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ON TYPICAL BEHAVIOR OF THE TRAJECTORIES OF A RATIONAL MAPPING OF THE SPHERE

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1. In this note we establish some typical (in the sense of Lebesgue measure λ) properties of the trajectories of a rational mapping f. Qualitatively, these properties mean that the behavior of the orbits of the critical points determines the behavior of almost all trajectories. For example, if the trajectories of all the critical points converge to stable cycles, then almost all the trajectories have this property. Consequently, the measure of the set $\Omega(f)$ of nonwandering points is equal to 0 in this case (the first results in this direction were obtained by Fatou [1]; in [2] Brolin proved this theorem in the case when $\Omega(f)$ is totally disconnected). It is established that if $\Omega(f) \neq S^2$, then any invariant measure with positive characteristic exponent is singular to Lebesgue measure. Finally, it is shown that the property $\lambda(\Omega(f)) = 0$ is in a certain sense typical for second-degree polynomials.

2. Notation. Denote by $\Gamma_m = \{(z, w) \mid w = f^m z\}$ the Riemann surface of the function f^{-m}. Let π_m: $(z, w) \mapsto w$ be the natural projection of Γ_m on S^2, $\hat{f}^m$: $z \mapsto (z, f^m z)$ the corresponding diffeomorphism of S^2 onto Γ_m, and $\hat{f}^{-m}$: $\Gamma_m \to S^2$ the inverse diffeomorphism. We have $f^m = \pi_m \hat{f}^m$. The Euclidean metric on unit sphere $S^2 \subset \mathbf{R}^3$ is denoted by d, and the Lebesgue measure it generates by λ. The projection π_m induces a metric $\hat{d}$ on Γ_m. The balls in the metrics d and $\hat{d}$ will be denoted by $B(z, r)$ and $\hat{B}(z, r)$, respectively.

Let Z be the set of critical points of f, let $Z_m = \bigcup_{k=1}^m f^k Z$, $1 \leqslant m \leqslant \infty$, and let $\hat{Z}_m \subset \Gamma_m$ be the set of branch points of the surface Γ_m. We have that $\pi_m \hat{Z}_m = Z_m$. Finally, let $r_m(z) = \hat{d}(\hat{f}^m z, \hat{Z}_m)$.

A set Y is said to be a *wandering* set if $f^{-m} Y \cap Y = \varnothing$, $m \geqslant 1$. A point z is called a *wandering* point if it has a wandering neighborhood. The set of wandering points is denoted by $\mathfrak{B}$, and its complement (the set of nonwandering points) is denoted by Ω.

3. Typical behavior of orbits of points of the first kind. We say that a point z is *of the first kind* if its trajectory $\{f^m z\}$ does not land in the interior of the set Ω.

THEOREM 1. *For almost all points of the first kind*

$$\lim_{m \to \infty} r_m(z) = 0.$$

The next lemma, which follows from Koebe's distortion theorem, is the only property of rational mappings needed to prove Theorem 1.

LEMMA 1. *Let* $0 < \alpha < 1$. *There exists a number* $\omega(\alpha)$ *such that for any two measurable subsets* Y_1, Y_2 *in* $\hat{B}(\hat{f}^m z, \alpha r_m(z))$

$$\lambda(\hat{f}^{-m} Y_1)/\lambda(\hat{f}^{-m} Y_2) \leqslant \omega(\alpha)(\lambda(Y_1)/\lambda(Y_2))$$

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LEMMA 2. *For any $\delta > 0$ there exist a $\xi(\delta) > 0$ and a finite set Y_δ such that if $x \in \overline{\mathfrak{B}}$, then the disk $B(x, \delta)$ contains a wandering disk $B(y, \xi(\delta))$ centered at some point $y \in Y_\delta$.*

Let $\Gamma_m(\varepsilon) = (\pi_m^{-1}\overline{\mathfrak{B}}) \setminus \hat{B}(\hat{Z}_m, \varepsilon)$ and $X_m(\varepsilon) = \hat{f}^{-m}\Gamma_m(\varepsilon) \subset S^2$.

LEMMA 3. $\sum_{m=1}^{\infty} \lambda(X_m(\varepsilon)) < \infty$.

PROOF. For $\delta = \varepsilon/3$ we find a set $\hat{Y}_\delta \subset \Gamma_m(2\delta)$ such that $\pi_m \hat{Y}_\delta \subset Y_\delta$ and $\hat{B}(\hat{Y}_\delta, \delta) \supset \Gamma_m(\varepsilon)$. By Lemma 1,

$$\frac{\lambda(\hat{f}^{-m}\hat{B}(y, \delta))}{\lambda(\hat{f}^{-m}\hat{B}(y, \xi(\delta)))} \leqslant \omega\left(\frac{1}{2}\right) \frac{\lambda(\hat{B}(y, \delta))}{\lambda(\hat{B}(y, \xi(\delta)))} \equiv C(\delta), \qquad y \in \hat{Y}_\delta.$$

Consequently,

$$\lambda(X_m(\varepsilon)) \leqslant C(\delta) \sum_{y \in \hat{Y}_\delta} \lambda(\hat{f}^{-m}\hat{B}(y, \xi(\delta))) \leqslant C(\delta) \sum_{y \in Y_\delta} \lambda(f^{-m}B(y, \xi(\delta))).$$

But since $B(y, \xi(\delta))$ is a wandering disk, it follows that

$$\sum_{m=1}^{\infty} \lambda(f^{-m}B(y, \xi(\delta))) < \infty.$$

PROOF OF THEOREM 1. Obviously, $X(\varepsilon) = \overline{\lim}_{m \to \infty} X_m(\varepsilon) = \{z \mid z$ is a point of the first kind with $\overline{\lim}\, r_m(z) > \varepsilon\}$. It follows from Lemma 3 that $\lambda(X(\varepsilon)) = 0$.

Let X denote $\bigcup_{c \in Z} \omega(c)$, where $\omega(z)$ is the ω-limit set of the orbit of z. The next two results were announced in [3].

COROLLARY 1. *For almost all points of the first kind, $f^m z \to X$ as $m \to \infty$.*

COROLLARY 2. *Let $\Omega \neq S^2$. If the trajectory of each critical point converges to some stable cycle or "jumps" into an unstable cycle, then almost all the trajectories converge to stable cycles. Consequently, $\lambda(\Omega) = 0$ in this case.*

This result was obtained in [2] under the assumption that $X = \{a\}$, where a is a stable fixed point.

COROLLARY 3 [1]. *If the sequence $\{f^{m_k} z\}$ converges uniformly to $\{a\}$ in some domain $U \subset S^2$, then $a \in X$.*

4. Behavior of the orbits of points of the second kind. To describe the behavior of these orbits we need a known classification of the points of the sphere [1]. The whole sphere is the union of the set F of *irregular* points and the set $\mathfrak{R}$ of regular points. If $F \neq S^2$, then F is a perfect nowhere dense set. Further, $\mathfrak{R} = \mathfrak{R}_1 \cup \mathfrak{R}_2$, where in a neighborhood of a point $z \in \mathfrak{R}_1$ all the limit functions of the family $\{f^m\}$ are constant, while in a neighborhood of a point $z \in \mathfrak{R}_2$ all the limit functions are nonconstant. The orbit of a point $z \in \mathfrak{R}_2$ falls in a component σ of $\mathfrak{R}$ which is mapped conformally onto itself under the action of some power f^p. We call σ a *Siegel domain*.

THEOREM 2. *Let f be a polynomial. Then the following assertions are true*:

a) *The Siegel domains are simply connected.*

b) *The domain σ contains a periodic point ζ with period p such that $|Df^p(\zeta)| = 1$ and* arg *$Df^p(\zeta)$ is irrational.*

c) *There exists a conformal mapping φ of σ onto a disk such that $\varphi(f^p z) = \lambda\varphi(z)$, where $\lambda = Df^p(\zeta)$.*

d) *There are finitely many Siegel domains.*

e) $\Omega = F \cup \mathrm{Per}^s(f) \cup (\cup_1^N \sigma_i)$, *where* $\mathrm{Per}^s(f)$ *is the set of stable cycles, and* $\{\sigma_i\}_1^N$ *is the set of Siegel domains.*

f) $\partial\sigma_i \subset X$.

g) *The set of points of the second kind coincides with* $\mathfrak{R}_2$.

Thus, the orbit of a point of the second kind falls on a closed analytic curve S which is invariant under f^p, and $f^p | S$ is analytically conjugate to an irrational rotation of the circle. We mention also that there are no Siegel domains for a rational mapping in general position.

5. Singularity of invariant measures. Let $\rho_m(z)$ denote the radius of the maximal disk about z in which f^m is univalent. This defines a decreasing sequence of continuous functions. The zeros of ρ_m are critical points of f^m. Koebe's distortion theorem gives us the following assertion.

LEMMA 4. a) $r_m(z) \geqslant \rho(f^m z)\eta(\rho_{m+1}(z)/\rho_m(z))$, *where* $\eta(\alpha) = ((1-\alpha)/(1+\alpha))^2$.

b) $\rho_m(z) \leqslant C|Df^m(z)|^{-1}$.

c) $\{\rho_m\}$ *is a supermultiplicative sequence, i.e.,*

$$\rho_{m+n}(z) \geqslant K\rho_m(f^n z)\rho_n z.$$

Let $\chi(z) = \overline{\lim}_{m\to\infty} m^{-1}\ln|Df^m(z)|$ denote the *characteristic exponent* of f at the point z.

COROLLARY 4. *The* ω*-limit set of the orbit of each point of the first kind with positive characteristic exponent contains some critical point.*

COROLLARY 5. *If* X *does not contain any irregular critical points, then* $\chi(z) \leqslant 0$ *for almost all points of the first kind.*

We state our main result.

THEOREM 3. *Assume that* $F \neq S^2$. *Let* ν *be an invariant measure such that* $\chi(z) > 0$ *almost everywhere with respect to* ν. *Then* ν *is singular to Lebesgue measure.*

LEMMA 5. *The function* $\ln(1/\rho(z))$ *is* ν*-integrable.*

LEMMA 6. *Suppose that* $\varphi_k \to 0$ *in the measure* ν *and* $|\varphi_k(z)| \leqslant \varphi(f^k z)$, *where* $\varphi \in L_1(\nu)$. *Then* $\int|\varphi_k|\,d\nu \to 0$.

PROOF OF THEOREM 3. It follows from Lemma 4a), that $\ln(\rho_m/\rho_{m+1}) \to 0$ in the measure ν. Since

$$\ln(\rho_m(z)/\rho_{m+1}(z)) \leqslant \ln(1/\rho(f^m z)) + \ln(1/K)$$

(Lemma 4b)), we have $\int \ln(\rho_m/\rho_{m+1})\,d\nu \to 0$ (Lemmas 5 and 6). This gives us that

$$(1/m)\int \ln(1/\rho_m)\,d\nu \to 0.$$

On the other hand,

$$\underline{\lim}\,(1/m)\int \ln(1/\rho_m)\,d\nu \geqslant \underline{\lim}\,(1/m)\int \ln|Df^m(z)|\,d\nu \geqslant \int \chi(z)\,d\nu > 0$$

[Lemma 4b) and Fatou's lemma], a contradiction.

COROLLARY 6. *If* $F \neq S^2$, *then any invariant ergodic measure with positive entropy is singular to Lebesgue measure.*

Indeed, an ergodic measure with positive entropy has a positive characteristic exponent [4].

6. Consider the family $f_w(z) = z^2 + w$, $w \in \mathbf{C}$. We associate with it the sequence of polynomials $\varphi_m(w) = f_w^m(0)$, which follows along the trajectory of the critical point 0. Denote by K the set of irregular points of the sequence $\{\varphi_m\}$.

THEOREM 4. *The following hold for a typical* $w \in K$ *(in the category sense)*:

a) $f_w^m z \to \infty$ λ-*almost everywhere*.

b) *A critical point is irregular*.

c) *A critical point moves over the set* $\Omega(f_w)$ *in a topologically transitive way*.

Parts b) and c) of Theorem 4 extend to any holomorphic families of rational mappings. Theorem 4 shows that the property $\lambda(\Omega) = 0$ can be combined with very complicated behavior of a critical point.

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