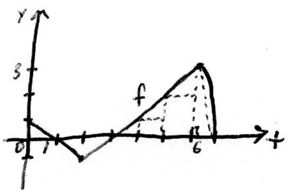
H.W. #5 Solution Key:

5.4 #2, 4, 6, 8, 10, 12, 16, 22

2-) Rough Sketch of $f(t)$:



(a-) Using $g(x)=\int_0^x f(t)\,dt$, we get: $g(0)=\int_0^0 f(t)\,dt=0$, since no area.

$g(1)=g(0)+\int_0^1 f(t)\,dt=0+\frac{1}{2}=\frac{1}{2}$, since area $=\frac{1}{2}b\cdot h=\frac{1}{2}\cdot 1\cdot 1=\frac{1}{2}$

$g(2)=g(1)+\int_1^2 f(t)\,dt=\frac{1}{2}-\frac{1}{2}=0$, since $\int_1^2 f(t)\,dt=-$(Area of right Δ)

$g(3)=g(2)+\int_2^3 f(t)\,dt=0-\frac{1}{2}=-\frac{1}{2}$

$g(4)=g(3)+\int_3^4 f(t)\,dt=-\frac{1}{2}+\frac{1}{2}=0$

$\int_0^4 f(t)\,dt=0$ The sum of areas from $t=0$ to $t=4$ is zero

$g(5)=g(4)+\int_4^5 f(t)\,dt=0+(1+\frac{1}{2})=\frac{3}{2}$, since; (Area of right Δ) $=\frac{1}{2}$ & (Area of square) $=1$

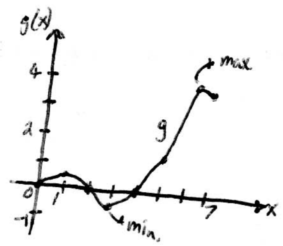
$g(6)=g(5)+\int_5^6 f(t)\,dt=\frac{3}{2}+(2+\frac{1}{2})=4$, since; (Area of right Δ) $=\frac{1}{2}$ & (Area of $\square$) $=2$

(b-) We can clearly see that the area of $f(t)$ from $t=6$ to $t=7$ is slightly greatly than the area of the right Δ drawn in dashed lines. In other words the area of $f(t)$ from $t=6$ to $t=7$ is slightly greater than 1.5 $(A=\frac{1}{2}(3)(1))$, so we can estimate the area to be 1.8 } $g(7)=2+1.8=3.8$

(c-) Using the information from pt. a: max. value of $g(x)$: $g(6)=4$ (at $x=6$)

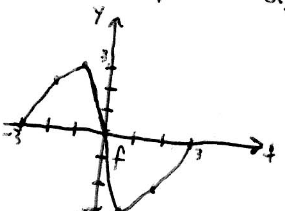
min. value of $g(x)$: $g(3)=-\frac{1}{2}$ (at $x=3$)

(d-) Rough sketch of $g(x)$:



4-) Using the same criteria as in problem #2;

Graph of $f(t)$:



$g(x)=\int_0^x f(t)\,dt$

From the symmetry of the graph, you should infer that the sum of areas from $t=-3$ to $t=3$ is zero, or $\int_{-3}^3 f(t)\,dt=0$

a,b-) $g(-3)=\int_{-3}^{-3} f(t)\,dt=0$ and $g(3)=g(-3)+\int_{-3}^{-2} f(t)\,dt+\int_{-2}^{-1} f(t)\,dt+\int_{-1}^{0} f(t)\,dt+\int_0^1 f(t)\,dt+\int_1^2 f(t)\,dt+\int_2^3 f(t)\,dt$

$g(3)=0+0.7+2.6+1.8-0.7-2.6-1.8=\underline{0}$

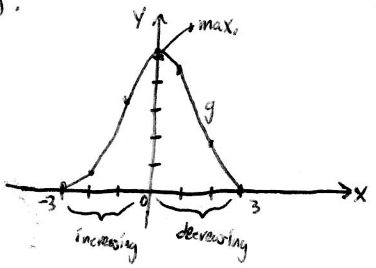
Plugging in our values, we get: $g(-3)=0$; $g(-2)=0.7$; $g(-1)=3.3$; $g(0)=5.1$; $g(1)=4.4$; $g(2)=1.8$; $g(3)=0$

c-) Using our values from pts. a & b: $(g(-3)=0; g(-2)=0.7; g(-1)=3.3; g(0)=5.1)$

$g(x)$ is increasing from $x=-3$ to $x=0$.

d-) $g(x)$ has its maximum value at $x=0$ $(g(0)=5.1)$.

e-) Graph of $g(x)$:



6-) We could use FTC pt. 1, since for $g(x)=\int_0^x (1+\sqrt{t})dt$, $\underline{f(t)=1+\sqrt{t}}$ is continuous from $[0,\infty)$.

a-) $g(x)=\int_0^x (1+\sqrt{t})dt \rightarrow g'(x)=\underline{1+\sqrt{x}}$ (Using $a=0 \rightarrow$ "If $g(x)=\int_0^x f(t)dt$, then $g'(x)=f(x)$")

b-) Using FTC pt. 2, $g(x)=\int_0^x (1+t^{1/2})dt = t+\frac{2}{3}t^{3/2}\Big|_0^x = x+\frac{2}{3}x^{3/2}$

To confirm pt. a, we differentiate $g(x) \rightarrow g'(x)=1+\frac{2}{3}(\frac{3}{2})x^{1/2}=\underline{1+\sqrt{x}}$

8-) Since $f(t)=\ln t$ is continuous from $[1,\infty)$, FTC pt. 1 says:

$g(x)=\int_1^x \ln t\,dt \rightarrow g'(x)=f(x)=\ln x$

10-) Using $\int_a^b f(x)dx = -\int_b^a f(x)dx$:

$F(x)=\int_x^{10} \tan\theta\,d\theta = -\int_{10}^x \tan\theta\,d\theta$

$\boxed{\text{Using } f(\theta)=\tan\theta\text{, we have } F'(x)=f(x)=-\tan x}$

Is this correct? No

Why? $\boxed{f(\theta)=\tan\theta \text{ is not continuous from } [10,\infty)}$

It isn't necessary, but we can modify this problem by choosing $x=10.9$, since: $\boxed{\tan\theta \text{ is continuous for: } \frac{5\pi}{2}<\theta<\frac{7\pi}{2} \rightarrow 7.854<\theta<10.996}$

If this was the case, then $f(\theta)=\tan\theta$ is continuous from $[10,10.9]$, so: $F'(x)=f(x)=\tan x$

12-) Using the chain rule $(\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx})$ with $u=x^2$:

$$\frac{d}{dx}\int_0^{x^2}\sqrt{1+r^3}\,dr = \frac{d}{dx}\int_0^u \sqrt{1+r^3}\,dr \rightarrow \frac{dh}{dx}=\frac{dh}{du}\cdot\frac{du}{dx}$$

$$\frac{dh}{dx}=\left(\frac{d}{du}\int_0^u\sqrt{1+r^3}\,dr\right)\left(\frac{d}{dx}x^2\right)$$

Using FTC pt. 1: $h'(x)=\frac{dh}{dx} = 2x\sqrt{1+u^3} = 2x\sqrt{1+x^6}$

16-) Using $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$:

$$y = \int_{\sin x}^{\cos x} (1+v^2)^{10} dv = \int_{\sin x}^{0} (1+v^2)^{10} dv + \int_0^{\cos x} (1+v^2)^{10} dv$$

We want to get both integrals in "$g(x) = \int_a^x f(t)dt$" form, so we could apply FTC pt. 1 :

$$y = -\int_0^{\sin x} (1+v^2)^{10} dv + \int_0^{\cos x} (1+v^2)^{10} dv$$

Now, we apply the Chain Rule ;

$$y = -\int_0^{u_1} (1+v^2)^{10} dv + \int_0^{u_2} (1+v^2)^{10} dv \; ; \boxed{u_1 = \sin x \;\&\; u_2 = \cos x}$$

$$y' = -\left(\frac{d}{du_1} \int_0^{u_1} (1+v^2)^{10} dv\right)\left(\frac{du_1}{dx}\right) + \left(\frac{d}{du_2}\left(\int_0^{u_2} (1+v^2)^{10} dv\right)\right)\left(\frac{du_2}{dx}\right)$$

Using FTC pt. 1 ;

$$y' = -(1+u_1^2)^{10} \frac{du_1}{dx} + (1+u_2^2)^{10} \frac{du_2}{dx}$$

$$= -(1+\sin^2 x)^{10} \cdot \cos x + (1+\cos^2 x)^{10}(-\sin x)$$

There fore, $y' = -\left(\cos x (1+\sin^2 x)^{10} + \sin x (1+\cos^2 x)^{10}\right)$

22-)

a) Using $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ and "$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$" ;

$$\frac{1}{2}\sqrt{\pi}\left[\operatorname{erf}(b) - \operatorname{erf}(a)\right] = \int_0^b e^{-t^2} dt - \int_0^a e^{-t^2} dt = \int_a^0 e^{-t^2} dt + \int_0^b e^{-t^2} dt = \int_a^b e^{-t^2} dt$$

b-) Plugging $y = e^{x^2} \operatorname{erf}(x)$ into the differential equation:

$$\left(e^{x^2} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt\right)' = 2x e^{x^2} \cdot \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt + \frac{2}{\sqrt{\pi}}$$

$$\frac{d}{dx}\left(e^{x^2} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt\right) = 2x e^{x^2} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt + e^{x^2} \frac{2}{\sqrt{\pi}} \frac{d}{dx} \int_0^x e^{-t^2} dt$$

Using FTC pt. 1 ;

$$\boxed{\left(e^{x^2} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt\right)' = \frac{4x e^{x^2}}{\sqrt{\pi}} \int_0^x e^{-t^2} dt + \frac{2}{\sqrt{\pi}}}$$