6. Yu. M. Berezanskii and Yu. S. Samoilenko, Preprint No. 79.16, Inst. Mat., Kiev (1979).
7. Yu. L. Daletskii, in: Dokl. Akad. Nauk SSSR, 227, No. 4, 787-794 (1978).
8. N. N. Frolov, "Self-adjointness of elliptic operators with an infinite number of variables," Funkts. Anal. Prilozhen., 14, No. 1, 85-86 (1980).
9. L. Gross, "Logarithmic Sobolev inequalities," Am. J. Math., 97, No. 4, 1061-1083 (1975).
10. S. Albeverio and R. Hoegh-Krohn, "Energy forms, Hamiltonians, and distorted Brownian paths," J. Math. Phys., 18, No. 5, 907-917 (1977).

THE MAXIMUM-ENTROPY MEASURE OF A RATIONAL ENDOMORPHISM OF THE RIEMANN SPHERE

M. Yu. Lyubich

UDC 517.53

Let f(z) be a rational function of a complex variable regarded as an analytic endomorphism of the Riemann sphere S^2. In the present note an existence and uniqueness theorem is established for the maximum-entropy measure μ of the endomorphism f. We prove that the roots of the equation $f^m z = \varphi(z)$ are asymptotically equally distributed with respect to the measure μ, where φ is an arbitrary rational function, apart, possibly, from two exceptional constants. In particular, the full inverse images $f^{-n}c$ (where c is not an exceptional constant), as well as the periodic points of the endomorphism f, are asymptotically equidistributed according to the measure μ.

We shall construct a maximum-entropy measure by investigating a special operator in the space of continuous functions. Let A be a bounded operator in the complex Banach space $\mathfrak{B}$. We consider its subspaces: $\mathfrak{B}_u$, the closure of the linear hull of the eigenvectors of the operator A corresponding to unitary eigenvalues ($|\lambda| = 1$); $\mathfrak{B}_0 = \{\varphi \in \mathfrak{B} \mid \|A^m\varphi\| \to 0 \ (m \to \infty)\}$.

Definition. The operator A is called almost periodic if the orbit $\{A^m\varphi\}_{m=1}^{\infty}$ of any vector $\varphi \in \mathfrak{B}$ is strongly precompact.

Theorem on the Decomposition of a Unitary Discrete Spectrum (see [1]). If A is an almost periodic operator in the Banach space $\mathfrak{B}$, then we have the direct decomposition $\mathfrak{B} = \mathfrak{B}_0 + \mathfrak{B}_u$.

COROLLARY. Let the almost-periodic operator A have no unitary eigenvalues other than 1, and let the subspace of invariant vectors be one-dimensional (= Lin{h}). Then there exists an A^*-invariant functional $\mu \in \mathfrak{B}^*$, μ(h) = 1 such that for any vector $\varphi \in \mathfrak{B}$ $\|A^m\varphi - \mu(\varphi)h\| \to 0 \ (m \to \infty)$.

We apply this result to the following operator: $A: C(S^2) \to C(S^2)$ in the space of continuous functions:

$$A\varphi(z) = \frac{1}{n}\sum_{\zeta \in f^{-1}z} \varphi(\zeta)$$

$(n = \deg f,\ \varphi \in C(S^2),\ z \in S^2)$, where the roots of the equation $f\zeta = z$ are counted with their numbers of multiplicity. We shall use the concepts of an "irregular point" and an "exceptional point" as defined, e.g., in [2]. By F we denote the compact of irregular points. By **1** we denote the function identically equal to 1. We put $\|\varphi\|_K = \sup_{z \in K} |\varphi(z)|$.

THEOREM 1. There exists an A^*-invariant probability measure† μ on the sphere S^2 such that

$$\left\| A^m\varphi - \left(\int \varphi\, d\mu\right)\mathbf{1} \right\|_K \to 0 \quad (m \to \infty)$$

†By "measure" we shall always mean a complex Borel regular measure.

Tashkent State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 16, No. 4, pp. 78-79, October-December, 1982. Original article submitted July 24, 1981.

for any compact $K \subset S^2$ containing no exceptional points of the function f, and any $\varphi \in C(S^2)$.

We denote by δ_ζ a unit mass concentrated as the point ζ. Let $\varphi(\zeta)$ be a rational function. We consider the following measures:

$$\mu_{m,\varphi} = \frac{1}{n^m} \sum_{f^m\zeta=\varphi(\zeta)} \delta_\zeta, \quad \mu'_{m,\varphi} = \frac{1}{\chi_{m,\varphi}} \sum_{f^m\zeta=\varphi(\zeta)} \delta_\zeta,$$

where the roots of the equation $f^m\zeta = \varphi(\zeta)$ are counted, respectively, with and without their degrees of multiplicity, and $\chi_{m,\varphi}$ is the number of roots counted, ignoring their degrees of multiplicity.

THEOREM 2. For all rational functions φ, apart, possibly, from two exceptional constants, the measures $\mu_{m,\varphi}$ and $\mu'_{m,\varphi}$ converge weakly to a certain probability measure μ independent of φ.

If $\varphi \equiv c$, where the constant c is not exceptional, the convergence $\mu_{m,\varphi} \to \mu$ $(m \to \infty)$ follows from Theorem 1 with $K = \{c\}$. This result was obtained in [3] for a polynomial f by methods of potential theory, which apparently do not work in the general case of a rational function.

COROLLARY. The periodic points of a rational endomorphism are asymptotically equidistributed according to the measure μ.

Proposition. a) The carrier of the measure μ is the set F of irregular points. b) If $F \neq S^2$, the measure μ and the Lebesgue measure are mutually singular.

THEOREM 3. a) The measure μ is f-invariant. b) The dynamical system (f, μ) is exact. c) The entropy $h_\mu(f) = \log n$.

It was proved in [4] and [5] that the topologic entropy $h(f) = \ln n$. Thus, μ is the maximum-entropy measure of the endomorphism f. We remark that the mere existence of a maximum-entropy measure of a rational endomorphism can be deduced from fairly general considerations. Namely, asymptotically h-expansive endomorphisms were defined in [6], and for these the existence of a maximum-entropy measure was established.

THEOREM 4. Rational endomorphisms of the sphere are asymptotically h-expansive.

Remark. A rational endomorphism (and even its restriction to the set F) is, generally speaking, not h-expansive (for the definition, see [7]).

THEOREM 5. A rational endomorphism of the Riemann sphere has a unique maximum-entropy measure.

The following lemmas are used to prove Theorem 5:

LEMMA 1 (see [8]). Let $f: X \to X$ be a continuous endomorphism of a metric compact X, and let μ be an f-invariant ergodic probability measure. Let $Y \subset X$ and $\mu(Y) > 0$. Then $h_\mu(f) \leqslant h_f(Y)$, where $h_f(Y)$ is the topologic entropy of f with respect to Y, as defined in [7].

Let $E = \{1, \ldots, n\}$, $0 \leqslant \varkappa \leqslant 1$. We consider the set $G_m(\varkappa) \subset E^m$ of all sequences of length m in which the element 1 appears at least $\varkappa m$ times.

LEMMA 2. Let $\varkappa > 1/n$. Then there exist $K > 0$ and $0 < \theta < n$ such that $|G_m(\varkappa)| \leqslant K\theta^m$.

We shall say that a system of sets distinguishes the measures μ and ν if $\mu(Z) \neq \nu(Z)$ for some set of the system.

LEMMA 3. Let the measure ν be mutually singular with the measure μ constructed in Theorem 1. There exists such a natural r and such a covering $D = \{D_i\}_{i=1}^{n^r}$ of the sphere by closed sets that $D_i = \overline{D_i^0}$; 2) $D_i^0 \cap D_j^0 = \varnothing$ $(i \neq j)$; 3) $(\mu + \nu)(\bigcup \partial D_i) = 0$; 4) the intersection of D_i^0 and $f^{-r}z$ is composed of not more than one point $(1 \leqslant i \leqslant n^r,\ z \in S^2)$; 5) the covering D distinguishes μ from ν.

Added in Print. The results in this note were presented at the Fifteenth Voronezh Mathematical Winter School (January, 1981); the theses are deposited with the All-Union Scientific and Technical Information Institute (VINITI) (No. 5691, pp. 65-66).

LITERATURE CITED

1. K. Deleeuw and I. Glicksberg, "Applications of almost-periodic compactifications," Acta Math., 105, 63-97 (1961).
2. P. Montel, "Leçons sur les Familles Normales de Fonctions Analytiques et leurs Applications (Lectures on Normal Families of Analytic Functions and their Applications)," Gauthier-Villars, Paris (1927).
3. H. Brolin, Arkiv Math., 6, No. 2, 103-144 (1965).
4. M. Gromov, "Entropy of holomorphic maps," Preprint, University of California (1980).
5. M. Yu. Lyubich, Funkts. Anal., 15, No. 4, 83-84 (1981).
6. M. Misiurewicz, Bull. Acad. Pol. Sci., 21, 903-910 (1973).
7. R. Bowen, Trans. Am. Math. Soc., 164, 323-331 (1972).
8. R. Bowen, Trans. Am. Math. Soc., 184, 125-136 (1973).

INVARIANT ORDERINGS IN SIMPLE LIE GROUPS. THE SOLUTION TO É. B. VINBERG'S PROBLEM

G. I. Ol'shanskii

UDC 519.46

Let $\mathfrak{g}$ be a real simple noncompact Lie algebra; $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition; G and K simply connected Lie groups corresponding to the algebras $\mathfrak{g}$ and $\mathfrak{k}$. We assume that D = G/K is a bounded symmetric domain. It is known [1] that in this case the set Con of all closed convex G-invariant cones in $\mathfrak{g}$ distinct from {0} and $\mathfrak{g}$ is not empty. There is a maximal and a minimal cone, C_{max} and C_{min} in Con; they are unique up to a multiplication by −1. The set Con has been described in [2] and [3], [4] (also see Sec. 2).

Let $C \in$ Con and let P = P(C) be the closed semigroup in G topologically generated by the set exp C. It defines an invariant partial ordering in G for which $\{g | g \geqslant e\}$ coincides with P. The ordering is nontrivial if $P \neq G$. É. B. Vinberg [1] proved that $P \neq G$ and $P \cap P^{-1} = \{e\}$ for $C = C_{min}$ and raised the question whether it was true for all $C \in$ Con. S. Paneitz [3] proved that $P(C) \neq G$ for all $C \in$ Con if D is a classical domain of tubular type; the definition can be found in [5].

THEOREM. (i) $P(C) \neq G \Leftrightarrow C \subseteq \pm C_0$ (the cone $C_0 \in$ Con is defined in Sec. 3), with $C_0 = C_{max}$ for tubular D and $C_0 \neq C_{max}$ for nontubular D. (ii) If $P = P(C) \neq G$, then $P \cap P^{-1} = \{e\}$ and the "tangent cone" C(P) (cf. [1]) coincides with C.

If $\mathfrak{g} \neq sp(n, \mathbf{R})$, su (2, 1), EIII, then $C_0 \neq C_{min}$ and the existence of a continuum of invariant orderings in G follows from (ii).

1. Notation. $\mathfrak{z} \subset \mathfrak{k}$ is the one-dimensional center; $\mathfrak{h} \subset \mathfrak{k}$, Cartan subalgebra; $\mathfrak{h}_{Re} = i\mathfrak{h}$; W, Weyl group for $(\mathfrak{k}_C, \mathfrak{h}_C)$; $\Delta^+ \subset \mathfrak{h}_{Re}$, set of noncompact positive roots; $\{\alpha_1, \ldots, \alpha_r\} \subset \Delta^+$ family of pairwise orthogonal roots, where r = rank D; E_k and E_{-k}, root vectors corresponding to the roots $\pm\alpha_k (k = 1, \ldots, r)$ normed in such a way that $H_k = [E_k, E_{-k}] \in \mathfrak{h}_{Re}$; $X_k = E_k + E_{-k} \in \mathfrak{p}$, $i(E_k - E_{-k}) \in \mathfrak{p}$, $\alpha_k(H_k) = 2$; Z, element of $i\mathfrak{z}$ defined by the condition that $\alpha(Z) = 2 \ \forall \alpha \in \Delta^+$; $H_0 = Z - H_1 - \ldots - H_r$); and $\langle , \rangle$, Killing form.

2. Information from [1], [2]. Each $C \in$ Con lies in $\overline{G \cdot \mathfrak{h}}$ and is uniquely determined by the cone $c = iC \cap \mathfrak{h}_{Re}$ (we write $C \leftrightarrow c$); here c contains either Z (then we write $C \in Con^+$) or −Z. If $C^* = \{X \in \mathfrak{g} | -\langle X, Y \rangle \geqslant 0 \ \forall Y \in C\}$, $c^* = \{X \in \mathfrak{h}_{Re} | \langle X, Y \rangle \geqslant 0 \ \forall Y \in c\}$, then $C \leftrightarrow c$ implies $C^* \leftrightarrow c^*$. Let $c_{min} \subset \mathfrak{h}_{Re}$ be the cone spanned by Δ^+ and $c_{max} = c^*_{min}$; then $C_{min} \leftrightarrow c_{min}$ and $C_{max} \leftrightarrow c_{max}$. If $c \subset \mathfrak{h}_{Re}$ is a closed convex cone, then ($c \leftrightarrow C$ for some $C \in Con^+) \Leftrightarrow (Wc = c, c_{min} \subseteq c \subseteq c_{max})$.

3. The Definition of the Cone C_0. $C_0 \leftrightarrow c_0$, where c_0^* is the cone spanned by c_{min} and WH_0.

If D is of tubular type, then $H_0 = 0$ and $C_0 = C_{max}$. If D is not of tubular type, then $H_0 \neq 0$, $H_0 \in c_{max}$, $H_0 \notin c_{min}$, and $C_0 \neq C_{max}$.

Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 16, No. 4, pp. 80-81, October-December, 1982. Original article submitted June 4, 1981.

0016-2663/82/1604-0311$07.50