Homework #3 Solution Key:

3.1 #20, 24

20) $y = \dfrac{x^2 - 2\sqrt{x}}{x} \rightarrow y = x - 2x^{-\frac{1}{2}}$

The derivative of y w.r.t. x is: $\dfrac{dy}{dx} = y' = 1 - 2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} = 1 + \dfrac{1}{\sqrt{x^3}}$

24) $u = \sqrt[3]{t^2} + 2\sqrt{t^3} = t^{\frac{2}{3}} + 2t^{\frac{3}{2}}$

Applying the power rule $\left(\frac{d}{dx}x^n = nx^{n-1}\right)$ twice, we get:

The derivative of u w.r.t. t is: $\dfrac{du}{dt} = u' = \frac{2}{3}t^{-\frac{1}{3}} + 3t^{\frac{1}{2}} = \dfrac{2}{3\sqrt[3]{t}} + 3\sqrt{t}$

3.2 #4, 12

4) $g(x) = \sqrt{x}\, e^x = x^{\frac{1}{2}} e^x$

Applying the product rule $\left((ab)' = a'b + ab'\right)$ and $\frac{d}{dx}e^x = e^x$, we have:

$$g'(x) = \frac{1}{2}x^{-\frac{1}{2}}e^x + x^{\frac{1}{2}}e^x = e^x\left(\frac{1}{2\sqrt{x}} + \sqrt{x}\right)$$

12) $y = \dfrac{t^3 + t}{t^4 - 2} = \dfrac{t(t^2+1)}{t^4 - 2}$

Using the mnemonic: $\overbrace{\text{Low}}^{t^4-2}$-$\overbrace{\text{dee}}^{\frac{d}{dt}}$-$\overbrace{\text{high}}^{t(t^2+1)}$ minus $\overbrace{\text{high}}^{t(t^2+1)}$-$\overbrace{\text{dee}}^{\frac{d}{dt}}$-$\overbrace{\text{low}}^{t^4-2}$, all over the $\overbrace{\text{square of whats below}}^{(t^4-2)^2}$:

The derivative of y w.r.t. t is: $\dfrac{dy}{dt} = y' = \dfrac{(t^4-2)(t^2+1+2t^2) - t(t^2+1)\cdot(4t^3)}{(t^4-2)^2} = -\dfrac{t^6 + 3t^4 + 6t^2 + 2}{(t^4-2)^2}$

3.3 #2

2) The position of the particle is: $x(t) = \dfrac{t}{1+t^2},\ t \geq 0$

a) Velocity $= v(t) = x'(t) = \dfrac{dx}{dt} = \dfrac{(1+t^2) - t(2t)}{(1+t^2)^2} = \dfrac{1-t^2}{(1+t^2)^2} = \dfrac{(1-t)(1+t)}{(1+t^2)^2}$ in "m/s"

b) Velocity $v>0$ (rightward motion): $(1-t)(1+t) > 0 \rightarrow -1 < t < 1 \xrightarrow{\text{time is non-negative}} 0 < t < 1$

Velocity $v<0$ (leftward motion): $(1-t)(1+t) < 0 \rightarrow t > 1$ or $\overset{\text{reject}}{\cancel{t < -1}} \rightarrow t > 1$

c) From b), we know exactly when the particle changes direction, so:

Total distance traveled during the first 4s: $|x(4) - x(1)| + |x(1) - x(0)| = \left|\frac{4}{17} - \frac{1}{2}\right| + \left|\frac{1}{2} - 0\right| \approx 0.765\,m$

d) $a(t) = x''(t) = v'(t) = \dfrac{dv}{dt} = \dfrac{(1+t^2)^2(-2t) + (1-t^2)(4t)(1+t^2)}{(1+t^2)^4} = \dfrac{2t(t^4 - 2t^2 - 3)}{(1+t^2)^4} \rightarrow$ Conclusion: $a(t) = 0$, when $t = 0s$ or $t = \sqrt{3} \approx 1.$

f) Particle speeds up when $a>0$ and $v>0$ or $a<0$ and $v<0 \rightarrow$ $0<t<1$ and $t>1.73$

Particle slows down when $a<0$ & $v>0$ or $a>0$ & $v<0 \rightarrow$ $1<t<1.73$

3.4 #8, 10

8-) $y = \dfrac{1+\sin x}{x+\cos x}$

Using the "Quotient Rule":

The derivative of y w.r.t. x is: $y' = \dfrac{dy}{dx} = \dfrac{(x+\cos x)(\cos x) - \overbrace{(1+\sin x)(1-\sin x)}^{\substack{\cos^2 x \\ 1-\sin^2 x}}}{(x+\cos x)^2}$

$$\frac{dy}{dx} = \frac{x\cos x}{(x+\cos x)^2}$$

10-) Two ways in solving this problem:

i) Short way (Separate the fraction into two fractions):

$y = \dfrac{1-\sec x}{\tan x} = \cot x - \csc x$, where $\dfrac{1}{\tan x} = \cot x$ & $\dfrac{\sec x}{\tan x} = \csc x$.

$\dfrac{d}{dx}(\cot x - \csc x) = \dfrac{d}{dx}\cot x - \dfrac{d}{dx}\csc x = -\csc^2 x + \csc x \cot x = \csc x(\cot x - \csc x)$

ii-) Using the Quotient Rule (Long Way):

$\dfrac{dy}{dx} = \dfrac{\tan x(-\sec x\tan x) - (1-\sec x)(\sec^2 x)}{\tan^2 x} = -\sec x - (1-\sec x)(\csc^2 x)$; $\csc^2 x = \dfrac{\sec^2 x}{\tan^2 x}$

$\dfrac{dy}{dx} = \sec x\cot^2 x - \csc^2 x$; $\cot^2 x = \csc^2 x - 1$

Therefore $\dfrac{dy}{dx} = \csc x\cot x - \csc^2 x = \csc x(\cot x - \csc x)$; $\sec x\cot x = \csc x$

3.5 #10, 14, 16, 22

10-) Using the Chain rule $\left(\dfrac{dy}{dt} = \dfrac{dy}{du}\cdot\dfrac{du}{dt}\right)$ and in this case: "$\underline{u = 1+\tan t}$":

$f(t) = (1+\tan t)^{1/3} \rightarrow f'(t) = \frac{1}{3}(1+\tan t)^{-2/3}(\sec^2 t)$; $\frac{d}{dt}\tan t = \sec^2 t$

$$f'(t) = \frac{\sec^2 t}{3\sqrt[3]{(1+\tan t)^2}}$$

14-) $y = 3\cot(n\theta) \rightarrow \dfrac{dy}{d\theta} = \dfrac{dy}{du}\cdot\dfrac{du}{d\theta}$; "$\underline{u = n\theta}$"

Hence, $\dfrac{dy}{d\theta} = \overbrace{-3\csc^2(n\theta)}^{\frac{dy}{du}} \cdot \overbrace{n}^{\frac{du}{d\theta}} = -3n\csc^2(n\theta)$

16-) We will use the product rule and chain rule $(u=3x)$:

$$y = e^{-5x}\cos 3x \longrightarrow \frac{dy}{dx} = -5e^{-5x}\cos 3x + e^{-5x}(-3\sin 3x)$$

$$\frac{dy}{dx} = e^{-5x}(5\cos 3x + 3\sin 3x)$$

22-) Using the Quotient Rule, and chain rule:

$G(y) = \left(\frac{y^2}{y+1}\right)^5$ and using $u = \frac{y^2}{y+1}$, we have:

$$G'(y) = 5\left(\frac{y^2}{y+1}\right)^4 \cdot \left(\frac{(y+1)(2y) - y^2}{(y+1)^2}\right) = 5\left(\frac{y^2}{y+1}\right)^4 \cdot \left(\frac{y^2+2y}{(y+1)^2}\right)$$

Hence, $G'(y) = 5\left(\frac{y^9(y+2)}{(y+1)^6}\right)$

3.7 # 6, 12:

6-) $f(x) = \ln\sqrt[5]{x} \longrightarrow f(x) = \ln x^{\frac{1}{5}} = \frac{1}{5}\ln x$

$$f'(x) = \frac{1}{5}\underbrace{\frac{d}{dx}(\ln x)}_{\frac{1}{x}} = \frac{1}{5x}$$

12-) $F(y) = y\ln(1+e^y) \rightarrow$ Using the product rule & chain rule:

$$F'(y) = \ln(1+e^y) + \frac{y}{1+e^y}\cdot e^y = \boxed{\ln(1+e^y) + \frac{e^y y}{1+e^y}}$$

4.9 # 4, 8, 10, 16, 24

-Please note that all of these problems are just like finding the "indefinite integral" of a certain function. (Look at pg. 369 & the bottom of pg. 328).

4-) $f(x) = 2x + 3x^{1.7} \rightarrow$ Antiderivative: $F(x) = \frac{2x^2}{2} + \frac{3x^{2.7}}{2.7} + C = x^2 + \frac{3}{2.7}x^{2.7} + C$; C is a constant

8-) $g(x) = \frac{5 - 4x^3 + 2x^6}{x^6} \rightarrow g(x) = 5x^{-6} - 4x^{-3} + 2 \rightarrow$ Antiderivative: $G(x) = \frac{5x^{-5}}{-5} - \frac{4x^{-2}}{-2} + 2x + C$

$$G(x) = -x^{-5} + 2x^{-2} + 2x + C = -\frac{1}{x^5} + \frac{2}{x^2} + 2x$$

)-) $f(x) = 3e^x + 7\sec^2 x$

$F(x) = 3e^x + 7\tan x + C$, since the antiderivative of e^x is e^x

" " $\tan x$ is $\sec^2 x$

-) $f''(x) = 2 + x^3 + x^6$

Using the information on the bottom of pg. 328, and apply it twice:

$f'(x) = 2x + \frac{x^4}{4} + \frac{x^7}{7} + C$

$f''(x) = \frac{2x^2}{2} + \frac{x^5}{20} + \frac{x^8}{56} + Cx + D = x^2 + \frac{x^5}{20} + \frac{x^8}{56} + Cx + D$

4-) $f''(x) = 4 - 6x - 40x^3$, $\overset{(i)}{f(0) = 2}$, $\overset{(ii)}{f'(0) = 1}$

$f'(x) = 4x - \frac{6x^2}{2} - \frac{40x^4}{4} + C = -10x^4 - 3x^2 + 4x + C$

Using condition (ii-), we have:

$f'(0) = -10(0)^4 - 3(0)^2 + 4(0) + C = 1 \rightarrow$ Hence, $C = 1$

$f(x) = \frac{-10x^5}{5} - \frac{3x^3}{3} + \frac{4x^2}{2} + Cx + D \rightarrow \underline{f(x) = -2x^5 - x^3 + 2x^2 + Cx + D}$

Using condition (i-), we have:

$f(0) = \frac{-10(0)^5}{5} - \frac{3(0)^3}{3} + \frac{4(0)^2}{2} + C(0) + D = 2 \rightarrow$ Hence, $D = 2$

Using $C = 1$ & $D = 2$, we finally have;

$f(x) = -2x^5 - x^3 + 2x^2 + x + 2$