} \cos(\sqrt{3}t) + c_2 e^{-t} \sin(\sqrt{3}t)$.

$\square$

Problem 4. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and the surrounding temperature. Let $T(t)$ be the temperature of the object and T_s be the surrounding temperature. We get

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant. Suppose that the temperature of the object is 200°F in the beginning and 1 minute later, it has cooled down to 190°F in a room at 70°F. Find the time when the temperature of the object becomes 150°F.

Solution. The general solution to the differential equation $\frac{dT}{dt} = k(T - T_s)$ is obtained through separation of variables and integrating both sides:

Separate the variables:

$$\frac{dT}{T - T_s} = k\,dt$$

Integrate both sides:

$$\int \frac{1}{T - T_s}\,dT = \int k\,dt$$

This gives:

$$\ln|T - T_s| = kt + C$$

Exponentiating both sides to eliminate the logarithm:

$$|T - T_s| = e^{kt+C}$$

Further simplification gives:

$$|T - T_s| = Ce^{kt}$$

Where $C = e^C$ is the constant of integration.

Now, the equation $|T - T_s| = Ce^{kt}$ can be rewritten as:

$$T - T_s = Ce^{kt}$$

or

$$T = Ce^{kt} + T_s$$

This is the general solution to the differential equation $\frac{dT}{dt} = k(T - T_s)$.

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First, let's determine the value of the constant C using the initial condition:

Given $T(0) = 200°$F:

$$200 = Ce^{k\cdot 0} + 70$$

$$C = 130$$

So, the equation becomes:

$$T = 130e^{kt} + 70$$

By the assumption $T(1) = 190 \Rightarrow k = \ln(12/13)$ and $T = 130e^{\ln(12/13)t} + 70 = 130(12/13)^t + 70$

Now, to find the time when $T = 150°$F: we set $130(12/13)^t + 70 = 150 \Rightarrow t = \frac{\log(8/13)}{\log(12/13)} \approx 6.06561$ $\square$

Problem 5. Determine the interval of convergence of the series

(1) $\sum_{n=2}^{\infty} \frac{nx^n}{\ln n}$
(2) $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}}$
(3) $\sum_{n=0}^{\infty} n!x^n$

Solution. (1) Let $a_n = \frac{n \cdot x^n}{\ln n}$. We will use the ratio test to compute the radius of convergence for $\sum_{n=2}^{\infty} a_n$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)\cdot|x^{n+1}|\cdot \ln n}{n\cdot|x^n|\cdot\ln(n+1)}$$

$$= \lim_{n\to\infty} \frac{n+1}{n}\cdot\frac{\ln n}{\ln(n+1)}\cdot|x|$$

After using the l'Hospital's rule we get

$$= 1\cdot|x|$$

Thus, the power series converges if $|x| < 1$ and diverges if $|x| > 1$. The radius of convergence is 1. Analysis of the end-points: $x = 1$. The series becomes $\sum_{n=2}^{\infty} \frac{n}{\ln n}$. The limit $\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x}$ By l'Hospital's rule the last limit is equal to $\lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty \Rightarrow$ the series is divergent.

$x = -1$. The series becomes $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$. As in the case $x = 1$ the series is divergent.

The interval of convergence $-1 < x < 1$.

(2) Use Root Test of convergence If $\sum_{n=1}^{\infty} a_n$. If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$ the series is convergent. If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} > 1$ the series is divergent. In our case

$$|a_n|^{\frac{1}{n}} = \left|\frac{x^n}{n\sqrt{n}}\right|^{\frac{1}{n}} = \frac{1}{n^{\frac{\sqrt{n}}{n}}}|x| = \frac{1}{n^{\frac{1}{\sqrt{n}}}}|x|$$

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{\sqrt{n}}}} = \lim_{x\to\infty} \frac{1}{x^{\frac{1}{\sqrt{x}}}} = \frac{1}{e^{\lim_{x\to\infty} \frac{\ln x}{x^{1/2}}}}$$

We use l'Hospital's rule to compute the last limit:

$$\lim_{x\to\infty}\frac{\ln x}{x^{1/2}}=\lim_{x\to\infty}\cdot\frac{1/x}{1/2x^{-1/2}}=2\lim_{x\to\infty}x^{-1/2}=0$$

and

$$\lim_{n\to\infty}\frac{1}{n^{\frac{1}{\sqrt{n}}}}=1/e^0=1$$

We conclude that

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}}=|x|$$

The series converges for $|x|<1$ and diverges for $|x|>1$. The border line cases:

$x=1$. The series becomes $\sum_{n=1}^{\infty}\frac{1}{n^{\sqrt{n}}}$ If $n>4\Rightarrow\sqrt{n}>\sqrt{4}=2\Rightarrow n^{\sqrt{n}}>n^2$. After taking the reciprocals, we obtain the inequality $\frac{1}{n^{\sqrt{n}}}\leq\frac{1}{n^2}$. We conclude that

$$\sum_{n=4}^{\infty}\frac{1}{n^{\sqrt{n}}}<\sum_{n=4}^{\infty}\frac{1}{n^2}$$

The last series is convergent because it is p-series with $p=2$.

$x=-1$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{\sqrt{n}}}$. This is an alternating series $\sum_{n=1}^{\infty}a_n$ with $a_n=(-1)^n f(n)$. $f(x)=\frac{1}{x^{x^{1/2}}}$. $f'(x)=(e^{-x^{1/2}\ln(x)})'=x^{-\sqrt{x}}\left(-\frac{1}{\sqrt{x}}-\frac{\ln(x)}{2\sqrt{x}}\right)$. The derivative is negative for $x>1$. $f(x)$ is a decreasing function. From $x=1$ case we know that $\lim_{n\to\infty}\frac{1}{n^{\sqrt{n}}}=0$. By alternating series test $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{\sqrt{n}}}$ is convergent.

(3) $a_n=n!x^n$ We will be using the ratio test: $\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|=(n+1)|x|$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1\text{ only if }x=1$$

Interval of convergence is degenerate and consists of one point $x=0$.

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