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Some typical properties of the dynamics of rational maps

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0. Notation.
The rational functions of fixed degree n form a complex manifold. We consider an arbitrary connected submanifold M. Let $M \times \mathbf{P}^1$ be the direct product of M with the extended complex plane $\mathbf{P}^1$, and $\pi : M \times \mathbf{P}^1 \to M$ the natural projection. By X_m we denote the analytic subset of $M \times \mathbf{P}^1$ given by the equation $f^m z = z$ and by Y_m the closure of the set $X_m \setminus \bigcup_{d|m} X_d$. Let $N_m \subset M$ be the branching set of the projection $\pi | Y_m$. If $f \in N_m$, then f possesses a multiple periodic point (that is, $(f^m)'(z) = 1$). We put $K = \overline{\bigcup N_m}$ and $\sigma = M \setminus K$. By $P(f)$ we denote the closure of the set of periodic points of f. We say that $f \in M$ is *P-stable* if $f | P(f)$ and $g | P(g)$ are topologically conjugate for all g sufficiently close to f.

1. *P*-stability is typical.
Main Lemma. *Let U be a simply-connected domain in M such that $U \cap N_{m_k} = \phi$ for some subsequence $m_k \to \infty$. Then*

a) *$U \cap N_m = \phi$ for all m; any connected component of the surface Y_m is given by an equation $z = \varphi_{m,i}(f)$, where $\varphi_{m,i} : U \to \mathbf{P}^1$ is a holomorphic function;*

b) *the closure Φ of the family $\{\varphi_{m,i}\}$ has the following properties:*

1) *if $\varphi(f_0) = \psi(f_0)$, $(\varphi, \psi \in \Phi)$, then $\varphi = \psi$;*

2) *the family Φ is compact;*

3) *the family Φ is invariant under the transformation $Q : (Q\varphi)(f) = f(\varphi(f))$;*

c) *if $(f^m)'(\varphi_{m,i}(f)) \neq$ consr, then $|(f^m)'(\varphi_{m,i}(f))| < 1$ or $|(f^m)'(\varphi_{m,i}(f))| > 1$ in U;*

d) *the endomorphisms f and g of U are P-conjugate. The conjugating homeomorphism transforms $\varphi(f)$ into $\varphi(g)$.*

***Corollary* 1.** *The set of P-stable endomorphisms coincides with the set σ.*

***Corollary* 2.** *If $f \in K$, then there exists a sequence $f_m \to f$ such that $f_m \in N_m$.*

We say that $\Lambda \subset M$ is a *set of local uniqueness* if $\Lambda \cap U$ for any neighbourhood U intersecting Λ is not contained in a non-trivial analytic set. A set of local uniqueness is perfect.

***Corollary* 3.** *K is a set of local uniqueness.*

***Corollary* 4.** *If $f \in K$, then there exists a sequence $f_p \to f$ such that f_p has an attracting cycle of order p.*

***Corollary* 5.** *The set K is nowhere dense.*

Thus, P-stability is typical in any holomorphic family M of rational functions.

***Corollary* 6.** *We suppose that in the family M there is a function without neutral cycles. If $f \in M$ has a neutral cycle, then $f \in K$.*

***Lemma* 1.** *Let $\varphi : U \to \mathbf{P}^1$ be a holomorphic function in U for which $\varphi(f)$ does not belong to the orbits of critical points of f. Then $U \subset \sigma$.*

We call a cycle $\{z_i\}_0^{p-1}$ of f *absorbing* if it is contained in the orbit of some critical point. By S we denote the set of functions $f \in M$ that have (with respect to the parameter) a non-stable absorbing repellent cycle.

***Corollary* 7.** *S is a dense subset of K.*

2. Stability of the orbit of a critical point.
The results of this subsection are a development of a paper by G.M. Levin [1]. We consider a family f_x of rational functions holomorphically dependent on a point x of an analytic set Z. We consider a sequence $\psi_m : Z \to \mathbf{P}^1$ of holomorphic functions satisfying the equation $\psi_{m+1}(x) = f_x(\psi_m(x))$. A point $x \in Z$ is called *regular* ($x \in \mathfrak{R}$) if the sequence $\{\psi_m\}$ is normal (that is, precompact) in some neighbourhood of it. The study of our sequences is based on Montel's criterion for normality, which is stated as in the classical case. It is easy to show that if the orbit $\{\psi_m(x)\}$ converges to an attracting cycle of f_x, then $x \in \mathfrak{R}$.

***Lemma* 2.** *Suppose that $\psi_m \not\equiv \psi_k$ for $m \neq k$. If the orbit $\{\psi_m(x)\}$ is absorbed by a repellent cycle ($x \in T$) or a removable neutral cycle, then $x \in L \equiv Z \setminus \mathfrak{R}$.*

We denote $\{\psi_m(x)\}$ by $\Psi(x)$. By analogy with P-stability we define *Ψ-stability.*

***Proposition* 1.** *For a typical regular value of the parameter $x \in \mathfrak{R}$ the function f_x is Ψ-stable.*

Let us make Lemma 2 more precise.

***Corollary* 8.** *Let $\psi_m \not\equiv \psi_k$ for $m \neq k$, let z be a removable neutral periodic point of f_x, and $z \in \Psi(x)$. Then $x \in L$.*

From Montel's theorem it follows that T is dense in L. Any holomorphic function φ (with perhaps two exceptions) in a neighbourhood of $x \in L$ has a sequence $x_m \to x$ such that $\psi_m(x_m) = \varphi(x_m)$. These simple facts are useful in the construction of examples (see Proposition 2 and the example below). They also imply that L is a set of local uniqueness. By $F(f)$ we denote the Julia-Fatou set of f [2].

***Proposition* 2.** *For a typical irregular value of the parameter $x \in L$ (that is, outside a set of the first category) the orbit $\{\psi_m(x)\}$ is dense in $F(f_x)$.*

The main special case is the set $Z = C$ given in $M \times \mathbf{P}^1$ by the equation $Df(c) = 0$ and the sequence $\psi_m(f, c) = f^m c$ on C. In this case $\mathfrak{R}$ is dense in C (an application of Montel's theorem to the equations $\psi_m(f, c) = c$). The following result connects 1 and 2.

***Proposition* 3.** *For f to be P-stable it is necessary and sufficient that $(\pi | C)^{-1} f \subset \mathfrak{R}$.*

In conclusion we consider an example.

Example. It can be shown that in the one-parameter family $f_\omega(z) = 1 + \omega/z^2$ for a typical irregular value of the parameter $\omega \in K$ the Fatou-Julia set coincides with the whole sphere $\mathbf{P}^1$. Here (Proposition 2) the critical points 0 and ∞ move topologically transitively on the sphere. On the other hand, there exist values of the parameter $\omega \in K$ for which $F(f_\omega) = \mathbf{P}^1$, while $f_\omega | \omega(0)$ ($\omega(0)$ is the limit set of the trajectory $\{f_\omega^m 0\}$) is topologically conjugate to an arbitrary transitive shift on an invariant subset of a one-sided TMC with the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. We note that in all known examples ([2], [3]) the property $F(f) = \mathbf{P}^1$ was associated with the absorbing of the orbits of critical points by repellent cycles.

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Note added in proof. After this paper had gone to the printers the author received a preprint by Manet, Sade, and Sullivan, in which they also show that P-stability is typical.

References

[1] G.M. Levin, Irregular values of the parameter of polynomial mappings, Uspekhi Mat. Nauk SSSR **36**:6 (1981), 219–220. MR 83c:30019.
= Russian Math. Surveys **36**:6 (1981), 189–190.

[2] P. Montel, Leçons sur les familles normales, Gauthier-Villars, Paris 1927.
Translation: *Normal'nye semeistva analiticheskikh funktsii*, Moscow-Leningrad, ONTI 1936.

[3] J. Guckenheimer, Endomorphisms of the Riemann sphere, Proc. Symp. Pure Math. **14** (1970), 95–123. MR **43** # 500.

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