# MEASURABLE DYNAMICS OF THE EXPONENTIAL

M. Yu. Lyubich

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## 1. Formulation of the Results

We consider the exponential transformation f: $z \to e^z$ of the complex plane $\mathbf{C}$. By $f^n$ we denote its n-th iterate. The investigation of the trajectories $\{f^n z\}_{n=0}^{\infty}$ of the obtained dynamical system is a meaningful problem, which in the last years has attracted great interest. In this paper we investigate this system from the point of view of the typical properties with respect to the Lebesgue measure.

The starting point for us has been Sullivan's question on the ergodicity of the exponential with respect to the plane Lebesgue measure [1, p. 45]. Ergodicity means that there exists no partition of the plane $\mathbf{C}$ into two measurable invariant subsets of positive measure. We obtain a negative answer to this question.

THEOREM 1.1. The exponential transformation f: $z \to e^z$ of the complex plane is not ergodic. Each ergodic component has measure zero.

Thus, the transformation f has a continuum of ergodic components.*

A set $X \subset \mathbf{C}$ is said to be wandering if $f^n X \cap f^m X = \varnothing$ for $n > m \geqslant 0$. In [2] it is proved that f does not have wandering sets of positive measure on which all iterations $f^n$ are injective. As shown by the next result, the last condition is essential.

THEOREM 1.2. The exponential transformation f: $z \to e^z$ has a wandering set of positive measure.

In [2] one formulates the problem of the existence of an absolutely continuous invariant measure for f. A partial solution of this problem is given by

THEOREM 1.3. The exponential transformation f: $z \to e^z$ does not have an absolutely continuous invariant measure that is finite on compacta.

Our subsequent results give a detailed description of the behavior of a typical† trajectory of the exponential. The first step in this direction has been made in [3, 4]. In order to formulate the result of these papers, we introduce the following notations: $\omega_f(z)$ [or simply $\omega(z)$ if it is clear what mapping is considered] is the set of the points of the complex plane $\mathbf{C}$ which are limit points for the trajectories $\{f^n z\}_{n=0}^{\infty}$, $\alpha_n = f^n 0$.

THEOREM 1.4 [3, 4]. $\omega_f(z) = \{\alpha_n\}_{n=0}^{\infty}$ for almost all $z \in \mathbf{C}$.

Thus, a typical trajectory of the exponential is not recurrent [the trajectory of the point z is said to be recurrent if $z \in \omega(z)$ ]. We describe in detail the content of Theorem 1.4. For almost all points $z \in \mathbf{C}$ there exist sequences of natural numbers $\ell_s$, $k_s$, $t_s$ (depending on z) such that $k_s \to \infty$, $l_{s+1} = l_s + k_s + t_s + 2$ and $\left|f^{l_s+i} z - \alpha_i\right| \leqslant 1$ $(i = 0, 1, \ldots, k_s - 1)$, $\left|f^{l_s+k_s} z - \alpha_{k_s}\right| > 1$, $\operatorname{Re}\left(f^{l_s+k_s+i} z\right) \geqslant \alpha_{k_s+i-1}$ $(i = 1, \ldots, t_s)$,‡ $\operatorname{Re}\left(f^{l_s+k_s+t_s+1} z\right) \leqslant -\alpha_{k_s+t_s}$.

We consider the circumference $\mathbf{T}$; it is convenient to identify it with the quotient $\mathbf{R}/2\pi\mathbf{Z}$. In this case, to the continuous functions on $\mathbf{T}$ there correspond continuous $2\pi$-periodic func-

*An ergodic component of positive measure is a completely invariant set Z, mes Z > 0, on which the transformation f: Z → Z is ergodic. Ergodic components of measure zero are defined correctly within the framework of the theory of measurable partitions (see [8]) as elements of the finest measurable partition into completely invariant sets.
†Here and in the sequel, the typicalness of some property means that the property holds for almost all points with respect to the Lebesgue measure.
‡If $t_s = 0$, then this section of the trajectory is missing.

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tions on $\mathbf{R}$. Points $a_i \in \mathbf{R}$ will be necessarily considered as points of the circumference $\mathbf{T}$, without introducing special notations. We say that a sequence $\{a_i\}$ is uniformly distributed on the circumference $\mathbf{T}$ if for any continuous function $\varphi \in C(\mathbf{T})$ we have the equality

$$\lim_{j\to\infty} \frac{1}{j} \sum_{i=1}^{j} \varphi(a_i) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(x)\, dx. \tag{0}$$

Instead of continuous functions $\varphi$ one can take the characteristic functions $\chi_I$ of all possible intervals $I \subset \mathbf{T}$. In this case the equality (0) has the following intuitive meaning: the frequency of the hits of an interval I by the sequence $\{a_i\}$ is proportional to the length of the interval.

THEOREM 1.5. For almost all $z \in \mathbf{C}$ we have

a) the sequence $\{\arg f^n z \colon n \in \cup [l_s + k_s + 1,\ l_{s+1}]\}$ is uniformly distributed on the circumference $\mathbf{T}$,

b) $\lim_{n\to\infty} \frac{1}{n} \sum_{s=1}^{n} t_s = 1,$

c) $\lim_{s\to\infty} (k_s/s) = 3.5.$

The proofs in this paper have a constructive character. For their understanding we need very unpretentious preliminary information. Namely:

Misiurewicz's Theorem [5]. The Julia set J(f) coincides with the entire plane $\mathbf{C}$.

COROLLARY 1.1. For any domain D and any compactum K, not containing 0, there exists N such that $f^n D \supset K$ for $n \geqslant N$.

COROLLARY 1.2. The exponential transformation has a dense trajectory $\{f^n z\}_{n=0}^{\infty}$. The set of points with dense trajectory is massive.*

In fact, we shall make use only of Corollary 1.1. Corollary 1.2 is in a curious contrast with Theorem 1.4. This is another illustration of the difference between the category and metric points of view.

The basic analytic technique in the present paper is Koebe's distortion theorem. In order to formulate it, we introduce the notation: $B(a, \rho) = \{z\colon |z - a| < \rho\}$.

Koebe's Distortion Theorem [6]. Suppose that $\varphi(z)$ is a univalent holomorphic function in the circle $B(a, \rho)$; $0 < \eta < 1$. Then for $z \in B(a, \eta\rho)$ we have the estimates:

a) $\frac{|\varphi'(a)|\,\eta\rho}{(1+\eta)^2} \leqslant |\varphi(z) - \varphi(a)| \leqslant \frac{|\varphi'(a)|\,\eta\rho}{(1-\eta)^2},$

b) $\frac{1-\eta}{(1+\eta)^3} \leqslant \frac{|\varphi'(z)|}{|\varphi'(a)|} \leqslant \frac{1+\eta}{(1-\eta)^3},$

c) $\left|\arg \frac{\varphi'(z)}{\varphi'(a)}\right| \leqslant 2 \ln \frac{1+\eta}{1-\eta}.$

From part a) there follows

The 1/4-Theorem. The domain $\varphi B(a, \rho)$ contains a circle with center at the point $\varphi(a)$ and radius $\frac{1}{4} |\varphi'(a)| \rho$.

The role of the points $\alpha_n = f^n 0$ consists in the fact that the singular points of the inverse function $f^{-n}$ are $\alpha_0, \ldots, \alpha_{n-1}$. Therefore, if a simply connected domain U does not contain these points, then $f^{-n} U = \bigcup_{i=1}^{\infty} D_i$, where $D_i \cap D_j = \varnothing$ $(i \neq j)$, and the domain $D_i$ under the action of $f^n$ is mapped univalently onto U. The inverse function $\varphi\colon U \to D_i$ is called a (single-valued) branch of the function $f^{-n}$. We shall apply Koebe's theorem to the branches of the inverse functions. It has been used in a similar manner for the first time in [7]. Finally, we shall apply systematically Lebesgue's theorem on density points, while in the proof of Theorem 1.5 we shall apply the strong law of large numbers for a stationary sequence of independent random variables (the ergodic theorem for the Bernoulli shift).

*A set $X \subset \mathbf{C}$ is said to be massive if $\mathbf{C} \backslash X$ is of first Baire category.

First we prove Theorem 1.4 (in Sec. 3). Then in Sec. 4 we shall prove Theorems 1.1-1.3. Theorem 1.5 is proved in Sec. 5. It relates to Sec. 3 and does not depend on Sec. 4.

We introduce notations: $\mathbf{N}=\{0, 1, 2, \ldots\}$ is the natural series; $K(a, \rho)$ is the square with center at the point a and side $2\rho$. We shall write both $\exp z$ and $e^z$.

## 2. Preliminary Constructions and Estimates

LEMMA 2.1. For every $\varepsilon > 0$ there exists N such that for $k \geqslant N$ we have the estimates

$$\alpha_{k-1}\alpha_k \leqslant \prod_{i=1}^{k}(e^{-1}\alpha_i) \leqslant \prod_{i=1}^{k}(e\alpha_i) \leqslant \alpha_k^{1+\varepsilon}.$$

Proof. We select a such that $ex(\ln x)^{1+\varepsilon} \leqslant x^{1+\varepsilon}$ for $x \geqslant a$. Let $\alpha_n \geqslant a$. We show by induction on $k \geqslant n$ that $\prod_{i=n}^{k}(e\alpha_i) \leqslant \alpha_k^{1+\varepsilon}$. For k = n this is obvious. If this holds for k − 1, then $\prod_{i=n}^{k}(e\alpha_i) \leqslant e\alpha_{k-1}^{1+\varepsilon}\alpha_k = e\alpha_k(\ln\alpha_k)^{1+\varepsilon} \leqslant \alpha_k^{1+\varepsilon}$. Now we set $A = \prod_{i=1}^{n-1}(e\alpha_i)$ and we find $b \geqslant a$ such that $eAx \times (\ln x)^{1+\varepsilon} \leqslant x^{1+\varepsilon}$ for $x \geqslant b$. Let $\alpha_{k-1} \geqslant b$. Then $\prod_{i=1}^{k}(e\alpha_i) \leqslant A\alpha_{k-1}^{1+\varepsilon}e\alpha_k = Ae\alpha_k(\ln\alpha_k)^{1+\varepsilon} \leqslant \alpha_k^{1+\varepsilon}$. The right-hand inequality of the chain is proved.

In order to prove the left-hand inequality, we mention that $e^{-1}\alpha_i \geqslant 1$ for $i \geqslant 2$. Let $\alpha_{k-2} \geqslant e^4$. Discarding the factors $e^{-1}\alpha_i$ $(i = 2,\ldots,k-3)$, we obtain

$$\prod_{i=1}^{k}(e^{-1}\alpha_i) \geqslant (e^{-4}\alpha_{k-2})\,\alpha_{k-1}\alpha_k \geqslant \alpha_{k-1}\alpha_k.$$

LEMMA 2.2. For each point $z \in \mathbf{C}$ there exists an index $k = k(z)$ for which $|f^{i+1}z - \alpha_i| < 1$ $(i = 0,\ldots,k-1)$, $|f^{k+1}z - \alpha_k| \geqslant 1$.

Proof. If $\zeta \in B(\alpha_i, 1)$, then $|f\zeta| \geqslant |\zeta| \inf\{|f'(t)|: t \in B(\alpha_i, 1)\} \geqslant |\zeta| \exp(\alpha_i - 1) = |\zeta| e^{-1}\alpha_{i+1}$. We assume that $f^k(fz) \in B(\alpha_k, 1)$ $(k = 0, 1,\ldots)$. Then $1 > |f^k(fz)| \geqslant \prod_{i=1}^{k}(e^{-1}\alpha_i)|fz| \geqslant \alpha_k|fz|$ (the last inequality is satisfied for large k by virtue of Lemma 2.1). Since $fz \neq 0$, the inequality $1 > \alpha_k|fz|$ is violated for large k. The contradiction proves the lemma.

Now we define a mapping $S: \mathbf{C} \to \mathbf{C}$ in the following manner: $Sz = f^{k(z)+1}z$, where k(z) is defined in Lemma 2.2. This mapping will play an important role in the subsequent investigation. We mention that if $\operatorname{Re} z \geqslant 0$, then $Sz = fz$.

Let $k \in \mathbf{Z}$, $\mu \in \{\pm 1\}$. We define vertical strips $V_k$ and sets $V_k^\mu$ which are unions of rectangles:

$$V_k = \{x + iy: \alpha_{|k|-1} + 3\alpha_{|k|-2} \leqslant |x| \leqslant \alpha_{|k|} - 3\alpha_{|k|-1},\ \operatorname{sgn} x = \operatorname{sgn} k\},$$
$$V_k^\mu = \{x + iy \in V_k: |\cos y| \geqslant 2/\alpha_{|k|-1},\ \operatorname{sgn}(\cos y) = \mu\} \subset V_k.$$

LEMMA 2.3. There exists N such that for k > N we have the inclusions:

1) $SV_k^+ = fV_k^+ \subset V_{k+1}$;

2) $SV_k^- = fV_k^- \subset V_{-(k+1)}$;

3) $SV_{-k}^+ = f^{k+2}V_{-k}^+ \subset V_{k+2}$;

4) $SV_{-k}^- = f^{k+2}V_{-k}^- \subset V_{k+1}$.

Proof. 1, 2. Let $z = x + iy \in V_k^+ \cup V_k^-$. Then Sz = fz and $|\operatorname{Re}(Sz)| = e^x|\cos y| \geqslant \exp(\alpha_{k-1} + \alpha_{k-2})(2/\alpha_{k-1}) = 2\alpha_k > \alpha_k + 3\alpha_{k-1}$.* On the other hand, $|\operatorname{Re}(Sz)| \leqslant \exp(\alpha_k - \alpha_{k-1}) = \alpha_{k+1}/\alpha_k < \alpha_{k+1} - 3\alpha_k$. Since $\operatorname{sgn}(\operatorname{Re}(Sz)) = \mu$ for $z \in V_k^\mu$, the inclusions 1, 2 are proved.

3, 4. Let $z = x + iy \in V_{-k} \cup V_{-k}^+$. We assume that $|f^j(fz) - \alpha_j| < 1$ $(j = 0, 1, \ldots, i-1)$. Then $|f^i(fz) - \alpha_i| \leqslant |fz| \prod_{j=0}^{i-1} \max\{|f'(\zeta)|: \zeta \in B(\alpha_j, 1)\} \leqslant |fz| \prod_{j=0}^{i-1} \exp(\alpha_j + 1) = |fz| \prod_{j=1}^{i}(e\alpha_j)$.

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*We shall not state continuously that certain inequalities hold only for sufficiently large k.

Let ε > 0. By virtue of Lemma 2.1, the last expression does not exceed $|fz|\alpha_{i-2}^{1+\varepsilon/2}e^2\alpha_{i-1}\alpha_i \leqslant |fz|\alpha_{i-2}^{1+\varepsilon}\alpha_{i-1}\alpha_i$. Further, since $-\mathrm{Re}\, z \geqslant \alpha_{k-1}+3\alpha_{k-2}$, we have $|fz| \leqslant \exp(-\alpha_{k-1}-3\alpha_{k-2}) = \alpha_k^{-1}\alpha_{k-1}^{-3}$. Consequently,

$$|f^i(fz)-\alpha_i| \leqslant \alpha_k^{-1}\alpha_{k-1}^{-3}\alpha_{i-2}^{1+\varepsilon}\alpha_{i-1}\alpha_i. \tag{1}$$

For $i \leqslant k$ we obtain $|f^i(fz)-\alpha_i| \leqslant \alpha_{k-2}^{1+\varepsilon}\alpha_{k-1}^{-2} < 1$. By induction, the last inequality is satisfied for all i = 0, 1,...,k [and thus, also inequality (1) for i = 0, 1,...,k + 1].

On the other hand, $|f^{k+1}(fz)-\alpha_{k+1}| \geqslant |fz|\prod_{i=0}^{k}\min\{|f'(\zeta)| : \zeta \in B(\alpha_i, \quad 1)\} \geqslant |fz|\prod_{i=0}^{k}\exp(\alpha_i - 1) = |fz|\prod_{i=1}^{k+1}(e^{-1}\alpha_i)$. By virtue of Lemma 2.1, the last expression is not smaller than $|fz|\alpha_k\alpha_{k+1}$. Since $-\mathrm{Re}\, z \leqslant \alpha_k - \alpha_{k-1}$, we have $|fz|\alpha_k\alpha_{k+1} \geqslant \exp(-\alpha_k+\alpha_{k-1})\alpha_k\alpha_{k+1} = \alpha_k^2$. Thus, $|f^{k+2}z - \alpha_{k+1}| > \alpha_k^2 > 1$. This shows that $Sz = f^{k+2}z$ and

$$|Sz-\alpha_{k+1}| > \alpha_k^2. \tag{2}$$

Further, setting in inequality (1) i = k + 1, we find

$$|Sz-\alpha_{k+1}| \leqslant \alpha_{k+1}/\alpha_{k-1}^{2-\varepsilon} = o(\alpha_{k+1}/\alpha_{k-1}) \quad (k \to \infty). \tag{3}$$

In the circle $\left\{\zeta : |\zeta-\alpha_{k+1}| < \frac{1}{2}\alpha_{k+1}\right\}$ all the branches of the function $f^{-(k+1)}$ are defined (since this circle does not touch the points $\alpha_k$). We consider that one for which $f^{-(k+1)}\alpha_{k+1}=0$. From Koebe's theorem and from equality (3) there follows that $|\arg(f^{-(k+1)})'(Sz)| = o(1/\alpha_{k-1})$. Therefore, $|\arg(Sz-\alpha_{k+1}) - \arg(fz)| = o(1/\alpha_{k-1})$. Since arg (fz) = Im z = y, while $|\cos y| \geqslant 2/\alpha_{k-1}$, denoting arg $(Sz - \alpha_{k+1})$ by θ, we obtain

$$|\cos\theta| \geqslant 1/\alpha_{k-1}, \quad \mathrm{sgn}(\cos\theta) = \mathrm{sgn}(\cos y). \tag{4}$$

From (2), (4) we find

$$|\mathrm{Re}(Sz)-\alpha_{k+1}| \geqslant \alpha_k^2/\alpha_{k-1} > 3\alpha_k. \tag{5}$$

Finally, it is obvious that from (3) there follows

$$\alpha_k + 3\alpha_{k-1} < \mathrm{Re}(Sz) < \alpha_{k+2} - 3\alpha_{k-1}. \tag{6}$$

From (4)-(6) there follow the required inclusions 3, 4. Lemma 2.3 is proved.

LEMMA 2.4. If $z \in V_k^\mu$, then $|S'(z)| \geqslant \frac{1}{5}\alpha_{|k|-1}$.

Proof. By virtue of Lemma 2.3, the distance from the point $Sz = f^pz$ to the set $\{\alpha_n\}_{n=0}^\infty$ is not less than $3\alpha_{|k|-1}$. Therefore, in the circle $B=B(f^pz, 3\alpha_{|k|-1})$ all the branches of the inverse function $f^{-p}$ are single-valued. We consider those for which $f^{-p}(Sz) = z$. The domain $f^{-p}B$ does not contain vertical segments of length 2π. On the other hand, by the 1/4-theorem, this domain covers a circle of radius $\frac{3\alpha_{|k|-1}}{4}|(f^{-p})'(Sz)|$. Consequently, $|(f^p)'(z)| \geqslant \frac{3\alpha_{|k|-1}}{4\pi} \geqslant \frac{1}{5}\alpha_{|k|-1}$; this is what we intended to prove.

We prove a lemma which will play a key role in Sec. 4 (while in Sec. 3 it is not needed).

LEMMA 2.5. Let $z, \zeta \in V_n^\mu$, $Sz, S\zeta \in V_j$. If $|\mathrm{Re}\, z - \mathrm{Re}\,\zeta| > 2\alpha_{|n|-2}$, then $|\mathrm{Re}(Sz) - \mathrm{Re}(S\zeta)| > 2\alpha_{|j|-2}$ (for sufficiently large |n|).

Proof. Let $z = x+iy$, $\zeta = x_1 + iy_1$, $|x| > |x_1|$.

1. Let n > 0. By Lemma 2.3 we have |j| = n + 1. Moreover, $SV_n^\mu = fV_n^\mu$ and thus, $|\mathrm{Re}(Sz)| = e^x|\cos y| \geqslant e^x(2/\alpha_{n-1}) = 2\exp(x-\alpha_{n-2})$, $|\mathrm{Re}(S\zeta)| = e^{x_1}|\cos y_1| \leqslant e^{x_1}$. Consequently, $|\mathrm{Re}(Sz)| - |\mathrm{Re}(S\zeta)| \geqslant e^{x_1}(2\exp(x-x_1-\alpha_{n-2})-1) > e^{x_1} > e^{\alpha_{n-1}} > 2\alpha_{n-1} = 2\alpha_{|j|-2}$. The required inequality is proved.

2. Let n < 0, k = |n|. By Lemma 2.3, j = k + 1 or k + 2. We have $|fz| = \exp(-|x|)$, $\arg(fz) = y$; $|f\zeta| = \exp(-|x_1|)$, $\arg(f\zeta) = y_1$. We make use of the inequality (3) from the proof of Lemma 2.3: $|Sz-\alpha_{k+1}| = o\left(\frac{\alpha_{k+1}}{\alpha_{k-1}}\right)$ and similarly for the point ζ. We consider in the circle $\left\{\zeta : |\zeta-\alpha_{k+1}| < \frac{1}{2}\alpha_{k+1}\right\}$ the single-valued branch of the function $f^{-(k+1)}$ for which $f^{-(k+1)}\alpha_{k+1} = 0$. By Koebe's theorem

$$\frac{|S\zeta-\alpha_{k+1}|}{|Sz-\alpha_{k+1}|}=\frac{|f\zeta|}{|fz|}\left(1+o\left(\frac{1}{\alpha_{k+1}}\right)\right)\geqslant\frac{1}{2}\exp(|x|-|x_1|)\geqslant\frac{1}{2}\alpha_{k-1}^2. \tag{7}$$

We set $\theta=\arg(S\zeta-\alpha_{k+1})$. Then $|\mathrm{Re}(Sz)-\mathrm{Re}(S\zeta)|\geqslant|S\zeta-\alpha_{k+1}|\cos\theta-|Sz-\alpha_{k+1}|=|Sz-\alpha_{k+1}|\left(\frac{|S\zeta-\alpha_{k+1}|}{|Sz-\alpha_{k+1}|}\times\right.$ $\left.\cos\theta-1\right)$. By virtue of the inequalities (2), (4), (7), the last expression is not less than $\alpha_k^2\left(\frac{1}{2}\alpha_{k-1}^2\frac{1}{\alpha_{k-1}}-1\right)=\frac{1}{2}\alpha_{k-1}\alpha_k^2-1>2\alpha_k$. Since k = j − 1 or j − 2, the lemma is proved.

## 3. Typical Behavior of the Trajectories

We divide the strip $V_k$ into the rectangles $Q_j$ by the lines $x=l_s$, $y=\frac{\pi}{2}+2\pi s$, where $\pi\leq l_{s+1}-l_s\leqslant 2\pi$. The index j runs through some countable set $\Delta_k$. We set $\Delta=\bigcup_{|k|\geqslant N}\Delta_k$, where N is chosen so that the Lemmas 2.3-2.5 should hold. For $j\in\Delta_k$ we consider two rectangles $Q_j^\mu=Q_j\cap V_k^\mu$ ($\mu=\pm1$). The rectangles of this family are numbered by the index $s=(j,\ \mu)\in\Gamma_k=\Delta_k\times\{\pm1\}$. We set $\Gamma=\bigcup_{|k|\geqslant N}\Gamma_k$. The mapping S is continuous (and analytic) on the rectangles $Q_k^\mu$. We have

$$\mathrm{mes}\,(Q_j\setminus(Q_j^+\cup Q_j^-))\leqslant\frac{C_8}{\alpha_{|k|-1}}\quad(j\in\Delta_k). \tag{8}$$

Further, by $Z_k^\mu$ we denote the union of those rectangles $Q_j$ which are contained entirely in $SQ_j^\mu$. Let $P_j^\mu=Q_j^\mu\cap S^{-1}Z_j^\mu$. The set $SQ_j^\mu\setminus Z_j^\mu$ is contained in a $2\pi\sqrt{2}$-neighborhood of the boundary $\partial(SQ_j^\mu)$. By virtue of Lemma 2.4, the set $Q_j^\mu\setminus P_j^\mu$ is contained in a $\frac{5\cdot2\pi\sqrt{2}}{\alpha_{|k|-1}}<\frac{100}{\alpha_{|k|-1}}$-neighborhood of the boundary $\partial Q_j^\mu$ $(j\in\Delta_k)$ (i.e., $P_j^\mu$ is almost a rectangle). Therefore,

$$\frac{\mathrm{mes}\,(Q_j^\mu\setminus P_j^\mu)}{\mathrm{mes}\,Q_j^\mu}\leqslant\frac{C_9}{\alpha_{|k|-1}}\quad(j\in\Delta_k). \tag{9}$$

From the inequalities (8), (9) there follows that

$$\mathrm{mes}\,(Q_j\setminus(P_j^+\cup P_j^-))\leqslant\frac{C_{10}}{\alpha_{|k|-1}}\quad(j\in\Delta_k). \tag{10}$$

The image $SP_j^\mu$ is the union of several rectangles $Q_i$. Then it is the union of the almost rectangles $P_i^\mu$. Thus, the family of almost rectangles $P_i^\mu$ $((i,\mu)\in\Gamma)$ has the Markov property.

Let $\underline{i}=(i_0,\ldots,i_n)$, $\underline{\mu}=(\mu_0,\ldots,\mu_n)$, where $(i_l,\mu_l)\in\Gamma\equiv\Gamma^0$. We consider the set $P_{\underline{i}}^{\underline{\mu}}=\{z:S^lz\in P_{i_l}^{\mu_l}\ (l=0,\ldots,n)\}$. By $\Gamma_k^n$ we denote the family of indices $(\underline{i},\underline{\mu})$ for which the set $P_{\underline{i}}^{\underline{\mu}}$ has a nonempty interior and is contained in $V_k$. Sometimes the sets $P_{\underline{i}}^{\underline{\mu}}$ themselves will be considered as the elements of the family $\Gamma_k^n$. Thus, $W\in\Gamma_k^n$ means that $W=P_{\underline{i}}^{\underline{\mu}}$, where $(\underline{i},\underline{\mu})\in\Gamma_k^n$. We set $\Gamma^n=\bigcup_{|k|\geqslant N}\Gamma_k^n$. From the Markov property of the family Γ there follows that if $W\in\Gamma^n$, then $S^nW\in\Gamma$. Consequently, $S^{n+1}W$ is the union of the squares $Q_j$ $(j\in\Delta)$. We set $Y^n=\bigcup_{(\underline{i},\underline{\mu})\in\Gamma^n}P_{\underline{i}}^{\underline{\mu}}$.

LEMMA 3.1. There exists a constant $A_6$ such that if $W\in\Gamma_k^n$, then $\frac{\mathrm{mes}\,(W\setminus Y^{n+1})}{\mathrm{mes}\,W}\leqslant\frac{A_6}{\alpha_{|k|+n}}$.

Proof. The set $S^{n+1}W$ is the union $\bigcup_{j\in L}Q_j$, where L is some subset in Δ. By Lemma 2.3, the set $S^{n+1}W$ is contained in $V_l$, where $|l|\geqslant|k|+n+1$, and thus, it is situated at distance considerably larger than 2π from the set $\{\alpha_i\}_{i=0}^\infty$. We denote by φ the inverse mapping $S^{-(n+1)}$: $\bigcup_{j\in L}Q_j\to W$. By Koebe's theorem, for $z,\zeta\in Q_j$ we have

$$\frac{|\varphi'(z)|}{|\varphi'(\zeta)|}\leqslant B, \tag{11}$$

where B is an absolute constant (independent of k, n, W, j, z, ζ). From the inequalities (10), (11) there follows that

$$\frac{\mathrm{mes}\,(\varphi\,(Q_j\setminus(P_j^+\cup P_j^-))}{\mathrm{mes}\,(\varphi Q_j)}\leqslant B^2\frac{\mathrm{mes}\,(Q_j\setminus(P_j^+\cup P_j^-))}{\mathrm{mes}\,Q_j}\leqslant\frac{A_6}{\alpha_{|l|-1}}.$$

Since $W\setminus Y^{n+1} = \bigcup_{j\in L} \varphi(Q_j\setminus(P_j^+ \cup P_j^-))$, the last inequality implies what had to be proved.

Now we consider the invariant set $Y^\infty = \bigcap_{n=0}^{\infty} Y^n$, which will play a very important role in the sequel.

LEMMA 3.2. Let K be a rectangle whose sides are parallel to the coordinate axis and have length at least h > 0. Assume that K is contained in $V_k$. Then we have the estimate

$$\frac{\operatorname{mes}(K\setminus Y^\infty)}{\operatorname{mes} K} \leqslant \frac{A_7(h)}{\alpha_{|k|-1}},$$

where $A_7(h) = \begin{cases} B_7 & (h \geqslant 1), \\ B_7/h & (h < 1) \end{cases}$ and the constant $B_7$ does not depend on h.

Proof. First we prove this inequality for the rectangles $Q_j$ $(j \in \Delta_k)$. Let $Y_j^n = Q_j \cap Y^n$ $(0 \leqslant n \leqslant \infty)$. We mention that $Y_j^0 = P_j^+ \cup P_j^-$. According to the previous lemma, $\operatorname{mes}(Y_j^n\setminus Y_j^{n+1}) \leqslant A_6/\alpha_{|k|+n}$. Summing with respect to n, we find $\operatorname{mes}(Y_j^0\setminus Y_j^\infty) \leqslant A_6 \sum_{n=0}^{\infty} (1/\alpha_{|k|+n}) \leqslant C/\alpha_{|k|}$. Making use of the inequality (10), we obtain what we intended to prove:

$$\operatorname{mes}(Q_j\setminus Y^\infty) \leqslant C_{12}/\alpha_{|k|-1}. \tag{12}$$

Now we proceed to the general case. We start to proceed in the same way as at the proof of the inequality (10). We divide K into rectangles $K_i$ by the lines $y = \pi/2 + \pi n$ $(n = 1,\ldots, \ell)$. Let a be the length of the horizontal sides of the rectangles $K_i$ and let $b_i$ be the lengths of their vertical sides ($b_i = \pi$ for $i \neq 1, \ell$). We consider also the rectangles $N_i = K_i \cap (V_k^+ \cup V_k^-)$. Let $Z_i$ be the union of the rectangles $Q_j$ $(j \in \Delta)$, contained entirely in the image $SN_i$; let $L_i = N_i \cap S^{-1}Z_i$. In the same way as for the proof of the inequality (10), we can see that the set $K_i \setminus L_i$ is in the $\varepsilon_k$-neighborhood of the boundary $\partial K_i$, where $\varepsilon_k = C/\alpha_{|k|-1}$. Therefore,

$$\frac{\operatorname{mes}(K_i\setminus L_i)}{\operatorname{mes} K_i} \leqslant \frac{C}{\alpha_{|k|-1}} \frac{2(a+b_i)}{ab_i} = \frac{2C}{\alpha_{|k|-1}}\left(\frac{1}{a} + \frac{1}{b_i}\right). \tag{13}$$

For $i \neq 1, \ell$ the last expression does not exceed $\frac{2C}{\alpha_{|k|-1}}\left(\frac{1}{h} + \frac{1}{\pi}\right)$ and, consequently,

$$\operatorname{mes}\left(\bigcup_{i=2}^{l-1} K_i \setminus \bigcup_{i=2}^{l-1} L_i\right) \leqslant \frac{2C}{\alpha_{|k|-1}}\left(\frac{1}{h} + \frac{1}{\pi}\right) \operatorname{mes} \bigcup_{i=2}^{l-1} K_i. \tag{14}$$

For $i = 1, \ell$ we make use of the fact that $(\operatorname{mes} K_i)/b_i = (\operatorname{mes} K)/b$ and of inequality (13):

$$\operatorname{mes}(K_i\setminus L_i) \leqslant \frac{2C}{\alpha_{|k|-1}}\left(\frac{1}{a} + \frac{1}{b}\right) \operatorname{mes} K \leqslant \frac{4C}{h\alpha_{|k|-1}} \operatorname{mes} K. \tag{15}$$

According to (14), (15), we have

$$\frac{\operatorname{mes}(K\setminus L)}{\operatorname{mes} K} \leqslant \frac{2C}{\alpha_{|k|-1}}\left(\frac{3}{h} + \frac{1}{\pi}\right), \tag{16}$$

where $L = \bigcup_{i=1}^{l} L_i$. Assume now that $Q_j^i$ are rectangles of the family $\Delta$, contained in $SL_i$; let $\varphi_{ij} = S^{-1}: Q_j^i \to L_i$ be a branch of the inverse function. For it we have the estimate of Koebe's theorem $|\varphi'_{ij}(z)|/|\varphi'_{ij}(\zeta)| \leqslant B$, where B does not depend on i, j. Consequently,

$$\frac{\operatorname{mes}(\varphi_{ij}(Q_j^i\setminus Y^\infty))}{\operatorname{mes}(\varphi_{ij}Q_j^i)} \leqslant B^2 \frac{\operatorname{mes}(Q_j^i\setminus Y^\infty)}{\operatorname{mes} Q_j^i} \leqslant \frac{B^2 C_{12}}{\pi^2} \frac{1}{\alpha_{|k|-1}}$$

[the last inequality by virtue of (12)]. Summing with respect to i, j, we obtain $\frac{\operatorname{mes}(L\setminus Y^\infty)}{\operatorname{mes} L} \leqslant \frac{\widetilde{C}}{\alpha_{|k|-1}}$. From the last inequality and (16) we obtain the required result. The lemma is proved.

Lemma 3.2 shows that the set $Y^\infty$ is very thick in any rectangle contained in $V_k$ (provided its dimensions are not comparable with $1/\alpha_{|k|-1}$).

We introduce some notations and terminology. We recall that K(z, r) is a square with center at the point z and side 2r. Let D be a domain, let $z \in D$, let B(z, r) be the circle inscribed in D, and let B(z, R) be the circle circumscribed to D. By the distortion of the domain D (relative to the point z) we shall mean the ratio R/r. If X is a measurable set, then by the density (upper, lower) of the set X at the point z we mean $\lim \frac{\operatorname{mes}(X \cap B(Z,R))}{\pi R^2}$ (resp. $\overline{\lim}$, $\underline{\lim}$). Lebesgue's theorem on density points asserts that if mes X > 0, then the density of the set X at almost all points $z \in X$ is equal to 1.

Proof of Theorem 1.4. By Lemma 3.2 we have mes $Y^{\infty} > 0$. From Lemma 2.3 there follows that $|\operatorname{Re}(S^n z)| \to \infty (n \to \infty)$ for $z \in Y^{\infty}$. The latter means that $\omega_f(z) \subset \{\alpha_n\}_{n=0}^{\infty}$ for $z \in Y^{\infty}$.

Now we consider the set $Y = \bigcup_{n=-\infty}^{\infty} f^n Y^{\infty}$. It is completely invariant (i.e., $f^{-1}Y = Y$) and has a positive measure. We show that $\operatorname{mes}(Y \cap B(z, r)) > 0$ for every disk B = B(z, r). Indeed, otherwise $\operatorname{mes}\left(Y \cap \bigcup_{n=0}^{\infty} f^n B\right) = 0$. But by virtue of Misiurewicz's theorem we have $\bigcup_{n=0}^{\infty} f^n B = \mathbf{C}$. Contradiction.

Now we consider the completely invariant set $Z = \{z : \omega_f(z) \not\subset \{\alpha_n\}_{n=0}^{\infty}\}$. According to what has been proved, $Z \cap Y = \varnothing$. Let $z \in Z$. Then there exists a point $\zeta \neq \alpha_n$ such that $z_k = f^{n_k} z \to \zeta$ for some sequence $n_k \to \infty$. Let $\rho = \frac{1}{2} \min_n |\zeta - \alpha_n|$; let $f^{-n_k}$ be a single-valued branch of the inverse function in the circle B(ζ, 2ρ), for which $f^{-n_k} z_k = z$; $D_k = f^{-n_k} B(\zeta, \rho)$. Let $R_k$ and $r_k$ be the radii of the circles with center at the point z, inscribed in and circumscribed to the domain D. By Koebe's theorem we have $\min_{u,v \in B(\zeta,\rho)} \frac{|(f^{-n_k})'(u)|}{|(f^{-n_k})'(v)|} \geqslant \varkappa > 0$, where $\varkappa$ does not depend on k. Consequently,

1) $\frac{R_k}{r} \leqslant \frac{1}{\varkappa} \frac{\rho + |z_k - \zeta|}{\rho - |z_k - \zeta|} \leqslant a,$

2) $\frac{\operatorname{mes}(Y \cap D_k)}{\operatorname{mes} D_k} \geqslant \varkappa^2 \frac{\operatorname{mes}(Y \cap B(\zeta, \rho))}{\pi \rho^2} \geqslant b > 0,$

where the constants a, b do not depend on k.

We show that $R_k \to 0$. Indeed, if $R_{k(i)} \geqslant \delta > 0$ for some sequence $k(i) \to \infty$, then $r_{k(i)} \geqslant a^{-1}\delta \equiv r$ and thus, $D_k \supset B(z, r)$. But then $f^{n_{k(i)}} B(z, r) \subset B(\zeta, \rho)$. This contradicts the corollary to Misiurewicz's theorem: the set $f^n B(z, r)$ covers any compactum $K \subset \mathbf{C}^*$ for sufficiently large n. The contradiction shows what we intended to prove.

Further, from properties 1, 2 there follows that

$$\frac{\operatorname{mes}(Y \cap B(z, R_k))}{\pi R_k^2} \geqslant \frac{\operatorname{mes}(Y \cap D_k)}{a^2 (\pi r_k^2)} \geqslant \frac{\operatorname{mes}(Y \cap D_k)}{a^2 \operatorname{mes} D_k} \geqslant \frac{b}{a^2} > 0.$$

Thus, the upper density of the set Y at the point z is positive and, consequently, the lower density of the set Z at the point z is less than 1. By Lebesgue's theorem on the density points, we have mes Z = 0. We have proved that $\omega(z) \subset \{\alpha_n\}_{n=0}^{\infty}$ for almost all $z \in \mathbf{C}$.

Now we prove the inverse inclusion. We consider the invariant set

$$U_t = \{z\colon\ \omega(z) \subset \{\alpha_n\}_{n=0}^{\infty},\quad \operatorname{Re}(f^n z) \geqslant -t\ (n = 0, 1, \ldots)\},$$

where t > 0. Let π/2 < θ < π. The set $U_t$ does not intersect the sector $\{z\colon \theta < \arg z < 2\pi - \theta, |z| > t/\sin\theta\}$. Since $U_t$ is invariant, it does not intersect the union of the semistripes:

$$\Gamma_t = \bigcup_{n=-\infty}^{\infty} \{x + iy\colon\ \theta + 2\pi n < y < -\theta + 2\pi(n+1),\quad x > \ln(t/\sin\theta)\}.$$

Let $z \in U_t$, $z_n = f^n z = x_n + iy_n$, $0 < \alpha < \pi/2$. We show that there exists a sequence $n_k \to \infty$ such that $x_{n_k} \to +\infty$, $|y_{n_k}| > \alpha$. We consider two cases.

1. $x_n \to +\infty$. We assume that $|y_n| \leqslant \alpha$ for all n, starting somewhere. Let $c = \inf_{|y| \leqslant \alpha} \frac{\sin y}{y} > 0$. Then $|y_{n+1}| = e^{x_n} |\sin y_n| \geqslant c^{-1} e^{x_n} |y_n| \geqslant 2|y_n|$ for sufficiently large n. Consequently, $|y_n| \to \infty$. Contradiction.

2. $x_{p_k-1} \leqslant \gamma$ for some sequence $p_k \to \infty$. Then $|z_{p_k}| \leqslant e^{\gamma}$. We screen the sequence $\{p_k\}$ (without changing the notation) so that $z_{p_k} \to \zeta\,(k \to \infty)$. Since $z \in U_t$, we have $\omega(z) \subset \{\alpha_n\}_{n=0}^{\infty}$ and, thus, we have $\zeta = \alpha_\ell$ for some $\ell$. We find $\beta$ such that $e^x \cos\alpha > 2x$ for $x \geqslant \beta$ and then $s \in \mathbf{N}$ such that $\alpha_{l+s} > \beta$. Since $z_{p_k} \to \alpha_l$, we have $z_{p_k+s} \to \alpha_{l+s}$ $(k \to \infty)$ and, consequently, $x_{p_k+s} > \beta$ for sufficiently large k. We assume that

$$|y_i| \leqslant \alpha \quad (p_k + s \leqslant i \leqslant p_k + s + j - 1). \tag{17}$$

We verify by induction on i that $x_{i+1} > 2x_i (p_k + s \leqslant i \leqslant p_k + s + j - 1)$. Indeed, $x_{i+1} = e^{x_i} \cos y_i \geqslant e^{x_i} \cos\alpha$. By virtue of the induction hypothesis, $x_i \geqslant x_{p_k+s} > \beta$ and thus, $e^{x_i} \cos\alpha > 2x_i$; this is what we intended to prove.

Thus, if inequality (17) holds, then $x_{p_k+s+j} \geqslant 2^j \beta$. If it is valid for all j, then $x_{p_k+s+j} \to +\infty$ $(j \to \infty)$, while $|y_{p_k+s+j}| \leqslant \alpha$, contradicting the considered case 1. Consequently, there exists a first value $j = m_k$ for which $|y_{n_k}| > \alpha$, where $n_k = p_k + s + m_k$. Since $|y_{p_k}| \to 0$, then $m_k \to \infty$ $(k \to \infty)$. Consequently, $x_{n_k} > 2^{m_k}\beta \to +\infty$ $(k \to \infty)$ and the required sequence $\{n_k\}$ is constructed.

Now we assume that $\alpha$ and $\theta$ are close to $\pi/2$ (it is sufficient that the inequality $1.5 \cdot \alpha > \theta$ should hold).* Let $\rho_k = \min\left(\frac{1}{2}|y_{n_k}|, \pi\right)$. Then, as one can easily see, every interval $(y_{n_k} - \rho_k, y_{n_k} + \rho_k)$ intersects the family of intervals $\bigcup_{n=-\infty}^{\infty} (\theta + 2\pi n, -\theta + 2\pi(n+1))$ and the length of the intersection is not less than some $\lambda > 0$, not depending on k. Consequently, $\operatorname{mes}\left(K(z_{n_k}, \rho_k) \cap \Gamma_t\right) / \operatorname{mes} K(z_{n_k}, \rho_k) \geqslant \lambda/\pi \equiv q$. Moreover, the twice as large squares $K(z_{n_k}, 2\rho_k)$ do not intersect the real line and, consequently, do not contain the points $\alpha_n$. Therefore, in these squares all the branches $f^{-n}$ of the inverse functions are defined. Let $f^{-n_k}$ be the branch defined in the square $K(z_{n_k}, 2\rho)$, for which $f^{-n_k} z_{n_k} = z$, $D_k = f^{-n_k} K(z_{n_k}, \rho_k)$. Applying Koebe's theorem, we can see that $D_k$ is a domain with bounded distortion, $\operatorname{diam} D_k \to 0$, and

$$\frac{\operatorname{mes}(D_k \setminus U_t)}{\operatorname{mes} D_k} \geqslant \varkappa^2 \frac{\operatorname{mes}\left(K(z_{n_k}, \rho_k) \cap \Gamma_t\right)}{\operatorname{mes} K(z_{n_k}, \rho_k)} \geqslant \frac{\varkappa^2 \lambda}{\pi} > 0$$

(we have used the fact that $U_t \cap \Gamma_t = \varnothing$). Thus, the lower density of the set $U_t$ at the point z is less than 1. By Lebesgue's theorem, we have $\operatorname{mes} U_t = 0$.

Since $U_t$ is an increasing family of sets, we have $\operatorname{mes} \bigcup_{t>0} U_t = 0$. But $\bigcup_{t>0} U_t = \{z\colon 0 \notin \omega(z) \subset \{\alpha_n\}_{n=0}^{\infty}\}$. The theorem is proved.

A set X is said to be absorbing if $\operatorname{mes}\left(\mathbf{C} \setminus \bigcup_{n=0}^{\infty} f^{-n} X\right) = 0$.

LEMMA 3.3. The set $Y^\infty$ is absorbing.

Proof. We consider the set $Z = \{z\colon 0 \in \omega(z)\} \setminus \left(\bigcup_{n=0}^{\infty} f^{-n} Y^\infty\right)$. By virtue of Theorem 1.4 it is sufficient to show that $\operatorname{mes} Z = 0$. Let $z \in Z$. Then there exists a sequence $n_j \to \infty$ such that $f^{n_j+1} z \to 0\ (j \to \infty)$. We set $x_j + iy_j = f^{n_j} z$. Then $x_j \to -\infty$ $(j \to \infty)$.

We define a sequence of radii in the following manner. If $z_j \in V_{-k}$, then $\rho_j = \pi$. Otherwise, $z_j$ is contained in some strip $\{x + iy\colon |x + \alpha_k| < 3\alpha_{k-1}\}$, where $k = k(j)$. Moreover, we set $\rho_j = 6\alpha_{k-1}$.

We consider the squares $K_j = K(z_j, \rho_j)$. The intersection $W_j = K_j \cap \left(\bigcup_{k>0} V_{-k}\right)$ contains a rectangle, whose vertical side is equal to $\rho_j$, while the horizontal side is at least $0.5\rho_j$. Consequently, $\operatorname{mes} W_j \geqslant \frac{1}{2} \operatorname{mes} K_j$. But by Lemma 3.2, $\operatorname{mes}(W_j \cap Y^\infty) \geqslant \frac{1}{2} \operatorname{mes} W_j$ for sufficiently large j. Thus, $\operatorname{mes}(K_j \cap Y^\infty) \geqslant \frac{1}{4} \operatorname{mes} K_j$.

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*We recall that $\alpha$, $\theta$ have been given arbitrarily in the intervals $(0, \pi/2)$ and $(\pi/2, \pi)$, respectively.

The squares $K(z_j, 2\rho_j)$ are contained in the left-hand semiplane and, consequently, all the branches of the inverse function $f^{-n}$ are defined in them. Let $f^{-n_j}$ be the branch in the square $K(z_j, 2\rho_j)$ for which $f^{-n_j}z_j = z$; $D_j = f^{-n_j}B_j$. Applying Koebe's theorem in the same way as it has been done at the proof of Theorem 1.4, we can see that $D_j$ is a domain with bounded distortion, $\operatorname{diam} D_j \to 0$, and

$$\frac{\operatorname{mes}(D_j \setminus Z)}{\operatorname{mes} D_j} \geqslant \gamma \frac{\operatorname{mes}(K_j \cap Y^\infty)}{\operatorname{mes} K_j} \geqslant \frac{\gamma}{4} > 0,$$

where the constant $\gamma$ does not depend on j. Thus, the lower density of the set Z at the point z is less than 1. By Lebesgue's theorem, we have $\operatorname{mes} Z = 0$ and the lemma is proved.

The proved lemma allows us to investigate the typical behavior of the trajectories only on the set $Y^\infty$ and on it we have sufficiently sharp estimates (Lemma 2.3).

## 4. Absence of Ergodicity and of an Absolutely Continuous Invariant Measure. The Existence of a Wandering Set

We construct on the set $Y^\infty$ a symbolic dynamics, associating to a point $z \in Y^\infty$ the sequence of integers $\underline{\varkappa}(z) = (\varkappa_0(z), \varkappa_1(z), \ldots)$ according to the rule $S^n z \in V_{\varkappa_n(z)}$. By virtue of Lemma 2.3, we have $|\varkappa_{n+1}(z)| > |\varkappa_n(z)|$. By $\tau$ we denote the shift $(\varkappa_0, \varkappa_1, \ldots) \mapsto (\varkappa_1, \varkappa_2, \ldots)$ in the space of integer-valued sequences. Clearly, $\underline{\varkappa}(Sz) = \tau(\underline{\varkappa}(z))$.

LEMMA 4.1. Let $z, \zeta \in Y^\infty$. If $S^n z = S^{n+m}\zeta$, then $\tau^m(\underline{\varkappa}(\zeta)) = \underline{\varkappa}(z)$.

Proof. Since $\tau^m(\underline{\varkappa}(\zeta)) = \underline{\varkappa}(S^m\zeta)$, replacing the point $\zeta$ by $S^m\zeta$, we reduce the problem to the case m = 0. This will be assumed in the sequel. We find the largest i such that

$$\varkappa_{n-j}(z) = \varkappa_{n-j}(\zeta) \quad (j = 0, 1, \ldots, i), \tag{18}$$

$$|\operatorname{Re}(S^{n-j}z) - \operatorname{Re}(S^{n-j}\zeta)| \leqslant 2\alpha_{|\varkappa_{n-j}(z)|-2} \quad (j = 0, \ldots, i). \tag{19}$$

We show that i = n [and thus, equality (18) is satisfied for all j = 0, 1,...,n, as required]. We assume that i < n. We denote $q = \varkappa_{n-i}(z)$.

We assume that the points $S^{n-i-1}z$ and $S^{n-i-1}\zeta$ are contained in the same set $V_k^\mu$. Then $\varkappa_{n-i-1}(z) = k = \varkappa_{n-i-1}(\zeta)$ and, by virtue of the selection of i, we must have the inequality $|\operatorname{Re}(S^{n-i-1}z) - \operatorname{Re}(S^{n-i-1}\zeta)| > 2\alpha_{|k|-2}$. From here, by Lemma 2.5, there follows the inequality $|\operatorname{Re}(S^{n-i}z) - \operatorname{Re}(S^{n-i}\zeta)| > 2\alpha_{|q|-2}$, in spite of the assumption (19). The contradiction shows that the points $S^{n-i-1}z$ and $S^{n-i-1}\zeta$ are contained in distinct sets $V_k^\mu$.

But by virtue of the assumptions (18), the points $S^{n-i}z$ and $S^{n-i}\zeta$ are contained in one set $V_q$. By Lemma 2.3, this is possible only for q > 0 in one of the three cases:

1) $S^{n-i-1}z \in V_{q-1}^+$, $S^{n-i-1}\zeta \in V_{-(q-2)}^+$;

2) $S^{n-i-1}z \in V_{q-1}^+$, $S^{n-i-1}\zeta \in V_{-(q-1)}^-$;

3) $S^{n-i-1}z \in V_{-(q-1)}^-$, $S^{n-i-1}\zeta \in V_{-(q-2)}^+$.

In the case 1 we have $S^{n-i}z = f(S^{n-i-1}z)$, while $S^{n-i}\zeta = f^q(S^{n-i-1}\zeta)$. We denote $v = f^{q-1}(S^{n-i-1}\zeta)$, $x + iy = S^{n-i-1}z$. From the definition of S we conclude that

$$|v - \alpha_{q-2}| < 1. \tag{20}$$

By the definition of the set $V_{q-1}^+$ we have

$$x \geqslant \alpha_{q-2} + 3\alpha_{q-3}, \quad \cos y \geqslant 2/\alpha_{q-2}. \tag{21}$$

From the inequalities (20), (21) there follows $\operatorname{Re}(S^{n-i}z) - \operatorname{Re}(S^{n-i}\zeta) \geqslant e^x \cos y - |e^v| \geqslant \alpha_{q-1}\alpha_{q-2}^3(2/\alpha_{q-2}) - e\alpha_{q-1} > 2\alpha_{q-2}$, which contradicts the assumption (19). The contradiction shows that case 1 is excluded.

Case 2 is entirely similar to case 1.

In case 3 we make use of inequality (3) from the proof of Lemma 2.3. We obtain $\alpha_q - S^{n-i}z < \alpha_q/2$, $S^{n-i}\zeta - \alpha_{q-1} < \alpha_{q-1}/2$. Consequently, $S^{n-i}z - S^{n-i}\zeta > (\alpha_q - 3\alpha_{q-1})/2 > 2\alpha_{q-2}$ and we have obtained again a contradiction with inequality (19). The lemma is proved.

We denote $W_k = V_k \cap Y^\infty$.

LEMMA 4.2. Let $z, \zeta \in W_k$. We assume that $f^p z = f^q \zeta$ for some $p, q \in \mathbf{N}$. Then a) $\underline{\varkappa}(z) = \underline{\varkappa}(\zeta)$, b) p = q.

Proof. We consider two nonintersecting sets $V_\infty = \bigcup_{|k|>N} V_k$, $B_\infty = \bigcup_{k=0}^{\infty} B(\alpha_k, 1)$. The trajectory $\{f^n z\}_{n=0}^{\infty}$ of any point $z \in Y^\infty$ is contained in $V_\infty \cup B_\infty$ and visits each of these sets infinitely often. We note that if n = 0, $L_1$, $L_2$,... are the moments at which the trajectory is in the set $V_\infty$, then $f^{L_i} z = S^i z$.

Let $u = f^p z = f^q \zeta$. We find the smallest moment $s \in \mathbf{N}$ such that $f^s u \in V_\infty$. Then $S^n z = f^s u = S^l \zeta$ for appropriate $n, l \in \mathbf{N}$. We assume, for the sake of definiteness, that $l \geqslant n$. By Lemma 4.1 we have $\underline{\varkappa}(z) = \tau^m \underline{\varkappa}(\zeta)$, where $m = l - n$. Consequently, $\varkappa_0(z) = \varkappa_m(\zeta)$. But for m > 0 we have $|\varkappa_m(\zeta)| > |\varkappa_0(\zeta)| = |k| = |\varkappa_0(z)|$. Thus, m = 0 and, consequently, $\underline{\varkappa}(z) = \underline{\varkappa}(\zeta)$, i.e., a) is proved. In addition, we have shown that $S^n z = S^n \zeta$ for an appropriate n.

Now we define natural numbers $l_i(z)$ in the following manner: $S^i z = f^{l_i(z)}(S^{i-1} z)$. Lemma 2.3 allows us to find $l_i(z)$ in terms of $\varkappa_{i-1}(z)$ and $\varkappa_i(z)$. Namely: 1) if $\varkappa_{i-1}(z) > 0$, then $l_i(z) = 1$; 2) if $\varkappa_{i-1}(z) < 0$, then two cases are possible: a) $\varkappa_i(z) = |\varkappa_{i-1}(z)| + 2$, then $l_i(z) = \varkappa_i(z)$; b) $\varkappa_i(z) = |\varkappa_{i-1}(z)| + 1$, then $l_i(z) = \varkappa_i(z) + 1$.

Since $\varkappa_i(z) = \varkappa_i(\zeta)$ for all $i \in \mathbf{N}$, we have $l_i(z) = l_i(\zeta) \equiv l_i$. We set $L = \sum_{i=1}^{n} l_i$, where n has been defined above. Then $f^L z = S^n z = f^{p+s} z$. Since the point z is not preperiodic, we have L = p + s. Similarly, L = q + s and part b) is proved.

Now we are completely ready for the proof of our fundamental results. We set $W_k^+ = V_k^+ \cap Y^\infty$, $W_k^- = V_k \cap Y^\infty$.

Proof of Theorem 1.1. Let $z \in W_k^+$, $\zeta \in W_k^-$. Then $\varkappa_1(z) > 0 > \varkappa_1(\zeta)$ and thus, $\underline{\varkappa}(z) = \varkappa(\zeta)$. By Lemma 4.2, the trajectories of the points z and do not intersect. Consequently, the invariants sets $U_k^+ = \bigcup_{n=0}^{\infty} f^n W_k^+$ and $U_k^- = \bigcup_{n=0}^{\infty} f^n W_k^-$ do not intersect. We consider the partition of the plane into two completely invariant sets $X_k^+ = \bigcup_{n=0}^{\infty} f^{-n} U_k^+$, $X_k^- = \mathbf{C} \setminus X_k^+$. Since $X_k^\pm \supset W_k^\pm$, it follows that both sets $X_k^\pm$ have positive measure (Lemma 3.2). The nonergodicity of the transformation f is proved.

Now we assume that the transformation f has an ergodic component Z of positive measure. Let $z \in Z \cap Y^\infty$, $\varkappa_n = \varkappa_n(z)$. We consider the square $K_n = K(S^n z, \pi)$ and we partition it into two sets: $K_n^+ = K_n \cap \{x + iy\colon \cos y > 0\}$, $K_n^- = K_n \cap \{x + iy\colon \cos y < 0\}$. We have $\operatorname{mes} K_n^\mu = \frac{1}{2} \operatorname{mes} K_n$ $(\mu = \pm 1)$. By Lemma 3.2, the measure of the set $K_n \cap W_{\varkappa_n}^\mu = K_n^\mu \cap Y^\infty$ is almost equal to $\frac{1}{2} \operatorname{mes} K_n^\mu$.

Further, from the ergodicity of the transformation f: Z → Z there follows that one of the sets $Z \cap X_{\varkappa_n}^\mu$ $(\mu = \pm 1)$ will be of zero measure. Moreover, $\operatorname{mes}\left(Z \cap K_n \cap W_{\varkappa_n}^{\mu_n}\right) = 0$. Therefore,

$$\frac{\operatorname{mes}(Z \cap K_n)}{\operatorname{mes} K_n} \leqslant \frac{\operatorname{mes}\left(K_n \setminus W_{\varkappa_n}^{\mu_n}\right)}{\operatorname{mes} K_n} \to \frac{1}{2} \quad (n \to \infty). \tag{22}$$

Let $S^n z = f^{l_n} z$, and let $f^{-l_n}$ be the branch of the inverse function in the square $K(S^n z, 2\pi)$ for which $f^{-l_n}(S^n z) = z$; $D_n = f^{-l_n} K_n$. Applying Koebe's theorem, we can see that $D_n$ is a domain with bounded distortion and $\operatorname{diam} D_n \to 0$. Making use additionally of inequality (22), we obtain that the density $\operatorname{mes}(Z \cap D_n)/\operatorname{mes} D_n$ of the set Z in the domain $D_n$ is separated from 1, i.e., the lower density of the set Z at any point $z \in Z \cap Y^\infty$ is less than 1. By Lebesgue's theorem, we have $\operatorname{mes}(Z \cap Y^\infty) = 0$. From the invariance of Z and Lemma 3.3 there follows that mes Z = 0.

Proof of Theorem 1.2. A wandering set of positive measure is the set $W_k$. Indeed, the positivity of the measure follows from Lemma 3.2, while Lemma 4.2 (statement b) means that $f^p W_k \cap f^q W_k = \varnothing$ for $p > q \geqslant 0$.

Proof of Theorem 1.3. We show that $\lambda(B(\alpha_k, \varepsilon)) = \infty$ for every $k \in \mathbf{N}$, $\varepsilon > 0$. We consider the set $W_{-\infty} = \bigcup_{k \geqslant N} W_{-k}$. By Theorem 1.4, the return mapping T: $W_{-\infty} \to W_{-\infty}$, $Tz = S^n z$, where n = n(z) is the first moment for which $S^n z \in W_{-\infty}$, is defined almost everywhere. We show that the set $W_{-k}$ is wandering under the action of the transformation T. Indeed, let $T^n z = T^m \zeta$ $(z, \zeta \in W_{-k})$. Then, by Lemma 4.2 we have $\underline{\varkappa}(z) = \underline{\varkappa}(\zeta)$. But $T^n z = S^p z$, $T^m \zeta = S^q \zeta$ for appropriate p, q.

Assume, for the sake of definiteness, that $p \geqslant q$. If p > q, then $|\varkappa_p(z)| > |\varkappa_q(z)| = |\varkappa_q(\zeta)|$ and the equality $S^p z = S^q \zeta$ is not possible. Consequently, p = q. Thus, $T^n z = S^p z = S^p \zeta = T^m \zeta$. But the numbers n, m are uniquely found from p and the sequence $\underline{\varkappa} = \underline{\varkappa}(z) = \underline{\varkappa}(\zeta)$. Indeed, they are equal to the number of the negative components of the vector $(\varkappa_1, \varkappa_2, \ldots, \varkappa_p)$. Thus, n = m; this is what we intended to prove.

Now we show that $\lambda(T^n W_{-k}) \geqslant \lambda(W_{-k})$. Indeed, for a given n we partition $W_{-k}$ into the union $\bigcup_j X_{nj}$: $z \in X_{nj}$ if $T^n z = f^j z$. By Lemma 4.2, we have $f^j X_{nj} \cap f^i X_{ni} = \varnothing$ for i ≠ j and from the invariance of the measure λ there follows that $\lambda(f^j X) \geqslant \lambda(X)$ for every measurable set X. Thus, $\lambda(TW_{-k}) = \sum_j \lambda(f^j X_{nj}) \geqslant \sum_j \lambda(X_{nj}) = \lambda(W_{-k})$.

Further, according to Theorem 1.4 and Lemma 3.3 we have $\lambda\left(\mathbf{C} \setminus \bigcup_{n=0}^{\infty} f^{-n} W_{-\infty}\right) = 0$, from where $\lambda(W_{-\infty}) > 0$. But then $\lambda(W_{-k}) > 0$ for an appropriate k. Therefore, $\lambda\left(\bigcup_{n=l}^{\infty} T^n W_{-k}\right) = \sum_{n=l}^{\infty} \lambda(T^n W_{-k}) \geqslant \sum_{n=l}^{\infty} \lambda(W_{-k}) = \infty$ for every ℓ. But the set $\bigcup_{n=l}^{\infty} T^n W_{-k}$ is contained in the left semiplane $\{z\colon \operatorname{Re} z < -\alpha_\ell\}$. Consequently, $\operatorname{mes}\{z\colon \operatorname{Re} z < -M\} = \infty$ for arbitrarily large M. It remains to note that $f^{-(k+1)} B(\alpha_k, \varepsilon)$ contains some left semiplane and, thus, $\lambda(B(\alpha_k, \varepsilon)) = \lambda(f^{-(k+1)} B(\alpha_k, \varepsilon)) = \infty$. The theorem is proved.

Question. Is there an absolutely continuous invariant measure, whose density is smooth in $\mathbf{C} \setminus \{\alpha_n\}_{n=0}^{\infty}$ (at the points $\alpha_n$ it must have nonintegrable singularities)?

## 5. Distribution of the Arguments

We consider the set $Q_j^\infty = Q_j \cap Y^\infty$ $(j \in \Delta)$ and we define on it the conditional measure $\pi_j(X) = \operatorname{mes}(X \cap Q_j^\infty)/\operatorname{mes} Q_j^\infty$, where X is a measurable subset in $Q_j$. We introduce the following notations. If X is a subset of the circumference $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$, then |X| is its Lebesgue measure and $p(X) = \frac{1}{2\pi}|X|$ is its normalized Lebesgue measure. If $x \in \mathbf{R}$, then $x \in X$ means that $x \pmod{2\pi} \in X$. If $X_0, \ldots, X_n \in \mathbf{T}$, $l \in \mathbf{N}$, then we set $[X_0 \ldots X_n]_j^l = \{z \in Q_j\colon \operatorname{Im}(S^m z) \in X_m \quad (m = l, \ldots, l+n)\}$ (we shall omit the index ℓ = 0).

LEMMA 5.1. Let $I_0, \ldots, I_n$ be intervals on the circumference **T**; assume that $p(I_k) \geqslant h > 0$; $j \in \Delta_k$. Then

$$\left(1 - \frac{A_{11}}{\alpha_{|k|+l-1}}\right) \prod_{m=0}^{n} p(I_m) \leqslant \pi_j [I_0 \ldots I_n]_j^l \leqslant \left(1 + \frac{A_{11}}{\alpha_{|k|+l-1}}\right) \prod_{m=0}^{n} p(I_m),$$

where the constant $A_{11} = A_{11}(h)$ does not depend on j, ℓ, n, while the dependence on h is the same as in Lemma 3.2 (with other constants).

Proof. First for ℓ = 0 we carry out induction on n. We apply Lemma 3.2 to the rectangles $K = [I_0]_j$ and $Q_j$:

$$\left(1 - \frac{A_7}{\alpha_{|k|-1}}\right) \operatorname{mes} K \leqslant \operatorname{mes}(K \cap Y^\infty) \leqslant \left(1 + \frac{A_7}{\alpha_{|k|-1}}\right) \operatorname{mes} K,$$

$$\left(1 - \frac{A_7}{\alpha_{|k|-1}}\right) \operatorname{mes} Q_j \leqslant \operatorname{mes} Q_j^\infty \leqslant \left(1 + \frac{A_7}{\alpha_{|k|-1}}\right) \operatorname{mes} Q_j.$$

Dividing the first inequality by the second one, we obtain the basis of the induction:

$$\left(1 - \frac{a_1}{\alpha_{|k|-1}}\right) p(I_0) \leqslant \pi_j(K) \leqslant \left(1 + \frac{a_1}{\alpha_{|k|-1}}\right) p(I_0). \tag{23}$$

Now we assume that for $i \in \Delta_{k+1}$ we have

$$\left(1 - \frac{b}{\alpha_{|k|}}\right) \prod_{m=1}^{n} p(I_m) \leqslant \pi_i [I_1 \ldots I_n]_i \leqslant \left(1 + \frac{b}{\alpha_{|k|}}\right) \prod_{m=1}^{n} p(I_m), \tag{24}$$

where the constant b satisfies some additional lower estimate, given below. Further we shall proceed according to the same scheme as in Lemma 3.2. Let Z be the union of those rectangles $Q_i$ $(i \in \Delta)$ which are entirely included in SK and let $L = K \cap S^{-1} Z$. Assume that $Q_i \subset SQ_j$, $\varphi_i = S^{-1}\colon Q_i \to L$ is a branch of the inverse function. The rectangle $Q_i$ is contained in the strip $V_\ell$, where $|l| \geqslant |k|+1$ and, thus, it is situated at a distance not smaller than $\alpha_{|k|-1}$ from the set $\{\alpha_n\}_{n=0}^{\infty}$. Moreover, $Q_i$ has bounded dimensions (at most $2\pi \times 2\pi$). By Koebe's theorem we have

$$1-\frac{C}{\alpha_{|k|-1}} \leqslant \frac{|\varphi_i'(z)|}{|\varphi_i'(\zeta)|} \leqslant 1+\frac{C}{\alpha_{|k|-1}}.$$

Consequently,

$$\left(1-\frac{a_2}{\alpha_{|k|-1}}\right)\pi_i[I_1\ldots I_n]_i \leqslant \frac{\pi_j(\varphi_i[I_1\ldots I_n]_i)}{\pi_j(\varphi_i Q_i)} \leqslant \left(1+\frac{a_2}{\alpha_{|k|-1}}\right)\pi_i[I_1\ldots I_n]_i.$$

Summing with respect to i, taking into account the induction hypothesis (24), we obtain

$$\left(1-\frac{a_2}{\alpha_{|k|-1}}\right)\left(1-\frac{b}{\alpha_{|k|}}\right)\prod_{m=1}^{n} p(I_m) \leqslant \frac{\pi_j(L\cap[I_0\ldots I_n]_j)}{\pi_j(L)} \leqslant \left(1+\frac{a_2}{\alpha_{|k|-1}}\right)\left(1+\frac{b}{\alpha_{|k|}}\right)\prod_{m=1}^{n} p(I_m). \quad (25)$$

But for $\pi_j(L)$ the estimate (16) from the proof of Lemma 3.2 holds

$$\left(1-\frac{a_3}{\alpha_{|k|-1}}\right)\pi_j(K) \leqslant \pi_j(L) \leqslant \left(1+\frac{a_3}{\alpha_{|k|-1}}\right)\pi_j(K). \quad (26)$$

If b is selected in advance so that we have the estimate $\left(1+\frac{b}{\alpha_{|k|}}\right)\prod_{i=1}^{3}\left(1+\frac{a_i}{\alpha_{|k|-1}}\right) \leqslant 1+\frac{b}{\alpha_{|k|-1}}$,

then from the inequalities (25), (26), (23) there follows the required estimate

$$\left(1-\frac{b}{\alpha_{|k|-1}}\right)\prod_{m=0}^{n} p(I_m) \leqslant \pi_j[I_0\ldots I_n]_j \leqslant \left(1+\frac{b}{\alpha_{|k|-1}}\right)\prod_{m=0}^{n} p(I_m).$$

The case of an arbitrary $\ell$ is obtained from $\ell = 0$ by considering the mapping $S^l\colon Q_j^{\infty} \to \mathbf{C}$ and by carrying out exactly the same reasoning as above. We omit the details.

We consider the space $\Sigma_2^+$ of sequences $\varepsilon = \{\varepsilon_n\}_{n=0}^{\infty}$ of zeros and ones. Let I be an interval on the circumference and let $\chi_I$ be its characteristic function. We fix a rectangle $Q \equiv Q_j$. We construct the mapping $h_I\colon Q^{\infty} \to \Sigma_2^+$, associating to the point $z \in Q^{\infty}$ the sequence $\{\varepsilon_n(z) = \chi_I(\operatorname{Im}(S^n z))\}_{n=0}^{\infty}$, and we carry over the measure $\pi \equiv \pi_j$ from $Q^{\infty}$ to $\Sigma_2^+$: $\mu_I = (h_I)_*\pi$. In addition, we consider on $\Sigma_2^+$ the Bernoulli measure $\nu_I$ (i.e., the measure corresponding to the scheme of independent trials), the probability of the appearance of unity being $p(I)$. Finally, by $\mathfrak{S}_l$ denote the σ-algebra on $\Sigma_2^+$, generated by the random variables $\varepsilon_n$ $(n = l, l+1, \ldots)$, $\mathfrak{S}_{\infty} = \bigcap_{l=0}^{\infty} \mathfrak{S}_l$ is the tail σ-algebra.

A measure μ on $\Sigma_2^+$ will be said to be asymptotically Bernoulli if for some Bernoulli measure ν there exists a sequence $\varepsilon_\ell \to 0$ such that $(1-\varepsilon_l)\nu(X) \leqslant \mu(X) \leqslant (1+\varepsilon_l)\nu(X)$ for every set $X \in \mathfrak{S}_l$.

LEMMA 5.2. For every interval I on **T**, the measure $\mu_I$ is asymptotically Bernoulli. More exactly, if $p(I) \geqslant h$, $Q \subset V_k$, then for every set $X \in \mathfrak{S}_l$ we have the estimates

$$\left(1-\frac{A_{11}}{\alpha_{|k|+l-1}}\right)\nu_I(X) \leqslant \mu_I(X) \leqslant \left(1+\frac{A_{11}}{\alpha_{|k|+l-1}}\right)\nu_I(X),$$

where $A_{11} = A_{11}(h)$ is the constant from Lemma 5.1.

Proof. We set $I_0 = \mathbf{T}\backslash I$, $I_1 = I$. By $[\varepsilon_0\ldots\varepsilon_n]^l$ we denote the cylinder in the space $\Sigma_2^+$, consisting of those sequences $x = \{x_k\}_{k=0}^{\infty}$ for which $x_i = \varepsilon_{i-l}$ $(i = l, \ldots, l+n)$. We have $\mu_I[\varepsilon_0\ldots\varepsilon_n]^l = \pi[I_{\varepsilon_0}\ldots I_{\varepsilon_n}]^l$, $\nu_I[\varepsilon_0\ldots\varepsilon_n]^l = \prod_{k=0}^{n} p(I_{\varepsilon_k})$. Now from Lemma 5.1 we obtain the required inequality for cylindrical sets $X = [\varepsilon_0\ldots\varepsilon_n]^l$. Since these sets generate the entire σ-algebra $\mathfrak{S}_l$, the lemma is proved.

COROLLARY 5.1. The measures $\mu_I$ and $\nu_I$ coincide on the tail σ-algebra: $\mu_I(X) = \nu_I(X)$ for $X \in \mathfrak{S}_{\infty}$.

We proceed to the proof of Theorem 1.5, formulated in the introduction. We shall make use of the notations $\ell_s$, $k_s$, $t_s$, introduced there.

Proof of Theorem 1.5. Since the set $Y^{\infty}$ is absorbing (Lemma 3.3), it is sufficient to verify the typical properties of the trajectories on $Y^{\infty}$. In turn, it is sufficient to do this on the set $Q^\infty = Q \cap Y^\infty$, where Q is an arbitrary rectangle of the family $\Delta$.

By the definition of the transformation S we have $f^{l_s+k_s}z = S(f^{l_s-1}z)$. Therefore, it is convenient to formulate Theorem 1.5 in terms of the transformation S: for almost all $z \in Q^\infty$ the sequence $\{\arg(f \circ S^n z) = \operatorname{Im}(S^n z) \pmod{2\pi}\}_{n=0}^\infty$ is uniformly distributed on the circumference $\mathbf{T}$. In other words, for every interval $I \subset \mathbf{T}$, almost everywhere relative to the measure $\pi$, we have the equality $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_I(\operatorname{Im}(S^i z)) = p(I)$. Passing to the space $\Sigma_2^+$, we obtain the following formulation: for almost all sequences $\underline{\varepsilon} \in \Sigma_2^+$, relative to the measure $\mu_I$, we have the equality

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varepsilon_i = p(I). \tag{27}$$

We consider the set X, consisting of all sequences for which equality (27) is satisfied. This set belongs to the tail σ-algebra $\mathfrak{S}_\infty$. Making use of the corollary to Lemma 5.2, we obtain $\mu_I(X) = \nu_I(X)$. But by the strong law of large numbers, for a sequence of independent random variables we have $\nu_I(X) = 1$. Part a) of Theorem 1.5 is proved.

Since almost all trajectories return to the left semiplane, we can assume without loss of generality that the rectangle Q is contained in the left semiplane. Moreover, we can set $l_0 = 1$. Let $I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \subset \mathbf{T}$. We consider the sequence $\gamma_s = \chi_I\left(\operatorname{Im}\left(f^{l_s-1}z\right)\right)$. The sequence $\underline{\gamma} = \{\gamma_s\}$ is constructed from the sequence $\underline{\varepsilon} = (h_I)_* z$ in the following manner. Since $\ell_0 = 1$, we have $\gamma_0 = \varepsilon_0$. If $\gamma_s = \varepsilon_{p_s}$ and $\varepsilon_\ell$ is the first zero coordinate of the sequence $\underline{\varepsilon}$, occurring after $\varepsilon_{p_s}$, then $\gamma_{s+1} = \varepsilon_{\ell+1}$. It is easy to show that $\{\gamma_s\}$ is a stationary sequence of independent (relative to the Bernoulli measure $\nu_I$) random variables with mathematical expectation 1/2. By the law of large numbers we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} \gamma_s = \frac{1}{2} \tag{28}$$

almost everywhere with respect to the measure $\nu_I$ and, therefore, also with respect to the measure $\mu_I$.

By $E_s = E_s(z)$ we denote the segment $[l_s + k_s + 1, l_{s+1} - 1]$ of the natural series and let $E = \cup E_s$. From part a), proved above, and from equality (28) there follows that the mean value of the sequence $\{\chi_I(\arg(f^n z)) : n \in E\} = \{\chi_I(\operatorname{Im}(f^{n-1}z)) : n \in E\}$, obtained by discarding $\underline{\gamma}$ from $\underline{\varepsilon}$, is also equal to 1/2. This means that, on the average, the trajectory $\{f^n z : n \in E\}$ is in the left semiplane as often as in the right semiplane. But on the s-th segment $\{f^n z : n \in E_s\}$ of this trajectory, exactly one point $f^{l_{s+1}-1}z$ is in the left semiplane, while $t_s$ are points in the right semiplane. Consequently,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} t_s = 1, \tag{29}$$

and part b) is proved.

Further, from Lemma 2.3 there follows successively that $f^{l_s+k_s}z \in V_{k_s+\gamma_s}$, $f^{l_s+k_s+t_s+1}z \in V_{k_s+\gamma_s+t_s+1}$ and, finally, $k_{s+1} = k_s + \gamma_s + t_s + 2$. Taking the average with respect to s and using then the equalities (28), (29), we obtain the required equality

$$\lim_{n\to\infty} \frac{1}{n} k_n = \lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} (k_{s+1} - k_s) = \lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} (\gamma_s + t_s + 2) = 3.5.$$

The theorem is completely proved.

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# APPROXIMATION OF INTEGRAL CURVATURES OF CURVES IN $R^n$ AND $S^n$

I. F. Mainik UDC 513.73

In [1] Aleksandrov suggested a method for constructing a general theory of nonregular curves in $R^3$ on the basis of their approximation by polygonal arcs. The curvature and torsion are replaced by the integral curvature and integral torsion, which are defined as limits of sums of angles between segments and planes of polygonal arcs inscribed in the curve. It turned out to be fruitful to combine this method with the integral-geometric method of projections suggested by Reshetnyak [2].

In [3, 4] a theory of nonregular curves was constructed by a method not based directly on approximation by polygonal arcs. The integral curvatures were defined here as limits of sums of angles between k-dimensional tangent planes in their natural order of succession and condensation of points of tangency. In the present paper we consider questions of approximation of integral curvatures of curves in $R^n$ and $S^n$ by curvatures of polygonal arcs and secant planes.

We cite the basic concepts needed.

Let $P^k$ and $Q^k$ be oriented k-dimensional planes in $R^n$, $k < n$. We define the angle between them by the equation

$$\varphi(P^k, Q^k) = \int_{\mathfrak{N}_n^k} \nu^k(PP^k, PQ^k)\, d\mu_{k,n},$$

where $\mathfrak{N}_n^k$ is the manifold of k-dimensional planes in $R^n$, passing through one point, $\mu_{k,n}$ is the unit measure on $\mathfrak{N}_n^k$, invariant with respect to transformations of $\mathfrak{N}_n^k$, corresponding to motions of $R^n$ [5]. The function $\nu^k$ on $\mathfrak{N}_n^k$ is the following:

$$\nu^k(P) = \nu^k(PP^k, PQ^k) = 1,$$

if the orientations of the orthogonal projections $PP^k$ and $PQ^k$ of the planes $P^k$ and $Q^k$ on $P \in \mathfrak{N}_n^k$ are different;

$$\nu^k(P) = \nu^k(PP^k, PQ^k) = 0,$$

if the orientations of $PP^k$ and $PQ^k$ coincide. The orientations of the projections are determined in the natural way, $\nu^k$ is defined almost everywhere with respect to the measure $\mu_{k,n}$, k-dimensional planes in $S^n$ and the angle between them are defined in the natural way by identification with (k + 1)-dimensional planes in $R^{n+1}$, passing through the center.

Further the curvatures are understood as classes of equivalent parametrizations with the Frechet metric [6]. We define the k-dimensional tangent planes of a curve and indicatrices of k-dimensional tangent planes. We restrict ourselves for simplicity to the case when $K \subseteq R^n$ and the curve has no arc of dimension less than n − 1. Let $x(t_0)$ be a point of the curve

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