

AN INTRODUCTION INTO THE FEYNMAN PATH INTEGRAL

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Abstract. In this lecture a short introduction is given into the theory of the Feynman path integral in quantum mechanics. The general formulation in Riemann spaces will be given based on the Weyl- ordering prescription, respectively product ordering prescription, in the quantum Hamiltonian. Also, the theory of space-time transformations and separation of variables will be outlined. As elementary examples I discuss the usual harmonic oscillator, the radial harmonic oscillator, and the Coulomb potential.

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I INTRODUCTION

It was Feynman's (and Dirac's [20]) genius [32, 33] to realize that the integral kernel (propagator) of the time-evolution operator can be expressed as a sum over all possible paths connecting the points q' and q'' with weight factor $\exp [iS(q'', q'; T)/\hbar]$, where S is the action, i.e.

$$K(q'', q'; T) = \sum_{\text{all paths}} A e^{iS(q'', q'; T)/\hbar} \quad (1.1)$$

with some appropriate normalization A .

Surprisingly enough, the same calculus ("same" in the sense of a naïve analytical continuation) was already known to mathematicians due to Wiener in the study of stochastic processes. This *calculus* in functional space ("Wiener measure") attracted several mathematicians, including Kac (who mentioned being influenced by Feynman's work!), and was further developed by several authors, where best known is the work of Cameron and Martin. The standard reference concerning these achievements is the review paper of Gelfand and Yaglom [37], where all these early work was first critically discussed.

Unfortunately, the discussion between physicists and mathematicians remains near to nothing for quite a long time, except for [37]; the situation changed with Nelson [80], and nowadays there are many attempts to understand the path integral mathematically despite its pathologies of "infinite measure", "infinite sums of phases" with unit absolute values etc.

In particular, the work of Morette-DeWitt starting with her early paper [76] gave rise to a beautiful theory of the semiclassical expansion in powers of \hbar [18, 19, 77-79]. As it is known for quite a long time, the propagator can be expressed semi-classically as $e^{iS_{Cl}/\hbar}$, with S_{Cl} the classical action, times a prefactor. This prefactor is remarkably simple, namely one has

$$K_{WKB}(x'', x'; t'', t') = [g(x')g(x'')]^{-\frac{1}{4}} \left(\frac{1}{2\pi i \hbar} \right)^{D/2} \times \sqrt{\det \left(- \frac{\partial^2 S_{Cl}[x', x'']}{\partial x'_a \partial x''_b} \right)} \exp \left(\frac{i}{\hbar} S_{Cl}[x', x''] \right). \quad (1.2)$$

$g = \det(g_{ab})$ of some possible metric structure of a Riemannian space, and the determinant

$$M := \det \left(- \frac{\partial^2 S_{Cl}[x', x'']}{\partial x'_a \partial x''_b} \right) \quad (1.3)$$

is known as the Pauli-van Vleck-Morette determinant. The semiclassical (WKB-) solution of the Feynman kernel (we use the notions semiclassical and WKB simultaneously) is based on the fact that the harmonic oscillator, respectively, the general quadratic Lagrangian, is exactly solvable and its solution is only determined by the classical path and not on the summation over all paths. As it turns out, an arbitrary kernel can be expanded in terms of the classical paths as an expansion in powers of \hbar . The semiclassical kernel (at least its short time representation) is known since van Vleck, derived by the correspondence principle. Later on Pauli [83] has given a detailed discussion in his well-known lecture notes. More rigorously the short time propagator for the one-dimensional case was discussed by Morette [76] in 1951 and a few years later

by DeWitt [17] extending the previous work to D -dimensional curved spaces. Concerning the semi-classical expansion, every path integral with a Hamiltonian which is quadratic in its momenta can be expanded about the semiclassical approximation (1.2) giving a consistent and converging theory, compare DeWitt-Morette [18, 19, 74, 75, 77].

Supersymmetric quantum mechanics provides a very convenient way of classifying exactly solvable models in usual quantum mechanics, and a systematic way of addressing the problem of finding all exactly solvable potentials [56].

By e.g. Dutt et al. [16, 29] it was shown that there are a total of twelve different potentials. A glance on these potentials shows that their corresponding Schrödinger equation leads either to the differential equation of the confluent hypergeometric differential equation with eigen-functions proportional to Laguerre polynomials (bound states) and Whittaker functions (continuous states), respectively to the differential equation of the hypergeometric differential equation with eigen-functions proportional to Jacobi polynomials (bound states) and hypergeometric functions (continuous states). In Reference [16] this topic was nicely addressed and it was shown that in principle the radial harmonic oscillator and the (modified) Pöschl-Teller-potential solutions are sufficient to give the solutions of all remaining ones, together with the technique of space-time transformations as introduced by Duru and Kleinert [27, 28, 62] (see also reference [49], Ho and Inomata [55], Steiner [90], and Pak and Sökmen [82]) enables one to give the explicit path integral solution of the Coulomb $V^{(C)}(r) = -e^2/r$ ($r > 0$) and the Morse potential $V^{(M)}(x) = (\hbar^2 A^2/2m)(e^{2x} - 2\alpha e^x)$ ($x \in \mathbf{R}$) (compare e.g. reference [46] for a review of some recent results). The same line of reasoning is true for the path integral solution of the (modified) Pöschl-Teller potential [5-35] which give in turn the path integral solutions for the Rosen-Morse $V^{(RM)}(x) = A \tanh x - B/\cosh^2 x$ ($x \in \mathbf{R}$), the hyperbolic Manning-Rosen $V^{(MRa)}(r) = -A \coth r + B/\sinh^2 r$ ($r > 0$), and the trigonometric Manning-Rosen-like potential $V^{(MRb)}(x) = -A \cot x + B/\sin^2 x$ ($0 < x < \pi$), respectively [44].

These lecture notes are far from being a comprehensive introduction into the whole topic of path integrals, in particular if field theory is concerned. As old as they be, the books of Feynman and Hibbs [34] and Schulman [86] as still a **must** for becoming familiar with the subject. A more recent contribution is due to Kleinert [64]. Myself and F. Steiner are presently preparing extended lecture notes “Feynman Path Integrals” and a “Table of Feynman Path Integrals” [50, 51], which will appear next year.

Several reviews have been written about path integrals, let me note Gelfand and Jaglom [37], Albeverio et al. [1-3], DeWitt-Morette et al. [19, 79], Marinov [73], and e.g. for the topic of path integrals for Coulomb potentials [46].

The contents of the lecture is as follows. In the next Chapter I outline the basic theory of the Feynman path integral, i.e. the lattice definition according to the Weyl-ordering prescription in the Hamiltonian and a related prescription which is of use in several applications in my own work. Furthermore, the technique of canonical coordinate-, time- and space-time transformations will be presented. The Chapter closes with the discussion of separation of variables in path integrals.

In the third chapter I present some important examples of exact path integral evaluations. This includes, of course, the harmonic oscillator in its simplest form. I also discuss the radial path integral with its application of the exact treatment of the radial (time-dependent) harmonic oscillator. The chapter concludes with a comprehensive discussion of the Coulomb potential, including the genuine Coulomb problem, its D -

dimensional generalization, and a generalization with additional axially-symmetric terms.

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II GENERAL THEORY

1. The Feynman Path Integral

In order to set up the requirements of the path integral formalism we start with the generic case, where the time dependent Schrödinger equation in some D -dimensional Riemannian manifold M with metric g_{ab} and line element $ds^2 = g_{ab}dq^a dq^b$ is given by

$$\left[-\frac{\hbar^2}{2m}\Delta_{LB} + V(q) \right] \Psi(q; t) = \frac{\hbar}{i} \frac{\partial}{\partial t} \Psi(q; t). \quad (1.1)$$

Ψ is some state function, defined in the Hilbert space \mathcal{L}^2 - the space of all square integrable functions in the sense of the scalar product $(f_1, f_2) = \int_M \sqrt{g} f_1(q) f_2^*(q) dq$ [$g := \det(g_{ab})$, $f_1, f_2 \in \mathcal{L}^2$] and Δ_{LB} is the Laplace-Beltrami operator

$$\Delta_{LB} := g^{-\frac{1}{2}} \partial_a g^{\frac{1}{2}} g^{ab} \partial_b = g^{ab} \partial_a \partial_b + g^{ab} (\partial_a \ln \sqrt{g}) \partial_b + g^{ab}{}_{,a} \partial_b \quad (1.2)$$

(implicit sums over repeated indices are understood).

The Hamiltonian $H := -\frac{\hbar^2}{2m}\Delta_{LB} + V(q)$ is usually defined in some dense subset $D(H) \subseteq \mathcal{L}^2$, so that H is selfadjoint. In contrast to the time independent Schrödinger equation, $H\Psi = E\Psi$, which is an eigenvalue problem, and equation (1.1) which are both defined on $D(H)$, the unitary operator $U(T) := e^{-iTH/\hbar}$ describes the time evolution of arbitrary states $\Psi \in \mathcal{L}^2$ (time-evolution operator): H is the infinitesimal generator of U . The time evolution for some state Ψ reads: $\Psi(t'') = U(t'', t')\Psi(t')$ Rewriting the time evolution with $U(T)$ as an integral operator we get

$$\Psi(q''; t'') = \int \sqrt{g(q')} K(q'', q'; t'', t') \Psi(q'; t') dq', \quad (1.3)$$

where $K(T)$ is the celebrated Feynman kernel. Equations (1.1) and (1.3) are connected. Having an explicit expression for $K(T)$ in (1.3) one can derive in the limit $T = \epsilon \rightarrow 0$ equation (1.1). This, on the other hand proves that $K(T)$ is indeed the correct integral kernel corresponding to $U(T)$. A rigorous proof includes, of course, the check of the selfadjointness of H , i.e. $H = H^*$. Due to the semigroup property

$$U(t_2)U(t_1) = U(t_1 + t_2) \quad (1.4)$$

(H time-independent) we have

$$K(q'', q'; t'' + t') = \int K(q'', q; t'' + t) K(q, q'; t + t') dq \quad (1.5)$$

In the case of an Euclidean space, where $g_{ab} = \delta_{ab}$, S is just the classical action, $S_{Cl} = \int [\frac{m}{2}\dot{q}^2 - V(q)] dt = \int \mathcal{L}_{cl}(q, \dot{q}) dt$, and we get explicitly [$\Delta q^{(j)} := (q^{(j)} - q^{(j-1)})$, $q^{(j)} = q(t_j)$, $t_j = t' + j\epsilon$, $\epsilon = (t'' - t')/N$, $N \rightarrow \infty$, $D =$ dimension of the Euclidean space] :

$$K(q'', q'; T) = \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \hbar T} \right)^{\frac{ND}{2}} \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} dq^{(j)} \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} \Delta^2 q^{(j)} - \epsilon V(q^{(j)}) \right] \right\}.$$

$$\equiv \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}q(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{q}^2 - V(q) \right] dt \right\} \quad (1.6)$$

For an arbitrary metric g_{ab} things are unfortunately not so easy. The first formulation for this case is due to DeWitt [17]. His result reads:

$$\begin{aligned} & K(q'', q'; T) \\ &= \int_{q(t')=q'}^{q(t'')=q''} \sqrt{g} \mathcal{D}_{DeW} q(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} g_{ab}(q) \dot{q}^a \dot{q}^b - V(q) + \hbar^2 \frac{R}{6m} \right] dt \right\} \\ &:= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{ND/2} \prod_{j=1}^{N-1} \int \sqrt{g(q^{(j)})} dq^{(j)} \\ &\times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} g_{ab}(q^{(j-1)}) \Delta q^{a,(j)} \Delta q^{b,(j)} - \epsilon V(q^{(j-1)}) + \epsilon \frac{\hbar^2}{6m} R(q^{(j-1)}) \right] \right\} \quad (1.7) \end{aligned}$$

($R = g^{ab}(\Gamma_{ab,c}^c - \Gamma_{cb,a}^c + \Gamma_{ab}^d \Gamma_{cd}^c - \Gamma_{cb}^d \Gamma_{ad}^c)$ - scalar curvature; $\Gamma_{bc}^a = g^{ad}(g_{bd,c} + g_{dc,b} - g_{bc,d})$ - Christoffel symbols). Of course, we identify $q' = q^{(0)}$ and $q'' = q^{(N)}$ in the limit $N \rightarrow \infty$. Two comments are in order:

- 1) Equation (1.7) has the form of equation (1.6) but the corresponding $S = \int \mathcal{L} dt$ is not the classical action, respectively the Lagrangian \mathcal{L} is not the classical Lagrangian $\mathcal{L}(q, \dot{q}) = \frac{m}{2} g_{ab} \dot{q}^a \dot{q}^b - V(q)$, but rather an effective one:

$$S_{eff} = \int \mathcal{L}_{eff} dt \equiv \int (\mathcal{L}_{cl} - \Delta V_{DeW}) dt. \quad (1.8)$$

The quantum correction $\Delta V_{DeW} = -\frac{\hbar^2}{6m} R$ is indispensable in order to derive from the time evolution equation (1.3) the Schrödinger equation (1.1) (e.g. [24]). The appearance of a quantum correction ΔV is a very general feature for path integrals defined on curved manifolds; but, of course, $\Delta V \sim \hbar^2$ depends on the lattice definition.

- 2) A specific lattice definition has been chosen. The metric terms in the action are evaluated at the “prepoint” $q^{(j-1)}$. Changing the lattice definition, i.e. evaluation of the metric terms at other points, e.g. the “postpoint” $q^{(j)}$ or the “mid-point” $\bar{q}^{(j)} := \frac{1}{2}(q^{(j)} + q^{(j-1)})$ changes ΔV , because in a Taylor expansion of the relevant terms, **all** terms of $O(\epsilon)$ contribute to the path integral. This fact is particularly important in the expansion of the kinetic term in the Lagrangian, where we have $\Delta^4 q^{(j)} / \epsilon \sim O(\epsilon)$.

One of the basics requirements of the construction of the path integral is Trotter’s [93] product formula. Discussions and proofs can be found in many textbooks on functional integration and functional analysis (e.g. Reed and Simon [85], Simon [87]). Let us shortly notice a simple proof [87] :

Theorem: Let A and B be selfadjoint operators on a separable Hilbert space so that $A + B$, defined on $D(A) \dots D(B)$, is selfadjoint. Then

$$e^{iT(A+B)} = \text{s-lim}_{N \rightarrow \infty} \left(e^{i t A / N} e^{i t B / N} \right)^N. \quad (1.9)$$

If furthermore, A and B are bounded from below, then

$$e^{-t(A+B)} = \text{s-lim}_{N \rightarrow \infty} \left(e^{-tA/N} e^{-tB/N} \right)^N. \quad (1.10)$$

Proof ([80,87]): Let $S_T = e^{iT(A+B)}$, $V_T = e^{iTA}$, $W_T = e^{iTB}$, $U_T = V_T W_T$, and let $\Psi_T = S_T \Psi$ for some $\Psi \in \mathcal{H}$, with underlying Hilbert space \mathcal{H} . Then

$$\begin{aligned} \|(S_T - U_{T/N}^N) \Psi\| &= \left\| \sum_{j=0}^{N-1} U_{T/N}^j (S_{T/N} - U_{T/N})^{N-j-1} \Psi \right\| \\ &\leq N \sup_{0 \leq s \leq T} \|(S_T - U_{T/N}) \Psi\|. \end{aligned} \quad (1.11)$$

Let $\Phi \in D(A) \cap D(B)$. Then $s^{-1}(S_s - 1)\Phi \rightarrow i(A+B)\Phi$ for $s \rightarrow 0$ and

$$\begin{aligned} \frac{U_s - 1}{s} \Phi &= V_s(iB\Phi) + V_s \left(\frac{W_s - 1}{s} - iB \right) \Phi + \frac{V_s - 1}{s} \Phi \\ &\rightarrow iB\Phi + iA\Phi + 0 \end{aligned} \quad (1.12)$$

hence

$$\lim_{N \rightarrow \infty} \left[N \|(S_{T/N} - U_{T/N}) \Phi\| \right] \rightarrow 0, \quad \text{for each } \Phi \in D(A) \cap D(B). \quad (1.13)$$

Now let D denote $D(A) \cap D(B)$ with the norm $\|(A+B)\Phi\| + \|\Phi\| \equiv \|\Phi\|_{A+B}$. By hypothesis, D is a Banach space. With the calculations of equations (1.11-1.13), $\{N(S_{t/N} - U_{t/N})\}$ is a family of bounded operators from D to \mathcal{H} with $\sup_N \{N\|(S_{T/N} - U_{T/N})\Phi\|\} < \infty$ for each Φ . As a result, the uniform boundedness principle implies that

$$N\|(S_{T/N} - U_{T/N})\Phi\| \leq C\|\Phi\|_{A+B} \quad (1.14)$$

for some positive C . Therefore the limit (1.14) is uniform over compact subsets of D . Now let $\Psi \in D$. Then $s \rightarrow \Psi_s$ is a continuous map from $[0, T]$ into D , so that $\{\Psi_s | 0 \leq s \leq T\}$ is compact in D . Thus the right-hand side of equation (1.14) goes to zero as $N \rightarrow \infty$. The proof of equation (1.10) is similar.

The conditions on the operators A and B can be weakened with the requirement that they are fulfill some “positiveness” and possesses a particular self-adjoint extension.

Thus we see that ordering prescriptions in the quantum Hamiltonian and corresponding lattice formulations in the path integral are closely related (compare also [22, 23]). This can be formulated in a systematic way (e.g. [68]):

We consider a monomial in coordinates and momenta (classical quantities)

$$M(n, m) = q^{\mu_1} \dots q^{\mu_n} p^{\nu_1} \dots p^{\nu_m} \quad (1.15)$$

We want to have a correspondence rule according to

$$\exp \left[\frac{i}{\hbar} (uq + vp) \right] \rightarrow D_\Omega(u, v; \underline{q}, \underline{p}) \equiv \Omega(u, v) \exp \left[\frac{i}{\hbar} (u\underline{q} + v\underline{p}) \right] \quad (1.16)$$

to generate operators $\underline{q}, \underline{p}$ from coordinates q, p . This produces the mapping

$$M(n, n) \rightarrow \frac{1}{i^{n+m}} \left| \frac{\partial^{n+m} D_\Omega(u, v; \underline{q}, \underline{p})}{\partial u_{\mu_1} \dots \partial u_{\mu_n} \partial v_{\nu_1} \dots \partial v_{\nu_m}} \right|_{u=v=0}. \quad (1.17)$$

Some better known examples are displayed in the following table

Correspondence Rule	$\Omega(u, v)$	Ordering Rule
Weyl	1	$\frac{1}{2^n} \sum_{l=0}^n \binom{n}{l} \underline{q}^{n-l} \underline{p}^m \underline{q}^l$
Symmetric	$\cos \frac{u \cdot v}{2}$	$\frac{1}{2} (\underline{q}^n \underline{p}^m + \underline{p}^m \underline{q}^n)$
Standard	$\exp \left(-i \frac{u \cdot v}{2} \right)$	$\underline{q}^n \underline{p}^m$
Anti-Standard	$\exp \left(i \frac{u \cdot v}{2} \right)$	$\underline{p}^m \underline{q}^n$
Born-Jordan	$\sin \frac{u \cdot v}{2} / \frac{u \cdot v}{2}$	$\frac{1}{m+1} \sum_{l=0}^n \underline{p}^{m-l} \underline{q}^n \underline{p}^l$

We can make this correspondence explicit. Let us consider the Fourier integral

$$\begin{aligned} A(p, q) &= \int A(\widehat{u}, v) \exp \left[\frac{i}{\hbar} (uq + ivp) \right] dudv \\ A(\widehat{u}, v) &= \frac{1}{(2\pi\hbar)^D} \int A(p, q) \exp \left[-\frac{i}{\hbar} (uq + ivp) \right] dpdq \end{aligned} \quad (1.18)$$

We define the operator $A^\Omega(\underline{p}, \underline{q})$ via

$$A^\Omega(\underline{p}, \underline{q}) = \int A(\widehat{u}, v) \Omega(u, v) \exp \left[\frac{i}{\hbar} (u\underline{q} + iv\underline{p}) \right] dudv, \quad (1.19)$$

which gives

$$\mathbb{A}^\Omega(\underline{p}, \underline{q}) = \frac{1}{(2\pi\hbar)^D} \int A(p, q) \Omega(u, v) \exp \left[-\frac{i}{\hbar} u(q - \underline{q}) - \frac{i}{\hbar} v(p - \underline{p}) \right] dudvdpdq. \quad (1.20)$$

The inverse transformation is denoted by $A^\Theta(p, q)$ and has the form

$$A^\Theta(p, q) = \frac{1}{(2\pi\hbar)^D} \text{Tr} \int \mathbb{A}(\underline{p}, \underline{q}) [\Omega(u, v)]^{-1} \exp \left[\frac{i}{\hbar} u(q - \underline{q}) + i \frac{i}{\hbar} v(p - \underline{p}) \right] dudv. \quad (1.21)$$

In particular, the δ -function has the correspondence operator

$$\Delta^\Omega(p - \underline{p}, q - \underline{q}) = \frac{1}{(2\pi\hbar)^D} \int \Omega(u, v) \exp \left[-\frac{i}{\hbar} u(q - \underline{q}) - \frac{i}{\hbar} v(p - \underline{p}) \right] dudv. \quad (1.22)$$

Let us turn to the Hamiltonian. Given a classical Hamiltonian $\mathcal{H}_{cl}(p, q)$ the quantum Hamiltonian is calculated as

$$\begin{aligned} \mathbb{H}(\underline{p}, \underline{q}) &= \int \exp \left(\frac{i}{\hbar} u \underline{p} + \frac{i}{\hbar} v \underline{q} \right) \Omega(u, v) \widehat{\mathcal{H}_{cl}}(u, v) dudv \\ \widehat{\mathcal{H}_{cl}}(u, v) &= \frac{1}{(2\pi\hbar)^{2D}} \int \exp \left(-\frac{i}{\hbar} u p - \frac{i}{\hbar} v q \right) \mathcal{H}_{cl}(p, q) dpdq. \end{aligned} \quad (1.23)$$

Obviously, by proposing a particular classical Hamiltonian depending on variables q' and q'' , respectively, produces a particular Hamiltonian function,

$$H(p, q'', q') = \frac{1}{(2\pi\hbar)^D} \int \Omega(q'' - q', y) e^{ivy/\hbar} \mathcal{H}_{cl}[p, \frac{1}{2}(q'' + q') - v] dudv \quad (1.24)$$

For $\Omega = 1$ and $\Omega = \cos \frac{uv}{2}$, respectively, we obtain Hamiltonian functions according to

$$\begin{aligned} H(p, q'', q') &= \mathcal{H}(p, \frac{1}{2}(q' + q'')) \\ H(p, q'', q') &= \frac{1}{2} [\mathcal{H}_{cl}(p, q'') + \mathcal{H}_{cl}(p, q')] \end{aligned} \quad (1.25)$$

which are the matrix elements

$$\langle q' | \mathbb{H}(\underline{p}, \underline{q}) | q'' \rangle \quad (1.26)$$

of the quantum Hamiltonians which are Weyl- and symmetrically ordered, respectively. Choosing $\Omega(u, v) = \exp[i(1 - 2u)uv]$ [54] yields

$$H(p, q; u) = \frac{q}{2m} g^{ab}(q) p_a p_b + \frac{i\hbar}{m} (\frac{1}{2} - u) p_a g^{ab},_b - \frac{\hbar^2}{2m} (\frac{1}{2} - u)^2 g^{ab},_{ab} + \Delta V_{Weyl}. \quad (1.27)$$

with the well-defined quantum potential

$$\Delta V_{Weyl} = \frac{\hbar^2}{8m} (g^{ab} \Gamma_{ac}^d \Gamma_{bd}^c - R) = \frac{\hbar^2}{8m} [g^{ab} \Gamma_a \Gamma_b + 2(g^{ab} \Gamma_a),_b + g^{ab},_{ab}] \quad (1.28)$$

$u = \frac{1}{2}$ corresponds to the Weyl prescription and is clearly emphasized.

In the next two Sections we discuss two particular ordering prescriptions and the corresponding matrix elements which will appear as the relevant quantities to be used in path integrals on curved spaces, it will be the Weyl-ordering prescription (leading to the midpoint rule in the path integral) and a product ordering (leading to a product rule in the path integral).

Another approach due to Kleinert [63, 64] I will not discuss here.

2. Weyl-Ordering

A very convenient lattice prescription is the mid-point definition, which is connected to the Weyl-ordering prescription in the Hamiltonian H . Let us discuss this prescription in some detail. First we have to construct momentum operators [83]:

$$p_a = \frac{\hbar}{i} \left(\frac{\partial}{\partial q^a} + \frac{\Gamma_a}{2} \right), \quad \Gamma_a = \frac{\partial \ln \sqrt{g}}{\partial q^a} \quad (2.1)$$

which are hermitean with respect to the scalar product $(f_1, f_2) = \int f_1 f_2^* \sqrt{g} dq$. In terms of the momentum operators (2.1) we rewrite the quantum Hamiltonian H by using the Weyl-ordering prescription [49,69,72]

$$H(p, q) = \frac{1}{8m} (g^{ab} p_a p_b + 2p_a g^{ab} p_b + p_a p_b g^{ab}) + \Delta V_{Weyl}(q) + V(q). \quad (2.2)$$

Here a well-defined quantum correction appears which is given by [49,72,81]:

$$\Delta V_{Weyl} = \frac{\hbar^2}{8m} (g^{ab} \Gamma_{ac}^d \Gamma_{bd}^c - R) = \frac{\hbar^2}{8m} [g^{ab} \Gamma_a \Gamma_b + 2(g^{ab} \Gamma_a)_{,b} + g^{ab}_{,ab}] \quad (2.3)$$

The Weyl-ordering prescription is the most discussed ordering prescription in the literature. Let us start by defining it for powers of position- and momentum- operators \underline{q} and \underline{p} , respectively [69]:

$$(\underline{q}^m \underline{p}^r)_{Weyl} = \left(\frac{1}{2} \right)^m \sum_{l=0}^m \frac{m!}{l!(m-l)!} \underline{q}^{m-l} \underline{p}^r \underline{q}^l, \quad (2.4)$$

with the matrix elements (one-dimensional case)

$$\begin{aligned} \langle q'' | (\underline{q}^m \underline{p}^r)_{Weyl} | q' \rangle &= \left(\frac{1}{2} \right)^m \sum_{l=0}^m \frac{m!}{l!(m-l)!} q^{m-l} \langle q'' | \underline{p}^r | q' \rangle q^l \\ &= \int \frac{dp}{(2\pi\hbar)^D} p^r e^{ip(q''-q')/\hbar} \left(\frac{q' + q''}{2} \right)^m \end{aligned} \quad (2.5)$$

and all coordinate dependent quantities turn out to be evaluated at **mid-points**. Here was used that the matrix elements of the position $|q\rangle$ and momentum eigen-states $|p\rangle$ have the property

$$\langle q'' | q' \rangle = (g' g'')^{-1/4} \delta(q'' - q'), \quad \langle q | p \rangle = (2\pi\hbar)^{-D/2} e^{ipq/\hbar}. \quad (2.6)$$

This power rule is nothing but a special case of a more general prescription. Of course, well behaved operator valued functions are included.

We have the **Weyl-transform** of an operator $A(\underline{p}, \underline{q})$ as [72]:

$$A(p, q) \equiv \int dv e^{ipv} \langle q - \frac{v}{2} | A(\underline{p}, \underline{q}) | q + \frac{v}{2} \rangle, \quad (2.7)$$

with the inverse transformation

$$\begin{aligned} \underline{A}(\underline{p}, \underline{q}) &= \frac{1}{(2\pi\hbar)^D} \int dpdq A(p, q) \Delta(p, q) \\ \Delta(\underline{p}, \underline{q}) &= \int du e^{i pu} |p - \frac{u}{2}\rangle \langle p + \frac{u}{2}|. \end{aligned} \quad (2.8)$$

Let us consider some simple examples. First of all for some operator $f(\underline{p})$

$$\begin{aligned} &\int dv e^{i pv/\hbar} \langle q - \frac{v}{2} | f(\underline{p}) | q + \frac{v}{2} \rangle \\ &= \int dv dp' e^{i pv/\hbar} \langle q - \frac{v}{2} | f(\underline{q}) | p' \rangle \langle p' | q + \frac{v}{2} \rangle \\ &= \frac{1}{(2\pi\hbar)^D} \int dv dp' e^{i(p-p')v/\hbar} f(p') = f(p). \end{aligned} \quad (2.9)$$

Hence (and similarly)

$$f(\underline{p}) \iff f(p), \quad f(\underline{q}) \iff f(q). \quad (2.10)$$

Straightforwardly one shows the following correspondence

$$\begin{aligned} &F(\underline{q}) \underline{p}_i g^{ij}(\underline{q}) \underline{p}_j F(\underline{q}) \\ &\iff p_i p_j F^2(q) g^{ij} + \frac{1}{2} \left(\frac{1}{2} F^2(q) g^{ij}{}_{,ij}(q) + F_{,i}(q) F_{,j}(q) g^{ij}(q) - F_{,ij}(q) F(q) g^{ij}(q) \right) \end{aligned} \quad (2.11)$$

Special cases are

$$\begin{aligned} \underline{p}_i \underline{p}_j F(\underline{q}) &\iff \left(p_i - \frac{i\hbar}{2} \frac{\partial}{\partial q^i} \right) \left(p_j - \frac{i\hbar}{2} \frac{\partial}{\partial q^j} \right) F(q) \\ \underline{p}_i F(\underline{q}) \underline{p}_j &\iff \left(p_i - \frac{i\hbar}{2} \frac{\partial}{\partial q^i} \right) \left(p_j + \frac{i\hbar}{2} \frac{\partial}{\partial q^j} \right) F(q) \\ \underline{p}_i F(\underline{q}) \underline{p}_j &\iff \left(p_i + \frac{i\hbar}{2} \frac{\partial}{\partial q^i} \right) \left(p_j + \frac{i\hbar}{2} \frac{\partial}{\partial q^j} \right) F(q). \end{aligned} \quad (2.12)$$

For the case that $F(q)$ is some symmetric $N \times N$ matrix we have

$$\frac{1}{4} \left[\underline{p}_i \underline{p}_j F^{ij}(\underline{q}) + 2 \underline{p}_i F^{ij}(\underline{q}) p_j + F^{ij}(\underline{q}) \underline{p}_i \underline{p}_j \right] \iff p_i p_j F^{ij}(q). \quad (2.13)$$

There is, of course, a one-to-one correspondence between the function $A(p, q)$ and the operator $A(\underline{p}, \underline{q})$, called **Weyl-correspondence**. It gives a prescription how **Weyl-ordered** operators can be constructed by the classical counterpart. Starting now with an operator A , the Weyl-correspondence gives an unique prescription for the

construction of the path integral. We have for the Feynman kernel for an arbitrary $N \in \mathbf{N}$ [which is due to the semi-group property of $U(T)$, i.e. $U(t_1+t_2) = U(t_1)U(t_2)$]:

$$\begin{aligned} K(q'', q'; T) &= \left\langle q'' \left| \exp \left[-\frac{i}{\hbar} \mathbf{H}(\underline{p}, \underline{q}) \right] \right| q' \right\rangle \\ &= \left(\prod_{j=1}^{N-1} \int \sqrt{g^{(j)}} dq^{(j)} \right) \prod_{j=1}^N \left\langle q^{(j)} \left| \exp \left[-\frac{i}{\hbar} \frac{T}{N} \mathbf{H}(\underline{p}, \underline{q}) \right] \right| q^{(j-1)} \right\rangle. \end{aligned} \quad (2.14)$$

Observing

$$\begin{aligned} \langle q' | \Delta(\underline{p}, \underline{q}) | q'' \rangle &= \int du e^{i u q} \langle q' | p - \frac{u}{2} \rangle \langle p + \frac{u}{2} | q'' \rangle \\ &= \frac{1}{(2\pi\hbar)^D} \int du \exp \left[\frac{i}{\hbar} p(q' - q'') + \frac{i}{\hbar} u \left(q - \frac{q' + q''}{2} \right) \right] \\ &= e^{i p(q' - q'')/\hbar} \delta \left(q - \frac{q' + q''}{2} \right) \end{aligned} \quad (2.15)$$

and making use of the Trotter formula $e^{-it(A+B)} = s\text{-}\lim_{N \rightarrow \infty} (e^{-itA/N} e^{-itB/N})^N$ and the short-time approximation for the matrix element $\langle q'' | e^{-i\epsilon H} | q' \rangle$:

$$\begin{aligned} \langle q^{(j)} | \exp \left[-i\epsilon \mathbf{H}(\underline{p}, \underline{q})/\hbar \right] | q^{(j-1)} \rangle &= \frac{1}{(2\pi\hbar)^D} \langle q^{(j)} | \int dp dq e^{-ih(p,q)/\hbar} \int dudv \exp [i(q - \underline{q})u + i(p - \underline{p})v] | q^{(j-1)} \rangle \\ &= \frac{1}{(2\pi\hbar)^D} \int dp \exp \left[\frac{i\epsilon}{\hbar} p(q^{(j)} - q^{(j-1)}) - \frac{i\epsilon}{\hbar} H(p, \bar{q}^{(j)}) \right], \end{aligned} \quad (2.16)$$

where $\bar{q}^{(j)} = \frac{1}{2}(q^{(j)} + q^{(j-1)})$ is the mid-point coordinate.

Let us consider the classical Hamilton-function

$$\mathcal{H}_{cl}(p, q) = \frac{1}{2m} g^{ab}(q) [p_a - A_a(q)] [p_b - A_b(q)] + V(q). \quad (2.17)$$

The corresponding quantum mechanical operator is not clearly defined due to the factor ordering ambiguity. Let us try the manifestly hermitean operator

$$H(\underline{p}, \underline{q}) = \frac{1}{2m} g^{-\frac{1}{4}} [\underline{p}_a - A_a(\underline{q})] g^{\frac{1}{2}}(\underline{q}) g^{ab}(\underline{q}) [\underline{p}_b - A_b(\underline{q})] g^{-\frac{1}{4}}(\underline{q}) + V(\underline{q}). \quad (2.18)$$

The Weyl-transformed of H reads

$$\begin{aligned} H(p, q) &= \mathcal{H}_{cl}(p, q) + \frac{\hbar^2}{2m} \left[\frac{1}{4} g^{-\frac{1}{2}} (g^{\frac{1}{2}} g^{ab})_{,ab} \right. \\ &\quad \left. + \frac{1}{2} (g^{-\frac{1}{4}})_{,a} (g^{-\frac{1}{4}})_{,b} g^{\frac{1}{2}} g^{ab} - \frac{1}{2} (g^{-\frac{1}{4}})_{,ab} g^{\frac{1}{4}} g^{ab} \right] \end{aligned}$$

$$\begin{aligned}
&= \mathcal{H}_{cl}(p, q) + \frac{\hbar^2}{8m} [\Gamma_{la}^m \Gamma_{mb}^l g^{ab} - R] \\
&= \mathcal{H}_{cl}(p, q) + \frac{\hbar^2}{8m} [g^{ab} \Gamma_a \Gamma_b + 2(g^{ab} \Gamma_a)_{,b} + g^{ab}_{,ab}] \quad (2.19)
\end{aligned}$$

and, of course, ΔV_{Weyl} appears again. For the path integral all quantities have to be evaluated at $\bar{q}^{(j)}$. Note: The Weyl-transformed of the the Weyl-ordered operator (2.16) is just the classical Hamiltonian (2.17). Inserting all quantities in equation (2.13) we obtain the **Hamiltonian path integral**:

$$\begin{aligned}
K(q'', q'; T) &= [g(q')g(q'')]^{-\frac{1}{4}} \lim_{N \rightarrow \infty} \prod_{j=1}^{N-1} \int dq^{(j)} \cdot \prod_{j=1}^N \int \frac{dp^{(j)}}{(2\pi\hbar)^D} \\
&\times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N [\Delta q^{(j)} \cdot p^{(j)} - \epsilon \mathcal{H}_{eff}(p^{(j)}, \bar{q}^{(j)})] \right\}. \quad (2.20)
\end{aligned}$$

The effective Hamiltonian to be used in the path integral (2.20) reads,

$$H(p, \bar{q}) = \mathcal{H}_{eff}(p^{(j)}, \bar{q}^{(j)}) = \frac{1}{2m} g^{ab}(\bar{q}^{(j)}) p_a^{(j)} p_b^{(j)} + V(\bar{q}^{(j)}) + \Delta V_{Weyl}(\bar{q}^{(j)}). \quad (2.21)$$

With the help of the famous Gaussian integral

$$\int_{-\infty}^{\infty} e^{-p^2 x^2 \pm qx} dx = \frac{\sqrt{\pi}}{p} \exp\left(\frac{q^2}{4p^2}\right) \quad (2.22)$$

and, respectively, by its D -dimensional generalization

$$\int_{\mathbf{R}^n} dp e^{iq \cdot p - \frac{1}{2} g^{ab} p_a p_b} = (2\pi)^{D/2} \sqrt{\det(g_{ab})} \exp\left(-\frac{1}{2} g_{ab} q^a q^b\right), \quad (2.23)$$

we get by integrating out the momenta the **Lagrangian path integral** which reads ($MP = \mathbf{Mid-Point}$):

$$\begin{aligned}
K(q'', q'; T) &= [g(q')g(q'')]^{-\frac{1}{4}} \int_{q^{(t')}=q'}^{q^{(t'')}=q''} \sqrt{g} \mathcal{D}_{MP} q(t) \exp \left[\frac{i}{\hbar} \int_{t'}^{t''} \mathcal{L}_{eff}(q, \dot{q}) dt \right] \\
&:= [g(q')g(q'')]^{-\frac{1}{4}} \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{ND}{2}} \left(\prod_{j=1}^{N-1} \int dq^{(j)} \right) \\
&\times \prod_{j=1}^N \sqrt{g(\bar{q}^{(j)})} \exp \left\{ \frac{i}{\hbar} \left[\frac{m}{2\epsilon} g_{ab}(\bar{q}^{(j)}) \Delta q^{a,(j)} \Delta q^{b,(j)} - \epsilon V(\bar{q}^{(j)}) - \epsilon \Delta V_{Weyl}(\bar{q}^{(j)}) \right] \right\}. \quad (2.24)
\end{aligned}$$

Equation (2.24) is, of course, equivalent with equation (1.7). The mid-point prescription arises here in a very natural way, as a consequence of the Weyl-ordering prescription. This is a general feature that ordering prescriptions lead to **specific** lattices and that different lattices define **different** ΔV .

To proof that the path integral (2.24) is indeed the correct one, one has to show that with the corresponding short time kernel

$$K(q^{(j)}, q^{(j-1)}; \epsilon) = \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{D/2} [g(q^{(j-1)})g(q^{(j)})]^{-\frac{1}{4}} \sqrt{g(\bar{q}^{(j)})} \\ \times \exp \left\{ \frac{i}{\hbar} \left[\frac{m}{2\epsilon} g_{ab}(\bar{q}^{(j)}) \Delta q^{a,(j)} \Delta q^{b,(j)} - \epsilon V(\bar{q}^{(j)}) - \epsilon \Delta V_{Weyl}(\bar{q}^{(j)}) \right] \right\}. \quad (2.25)$$

and the time evolution equation (1.3) the Schrödinger equation (1.1) follows. For this purpose a Taylor expansion has to be performed in equation (1.3) yielding

$$\Psi(q'', t) + \epsilon \frac{\partial \Psi(q''; t)}{\partial t} = B_0 \Psi(q''; t) + B_{q^b} \frac{\partial \Psi(q''; t)}{\partial q'^b} + B_{q^a q^b} \frac{\partial^2 \Psi(q''; t)}{\partial q'^b \partial q'^a} + \dots, \quad (2.26)$$

where the coefficients in the expansion are given by

$$B_0 = \int dq' \sqrt{g(q')} K(q'', q'; \epsilon) \\ \simeq \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{D/2} g^{-\frac{1}{4}}(q'') e^{-i\epsilon[V(q'') + \Delta V(q'')]/\hbar} \\ \times \int dq' g^{\frac{1}{2}}(\bar{q}) g^{\frac{1}{4}}(q') \exp \left(\frac{i m}{2\epsilon \hbar} \Delta q^a g_b \Delta q^b \right) \quad (2.27)$$

$$B_{q^b} = \int dq' \sqrt{g(q')} K(q'', q'; \epsilon) \Delta q^b \\ \simeq \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{D/2} g^{-\frac{1}{4}}(q'') \int dq' g^{\frac{1}{2}}(\bar{q}) g^{\frac{1}{4}}(q') \exp \left(\frac{i m}{2\epsilon \hbar} \Delta q^a g_b \Delta q^b \right) \Delta q^b \quad (2.28)$$

$$B_{q^a q^b} = \int dq' \sqrt{g(q')} K(q'', q'; \epsilon) \Delta q^a \Delta q^b \\ \simeq \left(\frac{m}{2\pi i \epsilon} \right)^{D/2} g^{-\frac{1}{4}}(q'') \int dq' g^{\frac{1}{2}}(\bar{q}) g^{\frac{1}{4}}(q') \exp \left(\frac{i m}{2\epsilon \hbar} \Delta q^a g_b \Delta q^b \right) \Delta q^a \Delta q^b \quad (2.29)$$

From these representations it is clear that we need equivalence relations (in the sense of path integrals) for $\Delta q^a \Delta q^b$ etc. They are given by

$$\Delta q^a \Delta q^b \doteq \frac{i \epsilon \hbar}{m} g^{ab} \quad (2.30)$$

$$\Delta q^a \Delta q^b \Delta q^c \Delta q^d \doteq \left(\frac{i \epsilon \hbar}{m} \right)^2 [g^{ab} g^{cd} + g^{ac} g^{bd} + g^{ad} g^{bc}]. \quad (2.31)$$

$$\begin{aligned}
& \Delta q^a \Delta q^b \Delta q^c \Delta q^d \Delta q^e \Delta q^f \\
& \doteq \left(\frac{i\epsilon\hbar}{m} \right)^3 \left[g^{ab} g^{cd} g^{cd} + g^{ac} g^{bd} g^{ef} + g^{ad} g^{bc} g^{ef} \right. \\
& \quad + g^{ab} g^{ce} g^{df} + g^{ab} g^{cf} g^{de} + g^{cd} g^{ae} g^{bf} + g^{cd} g^{af} g^{be} + g^{ac} g^{be} g^{df} + g^{ac} g^{bf} g^{de} \\
& \quad \left. + g^{bd} g^{ae} g^{cf} + g^{bd} g^{af} g^{ce} + g^{ad} g^{be} g^{cf} + g^{ad} g^{bf} g^{ce} + g^{bc} g^{ae} g^{df} + g^{bc} g^{af} g^{de} \right].
\end{aligned} \tag{2.32}$$

We just show the identity (2.30), the proof of the remaining ones is similarly. Let us consider the integral

$$I(g_{cd}) = \int dq \exp \left(\frac{im}{2\epsilon\hbar} \Delta q^c g_{cd} \Delta q^d \right) = \sqrt{\frac{m}{2\pi i\epsilon\hbar g}} \tag{2.33}$$

with the g_{cd} as free parameters. Differentiation with respect to one of the parameters gives on the one hand:

$$\frac{\partial}{\partial g_{ab}} I(g_{cd}) = \frac{im}{2\epsilon\hbar} \int dq \exp \left(\frac{im}{2\epsilon\hbar} \Delta q^c g_{cd} \Delta q^d \right) \Delta q^a \Delta q^b \tag{2.34}$$

and on the other

$$\frac{\partial}{\partial g_{ab}} I(g_{cd}) = \sqrt{\frac{m}{2\pi i\epsilon\hbar}} \frac{\partial}{\partial g_{ab}} g^{-1/2} = -\frac{1}{2} \sqrt{\frac{m}{2\pi i\epsilon\hbar}} g^{-1/2} g^{ab} = -\frac{1}{2} g^{ab} I(g_{cd}). \tag{2.35}$$

Here we have used the formula for the differentiation of determinants: $\partial g / \partial g_{ab} = g g^{ab}$. Combining the last two equations yield (2.30).

Let us denote by $\xi = q'' - q'$ and $q = q''$. The various Taylor expanded contributions yield:

$$\begin{aligned}
& g^{1/4}(q - \xi) g^{1/2}(q - \xi) \simeq g^{3/4}(q) \left[1 - \Gamma_a \xi^a + \frac{1}{8} (4\Gamma_a \Gamma_b + 3\Gamma_{a,b}) \xi^a \xi^b \right]. \tag{2.36} \\
& \exp \left[\frac{im}{2\epsilon\hbar} g_{ab}(q - \xi) \xi^a \xi^b \right] \simeq \exp \left[\frac{im}{2\epsilon\hbar} g_{ab}(q) \xi^a \xi^b \right] \\
& \times \left[1 + \frac{m}{2i\epsilon\hbar} g_{ab} \Gamma_{cd}^b \xi^a \xi^c \xi^d - \frac{m}{8i\epsilon\hbar} \left(g_{av} \Gamma_{bc,d}^v + g_{av} \Gamma_{vd}^a \Gamma_{bc}^v + g_{uv} \Gamma_{ad}^u \Gamma_{bc}^v \right) \xi^a \xi^b \xi^c \xi^d \right. \\
& \quad \left. + \frac{1}{2} \left(\frac{m}{2i\epsilon\hbar} \right)^2 g_{av} g_{du} \Gamma_{bc}^v \Gamma_{ef}^u \xi^a \xi^b \xi^c \xi^d \xi^e \xi^f \right]. \tag{2.37}
\end{aligned}$$

Here the various derivatives of the metric tensor g_{ab} have been expressed by the

Christoffel symbols. Thus combining the last two equations yield:

$$\begin{aligned}
 & g^{1/4}(q - \xi)g^{1/2}(q - \xi) \exp \left[\frac{i m}{2\epsilon\hbar} g_{ab}(q - \xi)\xi^a \xi^b \right] \simeq g^{3/4}(q) \exp \left[\frac{i m}{2\epsilon\hbar} g_{ab}(q)\xi^a \xi^b \right] \\
 & \times \left[1 - \left(\Gamma_a \xi^a + \frac{m}{2i\epsilon\hbar} g_a d \Gamma_{bc}^d \right) \xi^a \xi^b \xi^c + \frac{1}{2} \left(\frac{m}{2i\epsilon\hbar} \right)^2 g_{av} g_{du} \Gamma_{bc}^v \Gamma_{ef}^u \xi^a \xi^b \xi^c \xi^d \xi^e \xi^f \right. \\
 & \left. - \frac{m}{8i\epsilon\hbar} \left(g_{av} \Gamma_{bc,d}^v + g_{au} \Gamma_{vd}^a \Gamma_{bc}^v + g_{uv} \Gamma_{ad}^u \Gamma_{bc}^v \right) \xi^a \xi^b \xi^c \xi^d + \frac{1}{8} (4\Gamma_a \Gamma_b + 3\Gamma_{a,b}) \xi^a \xi^b \right].
 \end{aligned} \tag{2.38}$$

Let us start with the B_{ab} -terms. We get immediately by equation (2.30):

$$B_{ab} \doteq - \frac{i\epsilon\hbar}{2m} g^{ab}. \tag{2.39}$$

Similarly:

$$B_a \doteq - \frac{i\epsilon\hbar}{2m} [\Gamma_a g^{ab} + (\partial_a g^{ab})] \tag{2.40}$$

For the ξ^2 - and ξ^4 -terms in B_0 we get:

$$\frac{1}{8} (4\Gamma_a \Gamma_b + 3\Gamma_{a,b}) \xi^a \xi^b \doteq \frac{i\epsilon\hbar}{8m} g^{ab} (4\Gamma_a \Gamma_b + 3\Gamma_{a,b}), \tag{2.41}$$

$$\begin{aligned}
 & - \frac{m}{8i\epsilon\hbar} \left(g_{av} \Gamma_{bc,d}^v + g_{au} \Gamma_{vd}^a \Gamma_{bc}^v + g_{uv} \Gamma_{ad}^u \Gamma_{bc}^v \right) \xi^a \xi^b \xi^c \xi^d \\
 & \doteq - \frac{i\epsilon\hbar}{8m} g^{ab} \left[8\Gamma_a \Gamma_b + \Gamma_{ab,c}^c + 2\Gamma_{a,b} \right. \\
 & \left. + g_{uv} g^{cd} (2\Gamma_{ad}^u \Gamma_{bc}^v + \Gamma_{ab}^u \Gamma_{cd}^v) + 5\Gamma_{ab}^c \Gamma_{bd}^c + 2\Gamma_{ac}^d \Gamma_{bd}^c \right].
 \end{aligned} \tag{2.42}$$

For the ξ^6 -terms equation (2.32) yields

$$\begin{aligned}
 & \frac{1}{2} \left(\frac{m}{2i\epsilon\hbar} \right)^2 g_{av} g_{du} \Gamma_{bc}^v \Gamma_{ef}^u \xi^a \xi^b \xi^c \xi^d \xi^e \xi^f \\
 & \doteq \frac{i\epsilon\hbar}{8m} g^{ab} \left[4\Gamma_a \Gamma_b + 4\Gamma_{ab}^c \Gamma_c + 4\Gamma_{ac}^d \Gamma_{bd}^c + g_{uv} g^{cd} (2\Gamma_{ac}^u \Gamma_{bd}^v + \Gamma_{ab}^u \Gamma_{cd}^v) \right].
 \end{aligned} \tag{2.43}$$

Therefore combining the relevant terms yields finally:

$$\begin{aligned}
 & \int g^{1/4}(q - \xi)g^{1/2}(q - \xi) \exp \left[\frac{i m}{2\epsilon\hbar} g_{ab}(q - \xi)\xi^a \xi^b \right] d\xi \\
 & \doteq \left(\frac{m}{2\pi i\epsilon\hbar} \right)^{-D/2} \left[1 + \frac{i\epsilon\hbar}{8m} g^{ab} (\Gamma_{a,b} - \Gamma_{ab}^c \Gamma_c + 2\Gamma_{ac}^d \Gamma_{bd}^c - \Gamma_{ab,c}^c) \right] \\
 & \doteq \left(\frac{m}{2\pi i\epsilon\hbar} \right)^{-D/2} \exp \left(\frac{i\epsilon}{\hbar} \Delta V_{Weyl} \right).
 \end{aligned} \tag{2.44}$$

Inserting all the contributions into equation (2.67) yields the Schrödinger equation (1.1).

Let us emphasize that the above procedure is nothing but a formal proof of the path integral. A rigorous proof **must** include at least two more ingredients

1) One must show that in fact

$$\lim_{N \rightarrow \infty} |[e^{-iTH/\hbar} - K(T)]\Psi| \rightarrow 0 \quad (2.45)$$

for all $\Psi \in \mathcal{H}$ (\mathcal{H} : relevant Hilbert space).

2) One must show that the domain \mathcal{D} of the infinitesimal generator of the kernel $K(T)$ is in fact identical with the domain of the Hamiltonian corresponding to the Schrödinger equation (1.1), i.e. the infinitesimal generator is the (selfadjoint) Hamiltonian.

It is quite obvious that these strong mathematical requirements hold only under certain assumptions on the potential involved in the Hamiltonian. Here Nelson [80] has shown the validity of the path integral for one dimensional path integrals. Some instructive proof can be furthermore found in the books of Simon [87] and Reed and Simon [85]. Also due to Albeverio et al. [1-3] is a wide range of discussion to formulate the Feynman path integral without the delay of go back to the definition of Wiener integrals.

3. Product-Ordering

In order to develop another useful lattice formulation for path integrals we consider again the generic case [43]. We assume that the metric tensor g_{ab} is real and symmetric and has $\text{rank}(g_{ab}) = D$, i.e. we have no constraints on the coordinates. Thus one can always find a linear transformation $C : q_a = C_{ab}y_b$ such that $\mathcal{L}_{Cl} = \frac{m}{2}\Lambda_{ab}\dot{y}^a\dot{y}^b$ with $\Lambda_{ab} = C_{ac}g_{cd}C_{db}$ and where Λ is diagonal. C has the form $C_{ab} = u_a^{(b)}$ where the $\vec{u}^{(b)}$ ($b \in \{1, \dots, d\}$) are the eigenvectors of g_{ab} and $\Lambda_{ab} = f_a^2 \delta_{ac} \delta_{bc}$ where $f_a^2 \neq 0$ ($a \in \{1, \dots, d\}$) are the eigenvalues of g_{ab} . Without loss of generality we assume $f_a^2 > 0$ for all $a \in \{1, \dots, d\}$. (For a time like coordinate q_a one might have e.g. $f_a^2 < 0$, but cases like this we want to exclude). Thus one can always find a representation for g_{ab} which reads,

$$g_{ab}(q) = h_{ac}(q)h_{bc}(q). \quad (3.1)$$

Here the $h_{ab} = C_{ac}f_c C_{cb} = u_c^{(a)} f_c u_c^{(b)}$ are real symmetric $D \times D$ matrices and satisfy $h_{ab}h^{bc} = \delta_a^c$. Because there exists the orthogonal transformation C equation (3.1) yields for the y -coordinate system (denoted by M_y):

$$\Lambda_{ab}(y) = f_c^2(y) \delta_{ac} \delta_{bc}. \quad (3.2)$$

equation (3.2) includes the special case $g_{ab} = \Lambda_{ab}$. The square-root of the determinant of g_{ab} , \sqrt{g} and the Christoffels Γ_a read in the q -coordinate system (denoted by M_q):

$$\sqrt{g} = \det(h_{ab}) =: h, \quad \Gamma_a = \frac{h_{,a}}{h}, \quad p_a = \frac{\hbar}{i} \left(\frac{\partial}{\partial q_a} + \frac{h_{,a}}{2h} \right). \quad (3.3)$$

The Laplace-Beltrami-operator expressed in the h^{ab} reads on M_q :

$$\Delta_{LB}^{M_q} = \left\{ h^{ac} h^{bc} \frac{\partial^2}{\partial q^a \partial q^b} + \left[\frac{\partial h^{ac}}{\partial q^a} h^{bc} + h^{ac} \frac{\partial h^{bc}}{\partial q^a} + \frac{h_{,a}}{h} h^{ac} h^{bc} \right] \frac{\partial}{\partial q^b} \right\} \quad (3.4)$$

and on M_y :

$$\Delta_{LB}^{M_y} = \frac{1}{f_a^2} \left[\frac{\partial^2}{\partial y_a^2} + \left(\frac{f_{b,a}}{f_b} - 2f_{a,a} \right) \frac{\partial}{\partial y_a} \right]. \quad (3.5)$$

With the help of the momentum operators (3.3) we rewrite the Hamiltonian in the "product-ordering" form

$$H = -\frac{\hbar^2}{2m} \Delta_{LB}^{M_q} + V(q) = \frac{1}{2m} h^{ac}(q) p_a p_b h^{bc}(q) + V(q) + \Delta V_{Prod}(q), \quad (3.6)$$

with the well-defined quantum correction

$$\Delta V_{Prod} = \frac{\hbar^2}{8m} \left[4h^{ac} h^{bc}{}_{,ab} + 2h^{ac} h^{bc} \frac{h_{,ab}}{h} + 2h^{ac} \left(h^{bc}{}_{,b} \frac{h_{,a}}{h} + h^{bc}{}_{,a} \frac{h_{,b}}{h} \right) - h^{ac} h^{bc} \frac{h_{,a} h_{,b}}{h^2} \right]. \quad (3.7)$$

On M_y the corresponding $\Delta V_{Prod}(y)$ is given by

$$\Delta V_{Prod}(y) = \frac{\hbar^2}{8m} \frac{1}{f_a^2} \left[\left(\frac{f_{b,a}}{f_b} \right)^2 - 4 \frac{f_{a,aa}}{f_a} + 4 \frac{f_{a,a}}{f_a} \left(2 \frac{f_{a,a}}{f_a} - \frac{f_{b,a}}{f_b} \right) + 2 \left(\frac{f_{b,a}}{f_b} \right)_{,a} \right]. \quad (3.8)$$

The expressions (3.7) and (3.8) look somewhat circumstantial, so we display a special case and the connection to the quantum correction ΔV_{Weyl} which corresponds to the Weyl-ordering prescription.

- 1) Let us assume that Λ_{ab} is proportional to the unit tensor, i.e. $\Lambda_{ab} = f^2 \delta_{ab}$. Then $\Delta V_{Prod}(y)$ simplifies into

$$\Delta V_{Prod}(y) = \hbar^2 \frac{D-2}{8m} \frac{(4-D)f_{a,a}^2 + 2f \cdot f_{,aa}}{f^4}. \quad (3.9)$$

This implies: Assume that the metric has or can be transformed into the special form $\Lambda_{ab} = f^2 \delta_{ab}$. If the dimension of the space is $D = 2$, then the quantum correction ΔV_{Prod} vanishes.

- 2) A comparison between (3.7) and (2.3) gives the connection with the quantum correction corresponding to the Weyl-ordering prescription:

$$\Delta V_{Prod} = \Delta V_{Weyl} + \frac{\hbar^2}{8m} (2h^{ac} h^{bc}{}_{,ab} - h^{ac}{}_{,a} h^{bc}{}_{,b} - h^{ac}{}_{,b} h^{bc}{}_{,a}). \quad (3.10)$$

In the case of equation (3.2) this yields:

$$\Delta V_{Prod}(y) = \Delta V_{Weyl} + \frac{\hbar^2}{4m} \frac{f_{a,a}^2 - f_a f_{a,aa}}{f_a^4} \quad (3.11)$$

These equations often simplify practical applications.

Next we have to consider the short-time matrix element $\langle q'' | e^{-iTH} | q' \rangle$ in order to derive the path integral formulation corresponding to our ordering prescription (3.6).

We proceed similar as in the previous section. We consider the short-time approximation to the matrix element $[\epsilon = T/N, g^{(j)} = g(q^{(j)})]$:

$$\begin{aligned} \langle q^{(j)} | e^{-i\epsilon H/\hbar} | q^{(j-1)} \rangle &\simeq \langle q^{(j)} | 1 - i\epsilon H/\hbar | q^{(j-1)} \rangle \\ &= \frac{[g^{(j)} g^{(j-1)}]^{-\frac{1}{4}}}{(2\pi\hbar)^D} \int e^{ip\Delta q^{(j)}} dp \\ &\quad - \frac{i\epsilon\hbar}{2m} \langle q^{(j)} | h^{ac} p_a p_b h^{bc} | q^{(j-1)} \rangle - \frac{i\epsilon}{\hbar} \langle q^{(j)} | V + \Delta V_{Prod} | q^{(j-1)} \rangle. \end{aligned} \quad (3.12)$$

Therefore we get for the short-time matrix element ($\epsilon \ll 1$):

$$\begin{aligned} \langle q^{(j)} | e^{-i\epsilon H/\hbar} | q^{(j-1)} \rangle &\simeq \frac{[g^{(j)} g^{(j-1)}]^{-\frac{1}{4}}}{(2\pi\hbar)^D} \int dp \\ &\quad \times \exp \left[\frac{i}{\hbar} p \Delta q^{(j)} - \frac{i\epsilon}{2m\hbar} h^{ac}(q^{(j)}) h^{bc}(q^{(j-1)}) p_a p_b - \frac{i\epsilon}{\hbar} V(q^{(j)}) - \frac{i\epsilon}{\hbar} \Delta V_{Prod}(q^{(j)}) \right]. \end{aligned} \quad (3.13)$$

The choice of the “post point” $q^{(j)}$ in the potential terms is not unique. A “pre-point”, “mid-point” or a “product-form”-expansion is also legitimate. However, changing from one to another formulation does not alter the path integral, because differences in the potential terms are of $O(\epsilon)$, i.e. of $O(\epsilon^2)$ in the short-time Feynman kernel and therefore do not contribute. The Trotter formula $e^{-iT(A+B)} := s - \lim_{N \rightarrow \infty} (e^{-iTA/N} e^{-iTB/N})^N$ states that all approximations in equations (3.12) to (3.13) are valid in the limit $N \rightarrow \infty$ and we get for the **Hamiltonian path integral in the “product form”-definition** [$h_{ab}^{(j)} = h_{ab}(q^{(j)})$]:

$$\begin{aligned} K(q'', q'; T) &= [g(q')g(q'')]^{-\frac{1}{4}} \lim_{N \rightarrow \infty} \left(\prod_{j=1}^{N-1} \int dq^{(j)} \times \prod_{j=1}^N \frac{dp^{(j)}}{(2\pi\hbar)^D} \right) \\ &\quad \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[p \Delta q^{(j)} - \frac{\epsilon}{2m} h^{ac,(j)} h^{bc,(j-1)} p_a^{(j)} p_b^{(j)} - \epsilon V(q^{(j)}) - \epsilon \Delta V_{Prod}(q^{(j)}) \right] \right\}. \end{aligned} \quad (3.14)$$

Performing the momentum integrations we get for the **Lagrangian path integral**

in the “product form”-definition: ($PF=$ Product-Form)

$$\begin{aligned}
 & K(q'', q'; T) \\
 &= \int_{q(t')=q'}^{q(t'')=q''} \sqrt{g} \mathcal{D}_{PF} q(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} h_{ac} h_{bc} \dot{q}^a \dot{q}^b - V(q) - \Delta V_{Prod}(q) \right] dt \right\} \\
 &:= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{ND}{2}} \prod_{j=1}^{N-1} \int \sqrt{g(q^{(j)})} dq^{(j)} \\
 &\times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} h_{ac}^{(j)} h_{bc}^{(j-1)} \Delta q^{a,(j)} \Delta q^{b,(j)} - \epsilon V(q^{(j)}) - \epsilon \Delta V_{Prod}(q^{(j)}) \right] \right\}. \quad (3.15)
 \end{aligned}$$

In the last step we have to check that the Schrödinger equation (1.1) can be deduced from the short-time kernel of equation (3.15). Because one can always transform from the q -coordinates to the y -coordinates, which is a linear orthogonal transformation and thus does not produce any quantum correction in the path integral (3.14) defined on M_y , we shall use in the following the representation of equation (3.2). Looking at the Hamiltonian formulation of the path integral (3.14) we see that the unitary transformation $y = C^{-1}q$ changes the metric term $p_a h^{ac} h^{cb} p_b \rightarrow p_a \Lambda^{ab} p_b$, i.e. in the correct feature, the measure $dq^{(j)} dp^{(j)} \rightarrow dy^{(j)} dp^{(j)}$ due to the Jacobean $J = 1$ so that the Feynman kernel is transformed just in the right manner.

We restrict ourselves to the proof that the short-time kernels of equations (2.25) and (3.15) are equivalent, i.e. we have to show ($\bar{y} = (y'' + y')/2$):

$$\begin{aligned}
 & [g(y')g(y'')]^{-\frac{1}{4}} \sqrt{g(\bar{y})} \exp \left[\frac{im}{2\epsilon\hbar} \Lambda_{ab}(\bar{y}) \Delta y^a \Delta y^b - \frac{i\epsilon}{\hbar} V(\bar{y}) - i\epsilon\hbar \Delta V_W(\bar{y}) \right] \\
 & \doteq \exp \left\{ \frac{im}{2\epsilon\hbar} f_a(y') f_a(y'') \Delta^2 y^a - \frac{i\epsilon}{\hbar} V(y'') - i\epsilon\hbar \Delta V_{Prod}(y'') \right\}. \quad (3.16)
 \end{aligned}$$

Clearly, $e^{-i\epsilon V(\bar{y})/\hbar} \doteq e^{-i\epsilon V(y'')/\hbar}$ for the potential term. It suffices to show that a Taylor expansion of the g and the kinetic energy terms on the left-hand side of equation (3.15) yield an additional potential $\Delta \tilde{V}$ given by

$$\Delta \tilde{V}(y) = \Delta V_{Prod}(y) - \Delta V_{Weyl}(y) = \frac{\hbar^2}{4m} \frac{f_{a,a}^2(y) - f_a(y) f_{a,aa}(y)}{f_a^4(y)}. \quad (3.17)$$

We consider the g -terms on the left-hand side of equation (3.16) and expand them in a Taylor-series around y' . This gives ($\xi_a = (y''_a - y'_a)$, $f_a(y') \equiv f_a$):

$$[g(y')g(y'')]^{-\frac{1}{4}} \sqrt{g(\bar{y})} \simeq \left[1 - \frac{1}{8} \frac{f_c f_{c,ab} - f_{c,a} f_{c,b}}{f_c^2} \xi^a \xi^b \right]. \quad (3.18)$$

Exploiting the path integral identity (2.30) we get by exponentiating the $O(\epsilon)$ -terms,

$$[g(y')g(y'')]^{-\frac{1}{4}} \sqrt{g(\bar{y})} \simeq \exp \left[-\frac{i\epsilon\hbar}{8m} \frac{f_a f_{a,bb} - f_{a,b}^2}{f_a^2 f_b^2} \right]. \quad (3.19)$$

Repeating the same procedure for the exponential term gives:

$$\begin{aligned} \exp \left[\frac{im}{2\epsilon\hbar} \Lambda_{ab}(\bar{y}) \xi^a \xi^b \right] &\simeq \exp \left[\frac{im}{2\epsilon\hbar} f_a(y') f_a(y'') \xi^a \xi^a \right] \\ &\times \left[1 - \frac{i\epsilon}{8m} (f_c f_{c,ab} - f_{c,a} f_{c,b}) \xi^a \xi^b \xi^c \xi^c \right]. \end{aligned} \quad (3.20)$$

We use the identity (2.30) to get

$$\begin{aligned} \exp \left[\frac{im}{2\epsilon\hbar} \Lambda_{ab}(\bar{y}) \xi^a \xi^b \right] &\simeq \exp \left[\frac{im}{2\epsilon\hbar} f_a(y') f_a(y'') \xi^a \xi^a \right] \\ &\times \exp \left[\frac{i\epsilon\hbar}{8m} \frac{f_a f_{a,bb} - f_{a,b}^2}{f_a^2 f_b^2} + \frac{i\epsilon\hbar}{4m} \frac{f_a f_{a,aa} - f_{a,a}^2}{f_a^4} \right]. \end{aligned} \quad (3.21)$$

Combining equations (3.20) and (3.21) yields the additional potential $\Delta\tilde{V}$ and equation (3.16) is proven. Thus we conclude that the path integral (3.15) is well-defined and is the correct path integral corresponding to the Schrödinger equation (1.1).

A combination of lattice prescriptions were in the metric g_{ab} in the kinetic term in the exponential and in the determinant g different lattices are used exist also in the literature. Let us note the result of McLaughlin and Schulman [70]:

$$\begin{aligned} K(q'', q'; T) &= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{ND/2} \prod_{j=1}^{N-1} \int \sqrt{g(q^{(j)})} dq^{(j)} \\ &\times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} g_{ab}(\bar{q}^{(j)}) \Delta q^{(j)a} q^{(j)b} - \epsilon U(\bar{q}^{(j)}) \right] \right\} \end{aligned} \quad (3.22)$$

with the effective potential

$$U(q) = V(q) - \frac{e}{c} A(q) \cdot \dot{q} - F(q), \quad (3.23)$$

$A(q)$ an additional vector potential term and $F(q)$ given by

$$\begin{aligned} F(q) &= -\hbar^2 F_{abcd} \left(g^{ab} g^{cd} + g^{ac} g^{bd} + g^{ad} g^{bc} \right) \\ F_{abcd} &= \frac{1}{48m} \left(g_{ab,cd} - 2g^{ef} \Gamma_{abe} \Gamma_{cdf} \right) \end{aligned} \quad (3.24)$$

$[\Gamma_{abc} = \frac{1}{2}(g_{ab,c} + g_{ac,b} - g_{bc,a})]$. According to D'Olivio and Torres [21] the effective potential can be rewritten as

$$U(q) = V(q) - \frac{e}{c} A(q) \cdot \dot{q} + \frac{\hbar^2}{8m} \left[R + g^{ab} (\Gamma_{ad}^c \Gamma_{bc}^d + \partial_b \Gamma_{ac}^c) \right]. \quad (3.25)$$

which is up the derivative term the result of Mizrahi [72].

4. Space-Time Transformation

Let us consider a one-dimensional path integral

$$K(x'', x'; T) = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left[\frac{i}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \dot{x}^2 - V(x) \right) dt \right]. \quad (4.1)$$

It is now assumed that the potential V is so complicated that a direct evaluation of the path integral is not possible. We want to describe a method how a transformed path integral can be achieved to calculate $K(T)$ or $G(E)$, respectively, nevertheless. This method is called “space-time” transformation. This technique was originally developed by Duru and Kleinert [27,28]. It was further evolved by Steiner [90], Pak and Sökmen [82], Inomata [59], Kleinert [62, 64] and Grosche and Steiner [49]. In the rigorous lattice derivation of the following formulæ we follow ourselves [49] and Pak and Sökmen [82].

Let us start by considering the Legendre-transformed of the general one-dimensional Hamiltonian:

$$H_E = -\frac{\hbar^2}{2m} \left(\frac{d^2}{dx^2} + \Gamma(x) \frac{d}{dx} \right) + V(x) - E \quad (4.2)$$

which is hermitean with respect to the scalar product

$$(f_1, f_2) = \int f_1(x) f_2^*(x) J(x) dx, \quad J(x) = e^{\int^x \Gamma(x') dx'}. \quad (4.3)$$

Introducing the momentum operator

$$p_x = \frac{\hbar}{i} \left(\frac{d}{dx} + \frac{1}{2} \Gamma(x) \right), \quad \Gamma(x) = \frac{d \ln J(x)}{dx} \quad (4.4)$$

H_E can be rewritten as

$$H_E = \frac{p_x^2}{2m} + V(x) + \frac{\hbar^2}{8m} [\Gamma^2(x) + 2\Gamma'(x)] - E \quad (4.5)$$

with the corresponding path integral

$$K_E(x'', x'; T) = e^{iTE/\hbar} K(x'', x'; T), \quad (4.6)$$

where $K(T)$ denotes the path integral of equation (4.1).

Let us consider the space-time transformation

$$x = F(q), \quad dt = f(x) ds \quad (4.7)$$

with new coordinate q and new “time” s . Let $G(q) = \Gamma[F(q)]$, then

$$\hat{H}_E = -\frac{\hbar^2}{2m} \frac{1}{F'^2(q)} \left[\frac{d^2}{dq^2} + \left(G(q) F'(q) - \frac{F''(q)}{F'(q)} \right) \frac{d}{dq} \right] + V[F(q)] - E. \quad (4.8)$$

With the constraint $f[F(q)] = F'^2(q)$ we get for the new Hamiltonian $\tilde{H} = f\hat{H}_E$:

$$\begin{aligned}\tilde{H} &= -\frac{\hbar^2}{2m} \left[\frac{d^2}{dq^2} + \left(G(q)F'(q) - \frac{F''(q)}{F'(q)} \right) \frac{d}{dq} \right] + f[F(q)][V(F(q)) - E] \\ &= -\frac{\hbar^2}{2m} \left[\frac{d^2}{dq^2} + \tilde{\Gamma}(q) \frac{d}{dq} \right] + f[F(q)][V(F(q)) - E],\end{aligned}\quad (4.9)$$

where $\tilde{\Gamma}(q) = G(q)F'(q) - F''(q)/F'(q)$. The corresponding measure in the scalar product and the hermitean momentum are

$$p_q = \frac{\hbar}{i} \left[\frac{d}{dq} + \frac{1}{2} \tilde{\Gamma}(q) \right], \quad J(q) = \sqrt{g(q)} = e^{\int^q \tilde{\Gamma}(q') dq'}.\quad (4.10)$$

The Hamiltonian \tilde{H} expressed in the position- q and momentum operator p_q is

$$\tilde{H} = \frac{p_q}{2m} + f[F(q)][V(F(q))_E] + \Delta V(q)\quad (4.11)$$

with the well-defined quantum potential

$$\Delta V(q) = \frac{\hbar^2}{8m} \left[3 \left(\frac{F''(q)}{F'(q)} \right)^2 - 2 \frac{F'''(q)}{F'(q)} + (G(q)F'(q))^2 + 2G'(q)F'(q) \right].\quad (4.12)$$

Note that for $G \equiv 0$, ΔV is proportional to the Schwartz derivative of the transformation F . The path integral corresponding to the Hamiltonian \tilde{H} is

$$\begin{aligned}\tilde{K}(q'', q'; s'') &= \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}q(s) \\ &\times \exp \left\{ \frac{i}{\hbar} \int_0^{s''} \left[\frac{m}{2} \dot{q}^2 - f[F(q)][V(F(q)) - E] - \Delta V(q) \right] ds \right\}.\end{aligned}\quad (4.13)$$

As it is easily checked it is possible to derive from the short time kernel of equation (4.13) via the time evolution equation

$$\tilde{\Psi}(q''; s) = \int \tilde{K}(q'', q'; s'') \tilde{\Psi}(q'; s) dq'\quad (4.14)$$

the time dependent Schrödinger equation

$$H\tilde{\Psi}(q; s) = i\hbar \frac{\partial}{\partial s} \tilde{\Psi}(q; s).\quad (4.15)$$

The crucial point is now, of course, the rigorous lattice derivation of $\tilde{K}(s'')$ and the relation between $\tilde{K}(s'')$ and $K(T)$. It turns out that \tilde{K} is given in terms of $K(T)$ by the equations

$$\begin{aligned}K(x'', x'; T) &= \frac{1}{2\pi i \hbar} \int_{-\infty}^{\infty} e^{-iTE/\hbar} G(x'', x'; E) dE \\ G(x'', x'; E) &= i[f(x')f(x'')]^{\frac{1}{4}} \int_0^{\infty} \tilde{K}(q'', q'; s'') ds''.\end{aligned}\quad (4.16)$$

This we want to justify!

Let us consider the path integral $K(T)$ in its lattice definition

$$K(x'', x'; T) = \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{ND/2} \prod_{j=1}^{N-1} \int dx^{(j)} \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} (x^{(j)} - x^{(j-1)})^2 - \epsilon V(\bar{x}^{(j)}) \right] \right\}. \quad (4.17)$$

We consider a D -dimensional path integral. To transform the coordinates x into the coordinates q we use the midpoint prescription expansion method. It reads that one has to expand any dynamical quantity in question $F(q)$ which is defined on the points $q^{(j)}$ and $q^{(j-1)}$ of the j^{th} -interval in the lattice version about the midpoints $\bar{q}^{(j)} = \frac{1}{2}(q^{(j)} + q^{(j-1)})$ maintaining terms up to order $(q^{(j)} - q^{(j-1)})^3$. This gives:

$$\begin{aligned} \Delta F(q^{(j)}) &\equiv F(q^{(j)}) - F(q^{(j-1)}) = F\left(\bar{q}^{(j)} + \frac{\Delta q^{(j)}}{2}\right) - F\left(\bar{q}^{(j)} - \frac{\Delta q^{(j)}}{2}\right) \\ &= \Delta q_m^{(j)} \frac{\partial F(q)}{\partial q_m} \Big|_{q=\bar{q}^{(j)}} + \frac{1}{24} \Delta q_m^{(j)} \Delta q_n^{(j)} \Delta q_k^{(j)} \frac{\partial^3 F(q)}{\partial q_m \partial q_n \partial q_k} \Big|_{q=\bar{q}^{(j)}} + \dots \end{aligned} \quad (4.18)$$

Introducing the abbreviations

$$F_{,m}^{(j)} \equiv \frac{\partial F(q)}{\partial q_m} \Big|_{q=\bar{q}^{(j)}}, \quad F_{,mnk}^{(j)} \equiv \frac{\partial^3 F(q)}{\partial q_m \partial q_n \partial q_k} \Big|_{q=\bar{q}^{(j)}} \quad (4.19)$$

we get therefore

$$\begin{aligned} \Delta^2 F^{(j)} &= \left[\Delta q_m^{(j)} F_{,m}^{(j)} + \frac{1}{24} \Delta q_m^{(j)} \Delta q_n^{(j)} \Delta q_k^{(j)} F_{,mnk}^{(j)} \right]^2 \\ &\simeq \Delta q_m^{(j)} \Delta q_n^{(j)} F_{,m}^{(j)} F_{,n}^{(j)} + \frac{1}{12} \Delta q_m^{(j)} \Delta q_n^{(j)} \Delta q_k^{(j)} \Delta q_l^{(j)} F_{,m}^{(j)} F_{,nkl}^{(j)} \\ &\doteq \Delta q_m^{(j)} \Delta q_n^{(j)} F_{,m}^{(j)} F_{,n}^{(j)} + \frac{1}{12} \left(\frac{i \epsilon \hbar}{m} \right)^2 F_{,m}^{(j)} F_{,nkl}^{(j)} F_{mnkl}^{-1}(q^{(j)}) \end{aligned} \quad (4.20)$$

according to equation (2.30) with the abbreviation

$$\begin{aligned} F_{mnkl}^{-1}(q^{(j)}) &= (F_{,m}^{(j)} F_{,n}^{(j)})^{-1} (F_{,k}^{(j)} F_{,l}^{(j)})^{-1} \\ &\quad + (F_{,m}^{(j)} F_{,k}^{(j)})^{-1} (F_{,l}^{(j)} F_{,n}^{(j)})^{-1} + (F_{,m}^{(j)} F_{,l}^{(j)})^{-1} (F_{,k}^{(j)} F_{,n}^{(j)})^{-1}. \end{aligned} \quad (4.21)$$

Furthermore we have to transform the measure. Because

$$\prod_{j=1}^{N-1} dx^{(j)} = \prod_{j=1}^{N-1} \left| \frac{\partial F(q^{(j)})}{\partial q^{(j)}} \right| dq^{(j)} \equiv \prod_{j=1}^{N-1} F_{,q}(q^{(j)}) dq^{(j)} \quad (4.22)$$

does not have a symmetric look, we must expand about $\bar{q}^{(j)}$. We get:

$$\begin{aligned}
\prod_{j=1}^{N-1} dx^{(j)} &= [F_{;q}(q')F_{;q}(q'')]^{-1/2} \prod_{j=1}^{N-1} dq^{(j)} \prod_{j=1}^N [F_{;q}(q^{(j)})F_{;q}(q^{(j-1)})]^{1/2} \\
&\simeq [F_{;q}(q')F_{;q}(q'')]^{-1/2} \prod_{j=1}^{N-1} dq^{(j)} \prod_{j=1}^N F_{;q}(\bar{q}^{(j)}) \\
&\quad \times \left[1 - \frac{1}{8} \left(\frac{F_{;q,m}(\bar{q}^{(j)})F_{;q,n}(\bar{q}^{(j)})}{F_{;q}^2(\bar{q}^{(j)})} - \frac{F_{;q,mn}(\bar{q}^{(j)})}{F_{;q}(\bar{q}^{(j)})} \right) \Delta q_m^{(j)} \Delta q_n^{(j)} \right] \\
&\doteq [F_{;q}(q')F_{;q}(q'')]^{-1/2} \prod_{j=1}^{N-1} dq^{(j)} \prod_{j=1}^N F_{;q}(\bar{q}^{(j)}) \\
&\quad \times \exp \left[-\frac{i\epsilon\hbar}{8m} (F_{;m}^{(j)} F_{;n}^{(j)})^{-1} \left(\frac{F_{;q,m}(\bar{q}^{(j)})F_{;q,n}(\bar{q}^{(j)})}{F_{;q}^2(\bar{q}^{(j)})} - \frac{F_{;q,mn}(\bar{q}^{(j)})}{F_{;q}(\bar{q}^{(j)})} \right) \right] \quad (4.23)
\end{aligned}$$

Thus we have the coordinate transformed path integral

$$\begin{aligned}
&\hat{K}(q'', q'; T) \\
&= [F_{;q}(q')F_{;q}(q'')]^{-1/2} \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{ND/2} \prod_{j=1}^{N-1} \int dq^{(j)} \\
&\quad \times \prod_{j=1}^N F_{;q}(\bar{q}^{(j)}) \exp \left\{ \frac{i}{\hbar} \left[\frac{m}{2\epsilon} \Delta q_m^{(j)} \Delta q_n^{(j)} F_{;m}(\bar{q}^{(j)}) F_{;n}(\bar{q}^{(j)}) - \epsilon V(\bar{q}^{(j)}) \right. \right. \\
&\quad \left. \left. - \frac{\epsilon \hbar^2}{8m} (F_{;m}^{(j)} F_{;n}^{(j)})^{-1} \left(\frac{F_{;q,m}(\bar{q}^{(j)})F_{;q,n}(\bar{q}^{(j)})}{F_{;q}^2(\bar{q}^{(j)})} - \frac{F_{;q,mn}(\bar{q}^{(j)})}{F_{;q}(\bar{q}^{(j)})} \right) \right. \right. \\
&\quad \left. \left. - \frac{\epsilon \hbar^2}{8m} F_{;m}^{(j)} F_{;nkl}^{(j)} F_{;mnkl}^{-1}(q^{(j)}) \right) \right] \Big\} \quad (4.24)
\end{aligned}$$

This path integral has the canonical form (2.24). In order to see this note that the factor $F_{;m}(\bar{q}^{(j)})F_{;n}(\bar{q}^{(j)})$ in the dynamical term can be interpreted as a metric g_{nm} appearing in an effective Lagrangian. Thus we can rewrite the transformed Lagrangian in terms of this metric (Gervais and Jevicki [38]): Firstly, we have $F_{;q}(q) = \sqrt{\det(g_{nm}(q))} \equiv \sqrt{g(q)}$. In the expansion of the determinant about the midpoints we then get

$$\begin{aligned}
&[F_{;q}(q^{(j)})F_{;q}(q^{(j-1)})]^{1/2} \\
&\simeq \sqrt{g(\bar{q}^{(j)})} \left\{ 1 + \frac{1}{16} \left[g^{mn}(\bar{q}^{(j)})g_{mn,kl} + g^{mn}{}_{,k}(\bar{q}^{(j)})g_{mn,l}(\bar{q}^{(j)}) \right] \Delta q_k^{(j)} \Delta q_k^{(j)} \right\}. \quad (4.25)
\end{aligned}$$

Using now successively the identities

$$\frac{g_{,l}(q)}{g(q)} = g^{mn}(q)g_{mn,l}(q), \quad F_{;kl}^m(q) = \Gamma_{kl}^m(q)F_{;l}^m(q), \quad \Gamma_{ml,k}^m(q) = \frac{1}{2} \left(\frac{g_{,l}(q)}{g(q)} \right)_{,k} \quad (4.26)$$

we can cast the quantum potential into the form

$$\Delta V(q) = \frac{\hbar^2}{8m} g^{mn}(q) \Gamma_{lm}^k(q) \Gamma_{km}^l(q) \quad (4.27)$$

which is just the Weyl-ordering quantum potential without curvature term. In the present case the curvature vanishes since we started in flat Euclidean space which remains flat during the transformation.

In particular equation (4.24) takes in the one-dimensional case the form [$F' \equiv dF/dq$]:

$$\begin{aligned} \hat{K}(q'', q'; T) &= [F'(q')F'(q'')]^{-\frac{1}{2}} \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{N/2} \prod_{j=1}^{N-1} \int dq^{(j)} \\ &\times \prod_{j=1}^N F'(\bar{q}^{(j)}) \exp \left\{ \frac{i}{\hbar} \left[\frac{m}{2\epsilon \hbar} \Delta q_m^{(j)} \Delta q_n^{(j)} F'^2(\bar{q}^{(j)}) - \epsilon V(\bar{q}^{(j)}) - \frac{\epsilon \hbar^2 F'^2(\bar{q}^{(j)})}{8m F'^4(\bar{q}^{(j)})} \right] \right\} \end{aligned} \quad (4.28)$$

which is the same result as taking for $D = 1$ in ΔV_{Weyl} the metric tensor $g_{ab} = F'^2$. The infinitesimal generator of the corresponding time-evolution operator is \hat{H}_E . To proceed further we perform a time transformation according to

$$dt = f(s)ds, \quad s' = s(t') = 0, \quad s'' = s(t'') \quad (4.29)$$

and we make the choice $f = F'^2$. Translating this transformation into the discrete notation requires special care. In accordance with the midpoint prescription we must first symmetrize over the interval $(j, j-1)$ to prefer neither of the endpoints over the other, i.e.

$$\Delta t^{(j)} = \epsilon = \Delta s^{(j)} F'(q^{(j)}) F'(q^{(j-1)}), \quad \Delta s^{(j)} = s^{(j)} - s^{(j-1)} \equiv \delta^{(j)}. \quad (4.30)$$

Expanding about midpoints yields

$$\epsilon \simeq \delta^{(j)} F'^2(\bar{q}^{(j)}) \left\{ 1 + \frac{\Delta^2 q^{(j)}}{4} \left[\frac{F'''(\bar{q}^{(j)})}{F'(\bar{q}^{(j)})} - \left(\frac{F''(\bar{q}^{(j)})}{F'(\bar{q}^{(j)})} \right)^2 \right] \right\}. \quad (4.31)$$

Insertion yields for each j^{th} -term [identify $V(x^{(j)}) \rightarrow V[F(q^{(j)})] \equiv V(q^{(j)})$]

$$\begin{aligned} &\left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{N}{2}} \prod_{j=1}^{N-1} dx^{(j)} \times \prod_{j=1}^N \exp \left\{ \frac{i}{\hbar} \left[\frac{m}{2\epsilon} \Delta^2 x^{(j)} - \epsilon V(x^{(j)}) + \epsilon E \right] \right\} \\ &= \prod_{j=1}^{N-1} F'(q^{(j)}) dq^{(j)} \prod_{j=1}^N \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{1}{2}} \\ &\quad \times \exp \left(\frac{i}{\hbar} \left\{ \frac{m}{2\epsilon} \left[F'^2(\bar{q}^{(j)}) \Delta^2 q^{(j)} + \frac{1}{12} F'(\bar{q}^{(j)}) F'''(\bar{q}^{(j)}) \Delta^4 q^{(j)} \right] - \epsilon V(x^{(j)}) + \epsilon E \right\} \right) \end{aligned}$$

$$\begin{aligned}
&= \left[F'(q')F'(q'') \right]^{-\frac{1}{2}} \prod_{j=1}^{N-1} dq^{(j)} \prod_{j=1}^N \left(\frac{m}{2\pi i \delta^{(j)} \hbar} \right)^{\frac{1}{2}} \\
&\quad \times \prod_{j=1}^N \exp \left[\frac{i}{\hbar} \left\{ \frac{m}{2\delta^{(j)}} \left[1 + \frac{\Delta^2 q^{(j)}}{4} \left(\frac{F'''(\bar{q}^{(j)})}{F'(\bar{q}^{(j)})} - \left(\frac{F''(\bar{q}^{(j)})}{F'(\bar{q}^{(j)})} \right)^2 \right) \right] \right. \right. \\
&\quad \times \left. \left. \left[\Delta^2 q^{(j)} + \frac{\Delta^4 q^{(j)}}{12} \frac{F'''(\bar{q}^{(j)})}{F'(\bar{q}^{(j)})} \right] - \delta^{(j)} F'^2(\bar{q}^{(j)}) [V(q^{(j)}) - E] \right\} \right] \\
&\doteq \left[F'(q')F'(q'') \right]^{-\frac{1}{2}} \left(\frac{m}{2\pi i \delta \hbar} \right)^{\frac{N}{2}} \prod_{j=1}^{N-1} dq^{(j)} \\
&\quad \times \exp \left[\frac{i}{\hbar} \left\{ \frac{m}{2\delta} \Delta^2 q^{(j)} - \delta^{(j)} F'^2(q^{(j)}) [V(q^{(j)}) - E] \right. \right. \\
&\quad \left. \left. - \frac{i\delta\hbar}{8m} \left[3 \left(\frac{F''(q^{(j)})}{F'(q^{(j)})} \right)^2 - 2 \frac{F'''(q^{(j)})}{F'(q^{(j)})} \right] \right\} \right]. \tag{4.32}
\end{aligned}$$

Let us assume that the constraint

$$\int_0^{s''} ds f[F(q(s))] = T \tag{4.33}$$

has for all admissible path a unique solution $s'' > 0$. Of course, since T is fixed, the “time” s'' will be path dependent. To incorporate the constraint let us consider the Green function $G(E)$:

$$\begin{aligned}
G(x'', x'; E) &= i \int_0^\infty dT e^{iTE/\hbar} K(x'', x'; T) \\
K(x'', x'; T) &= \frac{1}{2\pi i \hbar} \int_{-\infty}^\infty dE e^{-iTE/\hbar} G(x'', x'; E). \tag{4.34}
\end{aligned}$$

This incorporation is far from being trivial, in particular the fact that the new time-slicing $\delta^{(j)}$ is treated in the same way as the old time-slicing ϵ , i.e. $\delta^{(j)} \equiv \delta$ is considered as being fixed, but can however be justified in a more comprehensive way, see e.g. [6, 35, 59]. We now observe that $G(E)$ can be written e.g. in the following ways [62]:

$$G(x'', x'; E) = f(x'') \int_0^\infty \langle x'' | e^{-i s'' f(x)(H-E)/\hbar} | x' \rangle ds'' \tag{4.35a}$$

$$= f(x') \int_0^\infty \langle x'' | e^{-i s'' (H-E)f(x)/\hbar} | x' \rangle ds'' \tag{4.35b}$$

$$= \sqrt{f(x')f(x'')} \int_0^\infty \langle x'' | e^{-i s'' \sqrt{f(x)(H-E)}\sqrt{f(x)}/\hbar} | x' \rangle ds''. \tag{4.35c}$$

The last line give for the short-time matrix element in the usual manner

$$\langle x^{(j)} | e^{-i \delta^{(j)} f^{\frac{1}{2}}(x)(H-E)f^{\frac{1}{2}}(x)/\hbar} | x^{(j-1)} \rangle$$

$$\begin{aligned}
 & \simeq \langle x^{(j)} | 1 - i\delta^{(j)} f^{\frac{1}{2}}(x)(H - E)f^{\frac{1}{2}}(x)/\hbar | x^{(j-1)} \rangle \\
 & = \langle x^{(j)} | x^{(j-1)} \rangle - \frac{i\delta^{(j)}}{\hbar} \sqrt{f(x^{(j)})f(x^{(j-1)})} \langle x^{(j)} | H - E | x^{(j-1)} \rangle \\
 & = \langle F(q^{(j)}) | F(q^{(j-1)}) \rangle - \frac{i\delta^{(j)}}{\hbar} \sqrt{f[F(q^{(j)})]f[F(q^{(j-1)})]} \langle F(q^{(j)}) | \hat{H}_E | F(q^{(j-1)}) \rangle \\
 & = \langle q^{(j)} | q^{(j-1)} \rangle - \frac{i\delta}{\hbar} \langle q^{(j)} | \tilde{H} | q^{(j-1)} \rangle \\
 & = \frac{1}{2\pi} \int dp_q \exp \left[\frac{i}{\hbar} p_q \Delta q^{(j)} - \frac{i\delta}{2m\hbar} p_q^2 - \frac{i\delta}{\hbar} F'^2(\bar{q}^{(j)})(V(\bar{q}^{(j)}) - E) - \frac{i\delta}{\hbar} \Delta V(\bar{q}^{(j)}) \right] \\
 & = \left(\frac{m}{2\pi i \delta \hbar} \right)^{\frac{1}{2}} \exp \left[\frac{im}{2\delta \hbar} \Delta^2 q^{(j)} - \frac{\delta}{\hbar} F'^2(\bar{q}^{(j)})(V(\bar{q}^{(j)}) - E) - \frac{\delta}{\hbar} \Delta V(\bar{q}^{(j)}) \right], \tag{4.36}
 \end{aligned}$$

where the well-defined quantum potential ΔV is given by

$$\Delta V(q) = \frac{\hbar^2}{8m} \left[3 \frac{F'''(q)}{F'(q)} - 2 \left(\frac{F''(q)}{F'(q)} \right)^2 \right]. \tag{4.37}$$

Note that this quantum potential has the form of a Schwartz-derivative. This procedure justifies our symmetrization prescription of equations (4.30). Thus we arrive finally at the **transformation formulæ**:

$$\begin{aligned}
 K(x'', x'; T) &= \frac{1}{2\pi i \hbar} \int_{-\infty}^{\infty} dE e^{-iTE/\hbar} G[F(q''), F(q'); E] \\
 G[F(q''), F(q'); E] &= i[F'(q'')F'(q')]^{-\frac{1}{2}} \int_0^{\infty} ds'' \tilde{K}(q'', q'; s'')
 \end{aligned} \tag{4.38}$$

where the path integral $\tilde{K}(s'')$ is given by

$$\begin{aligned}
 \tilde{K}(q'', q'; s'') &= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \delta \hbar} \right)^{\frac{1}{2} N-1} \prod_{j=1}^{N-1} \int dq^{(j)} \\
 & \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\delta} \Delta^2 q^{(j)} - \delta F'^2(\bar{q}^{(j)})(V(\bar{q}^{(j)}) - E) - \delta \Delta V(\bar{q}^{(j)}) \right] \right\}. \tag{4.39}
 \end{aligned}$$

This is exactly the path integral (4.13) and thus we have proven it. Note that only for $D = 1$ we have the simple transformation property

$$\frac{im}{2\epsilon} \Delta^2 x^{(j)} \rightarrow \frac{im}{2\delta} \Delta^2 q^{(j)}. \tag{4.40}$$

For higher dimensional problem this may only possible for one coordinate kinetic term. Anyway, the requirement for higher dimensional problems are

$$F_{;q} = F_{,m} F_{,n} \tag{4.41}$$

for at least one pair $(m, n) \in (1, \dots, D)$.

Finally we consider a pure time transformation in a path integral. We consider equation (4.35c), i.e.

$$G(x'', x'; E) = \sqrt{f(x')f(x'')} \int_0^\infty ds'' \langle q'' | e^{-i s'' \sqrt{f} (H-E) \sqrt{f}/\hbar} | q' \rangle, \quad (4.42)$$

where we assume that the Hamiltonian H is product ordered. Then

$$\begin{aligned} G(x'', x'; E) &= \sqrt{f(x')f(x'')} \int_0^\infty ds'' \prod_{j=1}^{N-1} \int dq^{(j)} \prod_{j=1}^N \langle q^{(j)} | e^{-i \delta \sqrt{f} (H-E) \sqrt{f}/\hbar} | q^{(j-1)} \rangle \\ &= \frac{\sqrt{f(x')f(x'')}}{(g'g'')^{\frac{1}{4}}} \int_0^\infty ds'' \lim_{N \rightarrow \infty} \prod_{j=1}^{N-1} \int dq^{(j)} \\ &\quad \times \prod_{j=1}^N \int \frac{dp^{(j)}}{(2\pi\hbar)^D} \exp \left(\frac{i}{\hbar} \left\{ p^{(j)} \Delta q^{(j)} - \delta \sqrt{f(q^{(j)})f(q^{(j-1)})} \right. \right. \\ &\quad \left. \left. \times \left[\frac{h^{ac}(q^{(j-1)})h^{bc}(q^{(j)})}{2m} p_a^{(j)} p_b^{(j)} + V(q^{(j)}) + \Delta V_{Prod}(q^{(j)}) - E \right] \right\} \right) \\ &= (f'f'')^{\frac{1}{2}(1-D/2)} \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{ND/2} \prod_{j=1}^{N-1} \int \sqrt{\frac{g(q^{(j)})}{f^D(q^{(j)})}} dq^{(j)} \\ &\quad \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} \frac{h_{ac}(q^{(j-1)})h_{bc}(q^{(j)})}{\sqrt{f(q^{(j-1)})f(q^{(j)})}} \Delta q^{a,(j)} \Delta q^{b,(j)} \right. \right. \\ &\quad \left. \left. - i \epsilon f(q^{(j)}) (V(q^{(j)}) + \Delta V_{Prod}(q^{(j)}) - E) \right] \right\} \\ &= (f'f'')^{\frac{1}{2}(1-D/2)} \int_0^\infty \tilde{K}(q'', q'; s'') ds'' \end{aligned} \quad (4.43)$$

with the path integral

$$\begin{aligned} \tilde{K}(q'', q'; s'') &= \int_{q(t')=q'}^{q(t'')=q''} \sqrt{\tilde{g}} \mathcal{D}_{PF} q(t) \\ &\quad \times \exp \left\{ \frac{i}{\hbar} \int_0^{s''} \left[\frac{m}{2} \tilde{h}_{ac} \tilde{h}_{cb} \dot{q}^a \dot{q}^b - f(V(q) + \Delta V_{Prod}(q) - E) \right] ds \right\}. \end{aligned} \quad (4.44)$$

Here denote $\tilde{h}_{ac} = h_{ac}/\sqrt{f}$ and $\sqrt{\tilde{g}} = \det(\tilde{h}_{ac})$ and equation (4.44) is of the canonical product form. Note that for $D = 2$ the prefactor is “one”.

5. Separation of Variables

Let us assume that the potential problem $V(x)$ has an exact solution according to

$$\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left[\frac{i}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \dot{x}^2 - V(x) \right) dt \right] = \int dE_\lambda e^{-iE_\lambda T/\hbar} \Psi_\lambda^*(x') \Psi_\lambda(x''). \quad (5.1)$$

Here $\int dE_\lambda$ denotes a Lebesgue-Stieltjes integral to include discrete as well as continuous states. Now we consider the path integral

$$\begin{aligned} & K(z'', z', x'', x'; T) \\ &= \int_{z(t')=z'}^{z(t'')=z''} f^d(z) \prod_{i=1}^{d'} g_i(z) \mathcal{D}z_i(t) \int_{x(t')=x'}^{x(t'')=x''} \prod_{k=1}^d \mathcal{D}x_k(t) \\ &\times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \left(\sum_{i=1}^{d'} g_i(z) \dot{z}_i^2 + f^2(z) \sum_{k=1}^d \dot{x}_k^2 \right) - \left(\frac{V(x)}{f^2(z)} + W(z) + \Delta W(z) \right) \right] dt \right\} \\ &= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{ND}{2}} \prod_{j=1}^{N-1} \int f^d(z^{(j)}) \prod_{i=1}^{d'} g_i(z^{(j)}) dz_i^{(j)} \int \prod_{k=1}^d dx_k^{(j)} \\ &\times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} \left(\sum_{i=1}^{d'} g_i(z^{(j-1)}) g_i(z^{(j)}) \Delta^2 z_i^{(j)} + f(z^{(j-1)}) f(z^{(j)}) \sum_{k=1}^d \Delta^2 x_k^{(j)} \right) \right. \right. \\ &\quad \left. \left. - \epsilon \left(\frac{V(x^{(j)})}{f^2(z^{(j)})} + W(z^{(j)}) + \Delta W(z^{(j)}) \right) \right] \right\}. \quad (5.2) \end{aligned}$$

Here $(z, x) \equiv (z_i, x_k)$ ($i = 1, \dots, d'$; $k = 1, \dots, d$, $d' + d = D$) denote a D -dimensional coordinate system, g_i and f the corresponding metric terms, and ΔW the quantum potential. For simplicity I assume that the metric tensor g_{ab} involved has only diagonal elements, i.e. $g_{ab} = \text{diag}(g_1^2(z), g_2^2(z), \dots, g_{d'}^2(z), f^2(z), \dots, f^2(z))$. Of course, $\det(g_{ab}) = f^{2d} \prod_{i=1}^{d'} g_i^2 \equiv f^{2d} G(z)$. The indices i and k will be omitted in the following. We perform the time transformation

$$s = \int_{t'}^t \frac{d\sigma}{f^2[z(\sigma)]}, \quad s'' = s(t''), \quad s(t') = 0, \quad (5.3)$$

where the lattice interpretation reads $\epsilon/[f(z^{(j-1)})f(z^{(j)})] = \delta^{(j)} \equiv \delta$. Of course, we identify $z(t) \equiv z[s(t)]$ and $x(t) \equiv x[s(t)]$. The transformation formulæ for a pure time transformation are now

$$K(z'', z', x'', x'; T) = \frac{1}{2\pi i \hbar} \int_{-\infty}^{\infty} dE e^{-iET/\hbar} G(z'', z', x'', x'; E) \quad (5.4)$$

$$G(z'', z', x'', x'; E) = i[f(z')f(z'')]^{1-D/2} \int_0^\infty ds'' \tilde{K}(z'', z', x'', x'; s''), \quad (5.5)$$

where the transformed path integral $\tilde{K}(s'')$ is given by

$$\begin{aligned} & \tilde{K}(z'', z', x'', x'; s'') \\ &= \int_{z(0)=z'}^{z(s'')=z''} \frac{\sqrt{G(z)}}{f^{2-d}(z)} \mathcal{D}z(s) \int_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s) \\ & \times \exp \left\{ \frac{i}{\hbar} \int_0^{s''} \left[\frac{m}{2} \left(\frac{g^2(z)}{f^2(z)} \dot{z}^2 + \dot{x}^2 \right) - V(x) - f^2(z) \left(W(z) + \Delta W(z) \right) + f^2(z) E \right] ds \right\} \\ &= \int dE_\lambda e^{-iE_\lambda s''/\hbar} \Psi_\lambda^*(x') \Psi_\lambda(x'') \hat{K}(z'', z'; s''), \end{aligned} \quad (5.6)$$

with the remaining path integration

$$\begin{aligned} \hat{K}(z'', z'; s'') &= \int_{z(0)=z'}^{z(s'')=z''} \frac{\sqrt{G(z)}}{f^{2-d}(z)} \mathcal{D}z(s) \\ & \times \exp \left\{ \frac{i}{\hbar} \int_0^{s''} \left[\frac{m}{2} \frac{g^2(z)}{f^2(z)} \dot{z}^2 - f^2(z) \left(W(z) + \Delta W(z) \right) + f^2(z) E \right] ds \right\}. \end{aligned} \quad (5.7)$$

Of course, in the path integrals (5.6,5.7) the same lattice formulation is assumed as in the path integral (5.2). Note the difference in comparison with a combined space-time transformation where a factor $[f(z')f(z'')]^{1/4}$ would instead appear. We also see that for $D = 2$ the prefactor is identically “one”. We perform a second time transformation in $\hat{K}(s'')$ effectively reversing the first:

$$\sigma = \int_0^s f^2[z(\omega)] d\omega, \quad \sigma'' = s'' \quad (5.8)$$

with the transformation on the lattice interpreted as $\sigma^{(j)} = \delta^{(j)} f(z^{(j-1)}) f(z^{(j)})$. Therefore we obtain the transformation formulæ

$$\hat{K}(z'', z'; s'') = \frac{1}{2\pi i \hbar} \int_{-\infty}^\infty dE' e^{-iE' s''/\hbar} \hat{G}(z'', z'; E') \quad (5.9)$$

$$\hat{G}(z'', z'; E') = i[f(z')f(z'')]^{\frac{D-d}{2}-1} \int_0^\infty d\sigma'' e^{iE'\sigma''/\hbar} \tilde{K}(z'', z'; \sigma'') \quad (5.10)$$

with the transformed path integral given by

$$\begin{aligned} \tilde{K}(z'', z'; \sigma'') &= \int_{z(0)=z'}^{z(\sigma'')=z''} \sqrt{G(z)} \mathcal{D}z(\sigma) \\ & \times \exp \left\{ \frac{i}{\hbar} \int_0^{\sigma''} \left[\frac{m}{2} g^2(z) \dot{z}^2 - W(z) - \Delta W(z) + \frac{E'}{f^2(z)} \right] d\sigma \right\}. \end{aligned} \quad (5.11)$$

Plugging all the relevant formulæ into equation (5.5) yields

$$\begin{aligned}
 K(z'', z', x'', x'; T) &= [f(z')f(z'')]^{-D/2} \int dE_\lambda \Psi_\lambda^*(x') \Psi_\lambda(x'') \\
 &\times \frac{1}{2\pi\hbar} \int_0^\infty d\sigma'' \int_{-\infty}^\infty dE e^{-iE(\sigma''-T)/\hbar} \frac{1}{2\pi\hbar} \int_{-\infty}^\infty dE' \int_0^\infty ds'' e^{-is''(E_\lambda+E')/\hbar} \\
 &\times \int_{z(0)=z'}^{z(\sigma'')=z''} \sqrt{G(z)} \mathcal{D}z(\sigma) \exp \left\{ \frac{i}{\hbar} \int_0^{\sigma''} \left[\frac{m}{2} g^2(z) \dot{z}^2 - W(z) - \Delta W(z) + \frac{E'}{f^2(z)} \right] d\sigma \right\}.
 \end{aligned} \tag{5.12}$$

The $d\sigma'' dE$ -integration produces just $\sigma'' \equiv T$, whereas the $dE' ds''$ -integration can be evaluated by giving $E_\lambda + E$ a small negative imaginary part and applying the residuum theorem yielding $E_\lambda \equiv -E'$. Therefore we arrive finally at the identity [45]

$$\begin{aligned}
 K(z'', z', x'', x'; T) &= [f(z')f(z'')]^{-D/2} \int dE_\lambda \Psi_\lambda^*(x') \Psi_\lambda(x'') \\
 &\times \int_{z(t')=z'}^{z(t'')=z''} \sqrt{G(z)} \mathcal{D}z(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} g^2(z) \dot{z}^2 - W(z) - \Delta W(z) - \frac{E_\lambda}{f^2(z)} \right] dt \right\}.
 \end{aligned} \tag{5.13}$$

Note that this result can be short handed interpreted by inserting

$$\begin{aligned}
 &\left(\frac{m}{2\pi i \delta^{(j)} \hbar} \right)^{D/2} \exp \left[\frac{i}{\hbar} \left(\frac{m}{2\delta^{(j)}} \Delta^2 x^{(j)} - \delta^{(j)} V(x^{(j)}) \right) \right] \\
 &= \int dE_{\lambda^{(j)}} e^{-iE_{\lambda^{(j)}} \delta^{(j)}/\hbar} \Psi_{\lambda^{(j)}}^*(x^{(j-1)}) \Psi_{\lambda^{(j)}}(x^{(j)})
 \end{aligned} \tag{5.14}$$

with $\delta^{(j)} = \epsilon/[f(z^{(j-1)})f(z^{(j)})]$ for all j^{th} and applying the orthonormality of the Ψ_λ in each j^{th} -path integration. Equation (5.14) describes therefore a ‘‘short-cut’’ to establish equation (5.12) instead of performing a time transformation back and forth.

III Important Examples

In this Chapter I present some of the most important path integral solutions which are

- 1) The free particle as a simple example.
- 2) The harmonic oscillator in its basic form, where we allow in addition a time dependent frequency.
- 3) Path integration in polar coordinates. We shall discuss the various features of properly defined path integrals including the ‘‘Besselian functional measure’’. In particular the (time-dependent) radial harmonic oscillator will be exactly evaluated.
- 4) The Coulomb potential.

1. The Free Particle

For warming up we calculate the path integral for the free particle in an D -dimensional Euclidean space. From the representation

$$\begin{aligned}
 K(x'', x'; T) &= \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left(\frac{im}{2\hbar} \int_{t'}^{t''} \dot{x}^2 dt\right) \\
 &= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar}\right)^{\frac{ND}{2}} \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} dx^{(j)} \exp\left[\frac{im}{2\epsilon \hbar} \sum_{j=1}^N (x^{(j)} - x^{(j-1)})^2\right]. \quad (1.1)
 \end{aligned}$$

it is obvious that the various integrations separate into a D -dimensional product. All integrals are Gaussian. Starting with ($\mu = 1, \dots, D$):

$$\begin{aligned}
 K^{N=2}(x_\mu^{(2)}, x_\mu^{(0)}; 2\epsilon) &= \left(\frac{m}{2\pi i \epsilon \hbar}\right) \int_{-\infty}^{\infty} dx^{(1)} \exp\left[-\frac{m}{2i\epsilon \hbar} (x_\mu^{(2)} - x_\mu^{(1)})^2 - \frac{m}{2i\epsilon \hbar} (x_\mu^{(1)} - x_\mu^{(0)})^2\right] \\
 &= \sqrt{\frac{m}{4\pi i \hbar \epsilon}} \exp\left[-\frac{m}{4i\epsilon \hbar} (x_\mu^{(2)} - x_\mu^{(0)})^2\right], \quad (1.2)
 \end{aligned}$$

it is easy to show that

$$K^N(x_\mu^{(N)}, x_\mu^{(0)}; \epsilon N) = \sqrt{\frac{m}{2\pi i \hbar \epsilon N}} \exp\left[-\frac{m}{2i\hbar \epsilon N} (x_\mu^{(N)} - x_\mu^{(0)})^2\right]. \quad (1.3)$$

Thus we get in the limit $N \rightarrow \infty$ (note $\epsilon N = T$):

$$\begin{aligned} K(x'', x'; T) &= \left(\frac{m}{2\pi i \hbar T} \right)^{D/2} \exp \left[\frac{i m}{2\hbar T} (x'' - x')^2 \right] \\ &= \frac{1}{(2\pi)^D} \int_{-\infty}^{\infty} \exp \left[i p (x'' - x') - i T \frac{\hbar p^2}{2m} \right] dp. \end{aligned} \quad (1.4)$$

From this representation the normalized wave-functions and the energy spectrum can be read off ($p \in \mathbf{R}^D$):

$$\Psi(x) = \frac{e^{i p \cdot x}}{(2\pi)^{D/2}}, \quad E_p = \frac{p^2 \hbar^2}{2m} \quad (x \in \mathbf{R}^D) \quad (1.5)$$

which is the known result.

The corresponding energy-dependent Green function is given by ($D = 1$):

$$G^{(1)}(x'', x'; E) = i \int_0^{\infty} K(x'', x'; T) e^{i E T / \hbar} dT = \sqrt{-\frac{m}{2E}} \exp \left(i \frac{|x'' - x'|}{\hbar} \sqrt{2mE} \right). \quad (1.6)$$

For the D -dimensional case we obtain

$$\begin{aligned} G^{(D)}(x'', x'; E) &= 2i \left(\frac{m}{2\pi i \hbar} \right)^{D/2} \left(\frac{m}{2E} |x'' - x'|^2 \right)^{\frac{1}{2}(1-D/2)} K_{1-D/2} \left(-i \frac{|x'' - x'|^2}{\hbar} \sqrt{2mE} \right) \end{aligned} \quad (1.7)$$

$$= i \left(\frac{m}{2\hbar} \right)^{D/2} \left(\frac{m\pi^2}{2E} |x'' - x'|^2 \right)^{\frac{1}{2}(1-D/2)} H_{1-D/2}^{(1)} \left(\frac{|x'' - x'|}{\hbar} \sqrt{2mE} \right), \quad (1.8)$$

where use has been made of the integral representation [40, p.340]

$$\int_0^{\infty} dt t^{\nu-1} \exp \left(-\frac{a}{4t} - pt \right) = 2 \left(\frac{a}{4p} \right)^{\frac{\nu}{2}} K_{\nu}(\sqrt{ap}), \quad (1.9)$$

and $K_{\nu}(z) = \frac{i\pi}{2} e^{i\pi\nu/2} H_{\nu}^{(1)}(iz)$, $K_{\pm\frac{1}{2}}(z) = \sqrt{\pi/2z} e^{-z}$.

2. The Harmonic Oscillator

After this easy task we proceed to the path integral calculation of the harmonic oscillator. We consider the simple one-dimensional case with the Lagrangian

$$\mathcal{L}[x, \dot{x}] = \frac{m}{2} \dot{x}^2 - \frac{c(t)}{2} x^2 + b(t) x \dot{x} - e(t) x \quad (2.1)$$

Here we assume that the various coefficients may be time-dependent. The path integral has the form

$$K(x'', x'; t'', t') = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) e^{\frac{i}{\hbar} S[x]} = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) e^{\frac{i}{\hbar} \int_{t'}^{t''} \mathcal{L}[x, \dot{x}] dt} \quad (2.2)$$

In this quadratic Lagrangian, of course, an ordering problem appears. This is due to the stochastic nature of the path integral, the “time”-integral in the exponent is not a Riemann-integral but a so called “Itô-integral”. This we have already discussed in chapter II. The ordering ambiguity appears in the “b(t)”-term, where we have in the corresponding Hamiltonian a px -term. The consequence is that we have to use the mid-point formulation, i.e. $(\bar{x}^{(j)} = \frac{1}{2}(x^{(j)} + x^{(j-1)}))$, and $c^{(j)} = c(t^{(j)})$, etc., see also Schulman’s book [86]:

$$\begin{aligned} & K(x'', x'; t'', t') \\ &= \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 - \frac{c(t)}{2} x^2 + b(t)x\dot{x} - e(t)x \right] dt \right\} \\ &= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{N}{2}} \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} dx^{(j)} \\ & \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} \left(x^{(j)} - x^{(j-1)} \right)^2 - \epsilon \frac{c^{(j)}}{2} (\bar{x}^{(j)})^2 + b^{(j)} \bar{x}^{(j)} \left(x^{(j)} - x^{(j-1)} \right) - e^{(j)} \bar{x}^{(j)} \right] \right\}. \end{aligned} \quad (2.3)$$

In the following we use the notations in equations (2.2) and (2.3) as synonymous. Let us expand the “path” $x(t)$ about the classical path $x_{Cl}(t)$, i.e.:

$$x(t) = x_{Cl}(t) + y(t),$$

where $y(t)$ denotes a fluctuating path about the classical one. The classical path obeys, of course, the Euler Lagrange equations:

$$\frac{\delta \mathcal{L}[x_{Cl}, \dot{x}_{Cl}]}{\delta x_{Cl}} = \frac{d}{dt} \frac{\partial \mathcal{L}[x_{Cl}, \dot{x}_{Cl}]}{\partial \dot{x}_{Cl}} - \frac{\partial \mathcal{L}[x_{Cl}, \dot{x}_{Cl}]}{\partial x_{Cl}} = 0, \quad x_{Cl}(t') = x', \quad x_{Cl}(t'') = x''_{Cl}. \quad (2.4)$$

Expanding we obtain for the action

$$S[x] = S[x_{Cl}] + \frac{1}{2} \frac{\delta^2 S[x]}{\delta x^2} \Big|_{x=x_{Cl}} y^2 = S[x_{Cl}(t''), x_{Cl}(t')] + \int_{t'}^{t''} \left[\frac{m}{2} \dot{y}^2 - \frac{c}{2} y^2 + by\dot{y} \right] dt. \quad (2.5)$$

The linear functional variation vanishes due to the Euler-Lagrange equations (2.4). For the path integral we now find

$$K(x''x'; t'', t') = \exp \left\{ \frac{i}{\hbar} S[x_{Cl}(t''), x_{Cl}(t')] \right\} F(t'', t'),$$

$$F(t'', t') = \int_{y(t')=0}^{y(t'')=0} \mathcal{D}y(t) \exp \left[\frac{im}{2\hbar} \int_{t'}^{t''} (\dot{y}^2 - \omega^2(t)y^2) dt \right]. \quad (2.6)$$

Here we have used the abbreviations $m\omega^2(t) = c(t) + \dot{b}(t)$, $y' = y(t')$ and $y'' = y(t'')$. Now

$$F(t'', t') = \lim_{N \rightarrow \infty} F_N$$

$$= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{N}{2}} \int_{-\infty}^{\infty} dy^{(1)} \dots \int_{-\infty}^{\infty} dy^{(N-1)}$$

$$\times \exp \left\{ -\frac{m}{2i\epsilon\hbar} \sum_{j=1}^N \left[(y^{(j)} - y^{(j-1)})^2 - \epsilon^2 \omega^{(j)2} y^{(j)2} \right] \right\} \quad (2.7)$$

Let us introduce a $N - 1$ -dimensional vector $z = (x_1, \dots, x_{N-1})^T$ and the $(N - 1) \times (N - 1)$ matrix B :

$$B = \begin{pmatrix} 2 - \epsilon^2 \omega^{(1)2} & -1 & \dots & 0 & 0 \\ -1 & 2 - \epsilon^2 \omega^{(2)2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 - \epsilon^2 \omega^{(N-2)2} & -1 \\ 0 & 0 & \dots & -1 & 2 - \epsilon^2 \omega^{(N-1)2} \end{pmatrix}. \quad (2.8)$$

Thus we get

$$F_N = \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{N}{2}} \int d^{N-1}z \exp \left(-\frac{m}{2i\epsilon\hbar} z^T B z \right) = \left(\frac{m}{2\pi i \epsilon \hbar \det B} \right)^{\frac{1}{2}}. \quad (2.9)$$

Therefore

$$F(t'', t') = \left(\frac{m}{2\pi i \hbar f(t'', t')} \right)^{\frac{1}{2}}, \quad \text{where} \quad f(t'', t') = \lim_{N \rightarrow \infty} \epsilon \det B. \quad (2.10)$$

Our final task is to determine $\det B$. Let us consider the $j \times j$ matrix

$$B^{(j)} = \begin{pmatrix} 2 - \epsilon^2 \omega^{(1)2} & -1 & \dots & 0 & 0 \\ -1 & 2 - \epsilon^2 \omega^{(2)2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 - \epsilon^2 \omega^{(j-1)2} & -1 \\ 0 & 0 & \dots & -1 & 2 - \epsilon^2 \omega^{(j)2} \end{pmatrix} \quad (2.11)$$

One can show that the following recursion relations holds:

$$\det B^{(j+1)} = (2 - \epsilon^2 \omega^{(j+1)^2}) \det B^{(j)} - \det B^{(j-1)}$$

with $\det B^{(1)} = 2 - \epsilon^2 \omega^{(1)^2}$ and $\det B^{(0)} = 1$. Let us define $g^{(j)} = \epsilon \det B^{(j)}$, then we have

$$g^{(j+1)} - 2g^{(j)} + g^{(j-1)} = -\epsilon^2 \omega^{(j+1)^2} g^{(j)}.$$

Turning to a continuous notation we find for the function $g(t) = f(t, t')$ a differential equation:

$$\ddot{g}(t) + \omega^2(t)g(t) = 0, \quad \text{with } g(t') = 0, \quad \dot{g}(t') = 1. \quad (2.12)$$

The two last equation follow from $g(t') = g_0 = \lim_{\epsilon \rightarrow 0} \epsilon \det B^{(0)}$ and $\dot{g}(t') = \lim_{\epsilon \rightarrow 0} [g(t' + \epsilon) - g(t')]/\epsilon = \lim_{\epsilon \rightarrow 0} (\det B^{(1)} - \det B^{(0)}) = 1$. Finally we have to insert $g(t'') = f(t'', t')$ into equation (2.10).

At once we recover the free particle with $g = t'' - t' = T$.

The case of the usual harmonic oscillator with $\omega(t) = \omega$ (time-independent) is given by

$$g(t) = \frac{1}{\omega} \sin \omega T.$$

To get the path integral solution for the harmonic oscillator we must calculate its classical action. It is a straightforward calculation to show that it is given by:

$$S[x_{Cl}(t''), x_{Cl}(t')] = \frac{m\omega}{2 \sin \omega T} [(x'_{Cl})^2 + (x''_{Cl})^2] \cos \omega T - 2x'_{Cl}x''_{Cl}. \quad (2.13)$$

and we have for the Feynman kernel:

$$K(x'', x'; T) = \left(\frac{m\omega}{2\pi i \hbar \sin \omega T} \right)^{\frac{1}{2}} \exp \left\{ -\frac{m\omega}{2i\hbar} \left[(x'^2 + x''^2) \cot \omega T - 2 \frac{x'x''}{\sin \omega T} \right] \right\}. \quad (2.14)$$

By the use of the Mehler-formula [31, Vol.III, p.272]:

$$e^{-(x^2+y^2)/2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z}{2} \right)^n H_n(x) H_n(y) = \frac{1}{\sqrt{1-z^2}} \exp \left[\frac{4xyz - (x^2 + y^2)(1+z^2)}{2(1-z^2)} \right], \quad (2.15)$$

where H_n denote the Hermite-polynomials, we can expand the Feynman kernel according to (identify $x \equiv \sqrt{m\omega/\hbar} x'$, $y \equiv \sqrt{m\omega/\hbar} x''$ and $z = e^{-i\omega T}$):

$$K(x'', x'; T) = \sum_{n=0}^{\infty} e^{-iTE_n/\hbar} \Psi_n^*(x') \Psi_n(x'') \quad (2.16)$$

with energy-spectrum and wave-functions:

$$E_n = \hbar\omega(n + \frac{1}{2}), \quad (2.17)$$

$$\Psi_n(x) = \left(\frac{m\omega}{2^{2n}\pi\hbar n!} \right)^{\frac{1}{4}} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) \exp \left(-\frac{m\omega}{2\hbar} x^2 \right). \quad (2.18)$$

Equation (2.14) is as it stands only valid for $0 < \omega T < \pi$. Let us investigate it for larger times. Let

$$T = \frac{n\pi}{\omega} + \tau, \quad \text{with} \quad n \in \mathbf{N}_0; \quad 0 < \tau < \pi/\omega \quad (2.19)$$

then we have $\sin \omega T = e^{i\pi n} \sin \omega \tau$, $\cos \omega T = e^{i\pi n} \cos \omega \tau$ and equation (2.14) becomes

$$K(x'', x'; T) = \left(\frac{m\omega}{2\pi\hbar \sin \omega \tau} \right)^{\frac{1}{2}} \times \exp \left\{ -\frac{i\pi}{2} \left(\frac{1}{2} + n \right) + \frac{i m \omega}{2\hbar \sin \omega T} [(x'^2 + x''^2) \cos \omega T - 2x'x''] \right\} \quad (2.20)$$

which is the formula for the propagator with the Maslov correction (note $\sin \omega \tau = |\sin \omega T|$). Now let $\tau \rightarrow 0$, i.e. we consider the propagator at caustics. We obtain

$$\begin{aligned} K\left(x'', x'; \frac{n\pi}{\omega}\right) &= \lim_{\tau \rightarrow 0} \left(\frac{m}{2\pi i \hbar \tau} \right)^{\frac{1}{2}} \exp \left\{ -\frac{i\pi n}{2} + \frac{i m}{2\hbar \tau} [(x'^2 + x''^2) - 2e^{-i\pi n} x'x''] \right\} \\ &= \lim_{a \rightarrow 0} \frac{1}{\sqrt{\pi} a} \exp \left[-\frac{1}{a^2} (x' - (-1)^n x'')^2 - i \frac{\pi n}{2} \right] \\ &= \exp \left(-\frac{i}{2} \pi n \right) \delta(x' - (-1)^n x''). \end{aligned} \quad (2.21)$$

Let us finally determine the energy dependent Green function. We use the integral representation [40, p.729], $a_1 > a_2$, $\Re(\frac{1}{2} + \mu - \nu) > 0$):

$$\begin{aligned} \int_0^\infty \coth^{2\nu} \frac{x}{2} \exp \left[-\frac{a_1 + a_2}{2} t \cosh x \right] I_{2\mu}(t\sqrt{a_1 a_2} \sinh x) dx \\ = \frac{\Gamma(\frac{1}{2} + \mu - \nu)}{t\sqrt{a_1 a_2} \Gamma(1 + 2\mu)} W_{\nu, \mu}(a_1 t) M_{\nu, \mu}(a_2 t). \end{aligned} \quad (2.22)$$

Here $W_{\nu, \mu}(z)$ and $M_{\nu, \mu}(z)$ denote Whittaker-functions. We reexpress $e^x = \sqrt{\pi x/2} [I_{\frac{1}{2}}(x) + I_{-\frac{1}{2}}(x)]$ apply equation (2.22) for $\nu = E/2\hbar\omega$ and use some relations between Whittaker- and parabolic cylinder-functions to obtain

$$\begin{aligned} G(x'', x'; E) &= -\frac{1}{2} \sqrt{\frac{m}{\pi\hbar\omega}} \Gamma\left(\frac{1}{2} - \frac{E}{\hbar\omega}\right) \\ &\times D_{-\frac{1}{2} + \frac{E}{\hbar\omega}} \left[\sqrt{\frac{2m\omega}{\hbar}} (x' + x'' + |x'' - x'|) \right] D_{-\frac{1}{2} + \frac{E}{\hbar\omega}} \left[-\sqrt{\frac{2m\omega}{\hbar}} (x' + x'' - |x'' - x'|) \right]. \end{aligned} \quad (2.23)$$

From this representation we can recover by means of the expansion for the Γ -function

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^\infty e^{-t} t^{z-1} dt$$

and some relations of the parabolic cylinder-functions and the Hermite polynomials, the wave-functions of equation (2.18).

Let us note that we can use equations (2.14, 2.23) to obtain recursion relations for the Feynman kernel and the Green function, respectively, of the harmonic oscillator in D dimensions. Let $\vec{x} \in \mathbf{R}^n$ and define $\mu = \vec{x}'^2 + \vec{x}''^2$, $\nu = \vec{x}' \cdot \vec{x}''$, then

$$\begin{aligned} K^{(D)}(\vec{x}'', \vec{x}'; T) &= \frac{1}{2\pi} \frac{\partial}{\partial \nu} K^{(D-2)}(\vec{x}'', \vec{x}'; T) \\ &= \frac{1}{2\pi} e^{-i\omega T} \left(\frac{m\omega}{\hbar} - 2 \frac{\partial}{\partial \mu} \right) K^{(D-2)}(\vec{x}'', \vec{x}'; T). \end{aligned} \quad (2.24)$$

For the Green functions this gives

$$G^{(D)}(\vec{x}'', \vec{x}'; E) \equiv \tilde{G}^{(D)}(\mu|\nu; E) \quad (2.25)$$

$$\tilde{G}^{(D)}(\mu|\nu; E) = \frac{1}{2\pi} \frac{\partial}{\partial \nu} \tilde{G}^{(D-2)}(\mu|\nu; E) \quad (2.26)$$

$$\tilde{G}^{(D)}(\mu|\nu; E) = \frac{1}{2\pi} \left(\frac{m\omega}{\hbar} - 2 \frac{\partial}{\partial \mu} \right) \tilde{G}^{(D-2)}(\mu|\nu; E - \omega\hbar) \quad (2.27)$$

Introducing furthermore $\xi = \frac{1}{2}(|\vec{x}' + \vec{x}''| + |\vec{x}'' - \vec{x}'|)$, $\eta = \frac{1}{2}(|\vec{x}' + \vec{x}''| - |\vec{x}'' - \vec{x}'|)$ we obtain for $D = 1, 3, 5, \dots$

$$\begin{aligned} G^{(D)}(\vec{x}'', \vec{x}'; E) &= -\frac{1}{2} \sqrt{\frac{m}{\pi\hbar\omega}} \Gamma\left(\frac{1}{2} - \frac{E}{\hbar\omega}\right) \left(\frac{1}{2\pi}\right)^{\frac{D-1}{2}} \\ &\times \left[\frac{1}{\eta^2 - \xi^2} \left(\eta \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \eta} \right) \right]^{\frac{D-1}{2}} D_{-\frac{1}{2} + \frac{E}{\hbar\omega}} \left(\sqrt{\frac{2m\omega}{\hbar}} \xi \right) D_{-\frac{1}{2} + \frac{E}{\hbar\omega}} \left(-\sqrt{\frac{2m\omega}{\hbar}} \eta \right), \end{aligned} \quad (2.28)$$

respectively,

$$\begin{aligned} G^{(D)}(\vec{x}'', \vec{x}'; E) &= -\frac{1}{2} \sqrt{\frac{m}{\pi\hbar\omega}} \Gamma\left(\frac{D}{2} - \frac{E}{\hbar\omega}\right) \left(\frac{1}{2\pi}\right)^{\frac{D-1}{2}} \\ &\times \left[\frac{1}{\eta^2 - \xi^2} \left(\xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} \right) + \frac{m\omega}{\hbar} \right]^{\frac{D-1}{2}} D_{-\frac{D}{2} + \frac{E}{\hbar\omega}} \left(\sqrt{\frac{2m\omega}{\hbar}} \xi \right) D_{-\frac{D}{2} + \frac{E}{\hbar\omega}} \left(-\sqrt{\frac{2m\omega}{\hbar}} \eta \right). \end{aligned} \quad (2.29)$$

Let us note that the most general solution for the general quadratic Lagrangian is due to Grosjean and Goovaerts [52, 53].

3. The Radial Path Integral

3.1. The General Radial Path Integral

Radial path integrals have been first discussed by Edwards and Gulyaev [30] and Arthurs [4]. Edwards and Gulyaev discussed the two- and three-dimensional cases, Arthurs concentrated on $D = 2$. Further progress have been made by Peak and Inomata [84] who calculated the path integral for the radial harmonic oscillator including some simple applications. See also [49] for the ordering problematics which give rise to the various quantum potentials in the formulation of the radial path integral. In this section we derive the path integral for D -dimensional polar coordinates. We follow in our line of reasoning references [49] and [92], where these features have been first discussed in their full detail. Similar topics can be also found in Böhm and Junker [5] from a group theoretic approach. We will get an expansion in the angular momentum l , where the angle dependent part can be integrated out and a radial dependent part is left over: the radial path integral. We discuss some properties of the radial path integral and show that it possible to get from the short time kernel the radial Schrödinger equation.

The path integral in D -dimensions has the form:

$$\begin{aligned}
 K^{(D)}(x'', x'; T) &= \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 - V(x) \right] dt \right\} \\
 &= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{ND/2} \int dx^{(1)} \dots \int dx^{(N-1)} \\
 &\quad \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon \hbar} (x^{(j)} - x^{(j-1)})^2 - V(x^{(j)}) \right] \right\}. \tag{3.1}
 \end{aligned}$$

Now let $V(x) = V(|x|)$ and introduce D -dimensional polar coordinates [31, Vol.II, Chapter IX]:

$$\left. \begin{aligned}
 x_1 &= r \cos \theta_1 \\
 x_2 &= r \sin \theta_1 \cos \theta_2 \\
 x_3 &= r \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
 &\dots \\
 x_{D-1} &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{D-2} \cos \phi \\
 x_D &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{D-2} \sin \phi
 \end{aligned} \right\} \tag{3.2}$$

where $0 \leq \theta_\nu \leq \pi$ ($\nu = 1, \dots, D-2$), $0 \leq \phi \leq 2\pi$, $r = \left(\sum_{\nu=1}^D x_\nu^2 \right)^{1/2}$. Therefore $V(x) = V(r)$. We have to use the addition theorem:

$$\begin{aligned} \cos \psi^{(1,2)} &= \cos \theta_1^{(1)} \cos \theta_1^{(2)} \\ &+ \sum_{m=1}^{D-2} \cos \theta_{m+1}^{(1)} \cos \theta_{m+1}^{(2)} \prod_{n=1}^m \sin \theta_n^{(1)} \sin \theta_n^{(2)} + \prod_{n=1}^{D-1} \sin \theta_n^{(1)} \sin \theta_n^{(2)}, \end{aligned} \quad (3.3)$$

where $\psi^{(1,2)}$ is the angle between two D -dimensional vectors $x^{(1)}$ and $x^{(2)}$ so that $x^{(1)} \cdot x^{(2)} = r^{(1)} r^{(2)} \cos \psi^{(1,2)}$. The metric tensor in polar coordinates is

$$(g_{ab}) = \text{diag}(1, r^2, r^2 \sin^2 \theta_1, \dots, r^2 \sin^2 \theta_1 \dots \sin^2 \theta_{D-2}). \quad (3.4)$$

If $D = 3$ equation (3.3) reduces to:

$$\cos \psi^{(1,2)} = \cos \theta^{(1)} \cos \theta^{(2)} + \sin \theta^{(1)} \sin \theta^{(2)} \cos(\phi^{(1)} - \phi^{(2)}).$$

The D -dimensional measure dx expressed in polar coordinates is

$$\left. \begin{aligned} dx &= r^{D-1} dr d\Omega = r^{D-1} \prod_{k=1}^{D-1} (\sin \theta_k)^{D-1-k} dr d\theta_k \\ d\Omega &= \prod_{k=1}^{D-1} (\sin \theta_k)^{D-1-k} d\theta_k. \end{aligned} \right\} \quad (3.5)$$

$d\Omega$ denotes the $(D-1)$ -dimensional surface element on the unit sphere S^{D-1} and $\Omega(D) = 2\pi^{D/2}/\Gamma(D/2)$ is the volume of the D -dimensional unit (S^{D-1} -) sphere. The determinant of the metric tensor is given by

$$g = \det(g_{ab}) = \left(r^{D-1} \prod_{k=1}^{D-1} (\sin \theta_k)^{D-1-k} \right)^2 \quad (3.6)$$

With all this the path integral equation (3.1) yields:

$$\begin{aligned} K^{(D)}(r'', \{\theta''\}, r', \{\theta'\}; T) &= \\ &= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{ND/2} \int_0^\infty r_{(1)}^{D-1} dr_{(1)} \int d\Omega_{(1)} \cdots \int_0^\infty r_{(N-1)}^{D-1} dr_{(N-1)} \int d\Omega_{(N-1)} \\ &\quad \times \prod_{j=1}^N \exp \left\{ \frac{i m}{2\epsilon \hbar} [r_{(j)}^2 + r_{(j-1)}^2 - 2r_{(j)} r_{(j-1)} \cos \psi^{(j,j-1)}] - \frac{i \epsilon}{\hbar} V(r_{(j)}) \right\}. \end{aligned} \quad (3.7)$$

$\{\theta\}$ denotes the set of the angular variables. For further calculations we need the formula [40, p.980]:

$$e^{z \cos \psi} = \left(\frac{z}{2} \right)^{-\nu} \Gamma(\nu) \sum_{l=0}^{\infty} (l + \nu) I_{l+\nu}(z) C_l^\nu(\cos \psi), \quad (3.8)$$

(for some $\nu \neq 0, -1, -2, \dots$), where C_l^ν are Gegenbauer polynomials and I_μ modified Bessel functions. Equation (3.8) is a generalization of the well known expansion in three dimension where $\nu = \frac{1}{2}$ (remember $C_l^{\frac{1}{2}} = P_l$, [40, p.980]):

$$e^{z \cos \psi} = \sqrt{\frac{\pi}{2z}} \sum_{l=0}^{\infty} (2l+1) I_{l+\frac{1}{2}}(z) P_l(\cos \psi) \quad (3.9)$$

Note: It is in some sense possible to include the case $D = 2$, i.e. $\nu = 0$ if one uses $l \lim_{\lambda \rightarrow 0} \Gamma(\lambda) C_l^\lambda = 2 \cos l\psi$ [40, p.1030], yielding finally [40, p.970]:

$$e^{z \cos \psi} = \sum_{k=-\infty}^{\infty} I_k(z) e^{i k \psi}. \quad (3.10)$$

The addition theorem for the real surface (or hyperspherical) harmonics S_l^μ on the S^{D-1} -sphere has the form [31, Vol.II, Chapter IX]:

$$\sum_{\mu=1}^M S_l^\mu(\Omega^{(1)}) S_l^\mu(\Omega^{(2)}) = \frac{1}{\Omega(D)} \frac{2l+D-2}{D-2} C_l^{\frac{D-2}{2}}(\cos \psi^{(1,2)}) \quad (3.11)$$

with $\Omega = x/r$ unit vector in \mathbf{R}^d and the M linearly independent S_l^μ of degree l with $M = (2l+D-2)(l+D-3)!/l!(D-3)!$. The orthonormality relation is

$$\int d\Omega S_l^\mu(\Omega) S_{l'}^{\mu'}(\Omega) = \delta_{ll'} \delta_{\mu\mu'}. \quad (3.12)$$

Combining equations (3.8) and (3.11) we get the expansion formula

$$e^{z(\Omega^{(1)} \cdot \Omega^{(2)})} = e^{z \cos \psi^{(1,2)}} = 2\pi \left(\frac{2\pi}{z} \right)^{\frac{D-2}{2}} \sum_{l=0}^{\infty} \sum_{\mu=1}^M S_l^\mu(\Omega^{(1)}) S_l^\mu(\Omega^{(2)}) I_{l+\frac{D-2}{2}}(z). \quad (3.13)$$

Let $\nu = (D-2)/2 > 0$ in equation (3.8), then the j th term in equation (3.7) becomes

$$\begin{aligned} R_j &= \exp \left\{ \frac{i m}{2\epsilon \hbar} (r_{(j)}^2 + r_{(j-1)}^2) - \frac{i \epsilon}{\hbar} V(r_{(j)}) \right\} \exp \left[\frac{m}{2i \epsilon \hbar} r_{(j)} r_{(j-1)} \cos \psi^{(j,j-1)} \right] \\ &= \left(\frac{2i \epsilon \hbar}{m r_{(j)} r_{(j-1)}} \right)^{\frac{D-2}{2}} \Gamma \left(\frac{D-2}{2} \right) \exp \left[\frac{i m}{2\epsilon \hbar} (r_{(j)}^2 + r_{(j-1)}^2) - \frac{i \epsilon}{\hbar} V(r_{(j)}) \right] \\ &\quad \times \sum_{l_j=0}^{\infty} \left(l_j + \frac{D}{2} - 1 \right) I_{l_j + \frac{D-2}{2}} \left(\frac{m}{i \epsilon \hbar} r_{(j)} r_{(j-1)} \right) C_{l_j}^{\frac{D-2}{2}}(\cos \psi^{(j,j-1)}) \\ &= 2\pi \left(\frac{2\pi i \epsilon \hbar}{m r_{(j)} r_{(j-1)}} \right)^{\frac{D-2}{2}} \exp \left[\frac{i m}{2\epsilon \hbar} (r_{(j)}^2 + r_{(j-1)}^2) - \frac{i \epsilon}{\hbar} V(r_{(j)}) \right] \\ &\quad \times \sum_{l_j=0}^{\infty} I_{l_j + \frac{D-2}{2}} \left(\frac{m}{i \epsilon \hbar} r_{(j)} r_{(j-1)} \right) \sum_{\mu_j=1}^M S_{l_j}^{\mu_j}(\Omega_{(j)}) S_{l_j}^{\mu_j}(\Omega_{(j-1)}). \end{aligned} \quad (3.14)$$

With the help of equations (3.11) and (3.14) now (3.7) becomes:

$$\begin{aligned}
& K^{(D)}(x'', x'; T) \\
&= (r' r'')^{-\frac{D-2}{2}} \sum_{l=0}^{\infty} \sum_{\mu=1}^M S_l^\mu(\Omega') S_l^\mu(\Omega'') \\
&\quad \times \lim_{N \rightarrow \infty} \left(\frac{m}{i \epsilon \hbar} \right)^N \int_0^\infty r_{(1)} dr_{(1)} \cdots \int_0^\infty r_{(N-1)} dr_{(N-1)} \\
&\quad \times \prod_{j=1}^N \exp \left[\frac{i m}{2 \epsilon \hbar} (r_{(j)}^2 + r_{(j-1)}^2) - \frac{i \epsilon}{\hbar} V(r_{(j)}) \right] I_{l+\frac{D-2}{2}} \left(\frac{m}{i \epsilon \hbar} r_{(j)} r_{(j-1)} \right). \quad (3.15)
\end{aligned}$$

Therefore we can separate the radial part of the path integral:

$$K^{(D)}(r'', \{\theta''\}, r', \{\theta'\}; T) = \Omega^{-1}(D) \sum_{l=0}^{\infty} \frac{l+D-2}{D-2} C_l^{\frac{D-2}{2}}(\cos \psi^{(')}) K_l(r'', r'; T) \quad (3.16)$$

with

$$\begin{aligned}
& K_l^{(D)}(r'', r'; T) = (r' r'')^{-\frac{D-2}{2}} \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{N/2} \int_0^\infty dr_{(1)} \cdots \int_0^\infty dr_{(N-1)} \\
&\quad \times \prod_{j=1}^N \mu_l[r_{(j)} r_{(j-1)}] \cdot \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} (r_{(j)} - r_{(j-1)})^2 - \epsilon V(r_{(j)}) \right] \right\} \quad (3.17)
\end{aligned}$$

and the functional measure

$$\mu_l^{(D)}(z_{(j)}) = \sqrt{2\pi z_{(j)}} e^{-z_{(j)}} I_{l+\frac{D-2}{2}}(z_{(j)}), \quad (3.18)$$

where $z_{(j)} = (m/i\epsilon\hbar)r_{(j)}r_{(j-1)}$.

In the literature often use is been made of the asymptotic form of the modified Bessel functions

$$I_\nu(z) \simeq (2\pi z)^{-\frac{1}{2}} e^{z - (\nu^2 - \frac{1}{4})/2z} \quad (|z| \gg 1, \quad \Re(z) > 0). \quad (3.19)$$

Then the functional measure becomes (we ignore $\Re(z) = 0$):

$$\mu_l^{(D)}(z_{(j)}) \simeq \exp \left\{ -\frac{i \epsilon \hbar}{2m r_{(j)} r_{(j-1)}} \left[\left(l + \frac{D-2}{2} \right)^2 - \frac{1}{4} \right] \right\} \quad (3.20)$$

and $K_l^{(D)}$ reads:

$$\begin{aligned}
& K_l^{(D)}(r'', r'; T) = (r' r'')^{-\frac{D-1}{2}} \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{N/2} \int_0^\infty dr_{(1)} \cdots \int_0^\infty dr_{(N-1)} \\
&\quad \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} (r_{(j)} - r_{(j-1)})^2 - \hbar^2 \frac{(l + \frac{D-2}{2})^2 - \frac{1}{4}}{2m r_{(j)} r_{(j-1)}} - \epsilon V(r_{(j)}) \right] \right\}. \quad (3.21)
\end{aligned}$$

This last equation seems to suggest a Lagrangian formulation of the radial path integral:

$$k_l^{(D)}(r'', r'; T) \stackrel{?}{=} \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{r}^2 - \hbar^2 \frac{(l + \frac{D-2}{2})^2 - \frac{1}{4}}{2mr^2} - V(r) \right] dt \right\}. \quad (3.22)$$

with $k_l^{(D)}(r'', r'; T) = (r'r'')^{(D-1)/2} K_l^{(D)}(r'', r'; T)$. But nevertheless, equation (3.17) can be written in a similar manner with a nontrivial functional measure:

$$k_l^{(D)}(r'', r'; T) = \int_{r(t')=r'}^{r(t'')=r''} \mu_l^{(D)}[r^2] \mathcal{D}r(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{r}^2 - V(r) \right] dt \right\} \quad (3.23)$$

and the whole l -dependence is in μ_l , as defined in equation (3.18) in the lattice formulation. The asymptotic expansion of I_ν in equation (3.19) is problematic because it is only valid for $\Re(z) > 0$, but we have $\Re(z) = 0$. The complete expansions in this case reads [40, p.962]

$$\begin{aligned} I_\nu(z) &\simeq \frac{e^z}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2z)^k} \frac{\Gamma(\nu + k + \frac{1}{2})}{k! \Gamma(\nu - k + \frac{1}{2})} \\ &\quad + \frac{\exp[-z \pm (\nu + \frac{1}{2})i\pi]}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{1}{(2z)^k} \frac{\Gamma(\nu + k + \frac{1}{2})}{k! \Gamma(\nu - k + \frac{1}{2})} \\ &\simeq \frac{1}{\sqrt{2\pi z}} \left\{ \exp \left[z - \frac{\nu^2 - \frac{1}{4}}{2z} \right] + \exp \left[-z + \frac{\nu^2 - \frac{1}{4}}{2z} \pm i\pi \left(\nu + \frac{1}{2} \right) \right] \right\}. \end{aligned} \quad (3.24)$$

(“+” for $-\pi/2 < \arg(z) < 3\pi/2$, “-” for $-3\pi/2 < \arg(z) < \pi/2$). Langguth and Inomata [67] argued that the inserting of the expansion (3.20) in the path integral (3.17) can be justified (adopting an argument due to Nelson [80]), but things are not such as easy. Our arguments are as follows:

1) The Schrödinger equation

$$\left[-\frac{\hbar^2}{2m} \left(\frac{d^2}{dr^2} + \frac{D-1}{2r} \frac{d}{dr} \right) + \hbar^2 \frac{l(l+D-2)}{2mr^2} + V(r) \right] \Psi(r; t) = i\hbar \frac{\partial}{\partial t} \Psi(r; t) \quad (3.25)$$

can be derived from the short time kernel of the path integral (3.17).

2) For $r', r'' \rightarrow 0$ equation (3.24) has the right boundary conditions at the origin. We have for $z \rightarrow 0$:

$$\sqrt{2\pi z} e^{-z} I_\lambda(z) \simeq \frac{2}{(2\lambda)!!} z^{\lambda + \frac{1}{2}} [1 + O(z^2)]$$

and it follows:

$$\mu_l^{(D)}(z_{(j)}) \simeq \begin{cases} r'^{l+(D-1)/2} & (r' \rightarrow 0, r'' > 0) \\ r''^{l+(D-1)/2} & (r'' \rightarrow 0, r' > 0) \end{cases} \quad (3.26)$$

for all j . If $R_{n,l}^{(D)}$ and $e_{n,l}^{(D)}$ denote the (normalized) radial state functions and energy levels, respectively, of the D -dimensional problem, we have

$$K_l^{(D)}(r'', r'; T) = \sum_{n=0}^{\infty} R_{n,l}^{(D)}(r') R_{n,l}^{(D)}(r'') e^{-i T e_{n,l}^{(D)}/\hbar} \quad (3.27)$$

and thus for any but s -states - including the factor $r^{-(D-1)/2}$ in equation (3.27) - we have:

$$R_{n,l}^{(D)}(r) \simeq r^{l+(D-1)/2} r^{-(D-1)/2} = r^l \quad (3.28)$$

which is the correct boundary condition for $R_{n,l}^{(D)}$.

Both points are not true for the path integral (3.22), because

- 1) we did not find any obvious method for deriving the full Schrödinger equation (3.25), the angular dependence comes out wrong.
- 2) In the “Euclidean region” $T \rightarrow -i\tau$ ($\tau > 0$) the functional measure is vanishing like

$$\exp \left\{ -\hbar \frac{\tau}{N} \left[\frac{(l + \frac{D-2}{2})^2 - \frac{1}{4}}{2mr_1 r'} + \frac{(l + \frac{D-2}{2})^2 - \frac{1}{4}}{2mr_{(N-1)} r''} \right] \right\} \\ \times \exp \left\{ -\hbar \frac{\tau}{N} \sum_{j=1}^{N-1} \frac{(l + \frac{D-2}{2})^2 - \frac{1}{4}}{2mr_{(j)} r_{(j-1)}} \right\} \quad (3.29)$$

which is also vanishing for $r', r'' \rightarrow 0$, but not in the right manner. The points $r' = r'' = 0$ are essential singularities, whereas the correct vanishing is powerlike.

- 3) Equation (3.22) seems very suggestive, but it is quite useless in explicit calculations. Even in the simplest case $D = 3, l = 0, V = 0$, i.e.

$$K_l^{(D)}(r'', r'; T) = (r' r'')^{-\frac{D-1}{2}} \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{N/2} \\ \times \int_0^\infty dr_{(1)} \cdots \int_0^\infty dr_{(N-1)} \exp \left\{ \frac{i m}{2\epsilon \hbar} \sum_{j=1}^N (r_{(j)} - r_{(j-1)})^2 \right\} \quad (3.30)$$

cannot be calculated. The integrals turn out to be repeated integrals over errorfunctions which are not tractable. Also the method of Arthurs [4] fails. In this method one assumes that the integrations in the limit $\epsilon \rightarrow 0$ are effectively from $-\infty$ to $+\infty$. The path integral (3.30) then becomes

$$K_l^{(D)}(r'', r'; T) = (r' r'')^{-\frac{D-1}{2}} \exp \left[\frac{i m}{2\epsilon \hbar} (r' - r'')^2 \right]$$

which is wrong. The “mirror” term $K_l^{(D)}(r'', -r'; T)$ is missing.

Let us discuss some properties of the path integral (3.22)

$$k_l^{(D)}(r'', r'; T) = \int_{r(t')=r'}^{r(t'')=r''} \mu_l^{(D)}[r^2] \mathcal{D}r(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{r}^2 - V(r) \right] dt \right\}. \quad (3.31)$$

- 1) The whole l - and D -dependence is contained in $\mu_l^{(D)}$, as defined in (3.18):

$$\mu_l^{(D)}[z_{(j)}] = \sqrt{2\pi z_{(j)}} e^{-z_{(j)}} I_{l+\frac{D-2}{2}}(z_{(j)}). \quad (3.32)$$

Therefore we can conclude:

$$\mu_l^{(D)}[r^2] = \mu_{l+\frac{D-3}{2}}^{(3)}[r^2] \quad (3.33)$$

and all dimensional dependence of the path integral (3.31) can be deduced from the three dimensional case:

$$k_l^{(D)}(r'', r'; T) = k_{l+\frac{D-3}{2}}^{(3)}(r'', r', T). \quad (3.34)$$

From now on we will denote $k_l(T) = k_l^{(3)}(T)$.

- 2) The boundary conditions we have already discussed; we have for the radial wave functions $u_l^{(D)}$ and $R_l^{(D)}$ respectively:

$$u_l^{(D)}(r) \rightarrow r^{l+\frac{D-1}{2}}, \quad R_l^{(D)}(r) \rightarrow r^l \quad (r \rightarrow 0). \quad (3.35)$$

We therefore state: The radial path integral $k_l(T)$ can be written as a superposition of two one dimensional path integrals:

$$k_l(r'', r'; T) = k_l^{(1)}(r'', r'; T) - (-1)^l k_l^{(1)}(r'', -r'; T) \quad (3.36)$$

with

$$\begin{aligned} k_l^{(1)}(r'', r'; T) &= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{N}{2}} \int_{-\infty}^{\infty} dr_{(1)} \cdots \int_{-\infty}^{\infty} dr_{(N-1)} \\ &\times \prod_{j=1}^N \mu_l^{(D)}[r_{(j)} r_{(j-1)}] \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon \hbar} \Delta^2 r_{(j)} - \epsilon V(|r_{(j)}|) \right] \right\}. \end{aligned} \quad (3.37)$$

- 4) From equation (3.36) for the case $l = V = 0$ we have two free particle path integrals the solution being:

$$\begin{aligned} k_0^{V=0}(r'', r'; T) &= \left(\frac{m}{2\pi i \hbar T} \right)^{\frac{1}{2}} \left[\exp \left(-\frac{i m}{2T \hbar} (r'' - r')^2 \right) - \exp \left(\frac{i m}{2T \hbar} (r'' + r')^2 \right) \right] \\ &= \left(\frac{m}{2\pi i \hbar T} \right)^{\frac{1}{2}} \exp \left(-\frac{i m}{2T \hbar} (r'' - r')^2 \right) \left[1 - \exp \left(-\frac{i m}{\hbar T} r' r'' \right) \right], \end{aligned} \quad (3.38)$$

the second term is the so called “mirror” term. It is not allowed to drop it, even for very small $T \rightarrow 0$, since this violates the boundary condition which demands that $k_0^{V=0}$ vanishes for $r', r'' \rightarrow 0$. These two parts of the Bessel functions express nothing

else than the famous mirror principle. Actually, in the above Euclidean form $k_0^{V=0}$ is nothing else but the heat kernel on the half line $r \geq 0$, which has the representation:

$$K^{(3)}(r'', r'; T) = K^{(1)}(r'', r'; T) - K^{(1)}(r'', -r'; T) \quad (3.39)$$

where $K^{(1)}$ denotes the one-dimensional heat kernel in \mathbf{R} . In the language of diffusion processes: $r = 0$ is an absorbing boundary and the job is done by the functional measure $\mu_l^{(D)}$.

Let us discuss the radial path integral in the language of the Weyl-ordering and product-ordering rule [49]. We consider the Schrödinger equation in D -dimensions with a potential $V(|x|) = V(r)$:

$$i \hbar \frac{\partial}{\partial t} \Psi(r, \{\theta\}; t) = \left[-\frac{\hbar^2}{2m} \Delta_{LB} + V(r) \right] \Psi(r, \{\theta\}; t) \quad (3.40)$$

with Δ_{LB} the Laplace Beltrami operator. Introducing the D -dimensional Legendre operator:

$$\begin{aligned} L_{(D)}^2 = & \left[\frac{\partial^2}{\partial \theta_1^2} + (D-2) \cot \theta_1 \frac{\partial}{\partial \theta_1} \right] + \frac{1}{\sin^2 \theta_1} \left[\frac{\partial^2}{\partial \theta_2^2} + (D-3) \cot \theta_2 \frac{\partial}{\partial \theta_2} \right] + \dots \\ & + \frac{1}{\sin^2 \theta_1 \dots \sin^2 \theta_{D-3}} \left[\frac{\partial^2}{\partial \theta_{D-2}^2} + \cot \theta_{D-2} \frac{\partial}{\partial \theta_{D-2}} \right] + \frac{1}{\sin^2 \theta_1 \dots \sin^2 \theta_{D-2}} \frac{\partial^2}{\partial \phi^2}. \end{aligned} \quad (3.41)$$

the Laplace operator is:

$$\Delta_{LB} = \frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} L_{(D)}^2. \quad (3.42)$$

Rewriting the Hamiltonian $H = -\frac{\hbar^2}{2m} \Delta_{LB}$ yields:

$$\begin{aligned} H(p_r, r, \{p_\theta, \theta\}) = & \frac{p_r^2}{2m} \\ & + \frac{1}{2mr^2} \left[p_{\theta_1}^2 + \frac{1}{\sin^2 \theta_1} p_{\theta_2}^2 + \dots + \frac{1}{\sin^2 \theta_1 \dots \sin^2 \theta_{D-2}} p_\phi^2 \right] + \Delta V_{Weyl}(\{\theta\}) \end{aligned} \quad (3.43)$$

with

$$\Delta V_{Weyl}(r, \{\theta\}) = -\frac{\hbar^2}{8mr^2} \left[1 + \frac{1}{\sin^2 \theta_1} + \dots + \frac{1}{\sin^2 \theta_1 \dots \sin^2 \theta_{D-2}} \right]. \quad (3.44)$$

and the hermitean momenta

$$\left. \begin{aligned} p_r &= \frac{\hbar}{i} \left(\frac{\partial}{\partial r} + \frac{D-1}{2r} \right) \\ p_{\theta_\nu} &= \frac{\hbar}{i} \left(\frac{\partial}{\partial \theta_\nu} + \frac{D-1-\nu}{2} \cot \theta_\nu \right) \\ p_\phi &= \frac{\hbar}{i} \frac{\partial}{\partial \phi}. \end{aligned} \right\} \quad (3.45)$$

Because of the special nature of g_{ab} , the product ordering gives the same quantum potential as the Weyl-ordering, i.e. we have $\Delta V_{Weyl} = \Delta V_{prod}$. In particular we have the equivalence

$$\begin{aligned} \prod_{j=1}^{N-1} \sqrt{g^{(j)}} \cdot \exp \left\{ \frac{i\epsilon}{\hbar} \sum_{j=1}^N \mathcal{L}_{Cl}^N(r^{(j)}, \{\theta^{(j)}\}, r^{(j-1)}, \{\theta^{(j-1)}\}) \right\} \\ \doteq (g'g'')^{-\frac{1}{4}} \prod_{j=1}^N \sqrt{\bar{g}^{(j)}} \cdot \exp \left\{ \frac{i\epsilon}{\hbar} \mathcal{L}_{Cl}^N(\bar{r}^{(j)}, \{\bar{\theta}^{(j)}\}) \right\} \end{aligned} \quad (3.46)$$

where

$$\begin{aligned} \mathcal{L}_{Cl}^N(r^{(j)}, \{\theta^{(j)}\}, r^{(j-1)}, \{\theta^{(j-1)}\}) \\ = \frac{m}{2\epsilon^2} \left\{ (r^{(j)} - r^{(j-1)})^2 + \widehat{r^{(j)}}^2 \left[(\theta_1^{(j)} - \theta_1^{(j-1)})^2 + \widehat{\sin^2 \theta_1^{(j)}} (\theta_2^{(j)} - \theta_2^{(j-1)})^2 + \dots \right. \right. \\ \left. \left. \dots + (\widehat{\sin^2 \theta_1^{(j)}} \dots \widehat{\sin^2 \theta_{D-2}^{(j-1)}}) (\phi^{(j)} - \phi^{(j-1)})^2 \right] \right\} \end{aligned} \quad (3.47)$$

denotes a ‘‘classical Lagrangian’’ on the lattice. $\mathcal{L}_{Cl}^N(\bar{r}^{(j)}, \{\bar{\theta}^{(j)}\})$ is similar defined as (3.47), except that one has to take all trigonometrics at mid-points. The features are, of course, direct consequences of the product form definition as described in section II.3. With the correct Hamiltonian (3.43) we can consider the Hamiltonian path integral for the Feynman kernel K:

$$\begin{aligned} K^{(D)}(r'', \{\theta''\}, r', \{\theta'\}; T) \\ = (g'g'')^{-\frac{1}{4}} \int_{\substack{r(t')=r' \\ \{\theta(t')=\theta'\}}}^{\substack{r(t'')=r'' \\ \{\theta(t'')=\theta''\}}} \mathcal{D}r(t) \mathcal{D}p_r(t) \mathcal{D}\{\theta\}(t) \mathcal{D}\{p_\theta\}(t) \\ \times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[p_r \cdot \dot{r} + \sum_{\nu=1}^{D-1} p_{\theta_\nu} \cdot \dot{\theta}_\nu - \mathcal{H}_{eff}(p_r, r, \{p_\theta\}, \{\theta\}) \right] dt \right\} \\ = (g'g'')^{-\frac{1}{4}} \lim_{N \rightarrow \infty} \prod_{j=1}^{N-1} \int dr_{(j)} d\{\theta_{(j)}\} \prod_{j=1}^N \int \frac{dp_{r_{(j)}} d\{p_{\theta_{(j)}}\}}{(2\pi\hbar)^D} \\ \times \exp \left\{ \frac{i}{\hbar} \left[\sum_{j=1}^N p_{r_{(j)}} \Delta r_{(j)} + \{p_{\theta_{(j)}}\} \Delta \{\theta_{(j)}\} - \epsilon \mathcal{H}_{eff}(p_{r_{(j)}}, r_{(j)}, \{p_{\theta_{(j)}}\}, \{\theta_{(j)}\}) \right] \right\} \end{aligned} \quad (3.48)$$

with the effective Hamiltonian

$$\begin{aligned} \mathcal{H}_{eff}(p_{r(j)}, r(j), \{p_{\theta(j)}, \theta(j)\}) &= \frac{p_{r(j)}^2}{2m} \\ &+ \frac{1}{2mr_{(j)}^2} \left[p_{\theta_1^{(j)}}^2 + \frac{1}{\sin^2 \theta_1^{(j)}} p_{\theta_2^{(j)}}^2 + \cdots + \frac{1}{\sin^2 \theta_1^{(j)} \cdots \sin^2 \theta_{D-2}^{(j)}} p_{\phi^{(j)}}^2 \right] + \Delta V_{Prod}(\{\theta(j)\}) \end{aligned} \quad (3.49)$$

After the integration over all momenta we get ($\Delta V \equiv \Delta V_{Weyl} = \Delta V_{prod}$):

$$\begin{aligned} K^{(D)}(r'', r', \{\theta''\}, \{\theta'\}; T) &= \int_{\substack{r(t')=r' \\ \{\theta(t')=\theta'\}}^{\substack{r(t'')=r'' \\ \{\theta(t'')=\theta''\}}} \mathcal{D}r(t) D\Omega(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} [\mathcal{L}_{Cl}^N(r, \dot{r}, \{\theta, \dot{\theta}\}) - \epsilon \Delta V(r, \{\theta\})] dt \right\}, \\ &= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{ND}{2}} \prod_{j=1}^{N-1} \int_0^\infty dr_{(j)}^{D-1} dr_{(j)} \int d\Omega_{(j)} \\ &\quad \times \exp \left\{ \frac{i}{\hbar} \left[\sum_{j=1}^N \mathcal{L}_{eff}^N(r_{(j)}, \{\theta_{(j)}\}, r_{(j-1)}, \{\theta_{(j-1)}\}) - \epsilon \Delta V(r_{(j)}, \{\theta_{(j)}\}) \right] \right\} \end{aligned} \quad (3.50)$$

with the effective Lagrangian

$$\begin{aligned} \mathcal{L}_{eff}(r, \dot{r}, \{\theta, \dot{\theta}\}) &= \frac{m}{2} [\dot{r}^2 + r^2 \dot{\theta}_1^2 + r^2 \sin^2 \theta_1 \dot{\theta}_2^2 + \cdots + r^2 (\sin^2 \theta_1 \cdots \sin^2 \theta_{D-2}) \dot{\phi}^2] - V(r, \{\theta\}) \end{aligned} \quad (3.51)$$

defined in the same way as \mathcal{L}_{Cl}^N in equation (3.47).

The path integral (3.50) with the Lagrangian given by equation (3.47) is too complicated for explicit calculations. We therefore try to replace equation (3.47) under the path integral (3.50) by the following expression:

$$\mathcal{L}_{Cl}(r, \{\theta\}, \dot{r}, \{\dot{\theta}\}) \rightarrow \tilde{\mathcal{L}}_{Cl}(r, \{\theta\}, \dot{r}, \{\dot{\theta}\}) := \frac{m}{2} R^2 \dot{\Omega}^2 - V_c(\{\theta\}) \quad (3.52)$$

where V_c has to be determined and Ω denotes the D -dimensional unit vector on the S^{D-1} -sphere. Thus we try to replace \mathcal{L}_{Cl} by a simpler expression and hope that $V_c + \Delta V_{Weyl}$ is simple enough. We have

$$(\Omega^{(1)} - \Omega^{(2)})^2 = 2(1 - \cos \psi^{(1,2)}) \quad (3.53)$$

with the addition theorem (3.3). We shall use equation (3.53) to justify the replacement (3.52) and thereby derive an expression for V_c . We start with the kinetic term $(x^{(j)} - x^{(j-1)})^2$ expressed in the polar coordinates (3.2), $R = r$ (not fixed), and expand it in terms of Δr and $\Delta\theta_\nu$. After some tedious calculations we obtain the following identity

$$\begin{aligned} & \exp \left[\frac{i\epsilon}{\hbar} \mathcal{L}_{Cl}^N(r^{(j)}, \{\theta^{(j)}\}, r^{(j-1)}, \{\theta^{(j-1)}\}) \right] \\ & \doteq \exp \left[\frac{im}{\epsilon\hbar} (r^{(j)2} + r^{(j-1)2} - 2r^{(j)}r^{(j-1)} \cos \psi^{(j,j-1)}) - i\epsilon V_c(r^{(j)}, \{\theta^{(j)}\}) \right] \end{aligned} \quad (3.54)$$

with

$$V_c(r^{(j)}, \{\theta^{(j)}\}) = \frac{\hbar^2}{8mr^{(j)2}} \left[1 + \frac{1}{\sin^2 \theta_1^{(j)}} + \dots + \frac{1}{\sin^2 \theta_1^{(j)} \dots \sin^2 \theta_{D-2}^{(j)}} \right] \quad (3.55)$$

(V_c is the same whether or not $\Delta r^{(j)} = 0$). The result is that the potential V_c generated by these steps cancels exactly $\Delta V_{Weyl}(r, \{\theta\})$! Therefore we get:

$$K^{(D)}(r'', r', \{\theta''\}, \{\theta'\}; T) = \int_{\substack{r(t')=r' \\ \{\theta(t')=\theta'\}}}^{\substack{r(t'')=r'' \\ \{\theta(t'')=\theta''\}}} \mathcal{D}r(t) \mathcal{D}\Omega(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 - V(r) \right] dt \right\}, \quad (3.56)$$

where \dot{x}^2 has to be expressed in polar coordinates. In the lattice formulation \dot{x}^2 reads

$$\dot{x}^2 \rightarrow [r_{(j)}^2 + r_{(j-1)}^2 - 2r_{(j)}r_{(j-1)} \cos \psi_{(j,j-1)}] / \epsilon^2. \quad (3.57)$$

Therefore we arrive at equation (3.52) and both approaches are equivalent as it should be.

Let us note that equation (3.54) as derived in [49] is effectively a regularization of the highly singular terms in ΔV . This corresponds to the $1/r^2$ -terms in the radial path integral which have been regularized by the functional measure $\mu_l[r^2]$. However, the functional measure method gives a quite practical tool to regularize such singular terms.

3.2. The Radial Harmonic Oscillator

Let us now discuss the most important application of equation (3.17), namely the harmonic oscillator with $V(r) = \frac{1}{2}m\omega^2 r^2$. The calculation has first been performed by Peak and Inomata [84]. However, we present the more general case with time-dependent coefficients following Goovaerts [39]. This example will be of great virtue in the solution of various path integral problems.

We have to study

$$\begin{aligned}
& K_l(r'', r'; \tau) \\
&= (r' r'')^{\frac{1-D}{2}} \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{N/2} \int_0^\infty dr_{(1)} \cdots \int_0^\infty dr_{(N-1)} \\
&\quad \times \prod_{j=1}^N \left\{ \mu_l^{(D)}[r_{(j)} r_{(j-1)}] \cdot \exp \left[\frac{i m}{2\epsilon \hbar} (r_{(j)} - r_{(j-1)})^2 - \frac{i \epsilon}{2\hbar} m \omega^2 (t^{(j)}) r_{(j)}^2 \right] \right\} \\
&= (r' r'')^{\frac{2-D}{2}} \lim_{N \rightarrow \infty} \left(\frac{m}{i \epsilon \hbar} \right)^{N/2} \int_0^\infty r_{(1)} r_{(1)} \cdots \int_0^\infty r_{(N-1)} dr_{(N-1)} \\
&\quad \times \prod_{j=1}^N \left\{ \exp \left[\frac{i m}{2\epsilon \hbar} (r_{(j)}^2 + r_{(j-1)}^2) - \frac{i \epsilon}{2\hbar} m \omega^2 (t^{(j)}) r_{(j)}^2 \right] \cdot I_{l+\frac{D-2}{2}} \left(\frac{m}{i \epsilon \hbar} r_{(j)} r_{(j-1)} \right) \right\} \\
&= (r' r'')^{\frac{2-D}{2}} \lim_{N \rightarrow \infty} \left(\frac{\alpha}{i} \right)^{N/2} e^{i\alpha(r'^2 + r''^2)/2} \int_0^\infty r_{(1)} dr_{(1)} \cdots \int_0^\infty r_{(N-1)} dr_{(N-1)} \\
&\quad \times \exp \left[i(\beta_{(1)} r_{(1)}^2 + \beta_{(2)} r_{(2)}^2 + \cdots + \beta_{(N-1)} r_{(N-1)}^2) \right] \\
&\quad \times \left[I_{l+\frac{D-2}{2}}(-i\alpha r_{(0)} r_{(1)}) \cdots I_{l+\frac{D-2}{2}}(-i\alpha r_{(N-1)} r_{(N)}) \right] \\
&= (r' r'')^{\frac{2-D}{2}} \lim_{N \rightarrow \infty} K_l^N(T) \tag{3.58}
\end{aligned}$$

where $K_l^N(T)$ is defined by the iterated integrals. Furthermore we have set $\alpha = m/\epsilon \hbar$ and $\beta_{(j)} = \alpha[1 - \epsilon^2 m \omega^2 (t^{(j)})/2]$. We are now using [40, p.718]

$$\int_0^\infty x e^{-\gamma x^2} J_\nu(\alpha x) J_\nu(\beta x) dx = \frac{1}{2\gamma} e^{-(\alpha^2 + \beta^2)/4\gamma} I_\nu \left(\frac{\alpha\beta}{2\gamma} \right) \tag{3.59}$$

which is valid for $\Re(\nu) > -1$, $|\arg \sqrt{\gamma}| < \pi/4$ and $\alpha, \beta > 0$. With analytic continuation [84] one can show that

$$\int_0^\infty r e^{i\alpha r^2} I_\nu(-i\alpha r) I_\nu(-i\beta r) dr = \frac{i}{2\alpha} e^{(\alpha^2 + \beta^2)/4\alpha i} I_\nu \left(\frac{ab}{2\alpha i} \right) \tag{3.60}$$

is valid for $\nu > -1$ and $\Re(\alpha) > 0$. By means of equation (3.60) we obtain for $K_l^N(T)$:

$$\begin{aligned}
K_l^N(T) &= \left(\frac{\alpha}{i} \right)^{\frac{N}{2}} \exp \left(\frac{i\beta}{2} (r'^2 + r''^2) \right) \int_0^\infty r_{(1)} dr_{(1)} \cdots \int_0^\infty r_{(N-1)} dr_{(N-1)} \\
&\quad \times \exp \left[i(\beta_{(1)} r_{(1)}^2 + \beta_{(2)} r_{(2)}^2 + \cdots + \beta_{(N-1)} r_{(N-1)}^2) \right] \\
&\quad \times \left[I_{l+\frac{D-2}{2}}(-i\alpha r_{(0)} r_{(1)}) \cdots I_{l+\frac{D-2}{2}}(-i\alpha r_{(N-1)} r_{(N)}) \right] \\
&= \frac{\alpha_N}{i} \exp(i p_N r'^2 + i q_N r''^2) I_{l+\frac{D-2}{2}}(-i\alpha_N r' r'') \tag{3.61}
\end{aligned}$$

where the coefficients α_N, p_N and q_N are given by

$$\left. \begin{aligned}
 \alpha_N &= \alpha \prod_{k=1}^{N-1} \frac{\alpha}{2\gamma_k} \\
 p_N &= \frac{\alpha}{2} - \sum_{k=1}^{N-1} \frac{\alpha_k^2}{4\gamma_k} \\
 q_N &= \frac{\alpha}{2} - \frac{\alpha^2}{4\gamma_{N-1}} \\
 \alpha_1 &= \alpha, \quad \alpha_{k+1} = \alpha \prod_{j=1}^k \frac{\alpha}{2\gamma_j} \quad (k \geq 1) \\
 \gamma_1 &= \beta_1, \quad \gamma_{k+1} = \beta_{k+1} - \frac{\alpha^2}{4\gamma_k}.
 \end{aligned} \right\} \quad (3.62)$$

We must now determine these quantities. Let us start with the evaluation of γ_k . Putting

$$\frac{2\gamma_k}{\alpha} = \frac{y_{k+1}}{y_k} \quad (3.63)$$

we obtain

$$\gamma_{k+1} = \beta_{k+1} - \frac{\alpha^2}{4\gamma_k} \iff \frac{y_{k+1} - 2y_k + y_{k-1}}{\epsilon^2} + \omega_k^2 y_k = 0. \quad (3.64)$$

In the limit $N \rightarrow \infty$ this gives a differential equation for y

$$\ddot{y} + \omega^2(t)y = 0$$

with solution $y = \eta(t' + t) \equiv \eta(t)$. Since on one hand side

$$\begin{aligned}
 y_1 &\simeq y_0 + \epsilon \dot{y}_0 \rightarrow y_0 & (\epsilon \rightarrow 0), \\
 y_1 &\simeq \frac{2}{\alpha} y_0 \gamma_1 \rightarrow 2y_0, & (\epsilon \rightarrow 0),
 \end{aligned}$$

and on the other

$$\dot{y}_0 = \frac{y_1 - y_0}{\epsilon} = \frac{1}{\epsilon} \left(\frac{2}{\alpha} \gamma_1 - 1 \right) y_0 \rightarrow 1, \quad (\epsilon \rightarrow 1),$$

we have the boundary conditions

$$\eta(0) = 0, \quad \dot{\eta}(0) = 1. \quad (3.65)$$

Now let us expand $y_{k+1} \simeq \eta[(k+1)\epsilon] + O(\epsilon^3)$, we observe that

$$y_{k+1} - 2y_k + y_{k-1} + \epsilon^2 \omega^2(t' + k\epsilon)y_k = 0$$

is satisfied up to second order in ϵ . Therefore expanding (3.64)

$$\gamma_{k+1} = \frac{\alpha y_{k+2}}{2 y_{k+1}} = \frac{\alpha \eta[(k+2)\epsilon] + O(\epsilon^3)}{2 \eta[(k+1)\epsilon] + O(\epsilon^3)}.$$

Consequently

$$\begin{aligned} \lim_{N \rightarrow \infty} \alpha_N &= \lim_{N \rightarrow \infty} \frac{m}{\epsilon \hbar} \prod_{j=1}^{N-1} \frac{\eta(j\epsilon) + O(\epsilon^3)}{\eta[(j+1)\epsilon] + O(\epsilon^3)} \\ &= \lim_{N \rightarrow \infty} \frac{m}{\epsilon \hbar} \frac{\eta(0)}{\eta(\epsilon N)} = \frac{m}{\hbar \eta(T)}. \end{aligned} \quad (3.66)$$

Similarly

$$\begin{aligned} \lim_{N \rightarrow \infty} q_N &= \lim_{N \rightarrow \infty} \frac{m}{2\epsilon \hbar} \left(1 - \frac{\eta[(N-1)\epsilon] + O(\epsilon^3)}{\eta[\epsilon N] + O(\epsilon^3)} \right) \\ &= \lim_{N \rightarrow \infty} \frac{m}{2\hbar} \frac{\eta[N\epsilon] - \eta[t' + (N-1)\epsilon]}{\epsilon \eta[N\epsilon]} = \frac{m \dot{\eta}(T)}{2\hbar \eta(T)}. \end{aligned} \quad (3.67)$$

Finally we must calculate p_N . We have

$$\begin{aligned} \lim_{N \rightarrow \infty} p_N &= \lim_{N \rightarrow \infty} \left(\frac{\alpha}{2} - \sum_{k=1}^{N-1} \frac{\alpha_k^2}{4\gamma_k} \right) \\ &= \frac{m}{2\hbar} \lim_{N \rightarrow \infty} \left(\frac{1}{\epsilon} - \sum_{k=1}^{N-1} \frac{\epsilon}{\eta^2[(k+1)\epsilon]} \right) \\ &= \frac{m}{2\hbar} \lim_{N \rightarrow \infty} \left(\frac{1}{\epsilon} - \int_{\epsilon}^T \frac{dt}{\eta^2(t)} \right) \\ &= \frac{m}{2\hbar \eta(T)} \lim_{N \rightarrow \infty} \left[\frac{\eta(T)}{\epsilon} - \eta(T) \int_{\epsilon}^T \frac{dt}{\eta^2(t)} \right] \equiv \frac{m}{2\hbar \eta(T)} \xi(T). \end{aligned} \quad (3.68)$$

Furthermore we find

$$\begin{aligned} \lim_{N \rightarrow \infty} \xi(\epsilon) &= \lim_{N \rightarrow \infty} \frac{\eta(\epsilon)}{\epsilon} - \eta(\epsilon) \int_{\epsilon}^{\epsilon} \frac{dt}{\eta^2(t)} \\ \lim_{\epsilon \rightarrow 0} \frac{\eta(0) + \epsilon \dot{\eta}(0)}{\epsilon} &= \dot{\eta}(0) = 1 \\ \lim_{\epsilon \rightarrow 0} \dot{\xi}(\epsilon) &= \lim_{\epsilon \rightarrow 0} \left(\frac{\dot{\eta}(\epsilon)}{\epsilon} - \frac{1}{\eta(\epsilon)} \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{\dot{\eta}(0)[\eta(0) + \epsilon \dot{\eta}(0)] - \epsilon}{\epsilon[\eta(0) + \epsilon \dot{\eta}(0)]} = 0, \end{aligned}$$

and therefore $\xi(t)$ is satisfying the boundary conditions

$$\xi(0) = 1, \quad \dot{\xi}(0) = 0. \quad (3.69)$$

$\xi(t)$ is found to satisfy the differential equation

$$\ddot{\xi} + \omega^2(t)\xi = 0. \quad (3.70)$$

Therefore we have

$$P(T) = \lim_{N \rightarrow \infty} p_N = \frac{m}{2\hbar} \frac{\xi(T)}{\eta(T)} \quad (3.71)$$

and we have finally for the path integral of the radial harmonic oscillator with time-dependent frequency:

$$K_l(r'', r'; T) = (r' r'')^{\frac{2-D}{2}} \frac{m}{i\hbar\eta(T)} \exp \left[\frac{i m}{2\hbar} \left(\frac{\xi(T)}{\eta(T)} r'^2 + \frac{\dot{\eta}(T)}{\eta(T)} r'^2 \right) \right] I_{l+\frac{D-2}{2}} \left(\frac{m r' r''}{i\hbar\eta(T)} \right). \quad (3.72)$$

In particular for $\omega(t) = \omega = \text{const.}$:

$$\begin{aligned} \eta(t) &= \frac{1}{\omega} \sin \omega(t - t'), & \dot{\eta}(t) &= \cos \omega(t - t') \\ \xi(t) &= \cos \omega(t - t') \end{aligned}$$

which yields the radial path integral solution for the radial harmonic oscillator with time-independent frequency

$$K_l(r'', r'; T) = (r' r'')^{\frac{2-D}{2}} \frac{m\omega}{i\hbar \sin \omega T} \exp \left[\frac{i m \omega}{2\hbar} (r'^2 + r''^2) \cot \omega T \right] I_{l+\frac{D-2}{2}} \left(\frac{m\omega r' r''}{i\hbar \sin \omega T} \right). \quad (3.73)$$

It is possible to get the same result, if one starts with the D -dimensional path integral in Cartesian coordinates:

$$K(x'', x'; \tau) = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left\{ \frac{i m}{2\hbar} \int_{t'}^{t''} [\dot{x}^2 - \omega^2 x^2] dt \right\} \quad (3.74)$$

insert for every dimension the solution of the one-dimensional oscillator

$$K(x''_k, x'_k; T) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega T}} \exp \left\{ \frac{i m \omega}{2\hbar} \left[(x'_k)^2 + (x''_k)^2 \cot \omega T - \frac{2x'_k x''_k}{\sin \omega T} \right] \right\} \quad (3.75)$$

and uses equations (3.8) and (3.11).

The next step is to calculate with the help of equation (3.73) the energy-levels and state-functions. For this purpose we use the Hille-Hardy-formula [40, p.1038]

$$\frac{t^{-\alpha/2}}{1-t} \exp \left[-\frac{1}{2}(x+y) \frac{1+t}{1-t} \right] I_\alpha \left(\frac{2\sqrt{xyt}}{1-t} \right) = \sum_{n=0}^{\infty} \frac{t^n n! e^{-\frac{1}{2}(x+y)}}{\Gamma(n+\alpha+1)} (xy)^{\alpha/2} L_n^{(\alpha)}(x) L_n^{(\alpha)}(y). \quad (3.76)$$

With the substitution $t = e^{-2i\omega T}$, $x = m\omega r'^2/\hbar$ and $y = m\omega r''^2/\hbar$ in equation (3.73) we get finally:

$$K_l(r'', r'; T) = \sum_{N=0}^{\infty} e^{-iTE_N/\hbar} R_N^l(r') R_N^l(r'') \quad (3.77)$$

$$E_N = \omega\hbar \left(N + \frac{D}{2} \right) \quad (3.78)$$

$$R_N^l(r) = \sqrt{\frac{2m\omega}{\hbar r^{D-2}} \cdot \frac{\Gamma(\frac{N-l}{2} + 1)}{\Gamma(\frac{N+l+D}{2})}} \left(\frac{m\omega}{\hbar} r^2 \right)^{l + \frac{D-2}{2}} \exp\left(-\frac{m\omega}{\hbar} r^2 \right) L_{\frac{N-l}{2}}^{(l + \frac{D-2}{2})} \left(\frac{m\omega}{\hbar} r^2 \right). \quad (3.79)$$

The path integral for the harmonic oscillator suggests a generalization in the index l . This will be very important in further applications. For this purpose we consider equation (3.17) with the nontrivial functional measure $\mu_l^{(D)}(r^2) \equiv \mu_l[r^2]$:

$$K_l(r'', r'; T) = \frac{1}{r' r''} \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \mu_l[r^2] \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{r}^2 - V(r) \right] dt \right\} \quad (3.80)$$

The functional measure corresponds to a potential $V_l = \hbar^2 \frac{l(l+1)}{2mr^2}$ in the Schrödinger equation. Assuming that we can analytically continue in $l \rightarrow \lambda$ with $\Re(\lambda) > -1$, then we get for an arbitrary potential $V_\lambda(r) = \hbar^2 \frac{\lambda^2 - \frac{1}{4}}{2mr^2}$ naïvely inserted into the radial path integral

$$\int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{r}^2 - \hbar^2 \frac{\lambda^2 - \frac{1}{4}}{2mr^2} - \frac{m}{2} \omega^2 r^2 \right] dt \right\}. \quad (3.81)$$

This path integral must now be interpreted in terms of the functional measure in equation (3.18). Define ($z \equiv mr_{(j-1)} r_{(j)} / i\epsilon\hbar$):

$$\mu_\lambda[r^2] = \lim_{N \rightarrow \infty} \prod_{j=1}^N \sqrt{2\pi z_{(j)}} e^{-z_{(j)}} I_\lambda(z_{(j)}). \quad (3.82)$$

Then equation (3.81) **must be interpreted** with the help of equation (3.82) as

$$\begin{aligned} & \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{r}^2 - \hbar^2 \frac{\lambda^2 - \frac{1}{4}}{2mr^2} - \frac{m}{2} \omega^2 r^2 \right] dt \right\} \\ & := \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \mu_\lambda[r^2] \exp \left[\frac{im}{2\hbar} \int_{t'}^{t''} (\dot{r}^2 - \omega^2 r^2) dt \right] \\ & = \frac{\sqrt{r' r''} m \omega}{i \hbar \sin \omega T} \exp \left[\frac{im\omega}{2\hbar} (r'^2 + r''^2) \cot \omega T \right] I_\lambda \left(\frac{m\omega r' r''}{i \hbar \sin \omega T} \right). \end{aligned} \quad (3.83)$$

This important equation has been derived by Peak and Inomata [84] with the help of the three-dimensional radial harmonic oscillator, and by Duru [26] with the corresponding two-dimensional case. Duru considered the radial potential problem

$$V(r) = \frac{a}{r^2} + br^2 \quad (3.84)$$

with some numbers a and b . Identifying $a = (\hbar^2/2m)(\lambda^2 - \frac{1}{4})$ and $b = \frac{1}{2}m\omega^2$ it is simple calculation to express equation (3.83) in terms of a and b , which is omitted here. However, these authors did not discuss this path integral identity in terms of the functional measure language. That such a procedure is actually legitimate is beyond the scope of these notes. It was justified by Fischer, Leschke and Müller [35] for the radial path integral and for the Pöschl-Teller path integral as well (see below), where one also has to be careful with the appropriate Besselian functional measure. In the functional measure interpretation equation (3.83) can be found in references [49] and [92]. Equation (3.83) is very important in numerous applications.

Let us note the free particle case. In the limit $\omega \rightarrow 0$ we obtain in equation (3.73)

$$\begin{aligned} K_l^{(\omega=0)}(r'', r'; T) &= (r' r'')^{\frac{2-D}{2}} \frac{m}{i \hbar T} \exp \left[\frac{i m}{2 \hbar T} (r'^2 + r''^2) \right] I_{l+\frac{D-2}{2}} \left(\frac{m r' r''}{i \hbar T} \right) \\ &= (r' r'')^{\frac{2-D}{2}} \int_0^\infty dp p \exp \left(- \frac{i \hbar T p^2}{2m} \right) J_{l+\frac{D-2}{2}}(pr') J_{l+\frac{D-2}{2}}(pr'') \end{aligned} \quad (3.85)$$

with wave-functions and energy spectrum

$$\Psi_p(r) = r^{\frac{2-D}{2}} \sqrt{p} J_{l+\frac{D-2}{2}}(pr), \quad E_p = \frac{\hbar^2 p^2}{2m}. \quad (3.86)$$

The one-dimensional case gives (i.e. motion on the half-line)

$$\Psi_p(r) = r \sqrt{p} J_{\frac{1}{2}}(pr) = \sqrt{\frac{2}{\pi r}} \sin pr, \quad E_p = \frac{\hbar^2 p^2}{2m}. \quad (3.87)$$

However, there is an ambiguity in the boundary condition for $r = 0$ for $D = 1$. The present case here, i.e. $\Psi_p(0) = 0$, corresponds to a specific self adjoint extension of the Hamiltonian $H = -\hbar^2 d^2/dx^2$ for functions $\Psi \in L^2([0, \infty))$ on the half-line.

Finally we calculate the energy dependent Green function for the radial harmonic oscillator. Making use of the integral representation of the previous section we obtain

$$\begin{aligned} G_l(r'', r'; E) &= i \int_0^\infty e^{iET/\hbar} K_l(r'', r'; T) dt \\ &= \frac{\Gamma[\frac{1}{2}(l + \frac{D}{2} - \frac{E}{\hbar\omega})]}{\omega (r' r'')^{D/2} \Gamma(l + \frac{D}{2})} W_{\frac{E}{2\hbar\omega}, \frac{1}{2}(l + \frac{D-2}{2})} \left(\frac{m\omega}{\hbar} r_{>}^2 \right) M_{\frac{E}{2\hbar\omega}, \frac{1}{2}(l + \frac{D-2}{2})} \left(\frac{m\omega}{\hbar} r_{<}^2 \right). \end{aligned} \quad (3.88)$$

From this representation we can recover by means of the expansion for the Γ -function the wave-functions of equation (3.79).

The corresponding Green function for the free particle has the form

$$\begin{aligned} G_l^{(\omega=0)}(r'', r'; E) &= \frac{2m}{\hbar(r'r'')^{\frac{D-2}{2}}} \\ &\quad \times I_{l+\frac{D-2}{2}} \left[\frac{1}{i} \sqrt{\frac{mE}{2\hbar^2}} (r' + r'' - |r'' - r'|) \right] K_{l+\frac{D-2}{2}} \left[\frac{1}{i} \sqrt{\frac{mE}{2\hbar^2}} (r' + r'' + |r'' - r'|) \right] \end{aligned} \quad (3.89)$$

$$\begin{aligned} &= \frac{i\pi m}{\hbar(r'r'')^{\frac{D-2}{2}}} \\ &\quad \times J_{l+\frac{D-2}{2}} \left[\sqrt{\frac{mE}{2\hbar^2}} (r' + r'' - |r'' - r'|) \right] H_{l+\frac{D-2}{2}}^{(1)} \left[\sqrt{\frac{mE}{2\hbar^2}} (r' + r'' + |r'' - r'|) \right], \end{aligned} \quad (3.90)$$

where use has been made of the integral representation

$$\int_0^\infty \frac{dx}{x} \exp\left(-px - \frac{a+b}{2x}\right) I_\nu\left(\frac{a-b}{2x}\right) = 2I_\nu\left[\sqrt{p}(\sqrt{a}-\sqrt{b})\right] K_\nu\left[\sqrt{p}(\sqrt{a}+\sqrt{b})\right]. \quad (3.91)$$

4. Other Elementary Path Integrals

There are two further path integral solutions based on the $SU(2)$ [5, 25, 60] and $SU(1, 1)$ [5, 61] group path integration, respectively. The first yields the path integral identity for the solution of the Pöschl-Teller potential according to

$$\begin{aligned} K^{(PT)}(x'', x'; T) &= \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 - \frac{\hbar^2}{2m} \left(\frac{\alpha^2 - \frac{1}{4}}{\sin^2 x} + \frac{\beta^2 - \frac{1}{4}}{\cos^2 x} \right) \right] dt \right\} \\ &= \sum_{l=0}^{\infty} \exp \left[-\frac{i\hbar T}{2m} (\alpha + \beta + 2l + 1)^2 \right] \Psi_l^{(\alpha, \beta)*}(x') \Psi_l^{(\alpha, \beta)}(x'') \\ &= \sqrt{\sin 2x' \sin 2x''} \sum_{J=0, \frac{1}{2}}^{\infty} \exp \left[-\frac{i\hbar T}{2m} (2J + 1)^2 \right] \\ &\quad \times (2J + 1) D_{\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2}}^{J*}(\cos 2x') D_{\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2}}^J(\cos 2x'') \end{aligned} \quad (4.1)$$

with the wave-functions given by

$$\Psi_n^{(\alpha, \beta)}(x) = \sqrt{(2J + 1) \sin 2x} D_{\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2}}^J(\cos 2x) \quad (4.2a)$$

$$\begin{aligned}
 &= \left[2(\alpha + \beta + 2l + 1) \frac{l! \Gamma(\alpha + \beta + l + 1)}{\Gamma(\alpha + l + 1) \Gamma(\beta + l + 1)} \right]^{\frac{1}{2}} \\
 &\quad \times (\sin x)^{\alpha + \frac{1}{2}} (\cos x)^{\beta + \frac{1}{2}} P_n^{(\alpha, \beta)}(\cos 2x). \tag{4.2b}
 \end{aligned}$$

$$\begin{aligned}
 &= \left[\frac{\alpha + \beta + 2l + 1}{2^{\alpha + \beta + 1}} \frac{l! \Gamma(\alpha + \beta + l + 1)}{\Gamma(\alpha + l + 1) \Gamma(\beta + l + 1)} \right]^{\frac{1}{2}} \\
 &\quad \times \sqrt{2 \sin 2x} (1 - \cos 2x)^{\frac{\alpha}{2}} (1 + \cos 2x)^{\frac{\beta}{2}} P_n^{(\alpha, \beta)}(\cos 2x). \tag{4.2c}
 \end{aligned}$$

Here, of course we can analytically continue from integer values of m and n to, say, real numbers α and β , respectively.

Similarly we can state a path integral identity for the modified Pösch-Teller potential which is defined as

$$V^{(\eta, \nu)}(r) = \frac{\hbar^2}{2m} \left(\frac{\eta^2 - \frac{1}{4}}{\sinh^2 r} - \frac{\nu^2 - \frac{1}{4}}{\cosh^2 r} \right). \tag{4.3}$$

This can be achieved by means of the path integration of the $SU(1, 1)$ group manifold. One gets

$$\begin{aligned}
 K^{(MPT)}(r'', r'; T) &= \sum_{n=0}^{N_M} \Phi_n^{(\eta, \nu)*}(r') \Phi_n^{(\eta, \nu)}(r'') \exp \left\{ -\frac{i \hbar T}{2m} \left[2(k_1 - k_2 - n) - 1 \right]^2 \right\} \\
 &+ \int_0^\infty dp \Phi_p^{(\eta, \nu)*}(r') \Phi_p^{(\eta, \nu)}(r'') \exp \left(-\frac{i \hbar T}{2m} p^2 \right). \tag{4.4}
 \end{aligned}$$

Introduce the numbers k_1, k_2 defined by: $k_1 = \frac{1}{2}(1 \pm \nu)$, $k_2 = \frac{1}{2}(1 \pm \eta)$, where the correct sign depends on the boundary conditions for $r \rightarrow 0$ and $r \rightarrow \infty$, respectively. In particular for $\eta^2 = \frac{1}{4}$, i.e. $k_2 = \frac{1}{4}, \frac{3}{4}$, we obtain wave-functions with even and odd parity, respectively. The number N_M denotes the maximal number of states with $0, 1, \dots, N_M < k_1 - k_2 - \frac{1}{2}$. The bound state wave-functions read as:

$$\left. \begin{aligned}
 \Phi_n^{(k_1, k_2)}(r) &= N_n^{(k_1, k_2)} (\sinh r)^{2k_2 - \frac{1}{2}} (\cosh r)^{-2k_1 + \frac{3}{2}} \\
 &\quad \times {}_2F_1(-k_1 + k_2 + \kappa, -k_1 + k_2 - \kappa + 1; 2k_2; -\sinh^2 r) \\
 N_n^{(k_1, k_2)} &= \frac{1}{\Gamma(2k_2)} \left[\frac{2(2\kappa - 1) \Gamma(k_1 + k_2 - \kappa) \Gamma(k_1 + k_2 + \kappa - 1)}{\Gamma(k_1 - k_2 + \kappa) \Gamma(k_1 - k_2 - \kappa + 1)} \right]^{\frac{1}{2}}
 \end{aligned} \right\} \tag{4.5}$$

($\kappa = k_1 - k_2 - n$), (there is a factor “2” missing in reference [36]). Note the equivalent formulation

$$\tilde{\Phi}_n^{(\alpha, \beta)}(r) = \left[\frac{\beta n! \Gamma(n + \alpha + \beta + 1)}{2^{\alpha + \beta} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)} \right]^{\frac{1}{2}} (1 - x)^{\frac{\alpha}{2}} (1 + x)^{\frac{\beta - 1}{2}} P_n^{(\alpha, \beta)}(x), \tag{4.6}$$

with the substitutions $\alpha = 2k_2 - 1$, $\beta = 2(k_1 - k_2 - n) - 1 = 2\kappa - 1$, $x = 2/\cosh^2 r - 1$ with the incorporation of the appropriate measure term, i.e. $dr = \left[(1+x)\sqrt{2(1-x)}\right]^{-1} dx$. The scattering states are given by:

$$\left. \begin{aligned} \Phi_p^{(k_1, k_2)}(r) &= N_p^{(k_1, k_2)} (\cosh r)^{2k_1 - \frac{1}{2}} (\sinh r)^{2k_2 - \frac{1}{2}} \\ &\quad \times {}_2F_1(k_1 + k_2 - \kappa, k_1 + k_2 + \kappa - 1; 2k_2; -\sinh^2 r) \\ N_p^{(k_1, k_2)} &= \frac{1}{\Gamma(2k_2)} \sqrt{\frac{p \sinh \pi p}{2\pi^2}} \left[\Gamma(k_1 + k_2 - \kappa) \Gamma(-k_1 + k_2 + \kappa) \right. \\ &\quad \left. \times \Gamma(k_1 + k_2 + \kappa - 1) \Gamma(-k_1 + k_2 - \kappa + 1) \right]^{\frac{1}{2}}, \end{aligned} \right\} \quad (4.7)$$

$[\kappa = \frac{1}{2}(1 + ip)]$.

It is possible to state closed expressions for the (energy dependent) Green functions for the Pöschl-Teller and modified Pöschl-Teller potential, respectively. For the Pöschl-Teller potential it has the form (Kleinert and Mustapic [65])

$$\begin{aligned} G(x'', x'; E) &= \frac{m}{i\hbar} \sqrt{\sin 2x' \sin 2x''} \frac{\Gamma(m_1 - L_E) \Gamma(L_E + m_1 + 1)}{\Gamma(m_1 + m_2 + 1) \Gamma(m_1 - m_2 + 1)} \\ &\quad \times \left(\frac{1 - \cos 2x'}{2} \right)^{(m_1 - m_2)/2} \left(\frac{1 + \cos 2x'}{2} \right)^{(m_1 + m_2)/2} \\ &\quad \times \left(\frac{1 - \cos 2x''}{2} \right)^{(m_1 - m_2)/2} \left(\frac{1 + \cos 2x''}{2} \right)^{(m_1 + m_2)/2} \\ &\quad \times {}_2F_1 \left(-L_E + m_1; L_E + m_1 + 1; m_1 - m_2 + 1; \frac{1 - \cos 2x'}{2} \right) \\ &\quad \times {}_2F_1 \left(-L_E + m_1; L_E + m_1 + 1; m_1 + m_2 + 1; \frac{1 + \cos 2x''}{2} \right) \end{aligned} \quad (4.8)$$

with $m_{1/2} = \frac{1}{2}(\lambda \pm \kappa)$, $L_E = -\frac{1}{2} + \frac{1}{2}\sqrt{2mE}/\hbar$ and $x'' \geq x'$. A similar expression is valid for the modified Pöschl-Teller-potential Green function (see [65]) which is, however, omitted here.

5. The Coulomb Potential

The hydrogen atom is, of course, one of the most interesting subjects in quantum mechanics. In the beginnings of quantum mechanics and with the atom model of Rutherford it was a riddle how to tract a system which is from the classical physics point of view unstable and doomed to vanish into pure radiation. Its was Bohr's genius, postulating the famous rules that only a countable number of orbits are allowed satisfying the quantum condition $\oint pdq = nh$ ($n \in \mathbf{N}$). However, this "old" quantum mechanics was not sufficient, because it fails e.g. in the case of the helium-atom, and it was Pauli who solved the hydrogen problem in the terms of the "new" quantum mechanics developed by Schrödinger and Heisenberg. It is surprising, that Pauli did

not use a differential equation and solves the corresponding Eigenvalue problem as Schrödinger did later, instead he exploited in fact the “hidden” $SO(4)$ symmetry of the Kepler-Coulomb-problem. This symmetry gives classically rise to an conserved quantity, called Lenz-Runge vector. This additional symmetry allows also a separation of the Coulomb problem in parabolic coordinates. This vector points from the focal point of the orbit the perihel.

Ever since the success of quantum mechanics the hydrogen atom was **the** model to test the theory, may it be Dirac’s relativistic quantum mechanics, where first the fine-structure constant $\alpha = e^2 / \hbar c$ arises.

It was for a long time a really nuisance that this important physical system could not be treated by path integrals. Calculating wave functions and energy levels remains more or less a simple task in the operator language, but even the construction of the Green function (resolvent kernel) was impossible for a long time. It takes as long as 1979 as Duru and Kleinert [27, 28] finally applied a long-known transformation in astronomy (Kustaanheimo-Stiefel transformation [66]) in the path integral of the Coulomb problem and were successful. Even more, their idea of transforming simultaneously coordinates and the time-slicing opens new possibilities in solving the huge amount of unsolved path integral problems. In particular, Coulomb-related potentials, in the sense, that all these problems can be reformulated in terms of confluent hypergeometric functions, like the Morse-potential could be successfully treated. However, the original attempt of Duru and Kleinert was done in a more or less formal manner, and it does not takes a long time when it was refined by Inomata [58] and Duru and Kleinert [28].

Closely related to the original Coulomb problem is, of course, the $1/r$ -potential discussion in \mathbf{R}^D . The $D = 2$ case in discussed in this subsection, whereas the original Coulomb problem in the next.

As it turns out its Schrödinger equation for the Coulomb potential is separable in four coordinate systems:

- 1) Spherical coordinates:

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \quad (r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi). \\ z &= r \cos \theta \end{aligned} \tag{5.1}$$

Separation of variables in this coordinate system is, of course, not a specific feature of the Coulomb problem, but a common property of all radial potentials.

- 2) Parabolic coordinates:

$$\begin{aligned} x &= \xi \eta \cos \phi \\ y &= \xi \eta \sin \phi \quad (\xi, \eta \geq 0, 0 \leq \phi \leq 2\pi). \\ z &= \frac{1}{2}(\xi^2 - \eta^2) \end{aligned} \tag{5.2}$$

Here electric and magnetic fields can be introduced without spoiling separability.

3) Spheroidal coordinates:

$$\begin{aligned} x &= \sinh \xi \sin \eta \sin \phi \\ y &= \sinh \xi \sin \eta \cos \phi \quad (\xi \geq 0, 0 \leq \eta \leq \pi, 0 \leq \phi \leq 2\pi). \\ z &= \cosh \xi \cos \eta + 1 \end{aligned} \quad (5.3)$$

Here a further charge can be introduced and therefore these coordinates are suitable for the study of the hydrogen ion H_2^+ .

4) Sphericoconical coordinates:

$$\begin{aligned} x &= r \operatorname{sn}(\alpha, k) \operatorname{dn}(\beta, k') & r \geq 0, |\alpha| \leq K, |\beta| \leq 2K', \\ y &= r \operatorname{cn}(\alpha, k) \operatorname{cn}(\beta, k') & 0 \leq k, k' \leq 1, \quad k^2 + k'^2 = 1. \\ z &= r \operatorname{dn}(\alpha, k) \operatorname{sn}(\beta, k') \end{aligned} \quad (5.4)$$

$\operatorname{sn}(\alpha, k)$, $\operatorname{cn}(\alpha, k)$ and $\operatorname{dn}(\alpha, k)$ denote Jacobi elliptic functions (e.g. [40, pp.910]) of modulus k and with real and imaginary periods $4K$ and $4iK'$, respectively. Here the corresponding wave functions in α and β can be identified with the wave functions of a quantum mechanically asymmetric top.

We will discuss the path integral for the Coulomb system in the first two of these coordinate systems. For the usual polar- and parabolic coordinate system the path integration can be exactly performed. In the remaining two, however, the theory of special functions of these coordinate is poorly developed and no solution seems up to now available. Nevertheless it is possible to formulate the Coulomb-problem in these coordinate systems and point out some relations to other problems connected to these coordinates [46].

It is surprising that the Coulomb path integral was first solved in Cartesian coordinates (where the Kustaanheimo-Stiefel transformation works) and not in polar coordinates. Looking at the appropriate formulæ we see that even in the one-dimensional case (in polar coordinates) we need the one-dimensional realization of the Kustaanheimo-Stiefel transformation and a space-time transformation.

5.1. The $1/r$ -Potential in \mathbf{R}^2 [28, 57]

We consider the Euclidean two-dimensional space with the singular potential $V(r) = -Ze^2/r$ ($r = |x|$, $x \in \mathbf{R}^2$). Here, e^2 denotes the square of an electric charged Z its multiplicity (including sign). However, as already noted, this potential is not the potential of a point charge in \mathbf{R}^2 . The classical Lagrangian now has the form

$$\mathcal{L}(x, \dot{x}) = \frac{m}{2} \dot{x}^2 + \frac{Ze^2}{r}. \quad (5.5)$$

Of course, bound and continuous states can exist, depending on the sign of Z . The path integral is given by

$$K(x'', x'; T) = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left[\frac{i}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \dot{x}^2 + \frac{Ze^2}{r} \right) dt \right]. \quad (5.6)$$

Discussion of this path integral are due to Duru and Kleinert [27,28] and Inomata [58]. However, the lattice-formulation is not trivial for the $1/r$ -term. In fact, it is too singular for a path integral, respectively, a stochastic process and some regularization must be found. This is known for some time, and it turns out that the Kustaanheimo-Stiefel transformation does the job.

As it turns out one must perform a space-time transformation in order to solve the path integral (5.6). We perform first the transformation

$$x_1 = \xi^2 - \eta^2, \quad x_2 = 2\xi\eta, \quad u \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbf{R}^2, \quad (5.7)$$

which casts the original Lagrangian into the form

$$\mathcal{L}(u, \dot{u}) = 2m(\xi^2 + \eta^2)(\dot{\xi}^2 + \dot{\eta}^2) + \frac{Z e^2}{\xi^2 + \eta^2}. \quad (5.8)$$

The metric tensor (g_{ab}) and it's inverse (g^{ab}) are given by

$$(g_{ab}) = 4(\xi^2 + \eta^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (g^{ab}) = \frac{1}{4(\xi^2 + \eta^2)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (5.9)$$

with determinant $g = \det(g_{ab}) = 16(\xi^2 + \eta^2)^2$ and $dV = dx_1 dx_2 = 4(\xi^2 + \eta^2) d\xi d\eta$. The hermitean momenta corresponding to the scalar product

$$(\Psi_1, \Psi_2) = 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi d\eta (\xi^2 + \eta^2) \Psi_1(\xi, \eta) \Psi_2^*(\xi, \eta) \quad (5.10)$$

are given by

$$p_\xi = \frac{\hbar}{i} \left(\frac{\partial}{\partial \xi} + \frac{\xi}{\xi^2 + \eta^2} \right), \quad p_\eta = \frac{\hbar}{i} \left(\frac{\partial}{\partial \eta} + \frac{\eta}{\xi^2 + \eta^2} \right). \quad (5.11)$$

Following our general theory of Chapter II.5 we start by considering the Legendre transformed Hamiltonian

$$H_E = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) - \frac{Z e^2}{r} - E. \quad (5.12)$$

which gives

$$\hat{H}_E = -\frac{\hbar^2}{2m} \frac{1}{4(\xi^2 + \eta^2)} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) - \frac{Z e^2}{\xi^2 + \eta^2} - E. \quad (5.13)$$

Therefore the time-transformation is given by $\epsilon = f(\xi, \eta)\delta$ with $f(\xi, \eta) = 4(\xi^2 + \eta^2)$ and the space-time transformed Hamiltonian has the form

$$\begin{aligned} \tilde{H} &= -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) - 4Z e^2 - 4E(\xi^2 + \eta^2) \\ &= \frac{1}{2m} (p_\xi^2 + p_\eta^2) - 4Z e^2 - 4E(\xi^2 + \eta^2) \end{aligned} \quad (5.14)$$

with the momentum operators and vanishing quantum potential ΔV :

$$p_\xi = \frac{\hbar}{i} \frac{\partial}{\partial \xi}, \quad p_\eta = \frac{\hbar}{i} \frac{\partial}{\partial \eta}, \quad \Delta V = 0. \quad (5.15)$$

This two-dimensional transformation can be interpreted as a two-dimensional Kustaanheimo-Stiefel transformation [66]. It has the general feature that it performs a change of variables to “square-root” coordinates, note $r = \xi^2 + \eta^2$. The general properties of transformations like this states the following problem (see e.g. [28, 46] for some review of the relevant literature): For which values of D does exist a formula

$$(x_1^2 + \cdots + x_D^2)(y_1^2 + \cdots + y_D^2) = z_1^2 + \cdots + z_D^2, \quad (5.16)$$

where the z_i are homogeneous bilinear forms in x and y . Following a theorem of Hurwitz and Lam this type of transformation can only be realized in the space-dimensions $D = 1, 2, 4, 8$. The assumption about z now implies that

$$z = B(x) \cdot y \quad (5.17)$$

$$z_i = \sum_{k,l} (B^k)_i^l x_k y_l = \sum_l B(x)_i^l y_l. \quad (5.18)$$

For the special case $x = y$ one has

$$\sum_i z_i^2 = \left(\sum_j x_j^2 \right)^2 \quad (5.19)$$

and the matrix B satisfies the condition

$${}^t B(x) B(x) = |x|^2. \quad (5.20)$$

The one-dimensional case is “trivial”, but we shall use it in the calculation of the path integral for the $1/r$ -potential in polar coordinates. The $D = 2$ case we have just encountered. The four-dimensional variation of this transformation has been used a long time ago in astronomy for the purpose of regularizing the Kepler problem. The $D = 8$ case is degenerate. Here the matrix B has the form:

$$B(x) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ -x_2 & x_1 & x_4 & -x_3 & x_6 & -x_5 & x_8 & -x_7 \\ -x_3 & -x_4 & x_1 & x_2 & x_7 & -x_8 & -x_5 & x_6 \\ -x_4 & x_3 & -x_2 & x_1 & -x_8 & -x_7 & x_6 & x_5 \\ -x_5 & -x_6 & -x_7 & x_8 & x_1 & x_2 & x_3 & -x_4 \\ -x_6 & x_5 & x_8 & x_7 & -x_2 & x_1 & -x_4 & -x_3 \\ -x_7 & -x_8 & x_5 & -x_6 & -x_3 & x_4 & x_1 & x_2 \\ -x_8 & x_7 & -x_6 & -x_5 & x_4 & x_3 & -x_2 & x_1 \end{pmatrix} \quad (5.21)$$

and maps $\mathbf{R}^8 \rightarrow \mathbf{R}$ and is thus of no further use.

In the next Section (hydrogen-atom) we need the $D = 4$ case which is more involved and not one-to-one.

To incorporate the time transformation

$$s(t) = \int_{t'}^t \frac{d\sigma}{4r(\sigma)}, \quad s'' = s(t'') \quad (5.22)$$

and its lattice definition $\epsilon \rightarrow 4\bar{r}^{(j)} \Delta s^{(j)} = 4\bar{u}^{(j)2} \delta^{(j)}$ into the path integral (5.6) we now use from the set of equations (II.4.35), namely equation (II.4.35a). Observing that in the path integral (5.6) the measure changes according to

$$\prod_{j=1}^N \left(\frac{m}{2\pi i \hbar \Delta t^{(j)}} \right) \times \prod_{j=1}^{N-1} dx_1^{(j)} dx_2^{(j)} = \frac{1}{r''} \prod_{j=1}^N \left(\frac{m}{2\pi i \hbar \delta^{(j)}} \right) \times \prod_{j=1}^{N-1} d\xi^{(j)} d\eta^{(j)} \quad (5.23)$$

we arrive at the path integral transformations equations ($u \in \mathbf{R}^2$):

$$K(x'', x'; T) = \frac{1}{2\pi i \hbar} \int_{-\infty}^{\infty} dE e^{-iTE/\hbar} G(x'', x'; E) \quad (5.24)$$

$$G(x'', x'; E) = i \int_0^{\infty} ds'' \left[\tilde{K}(u'', u'; s'') + \tilde{K}(-u'', u'; s'') \right] \quad (5.25)$$

where the space-time transformed path integral \tilde{K} is given by

$$\begin{aligned} \tilde{K}(u'', u'; s'') &= e^{4iZ e^2 s''/\hbar} \int \mathcal{D}\xi(s) \int \mathcal{D}\eta(s) \\ &\times \exp \left\{ \frac{i}{\hbar} \int_0^{s''} \left[\frac{m}{2} (\dot{\xi}^2 + \dot{\eta}^2) + 4E(\xi^2 + \eta^2) \right] ds \right\}. \end{aligned} \quad (5.26)$$

Note that the factor $1/r''$ has been exactly canceled by means of equation (II.4.35a). This is a specific feature of this potential problem. Furthermore we have taken into account that our mapping is of the “square-root” type which gives rise to a sign ambiguity. “Thus, if one considers all paths in the complex $x = x_1 + ix_2$ -plane from x' to x'' , they will be mapped into two different classes of paths in the u -plane: Those which go from u' to u'' and those going from u' to $-u''$. In the cut complex x -plane for the function $u = \sqrt{x^2}$ these are the paths passing an even or odd number of times through the square root from $x = 0$ and $x = -\infty$. We may choose the u' corresponding to the initial x' to lie on the first sheet (i.e. in the right half u -plane). The final u'' can be in the right as well as the left half-plane and all paths on the x -plane go over into paths from u' to u'' and those from u' to $-u''$ [28]”. Thus the two contributions arise in equation (5.25). Equation (5.26) is interpreted as the path integral of a two-dimensional isotropic harmonic oscillator with frequency $\omega = \sqrt{-8E/m}$. Therefore we get

$$\tilde{K}(u'', u'; s'') = \frac{m\omega}{2i\hbar \sin \omega s''} \exp \left\{ \frac{4iZ e^2 s''}{\hbar} - \frac{m\omega}{2\pi i \hbar} \left[(u'^2 + u''^2) \cot \omega s'' - 2 \frac{u' \cdot u''}{\sin \omega s''} \right] \right\}. \quad (5.27)$$

Introducing now two-dimensional polar-coordinates

$$\xi = \sqrt{r} \cos \frac{\phi}{2}, \quad \eta = \sqrt{r} \sin \frac{\phi}{2}, \quad (r > 0, \phi \in [0, 2\pi]) \quad (5.28)$$

so that $u' \cdot u'' = \sqrt{r'r''} \cos(\phi'' - \phi)/2$. Thus

$$G(x'', x'; E) = \frac{m\omega}{\pi\hbar} \int_0^\infty \frac{ds''}{\sin \omega s''} \times \exp \left[\frac{4iZ e^2}{\hbar} s'' - \frac{m\omega}{2i\hbar} (r' + r'') \right] \cosh \left(\frac{m\omega \sqrt{r'r''}}{i\hbar \sin \omega s''} \cos \frac{\phi'' - \phi'}{2} \right) \quad (5.29)$$

Using the expansion

$$\cos \left(z \cos \frac{\phi}{2} \right) = \sum_{l=-\infty}^{\infty} e^{il\phi} I_{2l}(z) \quad (5.30)$$

we get for the kernel $G(x'', x'; E)$:

$$G(x'', x'; E) = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} e^{il(\phi'' - \phi')} G_l(r'', r'; E), \quad (5.31)$$

where the radial kernel $G_l(E)$ is given by

$$\begin{aligned} G_l(r'', r'; E) &= \frac{2m\omega}{\hbar} \int_0^\infty \frac{ds''}{\sin \omega s''} \exp \left[\frac{4iZ e^2 s''}{\hbar} - \frac{m\omega}{2i\hbar} (r' + r'') \cot \omega s'' \right] I_{2l} \left(\frac{m\omega \sqrt{r'r''}}{i\hbar \sin \omega s''} \right) \\ (\text{Set } \omega &= \frac{2ip\hbar}{m}, \text{ substitution } u = \frac{2p\hbar s''}{m} \text{ and Wick-rotation}) \\ &= \frac{2m}{\hbar} \int_0^\infty \frac{du}{\sinh u} \exp \left[\frac{2i}{ap} u + ip(r' + r'') \coth u \right] I_{2l} \left(\frac{2p\sqrt{r'r''}}{i \sinh u} \right) \\ (\text{Substitution } \sinh u &= 1/\sinh v) \\ &= \frac{2m}{\hbar} \int_0^\infty \left(\coth \frac{v}{2} \right)^{\frac{2}{ap}} \exp [ip(r' + r'') \cosh v] I_{2l}(-2ip\sqrt{r'r''} \sinh v) dv \\ (\text{Reinsert } \hbar p &= \sqrt{2mE}) \\ &= \frac{1}{\sqrt{r'r''}} \sqrt{-\frac{m}{2E}} \frac{1}{(2l)!} \Gamma \left(\frac{1}{2} + l - \frac{Z e^2}{\hbar} \sqrt{-\frac{m}{2E}} \right) \\ &\quad \times W_{\frac{Z e^2}{\hbar} \sqrt{-\frac{m}{2E}}, l} \left(\sqrt{-\frac{8mE}{\hbar^2}} r_{>} \right) M_{\frac{Z e^2}{\hbar} \sqrt{-\frac{m}{2E}}, l} \left(\sqrt{-\frac{8mE}{\hbar^2}} r_{<} \right). \quad (5.32) \end{aligned}$$

In the last step we have used (2.22) and $r_{>}, r_{<}$ denotes the larger (smaller) of r', r'' , respectively. The representation (5.32) shows that $G_l(E)$ has in the complex energy-plane poles which are given at the negative integers $n_l = 0, -1, -2, \dots$ at the argument

of the Γ -function and a cut on the real axis with a branch point at $E = 0$. Thus we get a discrete and continuous spectrum, respectively:

$$E_N = -\frac{mZ^2 e^4}{2\hbar^2(N - \frac{1}{2})^2}, \quad (N = 1, 2, \dots) \quad (5.33)$$

$$E_p = \frac{\hbar^2 p^2}{2m}, \quad (p \in \mathbf{R}). \quad (5.34)$$

Here we have introduced the principle quantum number $N := n_l + l + 1$. This is the well-known result.

To determine the discrete spectrum we consider the first line in equation (5.32), identify in the Hille-Hardy formula $t = e^{-2u}$, $x = m\omega r'$ and $y = m\omega r''$ and thus get

$$\begin{aligned} G_l(r'', r'; E) &= \frac{m}{\hbar} \sum_{n=0}^{\infty} \frac{(-2ip\sqrt{r'r''})^{2l}}{n+l+\frac{1}{2}-\frac{i}{ap}} \\ &\times \frac{n!}{(2l+n)!} \exp[-ip(r'+r'')] L_n^{(2l)}(-2ipr') L_n^{(2l)}(-2ipr''). \end{aligned} \quad (5.35)$$

Taking equation (5.35) and the n^{th} residuum gives the energy levels and the wave functions. Inserting into equation (5.29) thus yields the Green functions for the discrete levels for the two-dimensional $1/r$ -potential

$$G(x'', x'; E) = \hbar \sum_{N=1}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\Psi_{N,l}(r', \phi') \Psi_{N,l}^*(r'', \phi'')}{E_N - E} \quad (5.36)$$

and the wave functions of the discrete spectrum are given by:

$$\begin{aligned} \Psi_{N,l}(r, \phi) &= \left[\frac{(N-l-1)!}{\pi a^2 (N-\frac{1}{2})^3 (N+l-1)!} \right]^{\frac{1}{2}} \left(\frac{2r}{a(N-\frac{1}{2})} \right)^l \\ &\times \exp \left[-\frac{r}{a(N-\frac{1}{2})} - il\phi \right] L_{N-l-1}^{(2l)} \left(\frac{2r}{a(N-\frac{1}{2})} \right). \end{aligned} \quad (5.37)$$

Furthermore we have used the abbreviation $a = \hbar^2/(mZ e^2)$ (the first ‘‘Bohr-radius’’). The correct normalization to unity can be seen from the relation

$$\int_0^{\infty} x^{2l+\lambda+2} e^{-x} \left[L_{N-l-1}^{(2l+\lambda+1)}(x) \right]^2 dx = \frac{2(N+\frac{\lambda}{2})(N+l+\lambda)!}{(N-l-1)!}, \quad (5.38a)$$

respectively

$$\int_0^{\infty} x^{\mu+a} e^{-x} \left[L_n^{(\mu)}(x) \right]^2 dx = \frac{(2n+\mu+1)\Gamma(n+\mu+1)}{n!}. \quad (5.38b)$$

In order to determine the continuous wave functions we use the dispersion relation [40, p.987]

$$\begin{aligned} \int_{-\infty}^{\infty} dx e^{-2ix\rho} \Gamma\left(\frac{1}{2} + \nu + ix\right) \Gamma\left(\frac{1}{2} + \nu - ix\right) M_{ix,\nu}(\alpha) M_{ix,\nu}(\beta) \\ = \frac{2\pi\sqrt{\alpha\beta}}{\cosh\rho} \exp\left[-(\alpha + \beta) \tanh\rho\right] J_{2\nu}\left(\frac{2\sqrt{\alpha\beta}}{\cosh\rho}\right). \end{aligned} \quad (5.39)$$

and get ($\hbar\tilde{p} = \sqrt{2mE}$)

$$\begin{aligned} G_l(r'', r'; E) \\ = 2i\tilde{p} \int_0^{\infty} \frac{ds''}{\sin\omega s''} \exp\left[\frac{4iZ e^2}{\hbar} s'' - \tilde{p}(r' + r'') \cot\omega s''\right] I_{2l}\left(\frac{2\tilde{p}\sqrt{r'r''}}{\sin\omega s''}\right) \\ = \frac{i}{\pi\sqrt{r'r''}} \int_0^{\infty} ds'' \exp\left(\frac{4iZ e^2}{\hbar} s''\right) \\ \times \int_0^{\infty} \frac{|\Gamma(\frac{1}{2} + l + ip)|^2}{\Gamma^2(2l + 1)} \exp\left(-\frac{4ip\tilde{p}\hbar s''}{m} + \pi p\right) M_{ip,l}(-2i\tilde{p}r') M_{-ip,l}(2i\tilde{p}r'') dp \\ = \hbar \int_0^{\infty} \frac{dp}{\hbar^2 p^2/2m - E} \exp\left(\frac{\pi}{ap}\right) \\ \times \frac{\Gamma(\frac{1}{2} + l + \frac{i}{ap}) \Gamma(\frac{1}{2} + l - \frac{i}{ap})}{2\pi\sqrt{r'r''}(2l)!} M_{-\frac{i}{ap},l}(2ipr') M_{\frac{i}{ap},l}(-2ipr''). \end{aligned} \quad (5.40)$$

Here again the residuum at $E = \hbar^2 p^2/2m$ has been taken in the integral. Thus the wave functions of the continuous spectrum are given by ($\hbar p = \sqrt{2mE}$):

$$\Psi_{p,l}(r, \phi) = \sqrt{\frac{1}{4\pi^2 r}} \frac{1}{(2l)!} \Gamma\left(\frac{1}{2} + l + \frac{i}{ap}\right) \exp\left(\frac{\pi}{2ap} - i l \phi\right) M_{\frac{i}{ap},l}(-2ipr). \quad (5.41)$$

Note that

$$\left|\Gamma\left(\frac{1}{2} + l + \frac{i}{ap}\right)\right| M_{\frac{i}{ap},l}(-2ipr) = \left|\Gamma\left(\frac{1}{2} + l + \frac{i}{ap}\right)\right| M_{-\frac{i}{ap},l}(2ipr) \quad (5.42)$$

and the wave-functions are therefore basically real. These results of the $1/r$ potential for the discrete and continuous spectrum, respectively, are equivalent with the results of the operator approach. The complete Feynman kernel therefore reads

$$\begin{aligned} K(x'', x'; T) = \sum_{l=0}^{\infty} \sum_{N=1}^{\infty} e^{-iTE_N/\hbar} \Psi_{N,l}^*(r', \phi') \Psi_{N,l}(r'', \phi'') \\ + \sum_{l=0}^{\infty} \int_0^{\infty} dp e^{-iTE_p/\hbar} \Psi_{p,l}^*(r', \phi') \Psi_{p,l}(r'', \phi'') \end{aligned} \quad (5.43)$$

with wave functions as given in equations (5.37) and (5.41) and energy spectrum equations (5.33, 5.34).

5.2. The $1/r$ -Potential in \mathbf{R}^3 - The Hydrogen Atom [27, 28, 41, 55, 57, 58]

We consider the Euclidean three-dimensional space with the singular potential $V(r) = -Z e^2 / r$ ($r = |x|$, $x \in \mathbf{R}^3$) with Z and e^2 as in the previous Section. This potential corresponds in the space \mathbf{R}^3 , of course, to the potential of a point charge and is thus the kernel of the Poisson equation in \mathbf{R}^3 . The classical Lagrangian has the form

$$\mathcal{L}(x, \dot{x}) = \frac{m}{2} \dot{x}^2 + \frac{Z e^2}{r}. \quad (5.44)$$

The path integral is given by

$$K(x'', x'; T) = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left[\frac{i}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \dot{x}^2 + \frac{Z e^2}{r} \right) dt \right]. \quad (5.45)$$

This path integral was first successfully solved (actually the Green function or resolvent kernel) by Duru and Kleinert [27], followed by further contributions of Inomata [58] and Duru and Kleinert [28], Ho and Inomata [55] Grinberg, Marañón and Vucetich [41,42] and Kleinert [62].

The transformation in the two-dimensional case corresponds to the two-dimensional realization of the Kustaanheimo-Stiefel transformation and we must look for a generalization. However, we have a the relevant space \mathbf{R}^3 whereas the relevant Kustaanheimo-Stiefel transformation maps $\mathbf{R}^4 \rightarrow \mathbf{R}^3$ which means that the transformation in question is not one-to-one. We have namely [28]

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u_3 & u_4 & u_1 & u_2 \\ -u_2 & -u_1 & u_4 & u_3 \\ -u_1 & u_2 & u_3 & -u_4 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \quad (5.46)$$

or in matrix notation $x = A \cdot u$ with $x \in \mathbf{R}^3$ and $u \in \mathbf{R}^4$, respectively. Several modifications of the matrix A are used in the literature, e.g. [10,11]

$$A = \begin{pmatrix} u_3 & u_4 & u_1 & u_2 \\ -u_4 & u_3 & u_2 & -u_1 \\ -u_1 & -u_2 & u_3 & u_4 \\ -u_2 & u_1 & -u_4 & -u_3 \end{pmatrix}, \quad (5.46b)$$

and see [66] as well. The transformation has the property $A^t A = u_1^2 + u_2^2 + u_3^2 + u_4^2 = r = |x|$ and no Jacobean exist. However, a fourth coordinate can be described by

$$x_4(s) = 2 \int_{s'}^s (u_4 u_1' - u_3 u_2' + u_2 u_3' - u_1 u_4') d\sigma. \quad (5.47)$$

Therefore we must circumvent this problem. The idea [27,58] goes at follows: We consider the lattice formulation of equation (5.45) and introduce a factor “one” by

means of

$$\begin{aligned} 1 &= \sqrt{\frac{m}{2\pi i \hbar T}} \int_{-\infty}^{\infty} d\xi'' \exp \left[-\frac{m}{2i\hbar T} (\xi'' - \xi')^2 \right] \\ &= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{N}{2}} \prod_{j=1}^N \int_{-\infty}^{\infty} d\xi^{(j)} \exp \left[\frac{im}{2\hbar \epsilon} \Delta^2 \xi^{(j)} \right] \end{aligned} \quad (5.48)$$

which gives for equation (5.45)

$$\begin{aligned} K(x'', x'; T) &= \int_{-\infty}^{\infty} d\xi'' \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{2N} \prod_{j=1}^{N-1} \int d^3 x^{(j)} \int_{-\infty}^{\infty} d\xi^{(j)} \\ &\quad \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} (\Delta^2 x^{(j)} + \Delta^2 \xi^{(j)}) + \epsilon \frac{Z e^2}{\bar{r}^{(j)}} \right] \right\} \end{aligned} \quad (5.49)$$

with the to-be-defined quantity $\bar{r}^{(j)}$. We now realize the transformation $A(u)$ on midpoints [58] $\bar{u}_a^{(j)} = \frac{1}{2}(u_a^{(j)} + u_a^{(j-1)})$ (we use in the following the abbreviation $x_4 \equiv \xi$):

$$\Delta x_a^{(j)} = 2 \sum_{b=1}^4 A^{ab}(\bar{u}^{(j)}) \Delta u_b^{(j)} \quad (5.50)$$

We have

$$A^t(\bar{u}^{(j)}) \cdot A(\bar{u}^{(j)}) = \bar{u}_1^{(j)2} + \bar{u}_2^{(j)2} + \bar{u}_3^{(j)2} + \bar{u}_4^{(j)2} =: \bar{r}^{(j)} \quad (5.51)$$

which defines $\bar{r}^{(j)}$. Furthermore

$$\begin{aligned} \Delta^2 x_1^{(j)} + \Delta^2 x_2^{(j)} + \Delta^2 x_3^{(j)} + \Delta^2 x_4^{(j)} \\ = 4\bar{r}^{(j)} [\Delta^2 u_1^{(j)} + \Delta^2 u_2^{(j)} + \Delta^2 u_3^{(j)} + \Delta^2 u_4^{(j)}] =: 4\bar{r}^{(j)} \Delta^2 u^{(j)}. \end{aligned} \quad (5.52)$$

Infinitesimal this has the form $dx_a^{(j)} = 2A^{ab}(\bar{u}^{(j)}) du_a^{(j)}$ (sums over repeated indices understood) and the Jacobean of this transformation exists and is given by

$$\frac{\partial(x_1, x_2, x_3, \xi)}{\partial(u_1, u_2, u_3, u_4)} = 2^4 \bar{r}^{(j)2}. \quad (5.53)$$

Thus we arrive at the transformed classical Lagrangian

$$\mathcal{L}(u, \dot{u}) = 2mu^2 \dot{u}^2 + \frac{Z e^2}{u^2}. \quad (5.54)$$

We now repeat the reasoning of the previous Section by observing that under the time-transformation equation (5.22)

$$s(t) = \int_{t'}^t \frac{d\sigma}{4r(\sigma)}, \quad s'' = s(t'')$$

and its lattice definition $\epsilon \rightarrow 4\bar{r}^{(j)}\Delta s^{(j)} = 4\bar{u}^{(j)2}\delta^{(j)}$ the measure in the path integral transforms according to

$$\prod_{j=1}^N \left(\frac{m}{2\pi i \epsilon \hbar} \right)^2 \times \prod_{j=1}^{N-1} (2\bar{u}^{(j)})^4 d^4 u^{(j)} = \frac{1}{(4r'')^2} \prod_{j=1}^N \left(\frac{m}{2\pi i \hbar \delta^{(j)}} \right)^2 \times \prod_{j=1}^{N-1} d^4 u^{(j)}. \quad (5.55)$$

Respecting again equation (II.4.35a) which produces a factor $4r''$ we get the path integral transformation

$$\begin{aligned} K(x'', x'; T) &= \frac{1}{2\pi i \hbar} \int_{-\infty}^{\infty} dE e^{-iTE/\hbar} G(x'', x'; E) \\ G(x'', x'; E) &= i \int_0^{\infty} ds'' e^{4iZ e^2 s''/\hbar} \tilde{K}(u'', u'; s'') \end{aligned} \quad (5.56)$$

where the space-time transformed path integral \tilde{K} is given by

$$\begin{aligned} &\tilde{K}(u'', u'; s'') \\ &= \frac{1}{4r''} \int_{-\infty}^{\infty} d\xi'' \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \hbar \delta} \right)^{2N} \\ &\quad \times \prod_{j=1}^{N-1} \int d^4 u^{(j)} \exp \left[\frac{i}{\hbar} \sum_{j=1}^N \left(\frac{m}{2\delta} \Delta^2 u^{(j)} + 4\delta E \bar{u}^{(j)2} \right) \right] \\ &= \left(\frac{m\omega}{2\pi i \hbar \sin \omega s''} \right)^2 \int_{-\infty}^{\infty} \frac{d\xi''}{4r''} \exp \left\{ -\frac{m\omega}{2i\hbar} \left[(u'^2 + u''^2) \cot \omega s'' - 2 \frac{u' \cdot u''}{\sin \omega s''} \right] \right\}. \end{aligned} \quad (5.57)$$

Here we have used once again the known solution for the harmonic oscillator, where we have actually a four-dimensional isotropic harmonic oscillator with frequency $\omega = \sqrt{-8E/m}$. Note that the factor $1/r''$ has not been canceled as in the two-dimensional case.

Up to now we have not discussed the problem of an eventually quantum correction appearing in the transformation procedure. As usual we consider the Legendre transformed Hamiltonian

$$H_E = -\frac{\hbar^2}{2m} \Delta_{(3)} - \frac{Z e^2}{r} - E. \quad (5.58)$$

Let us write H_E in the coordinates:

$$\left. \begin{aligned} u_1 &= u \cos \alpha \cos \beta \\ u_2 &= u \cos \alpha \sin \beta \\ u_3 &= u \sin \alpha \cos \gamma \\ u_4 &= u \sin \alpha \sin \gamma \end{aligned} \right\} \begin{aligned} (u = |u| = \sqrt{r}, \alpha = \frac{\theta}{2}, \beta \pm \gamma = \phi) \\ (0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi) \end{aligned} \quad (5.59)$$

Then the Schrödinger equation in polar coordinates

$$\left[-\frac{\hbar^2}{2} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} L^2 \right) - \frac{Z e^2}{r} - E \right] \Psi(r, \theta, \phi) = 0 \quad (5.60)$$

is transformed into

$$\left[-\frac{\hbar^2}{2} \frac{1}{4s^2} \left(\frac{\partial^2}{\partial s^2} + \frac{3}{s} \frac{\partial}{\partial s} - 4K^2 \right) - \frac{Z e^2}{s^2} - E \right] \Psi(u) = 0 \quad (5.61)$$

with

$$K^2 = -\frac{\partial^2}{\partial \alpha^2} - (\tan \alpha - \cot \alpha) \frac{\partial}{\partial \alpha} - \frac{1}{\sin^2 \alpha} \frac{\partial^2}{\partial \beta^2} - \cos^2 \alpha \frac{\partial^2}{\partial \gamma^2} \quad (5.62)$$

is the Casimir-operator of the group $\text{SO}(4)$. Writing back into u_1, \dots, u_4 we get for H_E in the coordinates $q \in \mathbf{R}^4$:

$$\tilde{H} = 4u^2 \hat{H}_E = -\frac{1}{2m} \Delta_4 - 4Z e^2 - 4Eu^2$$

and with $p_{u_k} = -i\hbar \partial / \partial u_k$:

$$H_{eff}(p_u, q) = \frac{1}{2m} \sum_{k=1}^4 p_{u_k}^2 - 4Z e^2 - 4Eu^2. \quad (5.63)$$

and no quantum correction appears. Therefore equation (5.57) is correct.

To perform the ξ'' -integration in equation (5.57) we introduce polar coordinates

$$\begin{aligned} u_1 &= \sqrt{r} \sin \frac{\theta}{2} \cos \frac{\alpha + \phi}{2} \\ u_2 &= \sqrt{r} \sin \frac{\theta}{2} \sin \frac{\alpha + \phi}{2} \\ u_3 &= \sqrt{r} \cos \frac{\theta}{2} \cos \frac{\alpha - \phi}{2} \\ u_4 &= \sqrt{r} \cos \frac{\theta}{2} \sin \frac{\alpha - \phi}{2} \end{aligned} \quad \left(\begin{array}{l} 0 \leq \theta \leq \pi \\ 0 \leq \phi \leq 2\pi \\ 0 \leq \alpha \leq 4\pi \end{array} \right). \quad (5.64)$$

We have with equation (5.47)

$$\xi(s) = -\int_0^s (\alpha' - \cos \theta \phi') r(\sigma) d\sigma. \quad (5.65)$$

The polar coordinates give the expansion

$$\begin{aligned} u' \cdot u'' &= \sqrt{r' r''} \\ &\times \left[\sin \frac{\theta'}{2} \sin \frac{\theta''}{2} \cos \left(\frac{\alpha'' - \alpha' + \phi'' - \phi'}{2} \right) + \cos \frac{\theta'}{2} \cos \frac{\theta''}{2} \cos \left(\frac{\alpha' - \alpha'' + \phi'' - \phi'}{2} \right) \right] \end{aligned} \quad (5.66)$$

Using the expansion

$$e^{z \cos \phi} = \sum_{\nu=-\infty}^{\infty} e^{i\nu\phi} I_{\nu}(z) \quad (5.67)$$

twice, respecting due to the explicit form of $\xi(s)$ from equation (5.65) above

$$\int_{-\infty}^{\infty} \frac{d\xi(s'')}{r''} = \int_0^{4\pi} d\alpha \quad (5.68)$$

and that the integration over α'' yields $4\pi\delta_{\nu_1\nu_2}$ we get for the resolvent kernel:

$$\begin{aligned} G(x'', x'; E) &= \frac{i}{\pi} \int_0^{\infty} ds'' \left(\frac{m\omega}{2i\hbar \sin \omega s''} \right)^2 \exp \left[\frac{4iZ e^2 s''}{\hbar} - \frac{m\omega}{2i\hbar} (r' + r'') \cot \omega s'' \right] \\ &\times \sum_{\nu=-\infty}^{\infty} e^{i\nu(\phi'' - \phi')} I_{\nu} \left(\frac{m\omega\sqrt{r'r''}}{i\hbar \sin \omega s''} \sin \frac{\theta'}{2} \sin \frac{\theta''}{2} \right) I_{\nu} \left(\frac{m\omega\sqrt{r'r''}}{i\hbar \sin \omega s''} \cos \frac{\theta'}{2} \cos \frac{\theta''}{2} \right). \end{aligned} \quad (5.69)$$

The last sum can be exactly performed with the help of

$$\left(\frac{z_1 - z_2 e^{-i\phi}}{z_1 - z_2 e^{i\phi}} \right)^{\frac{\nu}{2}} I_{\nu} \left(\sqrt{z_1^2 + z_2^2 - 2z_1 z_2 \cos \phi} \right) = \sum_{n=-\infty}^{\infty} (-1)^n I_n(z_2) I_{\nu+n}(z_1) e^{in\phi}, \quad (5.70)$$

in particular

$$\sum_{\nu=-\infty}^{\infty} e^{i\nu\phi} I_{\nu}(z) I_{\nu}(z') = I_0 \left(\sqrt{z^2 + z'^2 + 2zz' \cos \phi} \right), \quad (5.71)$$

and the trigonometric expressions $\sin^2 \frac{\alpha}{2} = \frac{1}{2}(1 - \cos \alpha)$ and $\cos^2 \frac{\alpha}{2} = \frac{1}{2}(1 + \cos \alpha)$. Thus

$$\begin{aligned} G(x'', x'; E) &= \frac{i}{\pi} \int_0^{\infty} ds'' \left(\frac{m\omega}{2i\hbar \sin \omega s''} \right)^2 \\ &\times \exp \left[\frac{4iZ e^2 s''}{\hbar} - \frac{m\omega}{2i\hbar} (r' + r'') \cot \omega s'' \right] I_0 \left(\frac{m\omega\sqrt{r'r''}}{i\hbar \sin \omega s''} \cos \frac{\gamma}{2} \right), \end{aligned} \quad (5.72)$$

where $\cos \gamma = \cos \theta' \cos \theta'' + \sin \theta' \sin \theta'' \cos(\phi'' - \phi')$. The I_0 -Bessel function can be expanded in terms of Legendre-polynomials:

$$I_0 \left(z \cos \frac{\gamma}{2} \right) = \frac{2}{z} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \gamma) I_{2l+1}(z). \quad (5.73)$$

This relation can be derived from the general expansion

$$\left(\frac{kz}{2} \right)^{\mu-\nu} I_{\nu}(kz) = k^{\mu} \sum_{l=0}^{\infty} \frac{\Gamma(\mu+l)}{l!\Gamma(1+\nu)} (-1)^l (2l+\mu) {}_2F_1(-l, l+\mu; 1+\nu; k^2) I_{2l+\mu}(z). \quad (5.74)$$

Equation (5.73) is recovered with the identifications $\mu = 1$, $\nu = 0$, $k = \cos \frac{\gamma}{2}$ and $P_l(z) = (-1)^l {}_2F_1(-l, l+1; 1; \frac{z+1}{2})$. Thus the resolvent kernel can be expanded according to

$$\begin{aligned} G(x'', x'; E) &= G(r'', \theta'', \phi'', r', \theta', \phi'; E) \\ &= \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \gamma) G_l(r'', r'; E) \end{aligned} \quad (5.75)$$

$$= \sum_{l=0}^{\infty} \sum_{n=-l}^l Y_l^{n*}(\theta', \phi') Y_l^n(\theta'', \phi'') G_l(r'', r'; E), \quad (5.76)$$

where the radial Green function $G_l(E)$ is given by

$$\begin{aligned} G_l(r'', r'; E) &= \frac{2m\omega}{\hbar\sqrt{r'r''}} \int_0^{\infty} \frac{ds''}{\sin \omega s''} \exp \left[\frac{4iZe^2 s''}{\hbar} - \frac{m\omega}{2i\hbar} (r' + r'') \cot \omega s'' \right] I_{2l+1} \left(\frac{m\omega\sqrt{r'r''}}{i\hbar \sin \omega s''} \right) \\ &= \frac{1}{r'r''} \sqrt{-\frac{m}{2E}} \frac{1}{(2l+1)!} \Gamma \left(1+l + \frac{iZe^2}{\hbar} \sqrt{\frac{m}{2E}} \right) \\ &\quad \times W_{\frac{Ze^2}{\hbar} \sqrt{-\frac{m}{2E}}, l+\frac{1}{2}} \left(\sqrt{\frac{-8mE}{\hbar^2}} r > \right) M_{\frac{Ze^2}{\hbar} \sqrt{-\frac{m}{2E}}, l+\frac{1}{2}} \left(\sqrt{\frac{-8mE}{\hbar^2}} r < \right). \end{aligned} \quad (5.77)$$

From the poles of the Γ -function at $z = -n_r = 0, -1, -2, \dots$ we read off the bound state energy levels which are given by

$$E_N = -\frac{mZ^2 e^4}{2\hbar^2 N^2}, \quad (N = n_r + l + 1 = 1, 2, 3, \dots) \quad (5.78)$$

which are the well-known Balmer levels.

The determination of the wave functions is analogous to the previous two-dimensional case and thus we only state the result for the **bound state wave functions of the hydrogen atom**:

$$\Psi_{N,l,n}(r, \theta, \phi) = \frac{2}{N^2} \left[\frac{(N-l-1)!}{a^3(N+l)!} \right]^{\frac{1}{2}} \exp \left(-\frac{r}{aN} \right) \left(\frac{2r}{aN} \right)^l L_{N-l-1}^{(2l+1)} \left(\frac{2r}{aN} \right) Y_l^n(\theta, \phi). \quad (5.79)$$

A quite complicated attempt was made by Grinberg, Marañón and Vucetich [41] to determine these functions from the wave functions of the corresponding four-dimensional oscillator. The **continuous state wave functions of the hydrogen atom** read as:

$$\Psi_{p,l,n}(r, \theta, \phi) = \frac{1}{\sqrt{2\pi}(2l+1)!r} \Gamma \left(1+l + \frac{i}{ap} \right) \exp \left(\frac{\pi}{2ap} \right) M_{\frac{i}{ap}, l+\frac{1}{2}}(-2ipr) Y_l^n(\theta, \phi). \quad (5.80)$$

Of course, these wave functions form a complete set. The spectrum of the continuous states is, of course, given by

$$E_p = \frac{\hbar^2 p^2}{2m}, \quad (p > 0). \quad (5.81)$$

These results coincides with the operator approach. The complete Feynman kernel therefore reads

$$\begin{aligned} K(x'', x'; T) = & \sum_{l=0}^{\infty} \sum_{N=1}^{\infty} e^{-iTE_N/\hbar} \Psi_{N,l}^*(r', \phi') \Psi_{N,l}(r'', \phi'') \\ & + \sum_{l=0}^{\infty} \int_0^{\infty} dp e^{-iTE_p/\hbar} \Psi_{p,l}^*(r', \phi') \Psi_{p,l}(r'', \phi'') \end{aligned} \quad (5.82)$$

with wave functions as given in equations (5.79) and (5.80) and energy spectrum equations (5.78) and (5.81), respectively.

5.3. Coulomb Potential and $1/r$ -Potential in D Dimensions [15, 55, 91]

In this Section we discuss the $1/r$ -potential in D dimensions. For $D = 3$ we have, of course, the Coulomb problem. The whole calculation is quite similar as in the previous cases, we just have to use a Kustaanheimo-Stiefel transformation in one dimension.

The D -dimensional path integral problem was first discussed by Chetouani and Hammann [15], whereas the radial problem by Ho and Inomata [55] and [91].

We start with the D -dimensional path integral with the singular potential $V(r) = -q_1 q_2 / r$, where $r = |x|$ ($x \in \mathbf{R}^D$):

$$K(x'', x'; T) = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left[\frac{i}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \dot{x}^2 + \frac{q_1 q_2}{|x|} \right) dt \right]. \quad (5.83)$$

We denote the coupling by $q_1 \times q_2$ to emphasize the possibility that two ‘‘charges’’ q_1 and q_2 can interact with each other. As we have studied very explicitly in Section III.3.1, the angular variables can be integrated out yielding

$$K(x'', x'; T) = \sum_{l=0}^{\infty} S_l^{(D)}(\{\theta''\}) S_l^{(D)*}(\{\theta'\}) K_l(r'', r'; T), \quad (5.84)$$

where the radial kernel $K_l(T)$ is given by

$$\begin{aligned} & K_l(r'', r'; T) \\ &= (r' r'')^{\frac{1-D}{2}} \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp \left[\frac{i}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \dot{r}^2 - \hbar^2 \frac{(l + \frac{D}{2} - 1)^2 - \frac{1}{4}}{2mr^2} + \frac{q_1 q_2}{r} \right) dt \right] \\ &\equiv (r' r'')^{\frac{1-D}{2}} \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \mu_{l+\frac{D}{2}-1}[r^2] \exp \left[\frac{i}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \dot{r}^2 + \frac{q_1 q_2}{r} \right) dt \right]. \end{aligned} \quad (5.85)$$

The $S_l^{(D)}(\{\theta\})$ are the hyperspherical harmonics on the S^{D-1} -sphere. The corresponding Hamiltonian to the path integral (5.85) has the form

$$\begin{aligned} H &= -\frac{\hbar^2}{2m} \left(\frac{d^2}{dr^2} + \frac{D-1}{r} \frac{d}{dr} \right) + \hbar^2 \frac{l(l+D-2)}{2mr^2} - \frac{q_1 q_2}{r} \\ &= \frac{1}{2m} p_r^2 + \hbar^2 \frac{(l + \frac{D}{2} - 1)^2 - \frac{1}{4}}{2mr^2} - \frac{q_1 q_2}{r} \end{aligned} \quad (5.86)$$

with the momentum operators equation (3.45). We now perform the space-time transformation

$$s(t) = \int_{t'}^t \frac{d\sigma}{4r(\sigma)}, \quad s'' = s(t''), \quad r = F(u) = u^2 \quad (5.87)$$

with $f = 4u^2$, thus that $F'^2 = f$ and we get the space-time formed Hamiltonian

$$H = \frac{1}{2m} p_u^2 + \hbar^2 \frac{(2l + D - 2)^2 - \frac{1}{4}}{2mu^2} - 4Eu^2 - 4q_1 q_2 \quad (5.88)$$

with the momentum operator

$$p_u = \frac{\hbar}{i} \left(\frac{d}{du} + \frac{2D-3}{2u} \right). \quad (5.89)$$

Here we have applied the results of space-time transformations for one-dimensional Hamiltonians. Thus we get the transformation formulæ

$$\begin{aligned} K_l(r'', r'; T) &= \frac{1}{2\pi i \hbar} \int_{-\infty}^{\infty} dE e^{-iTE/\hbar} G_l(r'', r'; E) \\ G_l(r'', r'; E) &= 2i(u'u'')^{\frac{3}{2}-D} \int_0^{\infty} ds'' e^{4iq_1 q_2 s''/\hbar} \tilde{K}_l(u'', u'; s''), \end{aligned} \quad (5.90)$$

where the kernel $\tilde{K}_l(s'')$ is given by

$$\begin{aligned} &\tilde{K}_l(u'', u'; s'') \\ &= \int_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(s) \mu_{2l+D-2}[u^2] \exp \left[\frac{i}{\hbar} \int_0^{s''} \left(\frac{m}{2} \dot{u}^2 + 4Eu^2 \right) ds \right] \\ &= \sqrt{u'u''} \frac{m\omega}{i\hbar \sin \omega T} \exp \left[-\frac{m\omega}{2i\hbar} (u'^2 + u''^2) \cot \omega T \right] I_{2l+D-2} \left(\frac{m\omega u' u''}{i\hbar \sin \omega T} \right), \end{aligned} \quad (5.91)$$

with $\omega = \sqrt{-8E/m}$. The Green function we obtain in the usual way with the result

$$\begin{aligned} &G_l(r'', r'; E) \\ &= (r'r'')^{\frac{2-D}{2}} \frac{2m\omega}{\hbar} \int_0^{\infty} \frac{ds''}{\sin \omega s''} \\ &\quad \times \exp \left[\frac{4iq_1 q_2 s''}{\hbar} - \frac{m\omega}{2\hbar i} (r' + r'') \cot \omega s'' \right] I_{2l+D-2} \left(\frac{m\omega \sqrt{r'r''}}{i\hbar \sin \omega s''} \right) \\ &= (r'r'')^{\frac{1-D}{2}} \sqrt{-\frac{m}{2E}} \frac{\Gamma(l + \frac{D-1}{2} - \frac{q_1 q_2}{\hbar} \sqrt{-\frac{m}{2E}})}{(2l+D-2)!} \\ &\quad \times W_{\frac{q_1 q_2}{\hbar} \sqrt{-\frac{m}{2E}}, l + \frac{D-2}{2}} \left(\sqrt{-\frac{8mE}{\hbar^2}} r > \right) M_{\frac{q_1 q_2}{\hbar} \sqrt{-\frac{m}{2E}}, l + \frac{D-2}{2}} \left(\sqrt{-\frac{8mE}{\hbar^2}} r < \right). \end{aligned} \quad (5.92)$$

Thus it is easy to display the wave functions and spectrum for the discrete contribution for the D -dimensional $1/r$ -problem [$a = \hbar^2/(m|q_1q_2|)$, $n \in \mathbf{N}$]:

$$\begin{aligned} \Psi_{N,l}(r, \{\theta\}) &= \left[\frac{1}{2(N + \frac{D-3}{2})} \frac{(N-l-1)!}{(N+l+D-3)!} \right]^{\frac{1}{2}} \left(\frac{2}{a(N + \frac{D-3}{2})} \right)^{D/2} \left(\frac{2r}{a(N + \frac{D-3}{2})} \right)^l \\ &\quad \times \exp \left[-\frac{r}{a(N + \frac{D-3}{2})} \right] L_{N-l-1}^{(2l+D-2)} \left(\frac{2r}{a(N + \frac{D-3}{2})} \right) S_l^{(D)}(\{\theta\}) \end{aligned} \quad (5.93a)$$

$$\begin{aligned} &= (-1)^{N-l-1} \left(\frac{1}{N + \frac{D-3}{2}} \right) \left[\frac{1}{a(N+l+D-3)!(N-L-1)!} \right]^{\frac{1}{2}} \\ &\quad \times r^{-\frac{D-1}{2}} W_{N+\frac{D-3}{2}, l+\frac{D-2}{2}} \left(\frac{2r}{a(N + \frac{D-3}{2})} \right) S_l^{(D)}(\{\theta\}) \end{aligned} \quad (5.93b)$$

$$E_N = -\frac{mq_1^2q_2^2}{2\hbar^2(N + \frac{D-3}{2})^2}, \quad (N = 1, 2, \dots) \quad (5.94)$$

For the continuous spectrum we get similarly

$$\begin{aligned} \Psi_{p,l}(r, \{\theta\}) &= r^{\frac{1-D}{2}} \frac{\Gamma(l + \frac{D-1}{2} + \frac{i}{ap})}{\sqrt{2\pi}(2l+D-2)!} \\ &\quad \times \exp \left(\frac{\pi mq_1^2q_2^2}{2\hbar^2p} \right) M_{\frac{i}{ap}, l+\frac{D-2}{2}}(-2ipr) S_l^{(D)}(\{\theta\}) \end{aligned} \quad (5.95)$$

with $E_p = \frac{\hbar^2 p^2}{2m}$. The complete Green function thus has the form

$$\begin{aligned} G(x'', x'; E) &= \sum_{N=1}^{\infty} \sum_{l=0}^{\infty} \exp \left[i T \frac{mq_1^2q_2^2}{2\hbar^3(N + \frac{D-3}{2})^2} \right] \Psi_{N,l}^*(r', \{\theta'\}) \Psi_{N,l}(r'', \{\theta''\}) \\ &\quad + \sum_{l=0}^{\infty} \int_0^{\infty} \exp \left(-i \hbar T \frac{p^2}{2m} \right) \Psi_{p,l}^*(r', \{\theta'\}) \Psi_{p,l}(r'', \{\theta''\}). \end{aligned} \quad (5.96)$$

Finally we can state the following path integral identity ($\omega = \sqrt{-8E/m}$):

$$\begin{aligned} &\int \mathcal{D}r(t) \mu_\lambda[r^2] \exp \left[\frac{i}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \dot{r}^2 + \frac{\alpha}{r} \right) dt \right] \\ &= \frac{2m\omega\sqrt{r'r''}}{i\pi\hbar} \int_{-\infty}^{\infty} dE e^{-iTE\hbar} \\ &\quad \times \int_0^{\infty} \frac{ds''}{\sin \omega s''} \exp \left[\frac{4i\alpha s''}{\hbar} - \frac{m\omega}{2i\hbar} (r' + r'') \cot \omega s'' \right] I_{2\lambda} \left(\frac{m\omega\sqrt{r'r''}}{i\hbar \sin \omega s''} \right) \\ &= i\hbar \int_{-\infty}^{\infty} dE e^{-iTE/\hbar} \sqrt{-\frac{m}{2E}} \frac{\Gamma(\lambda + \frac{1}{2} - \frac{q_1q_2}{\hbar} \sqrt{-\frac{m}{2E}})}{\Gamma(2\lambda + 1)} \end{aligned}$$

$$\begin{aligned}
& \times W_{\frac{q_1 q_2}{\hbar} \sqrt{-\frac{m}{2E}}, \lambda} \left(\sqrt{-\frac{8mE}{\hbar^2}} r_{>} \right) M_{\frac{q_1 q_2}{\hbar} \sqrt{-\frac{m}{2E}}, \lambda} \left(\sqrt{-\frac{8mE}{\hbar^2}} r_{<} \right) \\
& = i \hbar \int_0^\infty dE e^{-iTE} \left\{ \sum_{n=0}^\infty \frac{1}{E_n - E} \Psi_n^*(r') \Psi_n(r'') + \int_{-\infty}^\infty \frac{dp}{E_p - E} \Psi_p^*(r') \Psi_p(r'') \right\}
\end{aligned} \tag{5.97}$$

The discrete state wave functions and energy spectrum are given by ($a = \hbar^2/mq_1q_2$)

$$\begin{aligned}
\Psi_n(r) &= \frac{1}{n + \lambda + \frac{1}{2}} \left[\frac{n!}{a \Gamma(n + 2\lambda + 1)} \right]^{\frac{1}{2}} \left(\frac{2r}{a(n + \lambda + \frac{1}{2})} \right)^{\lambda + \frac{1}{2}} \\
& \quad \times \exp \left[-\frac{r}{a(n + \lambda + \frac{1}{2})} \right] L_n^{(2\lambda)} \left(\frac{2r}{a(n + \lambda + \frac{1}{2})} \right)
\end{aligned} \tag{5.98a}$$

$$= \frac{(-1)^n}{n + \lambda + \frac{1}{2}} \left[\frac{1}{an! \Gamma(n + 2\lambda + 1)} \right]^{\frac{1}{2}} W_{n + \lambda + \frac{1}{2}, \lambda} \left(\frac{2r}{a(n + \lambda + \frac{1}{2})} \right) \tag{5.98b}$$

$$E_n = -\frac{m\alpha^2}{2\hbar^2(n + \lambda + \frac{1}{2})^2}. \tag{5.99}$$

For the continuous states we have similarly for the wave functions and the energy spectrum

$$\Psi_p(r) = \sqrt{\frac{1}{2\pi} \frac{\Gamma(\lambda + \frac{1}{2} - \frac{i}{ap})}{\Gamma(2\lambda + 1)}} \exp \left(\frac{\pi}{2ap} \right) M_{\frac{i}{ap}, \lambda}(-2ipr) \tag{5.100}$$

with $E_p = \frac{\hbar^2 p^2}{2m}$. In particular this gives the identities

$$\begin{aligned}
& \frac{m}{i\hbar k} \frac{\Gamma(\frac{1}{2} + \lambda - \mu)}{\Gamma(1 + 2\lambda)} W_{\mu, \lambda}(2ikr_{>}) M_{\mu, \lambda}(2ikr_{<}) \\
& = \frac{m}{\hbar k} \sum_n \frac{n!}{\Gamma(n + 2\lambda + 1)} \frac{1}{n + \lambda + \frac{1}{2} - \mu} \\
& \quad \times \exp \left[-k(r' + r'') \right] (4k^2 r' r'')^{\lambda + \frac{1}{2}} L_n^{(2\lambda)}(2kr') L_n^{(2\lambda)}(2kr'') \\
& + \frac{1}{2\pi} \frac{m}{\hbar k} \int_{-\infty}^\infty \frac{dp e^{\pi p}}{p - i\mu} \\
& \quad \times \frac{\Gamma(\frac{1}{2} + \lambda + ip) \Gamma(\frac{1}{2} + \lambda - ip)}{\Gamma^2(1 + 2\lambda)} M_{ip, \lambda}(-2ikr'') M_{-ip, \lambda}(2ikr'),
\end{aligned} \tag{5.101}$$

or alternatively (and explicitly, $\alpha = (q_1 q_2 / \hbar) \sqrt{-m/2E}$)

$$\begin{aligned}
& \sqrt{-\frac{m}{2E}} \frac{\Gamma(\frac{1}{2} + \lambda - \alpha)}{\Gamma(1 + 2\lambda)} W_{\alpha, \lambda} \left(\sqrt{-\frac{8mE}{\hbar^2}} r_{>} \right) M_{\alpha, \lambda} \left(\sqrt{-\frac{8mE}{\hbar^2}} r_{<} \right) \\
& = \hbar \sum_{n=1}^\infty \frac{1}{E_n - E} \Psi_n^*(r') \Psi_n(r'') + \hbar \int_0^\infty \frac{dp}{E_p - E} \Psi_p^*(r') \Psi_p(r'').
\end{aligned} \tag{5.102}$$

Let us note that it is not difficult to analyze the Green function by taking the n^{th} residuum in equation (5.92). Using the appropriate expansion for the Γ -function again we get

$$G(r'', r'; E) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{n + \frac{1}{2} + \lambda - \frac{q_1 q_2}{\hbar} \sqrt{-\frac{m}{2E_n}}} \sqrt{-\frac{m}{2E_n}} \frac{1}{\Gamma(2\lambda + 1)} \\ \times W_{n+\lambda+\frac{1}{2}, \lambda} \left(\frac{2r_{>}}{a(n + \lambda + \frac{1}{2})} \right) M_{n+\lambda+\frac{1}{2}, \lambda} \left(\frac{2r_{<}}{a(n + \lambda + \frac{1}{2})} \right) + \text{regular terms} \quad (5.103)$$

which gives using the relations of the Whittaker-functions with the Laguerre-polynomials the wave functions (5.92) and the energy spectrum (5.94). In particular we have for $D = 1$

$$\Psi_n(x) = \left(\frac{1}{an^3} \right)^{\frac{1}{2}} \frac{2r}{an} e^{-r/an} L_n^{(1)} \left(\frac{2r}{an} \right), \quad (5.104)$$

$$E_n = -\frac{mZ^2 e^4}{2\hbar^2 n^2}, \quad n \in \mathbf{N} \quad (5.105)$$

$$\Psi_p(x) = \frac{1}{\sqrt{2\pi}} \Gamma \left(1 - \frac{i}{ap} \right) e^{\pi/2ap} M_{\frac{i}{ap}, \frac{1}{2}}(-2ipr). \quad (5.106)$$

Because, the domains $x < 0$ and $x > 0$ are separated the wave-functions (5.104,5.106) are doubly degenerated.

We can study several special cases (see [15]):

i) We first consider $D = 2$ and get

$$G(x'', x'; E) \\ = \frac{(r' r'')^{-\frac{1}{2}}}{2\pi} \sqrt{\frac{m}{2E}} \sum_{l=-\infty}^{\infty} \frac{\Gamma(|l| - \frac{i q_1 q_2}{\hbar} \sqrt{\frac{m}{2E}})}{(2|l|)!} \\ \times e^{il(\phi'' - \phi')} W_{\frac{i q_1 q_2}{\hbar} \sqrt{\frac{m}{2E}}, |l|} \left(\sqrt{-\frac{8mE}{\hbar^2}} r_{>} \right) M_{-\frac{i q_1 q_2}{\hbar} \sqrt{\frac{m}{2E}}, |l|} \left(\sqrt{-\frac{8mE}{\hbar^2}} r_{<} \right) \\ = \frac{2m}{\hbar} \int_0^{\infty} \frac{du}{\sinh u} \exp \left[\frac{q_1 q_2}{\hbar} \sqrt{-\frac{2m}{E}} u - \frac{\sqrt{-2mE}}{\hbar} (r' + r'') \coth u \right] \\ \times \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} e^{il(\phi'' - \phi')} I_{2l} \left(\frac{2\sqrt{-2mE} \sqrt{r' r''}}{\hbar \sinh u} \right) \\ = \frac{m}{\pi \hbar} \int_0^{\infty} \frac{du}{\sinh u} \cosh \left(\frac{\sqrt{-8mE}}{\hbar \sinh u} \sqrt{r' r''} \cos \frac{\phi'' - \phi'}{2} \right) \\ \times \exp \left[\frac{q_1 q_2}{\hbar} \sqrt{-\frac{2m}{E}} u - \sqrt{\frac{-2mE}{\hbar^2}} (r' + r'') \cosh u \right]. \quad (5.107)$$

This is equivalent with equation (5.29).

ii) We consider D arbitrary. The Green function reads in the integral representation, where we expand in addition the hyperspherical harmonics in terms of Gegenbauer-polynomials

$$G(x'', x'; E) = \frac{2m}{\hbar} (r' r'')^{\frac{2-D}{2}} \int_0^\infty \frac{du}{\sinh u} \exp \left[\frac{q_1 q_2}{\hbar} \sqrt{-\frac{2m}{E}} u - \frac{\sqrt{-2mE}}{\hbar} (r' + r'') \coth u \right] \\ \times \sum_{l=0}^\infty \frac{(2l + D - 2) \Gamma(\frac{D-2}{2})}{4\pi^{D/2}} C_l^{\frac{D-2}{2}}(\cos \psi^{(1,2)}) I_{2l+D-2} \left(\frac{2\sqrt{-2mE} \sqrt{r' r''}}{\hbar \sinh u} \right). \quad (5.108)$$

The l -summation can be performed by means of equation (5.74) and taking into account that we have for the Gegenbauer-polynomials

$$C_n^\lambda(\cos \theta) = \frac{\Gamma(n + 2\lambda)}{n! \Gamma(2\lambda)} {}_2F_1 \left(2\lambda + n, -n; \lambda + \frac{1}{2}; \sin^2 \frac{\theta}{2} \right) \\ = (-1)^n \frac{\Gamma(n + 2\lambda)}{n! \Gamma(2\lambda)} {}_2F_1 \left(2\lambda + n, -n; \lambda + \frac{1}{2}; \cos^2 \frac{\theta}{2} \right). \quad (5.109)$$

Thus we get

$$G(x'', x'; E) \\ = \frac{2m}{\hbar} (r' r'')^{\frac{2-D}{2}} (4\pi)^{\frac{1-D}{2}} \int_0^\infty \frac{du}{\sinh u} \exp \left[\frac{q_1 q_2}{\hbar} \sqrt{-\frac{2m}{E}} u - \frac{\sqrt{-2mE}}{\hbar} (r' + r'') \coth u \right] \\ \times \sum_{l=0}^\infty (-1)^l (2l + D - 2) \frac{\Gamma(l + D - 2)}{l! \Gamma(\frac{D-1}{2})} \\ \times {}_2F_1 \left(l + D - 2, -l; \frac{D-1}{2}; \cos^2 \frac{\psi^{(1,2)}}{2} \right) I_{2l+D-2} \left(\frac{2\sqrt{-2mE} \sqrt{r' r''}}{\hbar \sinh u} \right) \\ = \frac{2m}{\hbar} \left(\frac{4\pi \hbar}{\sqrt{-2mE}} \right)^{\frac{1-D}{2}} \left(\sqrt{r' r''} \cos \frac{\psi^{(1,2)}}{2} \right)^{\frac{3-D}{2}} \\ \times \int_0^\infty \frac{du}{(\sinh u)^{\frac{D+1}{2}}} \exp \left[\frac{q_1 q_2}{\hbar} \sqrt{-\frac{2m}{E}} u - \frac{\sqrt{-2mE}}{\hbar} (r' + r'') \coth u \right] \\ \times I_{\frac{D-3}{2}} \left(\frac{2\sqrt{-2mE} \sqrt{r' r''}}{\hbar \sinh u} \cos \frac{\psi^{(1,2)}}{2} \right) \quad (5.110)$$

Alternatively this can be written as

$$G(x'', x'; E) \\ = \frac{2m}{\hbar} \left(\frac{4\pi \hbar}{\sqrt{-2mE}} \right)^{\frac{1-D}{2}} \left(\frac{r' r'' + x' \cdot x''}{2} \right)^{\frac{3-D}{4}} \int_0^\infty \frac{du}{(\sinh u)^{\frac{D+1}{2}}} \\ \times \exp \left[\frac{q_1 q_2}{\hbar} \sqrt{-\frac{2m}{E}} u - \frac{\sqrt{-2mE}}{\hbar} (r' + r'') \coth u \right] I_{\frac{D-3}{2}} \left(\frac{2\sqrt{-2mE}}{\hbar \sinh u} \sqrt{\frac{r' r'' + x' \cdot x''}{2}} \right) \\ = \frac{m}{\hbar} \left(\frac{2\pi \hbar}{\sqrt{-2mE}} \right)^{\frac{1-D}{2}} (x^2 - y^2)^{\frac{3-D}{4}} \int_0^\infty \frac{du}{(\sinh u)^{\frac{D+1}{2}}} \\ \times \exp \left[\frac{q_1 q_2}{\hbar} \sqrt{-\frac{2m}{E}} u - \frac{\sqrt{-2mE}}{\hbar} x \coth u \right] I_{\frac{D-3}{2}} \left(\frac{\sqrt{-2mE} \sqrt{x^2 - y^2}}{\hbar \sinh u} \right), \quad (5.111)$$

where $x = r' + r''$ and $y = |x'' - x'|$. For $D = 2$ we recover, of course, equations (5.29,5.107) with $I_{-\frac{1}{2}}(z) = (2/\pi z)^{\frac{1}{2}} \cosh z$.

$D = 1$ yields with $\psi^{(1,2)} \rightarrow 2\pi$:

$$\begin{aligned}
 & G(x'', x'; E) \\
 &= \frac{2m}{\hbar} \sqrt{x'x''} \int_0^\infty \frac{du}{\sinh u} \\
 &\quad \times \exp \left[\frac{q_1 q_2}{\hbar} \sqrt{-\frac{2m}{E}} u - \frac{\sqrt{-2mE}}{\hbar} (x' + x'') \coth u \right] I_1 \left(\frac{2\sqrt{-2mE} \sqrt{x'x''}}{\hbar \sinh u} \right) \\
 &= \sqrt{-\frac{m}{2E}} \Gamma \left(1 - \frac{q_1 q_2}{\hbar} \sqrt{-\frac{m}{2E}} \right) \\
 &\quad \times W_{\frac{q_1 q_2}{\hbar} \sqrt{-\frac{m}{2E}}, \frac{1}{2}} \left(\sqrt{-\frac{8mE}{\hbar^2}} r_{>} \right) M_{\frac{q_1 q_2}{\hbar} \sqrt{-\frac{m}{2E}}, \frac{1}{2}} \left(\sqrt{-\frac{8mE}{\hbar^2}} r_{<} \right). \quad (5.112)
 \end{aligned}$$

Furthermore one can show, by using the relation:

$$\left(\frac{d}{z dz} \right)^m \left(\frac{I_\nu(z)}{z^\nu} \right) = \frac{I_{\nu+m}(z)}{z^{\nu+m}}$$

that the D -dimensional Green function for the $1/r$ -potential is connected with the $(D - 2)$ -dimensional Green function by the relation

$$G_D(x|y; E) = -\frac{1}{2\pi y} \frac{\partial}{\partial y} G_{D-2}(x|y; E). \quad (5.113)$$

Therefore

$$G_D(x|y; E) = \left(\frac{-\partial}{2\pi y \partial y} \right)^{\frac{D-1}{2}} G_1(x|y; E), \quad D = 1, 3, 5, \dots, \quad (5.114)$$

$$G_D(x|y; E) = \left(\frac{-\partial}{2\pi y \partial y} \right)^{\frac{D-2}{2}} G_2(x|y; E), \quad D = 2, 4, 6, \dots \quad (5.115)$$

5.4. Axially Symmetric Coulomb-Like Potential [7-13, 46-48, 88, 89]

It is possible to consider the even more complicated potential problem, namely [46]

$$\mathcal{L}_{Cl} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q^2}{\sqrt{x^2 + y^2 + z^2}} - \frac{b\hbar^2}{2m} \frac{1}{x^2 + y^2} - \frac{c\hbar^2}{2m} \frac{z}{(x^2 + y^2)\sqrt{x^2 + y^2 + z^2}} \quad (5.116a)$$

$$= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) + \frac{q^2}{r} - \frac{b\hbar^2}{2mr^2 \sin^2 \theta} - \frac{c\hbar^2}{2m} \frac{\cos \theta}{r^2 \sin^2 \theta} \quad (5.116b)$$

$$= \frac{m}{2}(\xi^2 + \eta^2)(\dot{\xi}^2 + \dot{\eta}^2) + \frac{m}{2}\xi^2\eta^2\dot{\phi}^2 + \frac{2q^2}{\xi^2 + \eta^2} - \frac{b\hbar^2}{2m\xi^2\eta^2} - \frac{c\hbar^2}{m} \frac{\eta^2 - \xi^2}{\xi^2\eta^2(\xi^2 + \eta^2)}. \quad (5.116c)$$

Here I have displayed the corresponding classical Lagrangian in cartesian, polar and parabolic coordinates. It belongs to the class of potentials mentioned long ago by Makarov, Smorodinsky, Valiev and Winternitz [71] which are separable in a specific coordinate system and furthermore exactly solvable. This potential was discussed by Carpio-Bernido, Bernido and Inomata [10] starting from cartesian coordinates and using the four-dimensional realization of the Kustaanheimo-Stiefel transformation. However, this potential is separable in polar as well as parabolic coordinates and we shall give the explicit calculation for all of them.

1) Cartesian coordinates

In our line of reasoning we follow reference [10], however with some simplifications because we have already discussed the realization of the four-dimensional Kustaanheimo-Stiefel in the discussion of the hydrogen atom. We consider the path integral representation corresponding to equation (5.116b)

$$K(x'', x'; T) = \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{3N}{2}} \prod_{j=1}^{N-1} \int dx^{(j)} dy^{(j)} dz^{(j)} \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} (\Delta^2 x^{(j)} + \Delta^2 y^{(j)} + \Delta^2 z^{(j)}) + \epsilon \frac{q^2}{r^{(j)}} - \frac{\epsilon b \hbar^2}{2m} \frac{1}{r^{(j)2} - z^{(j)2}} - \frac{\epsilon c \hbar^2}{2m} \frac{z^{(j)}}{r^{(j)}(r^{(j)2} - z^{(j)2})} \right] \right\}. \quad (5.117)$$

Similarly to equation (5.45) we insert a factor “one”

$$1 = \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{\frac{N}{2}} \prod_{j=1}^N \int_{-\infty}^{\infty} d\xi^{(j)} \exp \left[\frac{i m}{2\epsilon \hbar} \Delta^2 \xi^{(j)} \right]. \quad (5.118)$$

We repeat the steps from equations (5.49) to (5.57), i.e. we realise the four-dimensional Kustaanheimo-Stiefel transformation on midpoints and arrive together with the transformation formulæ

$$K(x'', x'; T) = \frac{1}{2\pi i \hbar} \int_{-\infty}^{\infty} dE e^{-iTE/\hbar} G(x'', x'; E) \quad (5.119)$$

$$G(x'', x'; E) = i \int_0^{\infty} ds'' e^{4iq^2 s''/\hbar} \tilde{K}(u'', u'; s''),$$

with the transformed path integral

$$\tilde{K}(u'', u'; s'') = \frac{1}{4r''} \int_{-\infty}^{\infty} d\xi'' \tilde{K}_1(u''_1, u'_1, u''_2, u'_2; s'') \times \tilde{K}_2(u''_3, u'_3, u''_4, u'_4; s''), \quad (5.120)$$

and the kernels $K_1(s'')$, $K_2(s'')$, respectively, are given by

$$\begin{aligned} & \tilde{K}_1(u''_1, u'_1, u''_2, u'_2; s'') \\ &= \int_{u_1(t')=u'_1}^{u_1(t'')=u''_1} \mathcal{D}u_1(s) \int_{u_2(t')=u'_2}^{u_2(t'')=u''_2} \mathcal{D}u_2(s) \\ & \quad \times \exp \left\{ \frac{i}{\hbar} \int_0^{s''} \left[\frac{m}{2} (\dot{u}_1^2 + \dot{u}_2^2) + 4E(u_1^2 + u_2^2) - \frac{(b+c)\hbar^2}{2m(u_1^2 + u_2^2)} \right] ds \right\} \\ &= \int_{\rho_1(t')=\rho'_1}^{\rho_1(t'')=\rho''_1} \rho_1 \mathcal{D}\rho_1(s) \int_{\phi_1(t')=\phi'_1}^{\phi_1(t'')=\phi''_1} \mathcal{D}\phi_1(s) \\ & \quad \times \exp \left\{ \frac{i}{\hbar} \int_0^{s''} \left[\frac{m}{2} (\dot{\rho}_1^2 + \rho_1^2 \dot{\phi}_1^2) + 4E\rho_1^2 - \hbar^2 \frac{b+c-\frac{1}{4}}{2m\rho_1^2} \right] ds \right\} \end{aligned} \quad (5.121)$$

$$\begin{aligned} &= \frac{1}{2\pi \sqrt{\rho'_1 \rho''_1}} \sum_{\nu_1=-\infty}^{\infty} e^{i\nu_1(\phi''_1 - \phi'_1)} \\ & \quad \times \int_{\rho_1(t')=\rho'_1}^{\rho_1(t'')=\rho''_1} \mathcal{D}\rho_1(s) \exp \left\{ \frac{i}{\hbar} \int_0^{s''} \left[\frac{m}{2} \dot{\rho}_1^2 + 4E\rho_1^2 - \hbar^2 \frac{b+c+\nu^2-\frac{1}{4}}{2m\rho_1^2} \right] ds \right\} \\ &= \frac{m\omega}{2\pi i \hbar \sin \omega s''} \sum_{\nu_1=-\infty}^{\infty} e^{i\nu_1(\phi''_1 - \phi'_1)} \\ & \quad \times \exp \left[-\frac{m\omega}{2i\hbar} (\rho'_1{}^2 + \rho''_1{}^2) \cot \omega s'' \right] I_{\lambda_1} \left(\frac{m\omega \rho'_1 \rho''_1}{i\hbar \sin \omega s''} \right). \end{aligned} \quad (5.122)$$

Here we have introduced two-dimensional polar coordinates $u_1 = \rho_1 \cos \phi_1$, $u_2 = \rho_1 \sin \phi_1$, separated the ϕ_1 and the ρ_1 path integration in the usual way, applied

the solution of the radial harmonic oscillator and have used the abbreviations $\omega = \sqrt{-8E/m}$, $\lambda_1 = +\sqrt{\nu^2 + b + c}$. Similarly

$$\begin{aligned} \tilde{K}_2(u''_3, u''_3, u''_4, u''_4; s'') &= \frac{m\omega}{2\pi i \hbar \sin \omega s''} \sum_{\nu_2=-\infty}^{\infty} e^{i\nu_2(\phi''_2 - \phi'_2)} \\ &\times \exp \left[-\frac{m\omega}{2i\hbar} (\rho_2'^2 + \rho_2''^2) \cot \omega s'' \right] I_{\lambda_2} \left(\frac{m\omega \rho_2' \rho_2''}{i\hbar \sin \omega s''} \right) \end{aligned} \quad (5.123)$$

with $u_3 = \rho_2 \cos \phi_2$, $u_4 = \rho_2 \sin \phi_2$, $\lambda_2 = +\sqrt{\nu^2 + b - c}$. We identify variables [Cayley-Klein parameters, c.f. equation (5.59)]

$$\begin{aligned} \rho_1 &= \sqrt{r} \cos \frac{\theta}{2} & \rho_2 &= \sqrt{r} \sin \frac{\theta}{2} \\ \phi_1 &= \frac{\alpha + \phi}{2} & \phi_2 &= \frac{\alpha - \phi}{2} \end{aligned} \quad (5.124)$$

($0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \alpha \leq 4\pi$, $d\xi'' = r'' d\alpha''$). Collecting factors we thus obtain

$$\begin{aligned} G(x'', x'; E) &= \frac{i}{\pi} \int_0^\infty ds'' \left(\frac{m\omega}{2i\hbar \sin \omega s''} \right)^2 \exp \left[\frac{4iq^2 s''}{\hbar} - \frac{m\omega}{2i\hbar} (r' + r'') \cot \omega s'' \right] \\ &\times \sum_{\nu=-\infty}^{\infty} e^{i\nu(\phi'' - \phi')} I_{\lambda_1} \left(\frac{m\omega \sqrt{r' r''}}{i\hbar \sin \omega s''} \sin \frac{\theta'}{2} \sin \frac{\theta''}{2} \right) I_{\lambda_2} \left(\frac{m\omega \sqrt{r' r''}}{i\hbar \sin \omega s''} \cos \frac{\theta'}{2} \cos \frac{\theta''}{2} \right). \end{aligned} \quad (5.125)$$

(Compare with equation (5.69)). Using the “addition theorem” [31, Vol.II, p.99]:

$$\begin{aligned} &\frac{z}{2} J_\mu(z \cos \alpha \cos \beta) J_\nu(z \sin \alpha \sin \beta) \\ &= (\sin \alpha \sin \beta)^\nu (\cos \alpha \cos \beta)^\mu \\ &\times \sum_{l=0}^{\infty} (-1)^l (\mu + \nu + 2l + 1) \frac{\Gamma(\mu + \nu + l + 1) \Gamma(\nu + l + 1)}{l! \Gamma^2(\nu + 1) \Gamma(l + \mu + 1)} J_{\mu + \nu + 2l + 1}(z) \\ &\times {}_2F_1(-l, \mu + \nu + l + 1; \nu + 1; \sin^2 \alpha) {}_2F_1(-l, \mu + \nu + l + 1; \nu + 1; \sin^2 \beta). \end{aligned} \quad (5.126)$$

we get in the usual way by performing the s'' integration

$$\begin{aligned} &G(x'', x'; E) \\ &= \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu(\phi'' - \phi')} \left(\sin \frac{\theta'}{2} \sin \frac{\theta''}{2} \right)^{\lambda_1} \left(\cos \frac{\theta'}{2} \cos \frac{\theta''}{2} \right)^{\lambda_2} \\ &\times \sum_{l=0}^{\infty} (\lambda_1 + \lambda_2 + 2l + 1) \frac{\Gamma(\lambda_1 + \lambda_2 + l + 1) \Gamma(\lambda_1 + l + 1)}{l! \Gamma^2(\lambda_1 + 1) \Gamma(\lambda_2 + l + 1)} \end{aligned}$$

$$\begin{aligned}
 & \times {}_2F_1\left(-l, \lambda_1 + \lambda_2 + l + 1; \lambda_1 + 1; \sin^2 \frac{\theta'}{2}\right) {}_2F_1\left(-l, \lambda_1 + \lambda_2 + l + 1; \lambda_1 + 1; \sin^2 \frac{\theta''}{2}\right) \\
 & \times \frac{m\omega}{\sqrt{r'r''}\hbar} \int_0^\infty \frac{ds''}{\sin \omega s''} e^{4iq^2 s''/\hbar} \\
 & \times \exp\left[-\frac{m\omega}{2i\hbar}(r' + r'') \cot \omega s''\right] I_{\lambda_1 + \lambda_2 + 2l + 1}\left(\frac{m\omega\sqrt{r'r''}}{i\hbar \sin \omega s''}\right) \\
 & = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu(\phi'' - \phi')} \sum_{l=0}^{\infty} \frac{(\lambda_1 + \lambda_2 + 2l + 1)}{2} \frac{\Gamma(\lambda_1 + \lambda_2 + l + 1)!}{\Gamma(\lambda_1 + l + 1)\Gamma(\lambda_2 + l + 1)} \\
 & \times \left(\sin \frac{\theta'}{2} \sin \frac{\theta''}{2}\right)^{\lambda_1} \left(\cos \frac{\theta'}{2} \cos \frac{\theta''}{2}\right)^{\lambda_2} P_l^{(\lambda_1, \lambda_2)}(\cos \theta') P_l^{(\lambda_1, \lambda_2)}(\cos \theta'') \\
 & \times \frac{1}{r'r''} \sqrt{-\frac{m}{2E}} \frac{\Gamma[\frac{1}{2}(\lambda_1 + \lambda_2) + l + 1 - p]}{\Gamma(\lambda_1 + \lambda_2 + 2l + 2)} \\
 & \times W_{p, l + \frac{1 + \lambda_1 + \lambda_2}{2}}(-2ikr_>) M_{p, l + \frac{1 + \lambda_1 + \lambda_2}{2}}(-2ikr_<). \tag{5.127}
 \end{aligned}$$

with the abbreviations $p = q^2 \sqrt{-m/2E\hbar^2}$, $k = \sqrt{+2mE}/\hbar$. This is the result of reference [10]. The energy-levels are determined by the poles of the Γ -function and are given by

$$E_n = -\frac{mq^4}{2\hbar^2(n + l + 1 + \frac{\lambda_1 + \lambda_2}{2})^2}, \quad n \in \mathbf{N}_0. \tag{5.128}$$

The corresponding wave functions will be calculated below, where we separate the radial from the angular path integrations directly.

2) Polar coordinates

We consider the path integral corresponding the Lagrangian (5.116a):

$$\begin{aligned}
 & K(x'', x'; T) \\
 & = \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \int_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \sin \theta \mathcal{D}\theta(t) \int_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t) \\
 & \quad \times \exp\left\{\frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2}(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \right. \right. \\
 & \quad \left. \left. + \frac{q^2}{r} - \frac{b\hbar^2}{2mr^2 \sin^2 \theta} - \frac{c\hbar^2}{2m} \frac{\cos \theta}{r^2 \sin^2 \theta} + \frac{\hbar^2}{8m} \left(1 + \frac{1}{\sin^2 \theta}\right)\right] dt\right\} \\
 & = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu(\phi'' - \phi')} K_\nu(r'', r', \theta'', \theta'; T) \tag{5.129}
 \end{aligned}$$

with the kernel $K_\nu(T)$, which in turn is also separated:

$$K_\nu(r'', r', \theta'', \theta'; T)$$

$$\begin{aligned}
&= (r' r'' \sin \theta' \sin \theta'')^{-\frac{1}{2}} \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \int_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \mathcal{D}\theta(t) \\
&\quad \times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - \hbar^2 \frac{\nu^2 + b + c \cos \theta - \frac{1}{4}}{2mr^2 \sin^2 \theta} + \frac{q^2}{r} + \frac{\hbar^2}{8mr^2} \right] dt \right\} \\
&= (r' r'' \sin \theta' \sin \theta'')^{-\frac{1}{2}} \\
&\quad \times \int_{r(t')=r'}^{r(t'')=r''} r \mathcal{D}r(t) \int_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \mathcal{D}\theta(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{r}^2 + \frac{4m}{2} r^2 \left(\frac{\dot{\theta}}{2} \right)^2 \right. \right. \\
&\quad \left. \left. - \hbar^2 \frac{\nu^2 + b + c - \frac{1}{4}}{8mr^2 \sin^2 \theta} - \hbar^2 \frac{\nu^2 + b - c - \frac{1}{4}}{8mr^2 \cos^2 \theta} + \frac{q^2}{r} + \frac{\hbar^2}{8mr^2} \right] dt \right\} \\
&= \sum_{l=0}^{\infty} \Psi_l^{(\lambda_1, \lambda_2)}(\theta') \Psi_l^{(\lambda_1, \lambda_2)}(\theta'') K_{l\nu}(r'', r'; T), \tag{5.130}
\end{aligned}$$

the (Pöschl-Teller) wave functions

$$\begin{aligned}
\Psi_l^{(\lambda_2, \lambda_1)}(\theta) &= \left[\frac{\lambda_1 + \lambda_2 + 2l + 1}{2^{1+\lambda_1+\lambda_2}} \frac{l! \Gamma(\lambda_1 + \lambda_2 + l + 1)}{\Gamma(\lambda_1 + l + 1) \Gamma(\lambda_2 + l + 1)} \right]^{\frac{1}{2}} \\
&\quad \times (1 - \cos \theta)^{\frac{\lambda_1}{2}} (1 + \cos \theta)^{\frac{\lambda_2}{2}} P_l^{(\lambda_1, \lambda_2)}(\cos \theta), \tag{5.131}
\end{aligned}$$

($\lambda_{1,2}$ as before) and the radial path integral

$$K_{l\nu}(r'', r'; T) = \frac{1}{r' r''} \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{r}^2 + \frac{q^2}{r} - \hbar^2 \frac{\lambda^2 - \frac{1}{4}}{2mr^2} \right] dt \right\} \tag{5.132}$$

[$\lambda = \frac{1}{2}(\lambda_1 + \lambda_2 + 2l + 1)$]. This path integral is nothing but a Coulomb potential path integral, however with generalized angular momentum. Thus we just analytically continue the known result in l and obtain

$$G_\lambda(r'', r'; E) = \frac{1}{r' r''} \sqrt{-\frac{m}{2E}} \frac{\Gamma(\lambda + 1 - p)}{\Gamma(2\lambda + 2)} W_{p, \lambda + \frac{1}{2}}(-2i kr_>) M_{p, \lambda + \frac{1}{2}}(-2i kr_<). \tag{5.133}$$

The energy spectrum was already stated in equation (5.128) and the corresponding

wave functions are

$$\Psi_\lambda(r, \theta, \phi) = \frac{e^{i\nu\phi}}{\sqrt{2\pi}} \Psi_i^{(\lambda_1, \lambda_2)}(\theta) \times \left\{ \begin{aligned} & \left[\frac{2}{(n + \lambda + \frac{1}{2})^2} \left[\frac{n!}{a^3(n + \lambda + \frac{1}{2})\Gamma(n + 2\lambda + 2)} \right]^{\frac{1}{2}} \left(\frac{2r}{a(n + \lambda + \frac{1}{2})} \right) \right. \\ & \quad \times \exp\left(-\frac{r}{a(n + \lambda + \frac{1}{2})}\right) L_n^{(2\lambda+1)}\left(\frac{2r}{a(n + \lambda + \frac{1}{2})}\right) \\ & \left. \sqrt{\frac{1}{2\pi}} \frac{\Gamma(\lambda + \frac{1}{2} + \frac{i}{ap})}{r\Gamma(2\lambda + 2)} \exp\left(\frac{\pi}{2ap}\right) M_{\frac{i}{ap}, \lambda + \frac{1}{2}}(-2ipr) \right\} \quad (5.134) \end{aligned} \right.$$

and a denotes the Bohr radius.

3) Parabolic coordinates

The potential described in equation (5.116c) is also separable in parabolic coordinates, in the operator approach as well in the path integral formalism. We consider the path integral formulation corresponding to the Lagrangian (5.116c):

$$\begin{aligned} K(x'', x'; T) &\equiv K(\xi'', \xi', \eta'', \eta', \phi'', \phi'; T) \\ &= \int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t) (\xi^2 + \eta^2) \xi \eta \int_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t) \\ &\quad \times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} (\xi^2 + \eta^2) (\dot{\xi}^2 + \dot{\eta}^2) + \frac{m}{2} \xi^2 \eta^2 \dot{\phi}^2 \right. \right. \\ &\quad \left. \left. + \frac{2q^2}{\xi^2 + \eta^2} - \frac{\hbar^2(b - \frac{1}{4})}{2m\xi^2\eta^2} - \frac{c\hbar^2}{m} \frac{\eta^2 - \xi^2}{\xi^2\eta^2(\xi^2 + \eta^2)} \right] dt \right\} \\ &= \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu(\phi'' - \phi')} K_\nu(\xi'', \xi', \eta'', \eta'; T) \end{aligned} \quad (5.135)$$

with the kernel $K_\nu(T)$

$$\begin{aligned} K_\nu(\xi'', \xi', \eta'', \eta'; T) &= (\xi' \xi'' \eta' \eta'')^{-\frac{1}{2}} \int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}(\xi^2 + \eta^2) \\ &\quad \times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} (\xi^2 + \eta^2) (\dot{\xi}^2 + \dot{\eta}^2) \right. \right. \\ &\quad \left. \left. + \frac{2q^2}{\xi^2 + \eta^2} - \hbar^2 \frac{b + \nu^2 - \frac{1}{4}}{2m\xi^2\eta^2} - \frac{c\hbar^2}{m} \frac{\eta^2 - \xi^2}{\xi^2\eta^2(\xi^2 + \eta^2)} \right] dt \right\} \end{aligned} \quad (5.136)$$

This path integral is now tracked by a time-transformation according to

$$\epsilon = \widehat{\xi_{(j)}^2 + \eta_{(j)}^2}, \quad s(t) = \int_{t'}^t \frac{d\sigma}{\xi^2(\sigma) + \eta^2(\sigma)}, \quad (5.137)$$

therefore

$$K_\nu(\xi'', \xi', \eta'', \eta'; T) = \frac{1}{2\pi i \hbar} \int_{-\infty}^{\infty} dE e^{-iET/\hbar} G_\nu(\xi'', \xi', \eta'', \eta'; E) \quad (5.138)$$

$$G_\nu(\xi'', \xi', \eta'', \eta'; E) = i \int_0^\infty ds'' e^{2iq^2 s''/\hbar} \tilde{K}_\nu(\xi'', \xi', \eta'', \eta'; s'')$$

with (note that this is a two-dimensional time-transformation and the prefactor is “one”):

$$\tilde{K}_\nu(\xi'', \xi', \eta'', \eta'; s'') = \tilde{K}_1(\xi'', \xi'; s'') \times \tilde{K}_2(\eta'', \eta'; s''). \quad (5.139)$$

This gives for the kernels $\tilde{K}_{1,2}(s'')$

$$\begin{aligned} \tilde{K}_1(\xi'', \xi'; s'') &= \frac{1}{\sqrt{\xi' \xi''}} \int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mu_{\lambda_1}[\xi^2] \mathcal{D}\xi(s) \exp \left\{ \frac{i}{\hbar} \int_0^{s''} \left[\frac{m}{2} \dot{\xi}^2 + E \xi^2 \right] ds \right\} \\ &= \frac{m\omega}{i \hbar \sin \omega s''} \exp \left[-\frac{m\omega}{2i \hbar} (\xi'^2 + \xi''^2) \cot \omega s'' \right] I_{\lambda_1} \left(\frac{m\omega \xi' \xi''}{i \hbar \sin \omega s''} \right) \end{aligned} \quad (5.140)$$

with $\omega = \sqrt{-2E/m}$, $\lambda_1 = +\sqrt{\nu^2 + b + c}$. Similarly for $\tilde{K}_2(s'')$

$$\begin{aligned} \tilde{K}_2(\eta'', \eta'; s'') &= \frac{1}{\sqrt{\eta' \eta''}} \int_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mu_{\lambda_2}[\eta^2] \exp \left\{ \frac{i}{\hbar} \int_0^{s''} \left[\frac{m}{2} \dot{\eta}^2 + E \eta^2 \right] ds \right\} \\ &= \frac{m\omega}{i \hbar \sin \omega s''} \exp \left[-\frac{m\omega}{2i \hbar} (\eta'^2 + \eta''^2) \cot \omega s'' \right] I_{\lambda_2} \left(\frac{m\omega \eta' \eta''}{i \hbar \sin \omega s''} \right) \end{aligned} \quad (5.141)$$

($\lambda_2 = +\sqrt{\nu^2 + b - c}$). Integrating over s'' yields

$$G^{(\text{bound})}(x'', x'; E) = \hbar \sum_{\nu=-\infty}^{\infty} \sum_{n_1, n_2=0}^{\infty} \frac{\Psi_N^*(\xi', \eta', \phi') \Psi_N(\xi'', \eta'', \phi'')}{E_N - E} \quad (5.142)$$

with the energy spectrum (note $\omega = q^2/\hbar N$)

$$E_N = -\frac{mq^4}{2\hbar^2 N^2}, \quad N = n_1 + n_2 + 1 + \frac{\lambda_1 + \lambda_2}{2}. \quad (5.143)$$

The wave functions have the form

$$\begin{aligned} \Psi_{\nu, n_1, n_2}(\xi, \eta, \nu) &= \frac{e^{i\nu\phi}}{\sqrt{2\pi}} \left[\frac{2}{a^2 N^3} \cdot \frac{2n_1! n_2!}{\Gamma(n_1 + \lambda_1 + 1) \Gamma(n_2 + \lambda_2 + 1)} \right]^{\frac{1}{2}} \\ &\times \left(\frac{\xi}{aN} \right)^{\lambda_1} \left(\frac{\eta}{aN} \right)^{\lambda_2} \exp \left[-\frac{\xi^2 + \eta^2}{2aN} \right] L_{n_1}^{(\lambda_1)} \left(\frac{\xi^2}{aN} \right) L_{n_2}^{(\lambda_2)} \left(\frac{\eta^2}{aN} \right). \end{aligned} \quad (5.144)$$

For the continuous spectrum we obtain

$$G^{(cont.)}(x'', x'; E) = \hbar \sum_{\nu=-\infty}^{\infty} \int_0^{\infty} dp \int_{-\infty}^{\infty} d\zeta \frac{\Psi_{p,\zeta,\nu}^*(\xi', \eta', \phi') \Psi_{p,\zeta,\nu}(\xi'', \eta'', \phi'')}{E_p - E} \quad (5.145)$$

with

$$E_p = \frac{\hbar^2 p^2}{2m} \quad (5.146)$$

and the wave functions

$$\begin{aligned} \Psi_{p,\zeta,\nu}(\xi, \eta, \phi) = & \frac{e^{i\nu\phi}}{(2\pi)^{\frac{3}{2}}} \frac{\Gamma[\frac{1+\lambda_1}{2} + \frac{i}{2}(\zeta + \frac{i}{ap})] \Gamma[\frac{1+\lambda_2}{2} + \frac{i}{2}(\zeta - \frac{i}{ap})]}{\xi\eta\Gamma(1+\lambda_1)\Gamma(1+\lambda_2)} \\ & \times M_{-\frac{i}{2}(\zeta + \frac{i}{ap}), \frac{\lambda_1}{2}}(-ip\xi^2) M_{-\frac{i}{2}(\zeta - \frac{i}{ap}), \frac{\lambda_2}{2}}(-ip\eta^2). \end{aligned} \quad (5.147)$$

Finally we state the relation of the Green function in parabolic coordinates. We have

$$\begin{aligned} G(\xi'', \xi', \eta'', \eta', \phi'', \phi'; E) = & \frac{i}{2\pi} \left(\frac{m\omega}{i\hbar} \right)^2 \sum_{\nu=-\infty}^{\infty} e^{i\nu(\phi'' - \phi')} \\ & \times \int_0^{\infty} \frac{ds''}{\sin^2 \omega s''} I_{\lambda_1} \left(\frac{m\omega \xi' \xi''}{i\hbar \sin \omega s''} \right) I_{\lambda_2} \left(\frac{m\omega \eta' \eta''}{i\hbar \sin \omega s''} \right) \\ & \times \exp \left[\frac{2iq^2}{\hbar} s'' - \frac{m\omega}{2i\hbar} (\xi'^2 + \xi''^2 + \eta'^2 + \eta''^2) \cot \omega s'' \right]. \end{aligned} \quad (5.148)$$

Note the similarity to the u_1, u_2, u_3, u_4 approach. Actually the Kustaanheimo-Stiefel approach in cartesian coordinates produces a separation in parabolic coordinates. We switch back to polar coordinates by means of

$$\begin{aligned} \xi^2 = r + z = r(1 + \cos \theta) &= 2r \cos^2 \frac{\theta}{2} \\ \eta^2 = r - z = r(1 - \cos \theta) &= 2r \sin^2 \frac{\theta}{2} \end{aligned} \quad (5.149)$$

Using again the addition theorem (5.126) with a rescaling $s'' \rightarrow 2s''$, $\omega \rightarrow \omega/2$ so that $\omega = \sqrt{-8E/m}$ we recover the Green function (5.125).

Let us finally note that even more complicated Coulomb-like potentials can be exactly solved [47, 48] which are, however, only separable in parabolic coordinates.

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