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The dynamics of rational transforms: the topological picture[1]

M.Yu. Lyubich

CONTENTS

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[1] This translation incorporates some corrections and additions communicated by the author (Ed.).

LIST OF BASIC NOTATION

$\mathbf{C}$, the complex plane;
$\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, the Riemann sphere;
$\mathbf{C}^* = \mathbf{C} \setminus \{0\}$, the punctured plane (cylinder)
$\mathbf{U}_r = \{z: |z| < r\}$, $\mathbf{U} \equiv \mathbf{U}_1$, a disc;
$\mathbf{U}^* = \mathbf{U} \setminus \{0\}$, the punctured disc;
$\mathbf{H} = \{z: \operatorname{Im} z > 0\}$ the upper half-plane;
$\mathbf{A}(r_0, r_1) = \{z: r_0 < |z| < r_1\}$, an annulus;
$\mathbf{T}_r = \{z: |z| = r\}$, $\mathbf{T} \equiv \mathbf{T}_1$, a circle;
$\mathbf{T}^2$, the two-dimensional torus;
$\mathbf{R}$, the real line;
$\mathbf{Z}$, the ring of integers;
$\mathbf{N} = \{0, 1, \ldots\}$, the natural numbers;
$SL_2(k)$, the unimodular group over a ring k;
$PSL_2(k) = SL_2(k)/\{\pm I\}$, the projective unimodular group;
$\rho = \rho_{\bar{\mathbf{C}}}$, the spherical metric;
$\rho_{\mathbf{C}}$, the Euclidean metric;
ρ_V, the hyperbolic metric on a hyperbolic Riemann surface V;
σ_V (or simply σ) the standard conformal structure on a Riemann surface V;
f, a transform (rational as a rule);
f^n, the n-th iterate of a transform f;
d, the degree of a transform f;
Df, the differential of f;
$\omega(z)$, the limit set of the orbit $\{f^n z\}_{n=0}^{\infty}$;
$\omega_f(\omega_f')$, the union of the limit set $\omega(c)$ of the orbits of critical points (lying in $J(f)$);
$\Delta(\alpha)$, the attracting region of a cycle α; $\Delta_0(\alpha) = \Delta(\alpha) \setminus \bigcup_{n=0}^{\infty} f^{-n}\alpha$;
$D(\alpha)$, the immediate attracting region of a cycle α;
$D(\alpha_k)$, the component of the immediate attracting region containing the point α_k;
$\mathfrak{R}_d$, the manifold of rational functions of degree d;
$\mathfrak{M}$, a complex analytic submanifold of $\mathfrak{R}_d$;
Σ, the set of J-stable functions in $\mathfrak{M}$;
$\mathrm{Cr} = \{(f, c) \in \mathfrak{M} \times \bar{\mathbf{C}}: Df(c) = 0\}$;
qc(f), the set of rational functions that are quasi-conformally conjugate to f;
$S(f)$, the Riemann surface of a function f;
$T(f)$, the Teichmüller space of a function f;
$G(f)$, the group of quasi-conformal homeomorphisms of the sphere that commute with f;
Mod(f), the modular group of a function f;
M, the Mandelbrot set.

Introduction

In Comptes Rendus for 1906 a note by Fatou was published in which a surprising discovery was made: iterations of a very simple function $f(z) = z^2/(z^2 + 2)$ lead naturally to the appearance of a Cantor set, which was considered then as a very exotic object. Later Fatou and Julia undertook a thorough study of the dynamics of rational transforms, the results of which were presented in extensive memoirs that appeared in 1918-1920.

Let $f : \mathbf{C} \circlearrowleft$ be a rational transform (an endomorphism) of the Riemann sphere, and $f^n = f \circ \dots \circ f$ its n-iterate. A decisive step made by Fatou and Julia is the decomposition of the sphere into two invariant subsets: an open set on which the family of iterates $\{f^n\}$ is a normal Montel family (that is, locally equicontinuous in the spherical metric), and a perfect set coinciding with the closure of the set of repelling periodic points of f. We call the first of them the Fatou set (or normality set) and denote it by $F(f)$, and the second the Julia set $J(f)$. The orbits of points $z \in F(f)$ are Lyapunov stable. On the contrary, the dynamics on $J(f)$ is of unstable chaotic character. Even in the simplest cases the Julia set has unexpectedly intricate structure. The Julia set of the function $z^2 - 3$ is a Cantor set. The Julia set of the function $z^2 + \varepsilon$ for small $\varepsilon \neq 0$ is a simple closed curve, but this curve does not have a tangent at any point. And the Julia set of the function $z^2 - 1$ is a curve that decomposes the plane into countably many components.

The study of the dynamics on the set $F(f)$ was carried out to a great extent by the classicists with the help of the local theory developed by Schröder, Koenigs, Leau, and Böttcher in the 19th century and the very beginning of the 20th century. The latter essential progress is related to the results of Siegel (1942) that an analytic transform reduces to a rotation in a neighbourhood of a fixed point.

The creation of the theory of hyperbolic dynamical systems in the 60's and early 70's in the papers by Anosov, Smale, Sinai, and Bowen led to the fact that the intricate dynamics of rational endomorphisms ceased to be considered as something strange related to irreversibility. The papers by Yakobson and Guckenheimer, where the iterates of rational functions are studied by the methods of symbolic dynamics, date back to that time.

Recently the study of the dynamics of rational endomorphisms has become very popular. Great enthusiasm was caused by numerical experiments carried out by Mandelbrot in 1980, which resulted in the appearance of deep conjectures and beautiful pictures visually demonstrating the fact that the situation is non-trivial [75]. Soon papers by Douady and Hubbard, Sullivan, and Thurston appeared, which related the dynamics of rational endomorphisms to the theory of Kleinian groups and Teichmüller spaces. These relations cast a new light on the whole field and provided a key to many problems. In particular, Sullivan completed the description of the dynamics on the set $F(f)$.

The only exposition in Russian of the classical results of Fatou and Julia is a chapter of the book by Montel (1936). The results of the development of this field up to 1965 were presented in a survey by Brolin [45]. Recently a survey by Blanchard [43] has appeared, which can be used as a brief and refined introduction to the topic. Finally, a number of recent results are stated in a section of a survey by Yakobson [35], which has just appeared.

The present survey was first conceived as a complete exposition of the basic results of the theory of iterates of one-dimensional analytic transforms. However, this field is being developed so rapidly that it soon became clear that such an aim can never be attained. Therefore, we had to restrict ourselves to only topological properties of the dynamics of rational transforms. But even under such a limitation a number of essential results were omitted or mentioned casually.

The topic of the survey can naturally be divided into two parts, which correspond to the first and the second chapters of the survey: 1) the dynamics of an individual endomorphism, and 2) the character of the dependence of the dynamics on parameters. The contents of the first chapter are concentrated around the problem of the classification of periodic points, the description of the dynamics on the Fatou set $F(f)$ and, the clarification of the structure of the Julia set $J(f)$. The basic problem of the second chapter is a topological classification of rational endomorphisms. Here a clue is the famous Fatou conjecture (still unproved), which in modern terms sounds like this: a generic rational endomorphism satisfies Smale's Axiom A. We present recent essential progress in this direction: 1) a theorem on the structural stability of a generic rational endomorphism, and 2) the theory of quasi-conformal deformations of rational endomorphisms. This problem is meaningful and difficult even for the simplest quadratic family $f_w\colon z \mapsto z^2 + w$ considered in the concluding section.

We mention a number of topics closely related to the contents of the survey but not touched on in it.

1. *The solution of natural functional equations in the class of rational functions,* in particular, *the description of commuting rational functions.* This topic is considered in the classical papers by Julia [66], Fatou [62], and Ritt [84] (see also Baker [38], [39]).

2. *The ergodic theory of rational endomorphisms.* The first results in this direction were obtained by Brolin [45]; however, systematic research has been started only recently. To guide the interested reader we mention several papers [41], [73], [82], [83], [85], [91].

3. *Iterates of entire functions.* The rational endomorphisms exhaust all the analytic transforms of the unique elliptic Riemann surface, namely the Riemann sphere. Naturally, the problem arises of describing the dynamics of analytic transforms $f : V \circlearrowleft$ of other Riemann surfaces. If V is a hyperbolic Riemann surface, then there are no chaotic phenomena and the situation is quite simple. It is presented in § 1.2. The examination of a parabolic surface leads to iterates of entire functions. This has interesting specific features (and additional analytic difficulties), which become apparent even for the simplest function e^z. Having no opportunity of dwelling on this subject in more detail, we only mention some recent papers [13], [39], [52], [78].

The body of the survey contains some historical comments which, however, do not claim to be complete. The references are also far from complete. When stating a theorem we usually indicate the author of the corresponding result, but not of the proof. As a rule, the proofs differ essentially from the original.

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CHAPTER I

THE DYNAMICS OF AN INDIVIDUAL ENDOMORPHISM

We present briefly the contents of this chapter. In §1.2 we describe the dynamics of analytic transforms of hyperbolic Riemann surfaces. In §§1.3, 1.4, and 1.6 we define the Fatou set $F(f)$ and the Julia set $J(f)$ of a rational endomorphism, study their simplest properties, and give the first examples. In §§1.8–1.12 we describe five types of periodic components of the set $F(f)$. The first four of them are generated by non-repelling periodic points. We thus also classify such points. The components bear the names of the authors who studied the corresponding local problem (Schröder, Leau, Böttcher, Siegel). The fact that a rational endomorphism may have components of the 5th type (periodic doubly-connected domains) has been obtained recently by Herman. The results of §1.2 and Theorem 1.10 (§1.11) show that the five types mentioned above exhaust all the possibilities. Sullivan's theorem on the absence of wandering domains (§1.15) completes the description of the dynamics on the Fatou set $F(f)$. In §§1.13, 1.14 we prove theorems on the density of repelling periodic points and inverse images in the Julia set $J(f)$. Sections 1.16–1.18 are concerned with special classes of endomorphisms: 1) satisfying Axiom A; 2) polynomial; 3) those whose orbits of critical points are absorbed by cycles (in this case $J(f) = \bar{\mathbf{C}}$). In §1.19 we give a sufficient condition under which the Lebesgue measure of $J(f)$ is zero. In the concluding section §1.20 Newton's iteration process of locating the roots of a polynomial is considered from the point of view of the theory developed. In §§1.1, 1.5, and 1.7 for the convenience of the reader we outline the necessary supply of preliminaries from the theory of Riemann surfaces and quasi-conformal maps.

§1.1. The hyperbolic metric

1. A central fact of classical complex analysis is *the Koebe-Riemann uniformization theorem,* which claims that *any simply-connected Riemann surface is conformally equivalent either to the complex plane* $\mathbf{C}$, *or the Riemann sphere* $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, *or the unit disc* $\mathbf{U} = \{z: |z| < 1\}$. In the latter case it is often convenient to use the upper half-plane $\mathbf{H} = \{z: \operatorname{Im} z > 0\}$ instead of the disc. The plane $\mathbf{C}$, the punctured plane $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$, and the two-dimensional torus $\mathbf{T}^2$ are called *parabolic* Riemann surfaces: their universal covering space is conformally equivalent to the plane $\mathbf{C}$. The sphere $\bar{\mathbf{C}}$ is called an *elliptic* Riemann surface. The universal covering space of the other Riemann surfaces are conformally equivalent to the disc $\mathbf{U}$. These surfaces are called *hyperbolic.*

We denote by $SL_2(k)$ the group of (2×2)-matrices over the ring k with determinant equal to 1. The cases $k = \mathbf{C}$, $\mathbf{R}$, or $\mathbf{Z}$ are of importance for us. The quotient group of $SL_2(k)$ over the subgroup $\{\pm I\}$ is denoted by $PSL_2(\bar{\mathbf{C}})$. The group of conformal diffeomorphisms$^{(1)}$ $z \mapsto (az + b)/(cz + d)$ of the Riemann sphere $\bar{\mathbf{C}}$ onto itself is naturally isomorphic to $PSL_2(\bar{\mathbf{C}})$. The group of conformal diffeomorphisms of the plane $\mathbf{C}$ is the affine group $\mathrm{Aff}(\mathbf{C})$ of the transforms $z \mapsto az + b$. Finally, the group of conformal diffeomorphisms of the half-plane $\mathbf{H}$ is isomorphic to $PSL_2(\mathbf{R})$. Consequently, any hyperbolic Riemann surface is the quotient space of the half-plane $\mathbf{H}$ modulo the action of some discrete subgroup $G \subset PSL_2(\mathbf{R})$.

2. We consider the Poincaré metric $\rho_{\mathbf{H}}$ on $\mathbf{H}$ with linear element $d\rho_{\mathbf{H}} = |dz|/\operatorname{Im} z$. The half-plane endowed with the Poincaré metric is a model of the Lobachevskii plane. The group of motions of the Lobachevskii plane coincides with the group of conformal diffeomorphisms of the half-plane. Hence, the Poincaré metric can be transferred from $\mathbf{H}$ to any hyperbolic Riemann surface V. The resulting metric ρ_V is called the *hyperbolic metric on V.*

Conformally invariant Schwarz lemma (see [10], [69]). *Let* $f: V \to W$ *be an analytic map of hyperbolic Riemann surfaces. Then the following alternative holds*: *either* a) f *is strictly contractive in the hyperbolic metric, that is,* $\|Df(z)\| < 1$ $(z \in V)$, *or* b) f *is a covering.*

3. We now consider a hyperbolic domain V on the sphere $\bar{\mathbf{C}}$. *The hyperbolicity of* V *is equivalent to the fact that* $\bar{\mathbf{C}} \setminus V$ *contains more than two points.* We denote by ρ the spherical metric $\frac{2|dz|}{1+|z|^2}$ on the sphere $\bar{\mathbf{C}}$ and by $p_V(z)$ the coefficient of proportionality relating the linear elements of the hyperbolic and spherical metrics: $d\rho_V(z) = p_V(z)d\rho(z)$ $(z \in V)$.

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(1) We assume (without mentioning it specially) that conformal (quasi-conformal) homeomorphisms preserve orientation.

***Lemma* 1.1.** $p_V(z) \to \infty \ (z \to \partial V)$.

Proof. Otherwise there is a sequence $z_k \to a \in \partial V$ such that $p_V(z_k) \leqslant C$. We choose two more points $b, c \in \partial V$ and consider the domain $W = \bar{\mathbf{C}} \setminus \{a, b, c\}$. By the Schwarz lemma $p_V(z) \geqslant p_W(z)$ $(z \in V)$.

Without loss of generality we can assume that $(a, b, c) = (0, 1, \infty)$. The universal covering of W is the modular function $\lambda : \mathbf{H} \to W$. It admits a representation $\lambda(\zeta) = \varphi(e^{i\pi\zeta})$, where $\varphi(h) = 16h + \dots$ is holomorphic in $\mathbf{U}$ (see [4], §23). From this with the help of direct calculations, using the explicit form of the Poincaré metric, it follows that $p_W(z) \sim 2(|z| \log(1/|z|))^{-1} \to \infty \ (|z| \to 0)$, which contradicts our assumptions.

§1.2. Analytic transforms of hyperbolic Riemann surfaces

1. Let $f : V \circlearrowleft$ be an analytic transform (an endomorphism) of a Riemann surface V, and α a *fixed point* of it: $f\alpha = \alpha$. The *multiplier* λ of α is the derivative of f at this point (which does not depend on the choice of a local parameter). The point α is called *attracting* if $0 < |\lambda| < 1$, *superattracting* if $\lambda = 0$, *neutral* if $|\lambda| = 1$, and *repelling* if $|\lambda| > 1$. Two transforms $f: V \circlearrowleft$ and $g: W \circlearrowleft$ are called *topologically* (*conformally*) *conjugate* if there is a (conformal) homeomorphism $h : V \to W$ such that $h \circ f = g \circ h$. We denote by f^m the m-th iterate of f. The set $\{f^m z\}_{m=0}^{\infty}$ is called the *orbit* (or *trajectory*) of f.

The simplest transforms of hyperbolic Riemann surfaces are the motions of the Lobachevskii plane $f: z \mapsto \frac{az+b}{cz+d}$, where $A_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$. The transform f is called *hyperbolic* if $|\mathrm{tr}\, A_f| > 2$, *parabolic* if $|\mathrm{tr}\, A_f| = 2$, and *elliptic* if $|\mathrm{tr}\, A_f| < 2$. In the first case f is conformally conjugate to the transform $z \mapsto \lambda z$ $(0 < \lambda < 1)$, in the second case to the translation $z \mapsto z + 1$. Finally, in the last case f is conformally conjugate to the rotation $r_\lambda: z \mapsto \lambda z$ $(|\lambda| = 1)$ of the disc $\mathbf{U}$. If $\lambda = e^{2\pi i\theta}$, where θ is irrational, then r_λ is called an *irrational rotation*.

2. We introduce the notation $\mathbf{U}^* = \mathbf{U} \setminus \{0\}$ for the punctured disc, and $\mathbf{A}(r_0, r) = \{z\colon r_0 < |z| < r\}$ for an annulus.

***Theorem* 1.1.** *Let $f : V \circlearrowleft$ be an analytic transform of a hyperbolic Riemann surface V. Then we have one of the following possibilities*:

a) *f has an attracting or superattracting fixed point $\alpha \in V$ to which all the orbits $\{f^m z\}_{m=0}^{\infty}$ $(z \in V)$ converge*;

b) *all the orbits $\{f^m z\}$ tend to infinity, that is, $\rho_V(a, f^m z) \to \infty$ $(m \to \infty)$ for every $a \in V$*;

c) *f is conformally conjugate to an irrational rotation of* (i) *the disc* $\mathbf{U}$, (ii) *the punctured disc* $\mathbf{U}^*$, (iii) *the annulus* $\mathbf{A}(r, 1)$, *where* $0 < r < 1$;

d) *the transform f is a conformal homeomorphism of finite order*: $f^p = \mathrm{id}$ *for some* p.

Proof. The hyperbolic Riemann surface V is the quotient of the Lobachevskii plane $\mathbf{H}$ modulo some group G of motions. We denote by $\pi : \mathbf{H} \to V$ the corresponding covering. The transform f can be lifted to an analytic endomorphism $A\colon \mathbf{H} \circlearrowleft$, $\pi \circ A = f \circ \pi$. By the Schwarz lemma we have one of two possibilities: 1) A is strictly contractive in the Poincaré metric $\rho_{\mathbf{H}}$; 2) A is a motion of the Lobachevskii plane.

Let us consider the first case. Suppose that an orbit $\{A^m z\}_{m=0}^{\infty}$ does not tend to infinity. Then there is a number $k \in (0, 1)$ such that $\rho_{\mathbf{H}}(A^{m_e+1}z, A^{m_e}z) \leqslant k\rho_{\mathbf{H}}(A^{m_e}z, A^{m_e-1}z)$ for some subsequence $\{m_e\}$. Hence, $\rho_{\mathbf{H}}(A^{m+1}z, A^m z) \to 0\ (m \to \infty)$. It follows that any limit point of the orbit $\{A^m z\}_{m=0}^{\infty}$ is a fixed point. Since the transform A is strictly contractive, it can have only one fixed point α. Therefore, $A^m z \to \alpha (m \to \infty)$. But any other orbit $\{A^m \zeta\}_{m=0}^{\infty}$ also does not tend to infinity, since $\rho_{\mathbf{H}}(A^m z, A^m \zeta) \leqslant C$. Everything said above can be applied to $\{A^m \zeta\}_{m=0}^{\infty}$, and therefore, $A^m \zeta \to \alpha$ $(m \to \infty)$. Finally, since A is strictly contractive, α is either an attracting or superattracting fixed point. Thus, in the first case one of the possibilities a) or b) holds.

In the second case we consider the group Γ of motions of the Lobachevskii plane, which is obtained by joining A to G. Suppose that Γ is discrete and f has infinite order. For every $a \in \mathbf{H}$ we have

$$\rho_V(f^m z, \pi a) = \inf_{B \in G} \rho_{\mathbf{H}}(A^m z, Ba) = \rho_{\mathbf{H}}(B_m^{-1} A^m z, a)$$

for a suitable $B_m \in G$. Since the transforms $B_m^{-1}A^m \in \Gamma$ are pairwise disjoint, it follows that $\rho_{\mathbf{H}}(B_m^{-1}A^m z, a) \to \infty\ (m \to \infty)$.

Finally, suppose that Γ is not discrete. We then consider the Lie group $\bar{\Gamma}$. Let Γ_0 be the connected component of the unit in the group $\bar{\Gamma}$. Since $AGA^{-1} \subset G$, it follows that $\Gamma_0 G \Gamma_0^{-1} \subset G$. But since G is discrete and Γ_0 is continuous, we have $\Gamma_0 g \Gamma_0^{-1} \equiv g$. Thus, Γ_0 and G commute. But the centralizer of any element $h \in \mathrm{PSL}_2(\mathbf{R})$ different from the unit is a one-parameter subgroup passing through h. It follows that G is contained in a one-parameter group. Now $G = \{g^m\}_{m=-\infty}^{\infty}$, since G is discrete. If $g = \mathrm{id}$, then $V = \mathbf{U}$; if g is a parabolic transform, then $V = \mathbf{U}^*$; if g is a hyperbolic transform, then $V = \mathbf{A}(r, 1)$. It follows from the property $AgA^{-1} = g^k$ that f is conjugate either to a rotation of the corresponding domain, or to the transform $z \mapsto z^k$ of the punctured disc $\mathbf{U}^*$, or to the transform $z \mapsto r/z$ of the annulus $\mathbf{A}(r, 1)$. This yields c), d), e), respectively.

Corollary. *Let V be a hyperbolic Riemann surface, and $f\colon V \circlearrowleft$ an analytic transform of infinite order. Then f has at most one fixed point. If α is a fixed point, then it is attracting, superattracting, or neutral. In the first two cases $f^m z \to \alpha\ (m \to \infty)$ for every $z \in V$. In the third case f is conformally conjugate to an irrational rotation of the disc* $\mathbf{U}$.

We observe that transforms of finite order may have several fixed points. For example, on the hyperelliptic Riemann surface of genus g there is an involution with $2g + 2$ fixed points.

3. ***Theorem* 1.2.** *Let V be a hyperbolic domain on the sphere, and $f : V \circlearrowleft$ an analytic transform continuous up to the boundary. Suppose that the set of fixed points of f on ∂V is totally disconnected. Then in the case* b) *of Theorem* 1.1 *there is a fixed point $\alpha \in \partial V$ such that $f^m z \to \alpha$ $(m \to \infty)$ for every $z \in V$.*

Proof. In case b) of Theorem 1.1 all the orbits $\{f^m z\}_{m=0}^{\infty}$ tend uniformly to ∂V. We consider a smooth curve l in V connecting two points $z, \zeta \in V$. The non-Euclidean length of the curve $l_m = f^m l$ does not increase. In view of Lemma 1.1 the spherical length of the curve l_m tends to zero. Consequently, $\rho(f^m z, f^m \zeta) \to 0$ $(m \to \infty)$, that is, the asymptotic behaviour of all the orbits in V is the same. In addition, for $\zeta = fz$ we see that the limit points of the curve $L = \cup l_m$ are fixed points. Since the set $\omega(L)$ of limit points of L is connected, and the set of fixed points of the transform $f: \partial V \circlearrowleft$ is totally disconnected, $\omega(L)$ consists of only one point $\alpha \in \partial V$. It follows that $f^m z \to \alpha$ $(m \to \infty)$.

In the above theorem the condition that the set of fixed points is totally disconnected is essential. Consider, for example, the shaded spiral domain W in the disc $\mathbf{U}$ (Fig. 1). It has a unique non-single-point prime end ξ. Its support is the circle $\mathbf{T} = \partial \mathbf{U}$. Consider a conformal diffeomorphism $\varphi : W \to \mathbf{H}$ that maps the prime end ξ to ∞. We put g: $\zeta \longmapsto \zeta + 1$, $f = \varphi^{-1} \circ g \circ \varphi$: $W \circlearrowleft$. Then the limit set of any orbit $\{f^m z\}$ is the whole circle $\mathbf{T}$. Next, consider the invariant domain $V = \{z \in W\colon \operatorname{Im} \varphi(z) > 1\}$. We have $\rho_W(z, fz) \leqslant 1$ for $z \in V$. By Lemma 1.1 $\rho(z, fz) \to 0$ as z tends to $\mathbf{T}$, remaining in V. Therefore, putting $f \,|\, \mathbf{T} = \mathrm{id}$, we obtain a continuous transform $\overline{V} \circlearrowleft$.

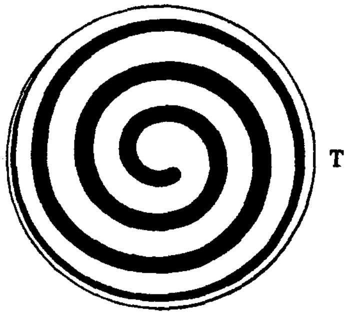

Fig. 1.

If $V = \mathbf{U}$, a disc, then no additional assumptions on the boundary properties of f are required.

Theorem of Denjoy and Wolff (see [7], §43). *If f: $\mathbf{U} \circlearrowleft$ is an analytic transform of the disc, then we have one of the following possibilities:*

a) *there is a point $\alpha \in \overline{\mathbf{U}}$ such that $f^m z \to \alpha$ $(m \to \infty)$ for every $z \in \mathbf{U}$;*

b) *f is conformally conjugate to a rotation of the disc.*

An analogous result holds for a domain $V \subset \overline{\mathbf{C}}$ bounded by finitely many Jordan curves.

The results of this section in general were known to the classicists (see [7], where there are historical comments). In our exposition we have followed Sullivan ([90], Part III). The reader interested in a more detailed investigation of analytic transforms of the disc can turn his attention to [40], [50].

§1.3. Montel's theorem. The Fatou set and the Julia set

1. In the classical papers by Julia and Fatou the key role was played by the notion of a normal family of meromorphic functions, which had been introduced by Montel at the beginning of the century. A meromorphic function in a domain D is an analytic map $D \to \bar{\mathbf{C}}$. We introduce the spherical metric on the Riemann sphere, and in the space of all analytic maps $D \to \bar{\mathbf{C}}$ we consider the topology of uniform convergence on compact subsets of D.

A family $\{f_i\}$ *of analytic maps* $D \to \bar{\mathbf{C}}$ *is called normal if it is precompact in the above topology.* Equivalently, $\{f_i\}$ is normal if it is equicontinuous on every compact subset of D. The basic criterion of normality is as follows.

Montel's theorem. *A family* $f_i : D \to \bar{\mathbf{C}}$ *of meromorphic functions omitting three values* $\alpha, \beta, \gamma \in \bar{\mathbf{C}}$ *is normal.*

Proof. We consider the hyperbolic metrics ρ_D and ρ_W on D and $W = \bar{\mathbf{C}} \setminus \{\alpha, \beta, \gamma\}$. The analytic transforms $f_i : D \to W$ are contractive with respect to these metrics. But the spherical metric ρ is subordinate to ρ_W (Lemma 1.1) and is equivalent to ρ_D on every compact set $K \subset D$. It follows that the family $f_i : K \to \bar{\mathbf{C}}$ is equicontinuous in the spherical metric.

In what follows we shall occasionally use the following simple fact:

Lemma **1.2.** *If a family* $\{f_i\}$ *of meromorphic functions is normal in a neighbourhood of* $a \in \bar{\mathbf{C}}$ *and* $|f_i(a)| \leqslant L$, *then* $|f_i'(a)| \leqslant M$.

2. We now consider an analytic transform $f\colon \bar{\mathbf{C}} \circlearrowleft$ of the Riemann sphere. It is given by a rational function $z \mapsto P(z)/Q(z)$. We denote by *d the degree of the transform f*, which is equal to max(deg P, deg Q). Every point of the sphere, except for a finite number, has d inverse images.

If $d = 1$, then $f\colon z \mapsto \frac{az+b}{cz+d}$, where

$$A_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C}).$$

If tr $A_f \in \mathbf{R}$, then f is conformally conjugate to a transform g with $A_g \in \mathrm{SL}_2(\mathbf{R})$. Such transforms have been classified in the preceding section. Otherwise f is called *loxodromic* and is conformally conjugate to the transform $z \mapsto \lambda z$ with $|\lambda| < 1$.

In what follows we assume, unless otherwise specified, that f is a rational endomorphism of the sphere $\bar{\mathbf{C}}$ *of degree* $d > 1$.

3. *The Fatou set* $F(f)$ (or *normality set*) of the transform f is the maximal subset on which the family of iterates f^m is normal. A set X is called *invariant* if $fX \subset X$, and *completely invariant* if $f^{-1}X = X$. It follows directly from the definitions that the Fatou set is open and completely invariant. If $z \in F(f)$, then the family $\{f^m\}_{m=0}^{\infty}$ is equicontinuous in a neighbourhood of z. It follows that the orbit of z is *Lyapunov stable*: for all $\varepsilon > 0$ there exists $\delta > 0$: $\rho(z, \zeta) < \delta \Rightarrow \rho(f^m z, f^m \zeta) < \varepsilon$ $(m = 0, 1, \ldots)$. In what follows we show (Corollary 3 of Theorem 1.15) that conversely, the stability of the orbit of z implies that $z \in F(f)$. If D is an invariant hyperbolic domain on the sphere $\bar{\mathbf{C}}$, then Montel's theorem yields $D \subset F(f)$.

The complement of the Fatou set is called *the Julia set*(1) $J(f) = \mathbf{C} \setminus F(f)$. The Julia set is closed and completely invariant. We mention the three simplest examples.

4. *Example* 1.1. $f\colon z \mapsto z^d$. The disc $\mathbf{U}$ is invariant under f and consequently, $\mathbf{U} \subset F(f)$. All the orbits in $\mathbf{U}$ converge to zero. Similarly, $\bar{\mathbf{C}} \setminus \bar{\mathbf{U}} \subset F(f)$ and $f^m z \to \infty$ $(m \to \infty)$ for $|z| > 1$. In a neighbourhood of the unit circle $\mathbf{T}$ some orbits converge to 0, while others converge to ∞. Consequently, $\mathbf{T} = J(f)$. We note that the dynamics on $\mathbf{T}$ is of complicated stochastic character. This is related to the fact that f preserves Lebesgue measure on $\mathbf{T}$ and is mixing (see [29]).

5. A perfect (that is without isolated points) totally disconnected compact metric space is called a *Cantor set*.

Example 1.2. $f\colon z \mapsto 2z - 1/z$ (Fig. 2). The upper and lower half-planes and the exterior of the unit disc are invariant under f. Hence, $J(f) \subset [-1, 1]$.

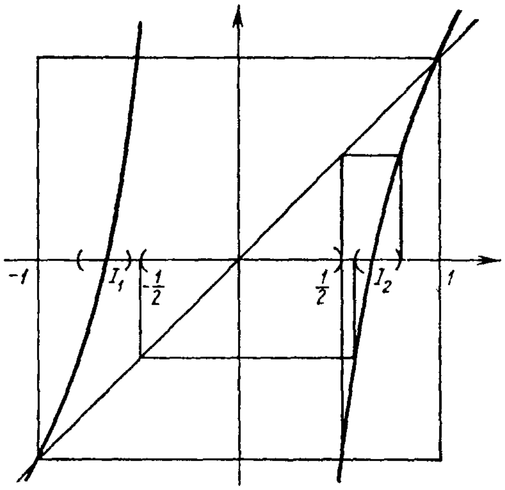

Fig. 2. The transform $z \to 2z - \frac{1}{z}$

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(1)The term "Julia set" and notation $J(f)$ are commonly used in recent papers. This cannot be said of the term "Fatou set" and notation $F(f)$, which we have borrowed from the survey by Blanchard [43].

We now consider the interval $I_0 = (-{}^1/_2, {}^1/_2)$. We have $fI_0 = \{x \in \mathbf{R} : |x| > 1\}$. It follows that $I_0 \subset F(f)$. Let $K_1 = [-1, -{}^1/_2]$, $K_2 = [{}^1/_2, 1]$. Then $J(f) \subset K_1 \cup K_2$ and f maps the K_i monotonically onto $[-1, 1]$. Therefore, there is an interval $I_i \subset K_i$ such that $fI_i = I_0$. If we cut off the interval I_i from K_i, we obtain two intervals K_{i1}, K_{i2}. They satisfy $J(f) \subset \bigcup_{i,\, j=1,\, 2} K_{ij}$ and f maps K_{ij} monotonically onto K_j. If we continue this construction, we obtain a family $K_{i_0 i_1 \dots i_{m-1}}$ ($i_l = 1, 2$) of 2^m intervals such that $J(f) \subset \bigcup K_{i_0 \dots i_{m-1}} \equiv K^m$ and f maps $K_{i_0 \dots i_{m-1}}$ monotonically onto $\mathcal{K}_{i_1 \dots i_{m-1}}$. Since $|f'(x)| \geqslant 3$ on $[-1, 1]$, we see that the lengths of the intervals $K_{i_0 \dots i_{m-1}}$ do not exceed $2/3^m$. Therefore, $K^\infty = \cap K^m$ is a Cantor set.

The Julia set $J(f)$ is contained in K^∞. Conversely, if $x \in K^\infty$, then $|f^m x| \leqslant 1$ and $|(f^m)'(x)| \geqslant 3^m$. By Lemma 1.2, $x \in J(f)$. Thus, $J(f) = K^\infty$ is a Cantor set in $[-1, 1]$.

6. *Example* 1.3. *The Ulam-von Neumann transform*⁽¹⁾ $f\colon z \mapsto 2z^2 - 1$. The interval $I = [-1, 1]$ is completely invariant. As in Example 1.1 it follows that $J(f) = I$. All the orbits converge to ∞ on the normality set $\bar{\mathbf{C}} \setminus I$. To study the dynamics on the Julia set we note that f is a Chebyshëv polynomial:

$$\cos 2\theta = f(\cos\theta). \tag{1.1}$$

This formula can be interpreted as follows. We consider the homeomorphism $\cos\colon [-\pi, 0] \to I$. In view of (1.1), cos is a conjugation of f and the saw-tooth transform $g\colon [-\pi, 0] \circlearrowleft$

$$g\colon \theta \mapsto \begin{cases} 2\theta & (-\pi/2 \leqslant \theta \leqslant 0), \\ -2\pi - 2\theta & (-\pi \leqslant \theta \leqslant -\pi/2). \end{cases}$$

The transform g preserves Lebesgue measure $d\theta$ and is mixing. Hence, f preserves the absolutely continuous measure $dx/\sqrt{1 - x^2}$ on I and is also mixing. This explains the stochastic character of the behaviour of almost all orbits on the Julia set, the phenomenon discovered by Ulam and von Neumann in 1947 with the help of one of the first computers [94].

The reader should not be misled by the above examples. They are good models of the dynamics on the Julia set $J(f)$ rather than its topological structure. As a rule, $J(f)$ does not lie on a smooth curve and has extremely complicated structure. The corresponding examples will be presented below.

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(1) The Ulam-von Neumann transform is often presented in the form $z \mapsto 4z(1 - z)$ or $z \mapsto z^2 - 2$, which reduces to $z \mapsto 2z^2 - 1$ by an affine conjugation.

§1.4. The simplest properties of the Julia set

1. ***Proposition* 1.1.** *The Julia set is non-empty.*

Proof. Suppose that the family $\{f^m\}$ is normal on the whole sphere $\bar{\mathbf{C}}$. Then there is a sequence $m_k \to \infty$ and a rational function g such that $f^{m_k} \to g$ uniformly on $\bar{\mathbf{C}}$. The latter is impossible, since $\deg f^{m_k} \to \infty$.

***Proposition* 1.2.** *The set $J(f)$ is nowhere dense or it coincides with the whole sphere.*

Proof. Suppose that $J(f)$ contains a domain D. Then by Montel's theorem $\bigcup_{m=0}^{\infty} f^m D$ is the whole sphere except perhaps for two exceptional points. But $J(f) \supset \overline{\bigcup_{m=0}^{\infty} f^m D} = \bar{\mathbf{C}}$.

2. A point $\alpha \in \bar{\mathbf{C}}$ is called *periodic* if $f^p\alpha = \alpha$ for some p. The number p is called *the period* of the periodic point α and $\{f^n\alpha\}_{n=0}^{p-1}$ is called a cycle. The smallest of the periods is called *the order* of a periodic point (cycle). *The multiplier* λ of a periodic point α (of its cycle) is the multiplier of the point α considered as a fixed point of the transform f^p, where p is the order. This definition does not depend on the choice of a point of the cycle. Depending on the value of $|\lambda|$ we define *attracting, superattracting, neutral, and repelling periodic points* (*of their cycles*) (see §1.2).

The attracting and superattracting cycles $\{\alpha_k\}_{k=0}^{p-1}$ are contained in the Fatou set $F(f)$. Indeed, if D is a sufficiently small disc centered at α_0, then $f^pD \subset D$. Therefore, the set $\bigcup_{k=0}^{p-1} f^kD$ is invariant and is consequently contained in $F(f)$. We note at once that any polynomial $f\colon z \mapsto a_0z^d + \dots + a_d$ has the superattracting fixed point ∞.

3. Let $a \in J(f)$. A point α is called *exceptional* (for the point a) if the family $\{f^m\}$ does not take the value α in a neighbourhood of a. For every $a \in J(f)$ there are at most two exceptional points α_i. The existence of exceptional points imposes essential restrictions on the rational function f.

Suppose that α is a unique exceptional point for some $a \in J(f)$. Then it is obvious that $f^{-1}\alpha = \{\alpha\}$. It follows that α is a superattracting fixed point and so $\alpha \in F(f)$. Consequently, α is an exceptional point for any $b \in J(f)$. We now show that the transform f is conformally conjugate to a polynomial transform. For if $\varphi\colon z \mapsto 1/(z - \alpha)$, then the rational function $g = \varphi \circ f \circ \varphi^{-1}$ has no poles in $\mathbf{C}$.

Similarly, if α_1, α_2 are two exceptional points for some $a \in J(f)$, then the α_i are either superattracting fixed points or form a superattracting cycle of the second order, $\alpha_i \in F(f)$. In addition, the α_i are exceptional for every $a \in J(f)$. If $z \mapsto (z - \alpha_1)/(z - \alpha_2)$, then $(\varphi \circ f \circ \varphi^{-1})(z) = cz^{\pm d}$.

We see that the notion of exceptional point in fact does not depend on the choice of $a \in J(f)$. Therefore, in what follows we shall not mention this dependence.

4. ***Proposition* 1.3.** *The Julia set is perfect.*

Proof. Let $a \in J(f)$. Suppose that a is not a periodic point. We consider a neighbourhood D of a. Since a^p is not an exceptional point, there are numbers $m > 0$, $\zeta \in D$ such that $f^m\zeta = a$. Since a is not a periodic point, we have $\zeta \neq a$. Finally, since the Julia set is completely invariant, we have $\zeta \in J(f)$.

Now let $a \in J(f)$ be a periodic point of order p. Then the multiplier λ is non-zero, and so a is a simple root of the equation $f^pz = a$. Therefore, there is a number $b \neq a$ that also satisfies this equation. The desired assertion now follows from the fact that the set of isolated points in $J(f)$ is completely invariant.

§1.5. Ramified coverings. The Riemann-Hurwitz formula

1. Let V and W be two-dimensional surfaces. A map $f\colon V \to W$ is called a *d-sheeted ramified covering* ($1 \leqslant d \leqslant \infty$) if every $a \in W$ has a neighbourhood D such that

1) $f^{-1}(D, a) = \bigcup_{i=1}^{d} (B_i, b_i)$, where the B_i are mutually disjoint neighbourhoods of the b_i;

2) there are homeomorphisms $\varphi_i\colon (B_i, b_i) \to (\mathbf{U}, 0)$, $\psi_i\colon (D, a) \to (\mathbf{U}, 0)$ such that $(\psi_i \circ f \circ \varphi_i^{-1})(z) = z^{k_i}$.

If $k_i > 1$ for some i, then b_i is called a *branch point* of f, k_i is called its *branch index*, and a is called *the projection of the branch point.*

2. The following properties of ramified coverings can easily be verified:

(i) *The inverse image of any point $y \in W$ consists of d points, taking account of multiplicities.*

(ii) *Suppose that the surface W is connected, and that $V = \cup V_i$ is a decomposition of V into connected components. Then $f\colon V_i \to W$ is a ramified covering. In particular, $f \mid V_i$ is onto.*

(iii) *Let D be a domain in W. Then $f\colon f^{-1}D \to D$ is a ramified covering.*

Properties (ii) and (iii) imply the next property.

(iv) *Let D be a simply connected domain in W, and B a connected component of the inverse image $f^{-1}D$. If B has no branch points, then $f\colon B \to D$ is a homeomorphism.*

3. We denote by χ_Y the Euler characteristic of a surface Y.

The Riemann-Hurwitz formula (see [28], Ch. VI). *Let $f\colon V \to W$ be a d-sheeted ramified covering of V over W, $d < \infty$. Suppose that f has finitely many branch points $c_1, \dots, c_l$ with branch indices $k_1, \dots, k_l$, respectively. Then*

$$\chi_V = d\chi_W - \sum_{i=1}^{l} (k_i - 1).$$

Corollary. *Suppose that a domain $V \subset \bar{\mathbf{C}}$ admits a finite-sheeted ramified covering $f : V \circlearrowleft$ with branch points. Then V is either simply-connected or infinitely-connected.*

4. ***Lemma* 1.3.** *Let $f: V \to W$ be an analytic map of Riemann surfaces.*

a) *For f to be a finite-sheeted ramified covering it is necessary and sufficient that f is proper* (*that is, the inverse images of compact sets are compact*).

b) *Suppose that V and W are domains on the sphere $\bar{\mathbf{C}}$ and that f is continuous on $\bar{V}$. Then f is a finite-sheeted ramified covering if and only if $f(\partial V) \subset \partial W$.*

Lemma 1.3 shows that a rational endomorphism f of degree d is a d-sheeted analytic ramified covering $\bar{\mathbf{C}} \circlearrowleft$. The branch points of this covering are critical points of f (a point c is called *critical* if $Df(c) = 0$). The projections of the branch points are *critical values.* In this case the Riemann–Hurwitz formula yields $2 = 2d - \sum(k_i - 1)$. But $\sum(k_i - 1)$ is the number of critical points of f, taking account of multiplicities. Thus, *a rational function of degree d has $2d-2$ critical points, taking account of multiplicities.*

We now suppose that the hyperbolic Riemann surfaces V and W are simply-connected. Then $\chi_V = \chi_W = 1$. By the Riemann–Hurwitz formula the d-sheeted ramified covering $f: V \to W$ has $d-1$ critical points, taking account of multiplicities.

Example 1.4. Any d-sheeted ramified covering $\mathbf{U} \circlearrowleft$ is determined by a finite Blaschke product $f: z \mapsto \lambda \prod_{i=1}^{d} \frac{z-a_i}{1-\bar{a}_i z}$ ($|a_i| < 1$, $|\lambda| = 1$). In addition, $f: \bar{\mathbf{C}} \setminus \bar{\mathbf{U}} \circlearrowleft$ is also a d-sheeted ramified covering. Consequently, both $\mathbf{U}$ and $\mathbf{C} \setminus \mathbf{U}$ contain $d-1$ critical points, while there are no critical points on $\mathbf{T}$. Fatou called such transforms *rational functions with fundamental disc* and he dedicated to them the whole of Chapter III of the first memoir [59]. In particular, Fatou showed that either $J(f) = \mathbf{T}$ or $J(f)$ is a Cantor subset of $\mathbf{T}$ (see Examples 1.1, 1.2, and also 1.7, 1.8 of §1.10).

5. Let D be a domain on the sphere $\bar{\mathbf{C}}$, and B_i the connected components of the inverse image $f^{-1}D$. Then f is a ramified covering of the domain B_i over D. If f is univalent on B_i, then the inverse map $f_i^{-1} : D \to B_i$ is defined, and is called a (*single-valued*) *branch of the inverse function.*

Suppose now that D is simply-connected. If B_i has no critical points of f, then $f: B_i \to D$ is a homeomorphism. Thus, a branch f_i^{-1} of the inverse function is well-defined. If in addition D has no critical values of f, then f is univalent on all the components B_i, and so there are d branches in D of the inverse function f^{-1}.

Let $\{c_i\}_{i=1}^{l}$ be critical points of f. It follows from the chain rule that $\bigcup_{i=1}^{l}\bigcup_{k=0}^{p-1} f^{-k}c_i$ is the set of critical points of f^p. The critical values of f^p are the points of the orbits $\{f^n c_i\}_{n=1}^{p}$. Suppose that these orbits do not go into a simply-connected domain D. Then d^n branches of the inverse function f^{-n} ($n = 1, 2, \dots, p$) are defined in D. In view of this, the behaviour of the orbits of critical points plays an essential role in the investigation of the dynamics of rational endomorphisms.

§1.6. Components of the Fatou set

1. Let D be a connected component of the Fatou set $F(f)$. First we note that *the domain D is hyperbolic,* since its complement $\bar{\mathbf{C}}\setminus D$ contains the continual Julia set.

Next, since $F(f)$ is invariant, we see that fD is contained in a component V of the set $F(f)$. The fact that $J(f)$ is invariant yields $f(\partial D) \subset \partial V$. In view of Lemma 1.3, $f: D \to V$ is a ramified covering. Consequently, *f maps D onto V.*

We now consider the inverse image $f^{-1}D$. Let B be a component of $F(f)$ intersecting $f^{-1}D$. Then obviously, $fB \subset D$ and consequently, $fB = D$. This shows that $f^{-1}D$ is the union of a finite number (not exceeding d) of components of $F(f)$.

***Proposition* 1.4.** *Suppose that the Fatou set $F(f)$ has a completely invariant component D. Then*

a) *the other components of $F(f)$ are simply-connected*;

b) *$J(f)$ is the boundary of D.*

Proof. We show that a domain V not intersecting ∂D is contained in $F(f)$. For either $V \subset D$ or $V \subset \bar{\mathbf{C}}\setminus D$. There is nothing to prove in the first case. In the second case the f^{-1}-invariance of D implies that $f^m V \subset \bar{\mathbf{C}}\setminus D$ ($m = 0, 1, 2, \dots$). By Montel's theorem, $V \subset F(f)$. Thus, $J(f) \subset \partial D$. The opposite inclusion is obvious and b) is now proved.

Next, let B be a component of $F(f)$, γ a simple Jordan curve contained in B, and V_1, V_2 the components of $\bar{\mathbf{C}}\setminus\gamma$. Then one of these components (say V_1) does not intersect $\bar{D}$. In view of what has just been proved, we have $V_1 \subset F(f)$. But then $V_1 \subset B$, as required.

2. We denote by $\mathfrak{N}$ the family of components of $F(f)$. The transform f induces a map $\hat{f}$ of $\mathfrak{N}$ onto $\mathfrak{N}$.

***Proposition* 1.5.** *Let $\mathfrak{L} \subset \mathfrak{N}$ be a family of components of $F(f)$ completely invariant under $\hat{f}$. Then consists of either one, or two, or countably many components.*

Proof. Suppose that $|\mathfrak{L}| < \infty$. Then $\hat{f}$ is an bijection, and so $\hat{f}^p = \text{id}$ for some p. Thus, all the components $D \in \mathfrak{L}$ are completely invariant under

$g = f^p$. Suppose that $| \mathfrak{L} | > 1$. Then in view of Proposition 1.4 all the components $D \in \mathfrak{L}$ are simply-connected. Since $g: D \circlearrowleft$ is an N-sheeted ramified covering ($N = d^p$), the Riemann–Hurwitz formula shows that D contains $N-1$ critical points of g. Since g has $2(N-1)$ critical points, we obtain $| \mathfrak{L} | = 2$. The assertion is now proved.

Corollary. *If the Fatou set is non-empty, then it consists of either one, or two, or countably many components.*

In the examples considered above the set $F(f)$ consists of one or two components. In §1.9 we present an example in which $F(f)$ consists of countably many components. In fact, it is this situation that is typical.

Our immediate goal is a detailed description of the dynamics on an invariant component D (§§1.8–1.12). Theorems 1.1 and 1.2 give a certain impression of it. Moreover, the case d) of Theorem 1.1 obviously cannot occur for a rational endomorphism of degree $d > 1$. The case $D = \mathbf{U}^*$ is also impossible, since the Julia set is perfect. The remaining cases occur.

§1.7. Quasi-conformal maps. The measurable Riemann theorem

1. Let V be a Riemann surface, and ω be a measurable Riemann metric on V. The metric ω can be reduced locally to the form $\gamma(z)|dz+\beta(z)\overline{dz}|^2$, where γ and β are measurable functions, $\gamma(z) > 0$, and $|\beta(z)| < 1$ almost everywhere. Moreover, $\beta(z) = k(z) \exp 2i\theta(z)$, where $(1+k(z))/(1-k(z)) = K(z)$ is the ratio of the axes of the infinitesimal ellipse $|dz+\beta(z)\overline{dz}| = 1$, and $\theta(z)$ is the direction of its major axis. The function $k(z)$ is defined globally on V and is called *a dilatation* of the metric ω. If $\|k(z)\|_\infty < 1$ (or equivalently $\|K(z)\|_\infty < \infty$), then we say that ω *has bounded dilatation.* Then $\|k(z)\|_\infty$ is called *the maximal dilatation of the metric* ω.

With the metric ω one can associate *the Beltrami differential,* a $(-1, 1)$-form which can be represented locally as $\beta(z)\overline{dz}/dz$. The Beltrami differentials of two metrics coincide if and only if the metrics are *proportional,* that is, there is a measurable function $\gamma(z)$ such that $\omega_2 = \gamma\omega_1$. A class of proportional Riemann metrics with bounded dilatation is called *a conformal structure on V. The standard conformal structure* (with zero Beltrami differential) on a Riemann surface V is denoted by σ_V (or simply σ when this does not lead to ambiguity).

2. The reader can become acquainted with the theory of quasi-conformal maps in [2], [5], [16]. We restrict ourselves to a brief exposition of the necessary information. Let $\varphi: V \to W$ be a quasi-conformal homeomorphism of Riemann surfaces.

(i) *The homeomorphism φ is differentiable almost everywhere.* It follows that φ acts naturally on measurable Riemann metrics: $\omega \mapsto \varphi_*(\omega)$.

(ii) *If a metric ω has bounded dilatation, then so does the metric $\varphi_*(\omega)$.*

Hence, φ acts on conformal structures: $\mu \mapsto \varphi_* \mu$. *If φ is a conformal homeomorphism, then the maximal dilatations of the structures μ and $\varphi_*\mu$ are equal.*

(iii) *If $\varphi_*\sigma_V = \sigma_W$, then φ is a conformal homeomorphism.*

(iv) *For every conformal structure μ on V there is a Riemann surface W and a quasi-conformal homeomorphism $\varphi : V \to W$ such that $\varphi_*\mu = \sigma_W$* (Morrey [79]).

The last theorem is especially important for us in the case $V = \bar{\mathbf{C}}$. Then it follows from the Koebe-Riemann theorem that $W = \bar{\mathbf{C}}$ and we obtain the following fact.

The measurable Riemann theorem (see Ahlfors and Bers [37]). *For every conformal structure μ on the sphere $\bar{\mathbf{C}}$ (the disc $\mathbf{U}$) there is a quasi-conformal homeomorphism $\varphi = \varphi^\mu : \bar{\mathbf{C}} \circlearrowleft (\mathbf{U} \circlearrowleft)$ such that*[1] $\varphi_*\mu = \sigma$.

Moreover, the homeomorphism $\varphi: \bar{\mathbf{C}} \circlearrowleft$ is uniquely determined by the following normalization: 0, 1, ∞ are fixed points of φ (0, 1 in the case of $\mathbf{U}$). A conformal structure μ on the sphere $\bar{\mathbf{C}}$ can be considered as a point of the unit ball in L^∞ (under the identification of μ with the function $\beta(z)$). In the space of homeomorphisms of $\bar{\mathbf{C}} \circlearrowleft$ we introduce the uniform topology.

(v) *The homeomorphism φ^μ depends continuously on μ.* Moreover, if μ is a smooth function of its parameters, then so is φ^μ.

§1.8. Attracting cycles. Schröder domains

1. We now prove a theorem which is the origin of the theory of iterates of analytic transforms.

Theorem 1.3 (Schröder [86], Koenigs [70]). *Let $f: z \mapsto \lambda z + bz^2 + \ldots$ be an analytic transform of a neighbourhood of the origin, $0 < |\lambda| < 1$. Then there is a conformal map $\varphi: z \mapsto z + cz^2 + \ldots$ of a neighbourhood of the origin onto the disc $\mathbf{U}$ that satisfies the Schröder equation $\varphi(fz) = \lambda\varphi(z)$.*

Proof. If ε is small, then the disc $\mathbf{U}_\varepsilon$ is f-invariant and so the set $V = \overline{\mathbf{U}_\varepsilon \setminus f\mathbf{U}_\varepsilon}$ is diffeomorphic to the annulus $\mathbf{A}[s, 1]$ (for every $s \in (0, 1)$). We consider a homeomorphism h_0 such that $h_0(fz) = sh_0(z)$ for $z \in \partial\mathbf{U}_\varepsilon$. It extends to a homeomorphism $h: \mathbf{U}_\varepsilon \to \mathbf{U}$ by the relation $h(f^m z) = s^m h_0(z)$ $(z \in V)$. Moreover, $h(fz) = sh(z)$ $(z \in \mathbf{U}_\varepsilon)$. Since f is a conformal transform, it follows that the metric $h_*\rho$ has bounded dilatation, and the corresponding conformal structure μ_0 is invariant under the transform $g_s: z \mapsto sz$. Therefore, the structure $\mu_m = (g_s^{-m})_*\mu_0$ extends the structure μ_0 to the disc $\mathbf{U}_{s^{-m}}$. Since the maximal dilatations of all structures μ_m are equal, the structure μ_0 extends to a structure μ on the whole sphere $\bar{\mathbf{C}}$. By the measurable

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(1) The case $V = \mathbf{U}$ can be obtained from that of $V = \bar{\mathbf{C}}$ by a symmetric reflection of the structure.

Riemann theorem(1) there is a quasi-conformal homeomorphism $\psi: \bar{\mathbf{C}} \circlearrowleft$, with fixed points 0, 1, ∞ such that $\psi_* \mu = \sigma$. We consider a quasi-conformal homeomorphism $g = \psi \circ g_s \circ \psi^{-1}: \bar{\mathbf{C}} \circlearrowleft$. Since $(g_s)_* \mu = \mu$, it follows that $g_* \sigma = \sigma$ and so g is a conformal transform. Since g leaves the points 0, ∞ invariant, we see that $g: z \mapsto qz$. We put $\varphi = \psi \circ h$. Then $\varphi_* \sigma = \sigma$, that is, φ is conformal in $\mathbf{U}_\varepsilon$. Finally, $\varphi(fz) = q\varphi(z)$, which yields $q = \lambda$.

A conformal solution of the Schröder equation is called a *Koenigs function.* A Koenigs function conjugates the transform f with a linear transform $z \mapsto \lambda z$ in a neighbourhood of the origin. These model transforms are pairwise conformally non-conjugate. Thus, Theorem 1.3 gives a conformal classification of the germs of analytic transforms in a neighbourhood of a fixed attracting point. On the other hand, we have proved in fact that all such transforms are pairwise quasi-conformally conjugate. We finally note that the classicists constructed the Koenigs function in the form $\lim_{m \to \infty} \lambda^{-m} (f^m z)$ (see [7], §37), or tried to find its Taylor expansion.

2. Again let f be a rational endomorphism, and $\alpha = \{\alpha_k\}_{k=0}^{p-1}$ an attracting cycle of order p. The set $\Delta(\alpha)$ of points whose orbits converge to α is called *the attracting region of the cycle.* An attracting region is open but is not connected in general (that is, it is not a domain in the usual sense). It is easy to show that $\Delta(\alpha)$ is the union of some components of the Fatou set $F(f)$.

In view of Proposition 1.5, $\Delta(\alpha)$ consists of either one, or two, or countably many components. The components $D(\alpha_k)$ of $\Delta(\alpha)$ (or, what is the same, of the Fatou set $F(f)$) containing the points α_k are called *Schröder domains.* In view of the Corollary of Theorem 1.1, the Schröder domains $D(\alpha_k)$ are pairwise disjoint. The set $D(\alpha) = \bigcup_{k=0}^{p-1} D(\alpha_k)$ is called *the immediate attracting region.*

Suppose now that α is a fixed point. The Schröder domain $D(\alpha)$ can be obtained as follows. We consider a small disc U_0 centered at α. Let U_1 be the component of $f^{-1}U_0$ containing U_0, U_2 the component of $f^{-1}U_1$ containing U_1, and so on. Then $D(\alpha) = \bigcup_{m=0}^{\infty} U_m$.

Suppose that the Koenigs function is defined in U_0. Then it can be extended analytically to U_1 by the Schröder equation $\varphi(z) = \lambda^{-1}\varphi(fz)$ $(z \in U_1)$. Next, φ extends analytically to U_2, U_3, ... and, consequently, to the whole domain $D(\alpha)$. It is easy to check that in addition φ is a countably-sheeted ramified covering of $\mathbf{C}$ by the Schröder domain $D(\alpha)$. The branch points of the covering φ are the critical points of f and their inverse images of all orders. We note that the Koenigs function cannot be continuously extended to any boundary point of $D(\alpha)$ (see [7]).

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(1) Under more accurate considerations the structure μ is generated by a smooth metric in $\mathbf{C}$ and the measurable Riemann theorem reduces to classical results.

3. We say that the orbit of c is absorbed by a cycle α if $f^m c \in \alpha$ for some m.

***Theorem* 1.4** (Julia [65], Fatou [60], §30). *The immediate attracting region $D(\alpha)$ of an attracting cycle α contains a critical point c of the function f whose orbit is not absorbed by α.*

Proof. Suppose first that α is a fixed point.

Method 1. We put $D_0(\alpha) = D(\alpha) \setminus \bigcup_{m=0}^{\infty} f^{-m}\alpha$. If D_0 has no critical points of f, then the Koenigs function φ is an unramified covering of $\mathbf{C}^*$ by $D_0(\alpha)$. This is impossible, since $D_0(\alpha)$ is a hyperbolic domain.

Method 2. Here we prove only the first part of the theorem. If $D(\alpha)$ has no critical points, then $f\colon D(\alpha) \circlearrowleft$ is an unramified covering. Consequently, f is locally isometric in the hyperbolic metric of $D(\alpha)$. On the other hand, $\|Df(\alpha)\| = |\lambda| < 1$.

If now $\alpha = \{\alpha_k\}_{k=0}^{p-1}$ is an attracting cycle, then $D(\alpha_0)$ contains a critical point of f^p, that is, the inverse image of order k of a critical point c of f, where $0 \leqslant k \leqslant p-1$. It follows that $D(\alpha_k)$ contains c. The theorem is now proved.

***Corollary* 1.** *A rational endomorphism $f : \bar{\mathbf{C}} \circlearrowleft$ of degree d has at most $2d-2$ attracting cycles. A polynomial endomorphism $f : \bar{\mathbf{C}} \circlearrowleft$ has at most $2d-2$ attracting cycles.*

***Conjecture* 1.1.** *There is a rational endomorphism of degree d with $2d-2$ attracting cycles.*

From the Riemann–Hurwitz formula we obtain the following result.

***Corollary* 2.** *Schröder domains are either simply-connected or infinitely-connected.*

4. Let α be an attracting cycle of order p, and φ the Koenigs function of the transform $f^p\colon D(\alpha_0) \circlearrowleft$. We extend φ to the whole attracting region $\Delta(\alpha)$ as follows. Let V be a component of $\Delta(\alpha)$ such that $f^m V = D(\alpha_0)$ and $f^k V \neq D(\alpha_0)$ for $k < m$. We put $\varphi(z) = \varphi(f^m z)$ $(z \in V)$. Then $\varphi : \Delta(\alpha) \to \mathbf{C}$ is a ramified covering. Its fibres are the classes of the following equivalence: $z \sim \zeta$ if $f^m z = f^m \zeta$ for some $m \geqslant 0$. We consider also the equivalence: $z \approx \zeta$ if $f^m z = f^n \zeta$ for some $m, n \geqslant 0$. The equivalence classes of $\approx$ are called the *large orbits* of f. In $\Delta(\alpha)$ we have $z \approx \zeta$ if and only if $\varphi(z) = \lambda^m \varphi(\zeta)$ for some $m \in \mathbf{Z}$.

We consider the torus $\mathbf{T}_\lambda^2$, the quotient of $\mathbf{C}^*$ modulo the action of the group $z \mapsto \lambda^m z$ $(m \in \mathbf{Z})$, $\pi : \mathbf{C}^* \to \mathbf{T}_\lambda^2$ being the natural projection. We put $\Delta_0(\alpha) = \Delta(\alpha) \setminus \bigcup_{m=0}^{\infty} f^{-m}\alpha$. In view of what we said above, we can define a ramified covering $\Phi = \pi \circ \varphi : \Delta_0(\alpha) \to \mathbf{T}_\lambda^2$. Its fibres are large orbits of the transform $f\colon \Delta_0(\alpha) \circlearrowleft$. Consequently, $\mathbf{T}_\lambda^2$ can naturally be identified with the space of large orbits of the transform under consideration. We mark points $d_0, \ldots, d_{k-1}$ on $\mathbf{T}^2_\lambda$ that correspond to large orbits of the critical points in $\Delta_0(\alpha)$. Thus, *with every cycle of Schröder domains we associate a torus with marked points.* Its construction is completely similar to that of the Riemann surface corresponding to a Kleinian group. Below we associate with every cycle of components of $F(f)$ a Riemann surface with marked points. The importance of these surfaces for iteration theory was discovered by Sullivan (see §2.5).

We consider finally a doubly-connected set $V = \overline{U \setminus f^p U}$, where U is a small disc centred at α_0. Every large orbit in $\Delta_0(\alpha)$ has one or two points in common with V, and in the latter case both points lie on ∂V. By analogy with the terminology of Kleinian groups, V is called *the fundamental domain* of the transform $f\colon \Delta_0(\alpha) \circlearrowright$. The torus $\mathbf{T}^2_\lambda$ can be obtained from the fundamental domain by identification of the components of the boundary (see Fig. 3).

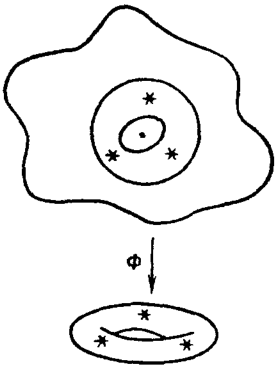

Fig. 3. The Schröder domain

§1.9. Superattracting cycles. Böttcher domains

1. ***Theorem* 1.5** (Böttcher [44]). *Let* $f\colon z \mapsto bz^k + dz^{k+1} + \ldots$ *be an analytic transform in a neighbourhood of the origin,* $k \geqslant 2$, $b \neq 0$. *Then there is a conformal map* $\varphi\colon z \mapsto z + cz^2 + \ldots$ *of a neighbourhood of the origin onto the disc* $\mathbf{U}$ *satisfying the Böttcher equation* $\varphi(fz) = \varphi(z)^k$.

Proof. We consider a small disc $\mathbf{U}_\varepsilon$ and denote by W the connected component of $f^{-1}\mathbf{U}_\varepsilon$ containing the origin. We put $V = \overline{W \setminus \mathbf{U}_\varepsilon}$. Let $r \in (0, 1)$. We consider a diffeomorphism $h_0\colon V \to \mathrm{A}[r^k, r]$ such that $h_0(fz) = h_0(z)^k$ $(z \in \partial W)$. The diffeomorphism h_0 extends to a quasi-conformal homeomorphism $h : W \to \mathbf{U}_r$ satisfying $h(fz) = h(z)^k$ $(z \in W)$. Next, the conformal structure $h_*\sigma$ extends naturally to a conformal structure μ on the disc $\mathbf{U}$ invariant under the transform $G\colon z \mapsto z^k$ in a neighbourhood of any non-zero point. By the measurable Riemann theorem there is a quasi-conformal homeomorphism $\psi\colon \overline{\mathbf{U}} \circlearrowright$ such that $\psi(0) = 0$, $\psi(1) = 1$, and $\psi_* \mu = \sigma$. Then the transform $g = \psi \circ G \circ \psi^{-1}$ is locally conformal outside the origin and is a k-sheeted covering $\mathbf{U}^* \circlearrowright$. Consequently, $g\colon z \mapsto z^k$.

A conformal solution of the Böttcher equation is called a *Böttcher function.* It conjugates f in a neighbourhood of a superattracting fixed point with the transform $z \mapsto z^k$. The classicists constructed a Böttcher function in the form $\lim\limits_{n\to\infty} \sqrt[k^n]{f^n z}$ (see [7], §40).

2. Now let $\boldsymbol{\alpha} = \{\alpha_i\}_{i=0}^{p-1}$ be a superattracting cycle of a rational endomorphism f. This means that one of the points of the cycle is critical. Just as in the case of an attracting cycle, we define the attracting region $\Delta(\alpha)$ and the immediate attracting region $D(\alpha)$. The components $D(\alpha_i)$ of the immediate attracting region of a superattracting cycle are called *Böttcher domains.*

***Theorem* 1.6.** *Let* $D(\alpha) = \bigcup\limits_{i=0}^{p-1} D(\alpha_i)$ *be a cycle of Böttcher domains. Then we have the following alternatives*:

a) *the transform* $f^p\colon D(\alpha_i) \circlearrowright$ *is conformally conjugate to the transform* $z \mapsto z^k$ *of the disc* $\mathbf{U}$;

b) *one of the domains* $D(\alpha_i)$ *contains a critical point whose orbit is not absorbed by the cycle* $\boldsymbol{\alpha}$.

***Lemma* 1.4.** *Let* $f\colon V \to W$ *be a k-sheeted covering of hyperbolic domains of the sphere, where* W *is simply-connected,* $k < \infty$. *Suppose that all the branch points of* f *lie in one fibre* $f^{-1}\alpha$. *Then* $f^{-1}\alpha$ *consists of one point* β *and there are conformal maps* $\varphi\colon (V, \beta) \to (\mathrm{U}, 0)$ *and* $\psi\colon (W, \alpha) \to (\mathrm{U}, 0)$ *such that* $\psi \circ f \circ \varphi^{-1}\colon z \mapsto z^k$.

Proof. The map $V \setminus f^{-1}\alpha \to W \setminus \{\alpha\}$ is an unramified covering. Therefore, the Euler characteristic of $V \setminus f^{-1}\alpha$ is zero. This is possible only if the domain V is simply-connected and $f^{-1}\alpha$ consists of one point. The rest is obvious.

Proof of Theorem 1.6. Suppose that b) does not hold. We consider a disc U_0 centered at $\alpha \equiv \alpha_i$ and not containing critical values of f^n different from α. We put $g = f^p$ and consider the component U_n of the inverse image $g^{-n} U$ containing α. All the branch points of the covering $g^n\colon U_n \to U$ lie over α. By Lemma 1.4 the domain U_n is simply-connected and contains the unique critical point α. But then the Böttcher domain $D(\alpha) = \bigcup\limits_{n=0}^{\infty} U_n$ has the same properties. We now consider the Riemann conformal map of $D(\alpha)$ onto $\mathbf{U}$. Under a suitable normalization it conjugates the transforms g and $z \mapsto z^k$.

3. The Böttcher domains, like the Schröder domains, are either simply-connected or infinitely-connected. The total number of cycles of the Schröder and Böttcher domains does not exceed $2d - 2$ ($d + 1$ in the case of polynomials). However, the topological pictures of the dynamics in the Schröder and Böttcher domains are essentially different.

We consider the foliation $\varphi^{-1}\xi$ in a neighbourhood of α_0, φ being the Böttcher function, and ξ the foliation of concentric circles in a neighbourhood of the origin. By means of f^{-m} this foliation extends to a foliation η on the whole attracting region $\Delta(\alpha)$. The foliation η has singularities at the critical points lying in $\Delta_0(\alpha) = \Delta(\alpha)\diagdown\{f^{-m}\alpha\}_{m=0}^{\infty}$ and their inverse images of all orders. Every large orbit in $\Delta_0(\alpha)$ densely fills out countably many fibres of the foliation η.

Sullivan ([90], Part III) has associated with every cycle of Böttcher domains a Riemann surface S with marked points as follows (Fig. 4). Suppose that $\Delta_0(\alpha)$ contains a critical point c. We consider the points $d_0 = \varphi(f^m c)$, $d_1 = \varphi(f^{m+1}c)$, where m is chosen so that $f^m c$ lies in a univalence domain of the Böttcher function φ. In the annulus $A[|d_0|, |d_1|]$ we mark the points $\{d_i\}_{i=0}^{l}$ such that the $\varphi^{-1}d_i$ lie on the orbits of critical points. We fix in addition the action of the rotation group[1]: $z \mapsto ze^{i\theta}(\theta \in \mathbf{R})$ on the annulus. We obtain the required Riemann surface S. But if $\Delta_0(\alpha)$ has no critical points, then S is the punctured disc $\mathbf{U}^*$ on which the rotation group acts.

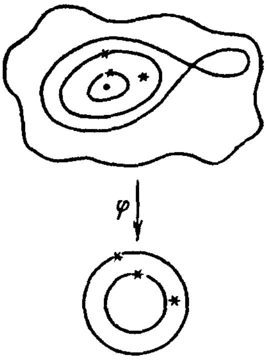

Fig. 4. A Böttcher domain

4. The presence of a superattracting cycle is a degenerate situation in the class of rational endomorphisms. However, it is very important, since ∞ is a superattracting fixed point of any polynomial transform.

***Proposition* 1.6.** *Let* $f\colon z \mapsto b_0 z^d + \ldots + b_d$ *be a polynomial. Then the immediate attracting region* $D(\infty)$ *is completely invariant*[2] *and* $J(f) = \partial D(\infty)$. *All bounded components of the Fatou set* $F(f)$ *are simply-connected.*

Proof. Let $f^{-1}D(\infty) = D(\infty) \bigcup\limits_{i=1} B_i$, where the B_i are some components of $F(f)$ different from $D(\infty)$. Then $fB_i = D(\infty)$, despite the fact that $f^{-1}(\infty) = \{\infty\}$. Consequently, $D(\infty)$ is completely invariant. The remaining assertions follow from Proposition 1.4.

Example 1.5. $f\colon z \mapsto z^2 - 1$. The points 0, −1 form a superattracting cycle of the second order. The immediate attracting region of this cycle consists of two components $D(0)$ and $D(-1)$. The transform f maps $D(-1)$ univalently onto $D(0)$. Consequently, there is a component $D \neq D(-1)$ of $F(f)$ that is also mapped onto $D(0)$ univalently. The component D also has two inverse

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[1] The rotation group arises because a z^k-invariant conformal structure on the disc $\mathbf{U}_r$ is invariant under rotations (see §2.5).

[2] And, consequently, coincides with the attracting region $\Delta(\infty)$.

images, the components D_1 and D_2, and so on. Thus, the attracting region of the cycle consists of countably many components (Fig. 5).

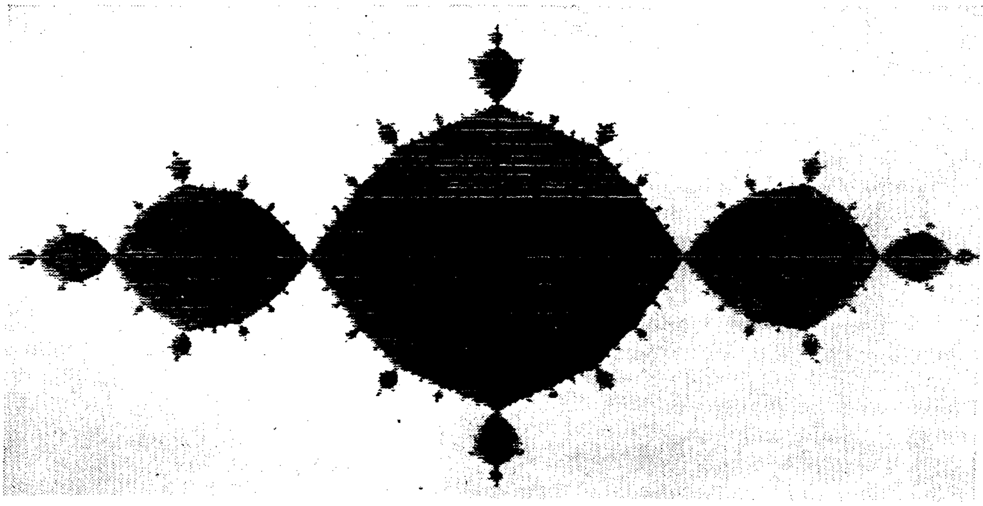

Fig. 5. The transform $z \mapsto z^2 - 1$

§1.10. Neutral rational cycles. The Leau flower

1. A neutral cycle $\alpha = \{\alpha_k\}_{k=0}^{p-1}$ is called *rational* if its multiplier λ is a root of 1 and *irrational* otherwise. In this section we describe a local picture of the dynamics in a neighbourhood of rational cycles. Most of the results presented below are contained in the papers by Leau [72] and Fatou ([59], Ch. 2).

Suppose that f is a function analytic in a neighbourhood of the fixed point $\alpha = 0$ with multiplier $\lambda = 1$, $f\colon z \mapsto z + az^2 + \ldots$. First we consider the non-degenerate case $a \neq 0$. The conjugation by means of the conformal transform $z \mapsto Az^{-1}$ with suitable A brings f in a neighbourhood of ∞ to the form

$$(1.2) \qquad g\colon \zeta \mapsto \zeta + 1 + \frac{b}{\zeta} + O\left(\frac{1}{|\zeta|^2}\right).$$

We consider the half-plane $P = \{\zeta\colon \operatorname{Re}\zeta > M\}$. If M is sufficiently large, then $\operatorname{Re}(g\zeta) > \operatorname{Re}\zeta + 1 - \varepsilon$ in P. Consequently, P is g-invariant and $\operatorname{Re}(g^m\zeta) \to +\infty$ $(m \to \infty)$ for $\zeta \in P$. Conversely, suppose that $|g^m\zeta| \to \infty$ $(m \to \infty)$. Then in view of (1.2), $\operatorname{Re}(g^{m+1}\zeta) > \operatorname{Re}(g^m\zeta) + 1 - \varepsilon$ for sufficiently large m, and so $g^m\zeta \in P$ beginning from some m.

The set $\Delta(\alpha) = \{z\colon f^m z \to \alpha\ (m \to \infty)\} \setminus \bigcup_{m=0}^{\infty} f^{-m}\alpha$ is called *the attracting region of a neutral rational fixed point.* For a function g of the form (1.2) we have established the following facts:

(i) *the transform g maps P into itself univalently*;

(ii) *the attracting region $\Delta(\infty)$ coincides with $\bigcup_{m=0}^{\infty} g^{-m} P$ and so it is open.*

We consider a curvilinear strip $Q = \overline{P \setminus gP}$. This strip is *a fundamental domain for* $g\colon \Delta(\infty) \circlearrowleft$. Every large orbit from $\Delta(\infty)$ has either one or two points in common with Q. In the latter case both points lie on different components of the boundary of the strip. The space of large orbits $\Delta(\infty)/\approx$ can be obtained by identification of the sides of the strip Q and so it is homeomorphic to a cylinder.

2. ***Theorem*** 1.7 (Leau [72], Fatou [59], Ch. 2). *There is a univalent analytic function φ in the half-plane that satisfies the Abel equation* $\varphi(gz) = \varphi(z) + 1$. *Moreover,* $\bigcup_{m=0}^{\infty} (\varphi(P) - m) = \mathbf{C}$.

Thus the transform $g\colon P \circlearrowleft$ is conformally conjugate to the translation $\zeta \mapsto \zeta + 1$. The conjugating transform φ is called the *Abel function.*

Proof. We consider the rectilinear strip $\Pi = \{\zeta\colon 0 \leqslant \operatorname{Re}\zeta \leqslant 1\}$. It follows easily from (1.2) that there is a quasi-conformal diffeomorphism $h_0\colon Q \to \Pi$ such that $h_0(g\zeta) = h_0(\zeta) + 1$ if $\zeta \in \partial P$. By the Abel equation the diffeomorphism h_0 can be extended to a quasi-conformal homeomorphism $h\colon P \to P_0 = \{\zeta\colon \operatorname{Re}\zeta \geqslant 0\}$ of the half-planes.

We now consider the conformal structure $\mu_0 = h_*\sigma$ on P_0. Since $g_*\sigma = \sigma$, we have $G_*\mu_0 = \mu_0$, where $G\colon \zeta \mapsto \zeta + 1$. It follows that μ_0 extends to a G-invariant conformal structure μ on the whole plane $\mathbf{C}$. By the measurable Riemann theorem there is a quasi-conformal homeomorphism $\psi\colon \mathbf{C} \circlearrowleft$ such that $\psi_*\mu = \sigma$. Hence, $T = \psi \circ G \circ \psi^{-1}$ is a conformal transform of $\mathbf{C}$ without fixed points, that is, $T\colon \zeta \mapsto \zeta + a$. Under a suitable normalization of ψ we have $a = 1$. The transform $\varphi = \psi \circ h$ is the Abel function.

Finally, $\bigcup_{m=0}^{\infty} (\varphi(P) - m) = \psi \bigcup_{m=0}^{\infty} (P_0 - m) = \mathbf{C}$.

Fatou constructed the Abel function by studying the asymptotics of the orbits $\{g^m\zeta\}$ in the half-plane P. It turns out that for g of the form (1.2) we have $g^m\zeta = m + b \log m + \varphi(\zeta) + o(1)$ $(m \to \infty)$, where g is the Abel function. In addition, Fatou obtained the asymptotics $\varphi(\zeta) = \zeta + O(\log|\zeta|)$ $(\zeta \in P,\ |\zeta| \to \infty)$. This implies that the intersections of the domain $\varphi(P)$ with the horizontal lines are rays. The orbits of $\zeta \mapsto \zeta + 1$ lie on these rays. Consequently, the orbits $\{g^m\zeta\}$ lie on analytic curves going to ∞ under a zero angle.

3. We note that everything said above remains valid in the case of a many-valued function g for which ∞ is an algebraic singularity and

$$(1.3) \qquad g\colon \zeta \mapsto \zeta + 1 + O(|\zeta|^{-\gamma}) \quad (|\zeta| \to \infty), \ \gamma > 0.$$

The following stipulations are to be made: a) a sequence $\{\zeta_m\}$ is called the orbit of g if $g\zeta_m = \zeta_{m+1}$ for some choice of the branch of g; b) Theorem 1.7 holds for any univalent branch of g in the half-plane P.

This remark enables us to study the degenerate case $f\colon z \mapsto z + az^{p+1} + \dots (a \neq 0)$. We consider the transform $\psi\colon z \mapsto Az^{-p}$ and the many-valued function $g = \psi \circ f \circ \psi^{-1}$ in a neighbourhood of ∞. With suitable A the function g is of the form (1.3). Thus, the picture described above takes place for g, and we obtain the following information about f.

We consider the inverse image $L = \psi^{-1}P$ of P, *the Leau flower*. It consists of p components $L_1, \dots, L_p$ called *the Leau petals*. The petal L_k is obtained by rotating L_1 through an angle $2\pi(k-1)/p$. The petal L_k is bounded by a simple curve analytic except for the origin and having a break with angle π/p at the origin.

The g-invariance of P implies the f-invariance of the flower L. Since $f(z) \sim z$ as $|z| \to 0$, in fact all the petals L_k are f-invariant. If $f^m z \to 0$ ($m \to \infty$), then the orbit $\{f^m z\}$ is absorbed by one of the petals L_k. The transform $f\colon L_k \circlearrowleft$ is conformally conjugate to the translation $\zeta \mapsto \zeta + 1$. The conjugating map coincides with $\varphi_k(z) = \varphi(z^{-1/p})$, where φ is the Abel function for g in P; $\bigcup\limits_{m=0}^{\infty} (\varphi_k(L_k) - m) = \mathbf{C}$. The orbit $\{f^m z\}$, as it tends to zero, lies on an analytic curve that enters the origin along the bisector of the petal.

All the transforms of the form $z \mapsto z + az^{p+1} + \dots$ ($a \neq 0$) are pairwise topologically conjugate [47] in a neighbourhood of the origin. The formal normal form of such a transform is $z \mapsto z - z^{p+1} + \mu z^{2p+1}$, where μ is a unique module. However, unlike the case of an attracting fixed point, μ is not a unique module of analytic equivalence (in a whole neighbourhood of a fixed point). For every μ there is a functional module that can be constructed with the help of the solutions of the Abel equation [8].

4. Next, let α be a rational fixed point whose multiplier is a q-th root of 1. Then α is the centre of a flower with sq petals, s being a natural number. The transform f rearranges these petals, splitting them into cycles of order q. All the properties of this flower follow easily from the case $\lambda = 1$ already considered.

We return to the case of a rational function. Let $\{\alpha_i\}_{i=0}^{p-1}$ be a rational cycle with multiplier λ that is a q-th root of 1. Then $l = qs$ Leau petals $L_{i1}, \dots, L_{il}$ cling to α_i. The transform f permutes these petals $L_{ik} \to L_{i+1,j}$ (where $j = j(i, k)$, $L_{pl} \equiv L_{0l}$), splitting them into cycles of order pq.

We consider the components D_{ik} of $F(f)$ containing L_{ik}. It is easy to see that these components are pairwise distinct. The transform f also permutes them, creating cycles of order pq. These components are called the *Leau domains* and their union is called *the immediate attracting region of a rational cycle* $\{\alpha_i\}$.

5. Just like Koenigs' function, the Abel function for $f^{pq}\colon L_{ik} \circlearrowleft$ can be extended analytically to the Leau domain D_{ik} with the help of the Abel equation. In addition, $\varphi(D_{ik}) = \bigcup\limits_{m=0}^{\infty} (\varphi(L_{ik}) - m) = \mathbf{C}$ and φ is a ramified covering of the Leau domain over $\mathbf{C}$. The branch points of the covering φ are the critical points of f and their inverse images of all orders.

Theorem 1.8 (Fatou [60], §30). *Every cycle of Leau domains contains a critical point of f.*

Proof. Otherwise the Abel function φ would be a conformal homeomorphism of the Leau domain onto $\mathbf{C}$. This is impossible, since the Leau domain is hyperbolic.

Corollary 1. *The total number of cycles of the Schröder, the Böttcher, and the Leau domain does not exceed* $2d-2$. *The total number of attracting, superattracting, and neutral rational cycles does not exceed* $2d-2$.

Corollary 2. *Leau domains are either simply-connected or infinitely-connected.*

The Abel function φ can be extended naturally to the whole large orbit V of the Leau domain. The large orbits of the transform f are the fibres of the ramified covering $\pi = \exp(2\pi i\varphi) : V \to \mathbf{C}^*$. Thus, π is a quotient map onto the space of large orbits $V/\approx$, which is therefore conformally equivalent to $\mathbf{C}^*$. We mark on $\mathbf{C}^*$ the projections $d_0, \ldots, d_{k-1}$ $(k \geqslant 1)$ of the branch points of π, that is, the images of the large orbits of critical points. We obtain a Riemann surface with marked points related to a cycle of the Leau domains (Fig. 6).

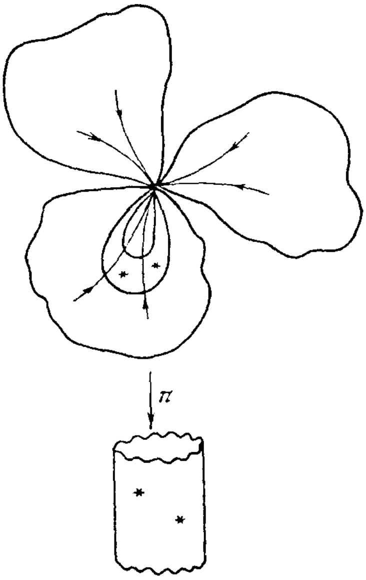

Fig. 6. The Leau flower

6. *Example* 1.6. $f: z \mapsto z + 1 - \frac{1}{z}$. This is a transform with fundamental disc: the upper and lower half-planes are f-invariant. The Julia set $J(f)$ is a Cantor set on the circle $\bar{\mathbf{R}}$ containing ∞. The Fatou set $F(f)$ coincides with the Leau domain of the fixed point ∞. Thus, we have an example of an infinitely-connected Leau domain.

Example 1.7. $f\colon z \mapsto z - \frac{1}{z}$. This is also a transform with fundamental disc. However, in this case $J(f) = \overline{\mathbf{R}}$, and $F(f)$ consists of two half-planes, which are the Leau domains for ∞.

§1.11. Neutral irrational cycles. Siegel discs

1. We consider an analytic transform $f\colon z \mapsto \lambda z + az^2 + \dots$ in a neighbourhood of the neutral fixed point $\alpha = 0$, $|\lambda| = 1$. A central problem relating to such transforms is the question of conformal conjugacy to the rotation $z \mapsto \lambda z$, that is, the question of solubility of the Schröder equation $\varphi(fz) = \lambda\varphi(z)$. This problem is important both for theory and for applications, in particular, in celestial mechanics. It is intimately related to the problem of the stability of an equilibrium state.

Proposition 1.7 (Siegel [14]). *Let $f\colon z \mapsto \lambda z + az^2 + \dots$ be an analytic transform of a neighbourhood V of the origin* (*onto another neighbourhood*), $|\lambda| = 1$. *The following properties are equivalent*:

a) *f is conformally conjugate to a rotation in a neighbourhood of the origin*;

b) *f is topologically conjugate to a rotation in a neighbourhood of the origin*;

c) *the origin is a Lyapunov stable position of equilibrium*;

d) *there is a neighbourhood $W \subset V$ such that $\bigcup_{m=0}^{\infty} f^m W \subset V$.*

Proof. d) ⇒ a) is the only non-trivial implication. We consider the invariant domain $D = \bigcup_{m=0}^{\infty} f^m W \subset V$. We consider a universal covering $\pi : \mathbf{U} \to D$ with $\pi(0) = 0$. The transform $f\colon D\circlearrowleft$ can be lifted to an analytic transform $g\colon \mathbf{U}\circlearrowleft$ such that $g(0) = 0$. Consequently, $|g'(0)| = |\lambda| = 1$. By the Schwarz lemma, $g(z) = \lambda z$. We now consider a disc $B \subset \mathbf{U}$ centered at the origin on which π is univalent. Then $\pi : B \to \pi(B)$ conformally conjugates $f\colon \pi(B)\circlearrowleft$ to a rotation of B.

Corollary 1. *Let α be a neutral periodic point of order p of a rational endomorphism f. For f^p to be conjugate to a rotation in a neighbourhood of α it is necessary and sufficient that $\alpha \in F(f)$.*

Proof. If $\alpha \in F(f)$, then α is a Lyapunov stable position of equilibrium.

Corollary 2. *Neutral rational cycles lie on the Julia set.*

Proof. We suppose the contrary. Then, in view of Corollary 1, f is a transform of finite order, which contradicts our assumption $\deg f > 1$.

2. If a neutral periodic point α lies on the Fatou set, then it (as well as its cycle) is called *Siegel*. A component D of $F(f)$ containing a Siegel point α is called *a Siegel disc*. The corollary of Theorem 1.1 immediately yields the following result.

***Proposition* 1.8.** *A Siegel disc D is simply-connected. The transform $f^p\colon D \circlearrowleft$ is conformally conjugate to a rational rotation $z \mapsto \lambda z$ of the disc* $\mathbf{U}$.

Following Sullivan ([90], Part III) we associate with the cycle $\{f^nD\}_{n=0}^{p-1}$ of a Siegel disc a Riemann surface S with boundary and marked points as follows (Fig. 7).

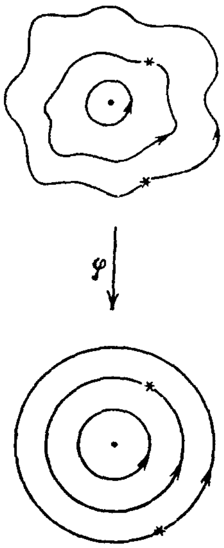

Fig. 7. A Siegel disc

Let $c_1, \dots, c_k$ be critical points whose orbits fall into the punctured disc $D \setminus \{\alpha\}$ and are pairwise disjoint; $a_i = f^n c_i$ is the first point of the orbit that lies in D. Let $\varphi : D \to \mathbf{U}$ be a conformal homeomorphism. We put $S = \mathbf{U}^* \cup \mathbf{T}$ and mark the points $d_i = \varphi(a_i)$ $(i = 1, \dots, k)$ in S, as well as some point $d_0 \in \mathbf{T}$. Finally, we fix an additional structure on S: the action of the rotation group.

3. The following classical theorem shows that rational endomorphisms may have Siegel discs.

***Theorem* 1.9** (Siegel [87], [14]). *Let $f\colon z \mapsto e^{2\pi i\theta}z + az^2 + \dots$ be an analytic transform in a neighbourhood of the origin. Suppose that there are constants C and ε such that for all integers m and l*

$$\left|\theta - \frac{m}{l}\right| \geqslant \frac{C}{l^{2+\varepsilon}}. \tag{1.4}$$

Then f is conformally conjugate to a rotation in a neighbourhood of the origin.

See [3] for a transparent proof of Siegel's theorem. The condition (1.4) holds for almost every multiplier $\lambda = e^{2\pi i\theta}$ with respect to Lebesgue measure on the circle $\mathbf{T}$. Thus, neutral irrational cycles can be contained in the Fatou set, and this situation is typical from the metric point of view. It turns out that from the category point of view the opposite situation is typical.

***Proposition* 1.9.** *We consider a one-parameter family* $f_\lambda: z \mapsto \lambda z + g(z)$ $(\lambda = e^{2\pi i\theta} \in \mathbf{T})$ *of transforms in a neighbourhood of the origin. Suppose that all the* f_λ *have infinite order. Then the set* Λ *of* $\lambda \in \mathbf{T}$ *for which* f_λ *is not conjugate to a rotation is a dense* G_δ *set.*

For example, if f_λ is a family of non-linear entire functions, then the condition of Proposition 1.9 is valid. The simplest family of this type is $f_\lambda: z \mapsto \lambda z + z^2$.

Proof. Let f_λ be defined in an ε-neighbourhood of the origin. We consider the set $X_m(\lambda) = \{z: \exists k \in [0, m] \mid f_\lambda^k z \mid > \varepsilon\}$ and the function $\rho_m(\lambda)$, which is the distance from $X_m(\lambda)$ to the origin. The function ρ_m is obviously upper semi-continuous. Consequently, $\rho = \inf\limits_m \rho_m$ is also upper semi-continuous. Hence, the set of zeros of ρ is of G_δ-type.

We now note that in view of Proposition 1.7 f_λ is not conjugate to a rotation if and only if $\lambda \in \Lambda$. But if λ is a root of 1, then the transform f_λ is not conjugate to a rotation, since it is of infinite order. It follows that Λ is dense in $\mathbf{T}$.

The first example of non-stable irrational fixed points was given by Pfeiffer [81] in 1917. Cremer [63] constructed such examples for an arbitrary multiplier λ satisfying $\underline{\lim} \mid \lambda^n - 1 \mid^{1/n} = 0$. The proof presented here is taken from [21]-[23], [68].

4. ***Theorem* 1.10** (Fatou [61], 241). *We consider an analytic transform* $f: z \mapsto \lambda z + bz^2 + \ldots$ *in a neighbourhood of the origin such that* $|\lambda| = 1$, $\lambda \neq 1$. *Suppose that a domain* V *satisfies* $fV \cap V \neq \emptyset$. *Then the orbit* $\{f^m V\}_{m=0}^{\infty}$ *cannot converge uniformly to zero.*

Proof. If λ is a root of 1, $\lambda \neq 1$, then it follows from the examination of the Leau flower that the condition $fV \cap V = \emptyset$ cannot hold. Consequently, $\frac{1}{2\pi} \arg \lambda$ is irrational. Let $\zeta \in V$. We consider a sequence $\varphi_m(z) = f^m z / f^m \zeta$ of holomorphic functions in V. Since the φ_m are univalent, do not vanish, and $\varphi_m(\zeta) = 1$, we see that the family $\{\varphi_m\}$ is normal (Koebe's distortion theorem [10]). We show that the limit functions of the sequence $\{\varphi_m\}$ differ from a constant. For otherwise some domains $f^{m_k} V$ could be seen from the origin at a small angle and the domain $f^{m_k+1} V \approx \lambda(f^{m_k} V)$ would not intersect $f^{m_k} V$, which contradicts our assumption. Consequently, all the domains $f^m V$ are seen from the origin at an angle not less than some $\theta > 0$, and have "thickness" of order of the distance to the origin. Then the density of the orbits of the rotation $z \mapsto \lambda z$ on the circle $\mathbf{T}$ implies that for some N and sufficiently large m the domains $f^m V, f^{m+1} V, \ldots, f^{m+N} V$ form a closed chain. Hence, the invariant domain $\{0\} \bigcup\limits_{m=0}^{\infty} f^m V$ is a neighbourhood of the origin. Consequently, the origin is a stable position of equilibrium. In view of Proposition 1.7, f reduces to a rotation in a neighbourhood of the origin. But then $f^m V \not\to 0$ $(m \to \infty)$, and we obtain a contradiction.

Corollary. *Let D be an invariant component of the normality set of a rational endomorphism f. Suppose that $f^m z \to \alpha \in D$ $(m \to \infty)$ in D. Then α is a neutral fixed point with multiplier $\lambda = 1$. Thus, if case* b) *of Theorem* 1.1 *holds in an invariant component D of the normality set, then D is a Leau domain.*

Conjecture 1.2. *If α is a neutral irrational fixed point of an analytic transform f, then $\{f^m V\}$ cannot converge to α (without the assumption $fV \cap V \neq \emptyset$). For rational functions and a certain class of entire functions this assertion does hold* ([89], [13], [39]).

5. To conclude this section we prove that a rational endomorphism has a finite number of neutral cycles. A scheme of proof suggested by Fatou is the following. If a function f_0 has l neutral cycles, then it can be perturbed in such a way that the new function f has at least $\left[\frac{l+1}{2}\right]$ attracting cycles and $\deg f = d$. In view of Corollary 1 of Theorem 1.4, we have $l \leqslant 2(2d-2)$.

Lemma 1.5. *Let $\lambda_1(w), \dots, \lambda_l(w)$ be non-constant holomorphic functions in a neighbourhood of the origin, $|\lambda_i(0)| = 1$ $(i = 1, \dots, l)$. Then there is an arbitrarily small w_0 such that $|\lambda_i(w_0)| < 1$ for at least $(i+1)/2$ functions λ_i.*

We present a proof of this lemma based on an idea of Drinfel'd. Let $\lambda_i(w) = \lambda_i(0) + b_i w^{k_i} + \dots$ $(b_i \neq 0)$. Then to a first approximation the set $\{w : |\lambda_i(w)| = 1\}$ is the union of $2k_i$ rays coming from the origin at equal angles. It follows that $\int_0^{2\pi} \varepsilon_i(re^{i\theta})d\theta \to 0$ as $r \to 0$, where $\varepsilon_i(w) = \operatorname{sign} \log |\lambda_i(w)|$. Consequently, $\int_0^{2\pi} \sum_{i=1}^{l} \varepsilon_i(re^{i\theta})d\theta \to 0$ $(r \to 0)$. Since $\sum \varepsilon_i$ is an integer-valued function, $\sum_{i=1}^{l} \varepsilon_i(re^{i\theta}) \leqslant 0$ on some arc $\{re^{i\theta}: \theta_1 < \theta < \theta_2\}$ for sufficiently small r. This yields the required assertion.

We now consider the set $\mathfrak{R}_d$ of rational functions of degree d. It can be embedded naturally in the complex projective space $\mathbf{C}P^{2d+1}$ as a domain. This embedding induces the structure of a complex analytic manifold on $\mathfrak{R}_d$. We consider a many-valued analytic function $\alpha: \mathfrak{R}_d \to \bar{\mathbf{C}}$ satisfying the algebraic equation $f^p(\alpha(f)) = \alpha(f)$ with fixed p. Since the sphere $\bar{\mathbf{C}}$ is compact, it follows that α can have only algebraic singular points. Consequently, any periodic point of a given $f_0 \in \mathfrak{R}_d$ can be converted into a periodic point of any other function $f \in \mathfrak{R}_d$ by analytic continuation along a suitable path.

We now consider the multiplier $\lambda(f)$ of a periodic point $\alpha(f)$ as a function on the manifold $\mathfrak{R}_d$. It follows from what we have said that every branch of λ is a many-valued function $\mathfrak{R}_d \to \bar{\mathbf{C}}$, which has only algebraic singularities on $\mathfrak{R}_d$. The branches of λ are not constant. For in the family

$f_w(z) = z^d + w$ ($w \in \mathbf{C}$) we have $\lambda(w) \to \infty$ ($w \to \infty$) for every finite periodic point. In the family $z^d/(1+wz^d)$ a perturbation of the fixed point at infinity has a non-zero multiplier.

Theorem 1.11 (Fatou [60], §30). *A rational endomorphism of degree d has at most $4d-4$ neutral cycles. Consequently, f has at most $4d-4$ cycles of Leau and Siegel domains.*

Proof. Let $\alpha_1, \ldots, \alpha_l$ be neutral cycles of a function f_0. Since the corresponding branches of the multipliers $\lambda_i(f)$ are not constant on $\mathfrak{R}_d$, it follows that f can be immersed in a one-parameter family f_w in which the multipliers $\lambda_i(w) \equiv \lambda_i(f_w)$ are not constant. Since the functions λ_i can have at most algebraic singularities at the origin, we see that for some N the functions $\mu_i(u) = \lambda_i(u^N)$ are single-valued in a neighbourhood of the origin. By Lemma 1.5 there is a function f_w that has at least $\left[\frac{l+1}{2}\right]$ attracting cycles.

Conjecture 1.3. *The total number of neutral cycles of a rational endomorphism of degree d does not exceed $2d-2$.*

§1.12. Arnol'd-Herman rings

The Schröder, Böttcher, Leau, and Siegel domains correspond to the cases a), b), and c) (i) of Theorem 1.1. The only possibility that this theorem leaves for an endomorphism $f^p : D \circlearrowleft$, where D is a periodic component of the Fatou set, is the case c) (iii), where the transform f^p: $D\circlearrowleft$ is conformally conjugate to an irrational rotation of the annulus $\mathrm{A}(r, 1)$ ($0 < r < 1$). Such a domain is called an Arnol'd-Herman ring. In view of Proposition 1.6, polynomial transforms have no Arnol'd-Herman rings.

Example 1.8 (Herman [67]). f: $z \mapsto e^{2\pi i\theta}z^2(1 - \bar{a}z)/(z - a)$. The circle $\mathbf{T}$ is f-invariant and the restriction $f : \mathbf{T}\circlearrowleft$ is a diffeomorphism. It follows from Arnol'd's theorem ([3], §12) that for suitable θ and a there is an invariant neighbourhood V of $\mathbf{T}$ on which f: $V\circlearrowleft$ is conformally conjugate to a rotation of an annulus. Let D be a component of the Fatou set containing V. It is obvious that D is neither a Schröder, nor Böttcher, nor Leau domain. The component D cannot also be a Siegel disc, since $\mathbf{U}$ contains the pole a, while the complement $\bar{\mathbf{C}}\setminus\bar{\mathbf{U}}$ contains the zero $\bar{a}^{-1}$. Consequently, D is an Arnol'd-Herman ring.

Theorem 1.12 (Sullivan [90], Part III). *The total number of cycles of Schröder domains and Arnol'd-Herman rings does not exceed $2d-2$.*

We prove this theorem in §2.5. Now we summarize the results of the last five sections.

Theorem 1.13. *Every periodic component D of the Fatou set $F(f)$ of a rational endomorphism f: $\bar{\mathbf{C}}\circlearrowleft$ is either a Schröder, or Böttcher, or Leau domain, or a Siegel disc, or an Arnol'd-Herman ring. The endomorphism f has a finite number of periodic components.*

Conjecture **1.4.** *A rational endomorphism of degree d has at most $2d-2$ cycles of components of the Fatou set.*

With every cycle $\{f^n D\}_{n=0}^{p-1}$ of Arnol'd–Herman rings we associate a Riemann surface with boundary and marked points. We define the points $a_i \in D$ $(i = 2, \ldots, k+1)$ by analogy with the case of a Siegel disc (§1.11, subsection 2). We consider a conformal homeomorphism $\varphi : D \to \mathbf{A}(1, r)$ and put $S = \mathbf{A}[1, r]$, $d_i = \varphi(a_i)$ $(i = 2, \ldots, k+1)$. In addition, we mark the points d_0, d_1 on the components of the boundary of the annulus S. Finally, we recall that the rotation group acts on S.

§1.13. The density of repelling cycles in $J(f)$

The density theorem rests on the following modification of Montel's theorem.

Lemma **1.6.** *We consider a family $\{f_t\}$ of meromorphic functions in a domain D. Suppose that there are 3 meromorphic functions g_j $(j = 0, 1, 2)$ in D with the following property: the equations $f_i(z) = g_j(z)$, $g_k(z) = g_j(z)$ have no roots in D. Then the family $\{f_t\}$ is normal.*

This lemma reduces to Montel's theorem for the family $\left\{\frac{f_t-g_1}{f_t-g_2}\,\frac{g_0-g_2}{g_0-g_1}\right\}$, which has 3 exceptional values 0, 1, ∞.

Theorem **1.14** (Julia [65], Fatou [60], §27). *The Julia set $J(f)$ is the closure of the set of repelling periodic points of f.*

Proof. Let α be a repelling periodic point of order p. Without loss of generality we may assume that $\alpha \neq \infty$. Then $(f^{pm})'(\alpha) = \lambda^m \to \infty$, $|f^{pm}\alpha| = |\alpha| < \infty$. In view of Lemma 1.2, the family $\{f^{pm}\}_{m=1}^{\infty}$ is not normal in a neighbourhood of α. Thus, repelling cycles lie on the Julia set.

Next, let $a \in J(f)$. We want to approximate a by repelling periodic points. Since $J(f)$ is a perfect set, we may assume that a is not periodic and is not a critical value of f. Consequently, a has two different inverse images a_1, a_2, where $a_i \neq a$, and in a neighbourhood D of a there are univalent branches g_1, g_2 of f^{-1} such that $g_i(a) = a_i$. In addition, $g_1 D \cap g_2 D = \emptyset$. We put $g_0(z) \equiv z$. If D is a sufficiently small neighbourhood, then $g_j D \cap g_0 D = \emptyset$ $(j = 1, 2)$. Consequently, the equations $g_k(z) = g_j(z)$ $(k \neq j)$ have no roots in D.

We now consider an arbitrary neighbourhood $V \subset D$ of a. By Lemma 1.6, in V there is a root α of some equation $f^p z = g_j(z)$. The point α is periodic (with period p for $j = 0$ and $p+1$ for $j = 1, 2$). Finally, since there are only finitely many attracting and neutral periodic points, we see that α is repelling provided that the neighbourhood V is sufficiently small. The theorem is now proved.

Corollary. $J(f^m) = J(f)$ $(m = 2, 3, \ldots)$.

§1.14. Further properties of $f\colon J(f) \circlearrowleft$: the density of inverse images, mixing

1. ***Theorem*** 1.15 (Julia [65], Fatou [60], §27). *Let $a \in J(f)$, let V be a neighbourhood of a, and let K be a compact set without exceptional points. Then there is an integer N such that $f^m V \supset K$ for $m \geqslant N$.*

Proof. By Theorem 1.14 the neighbourhood V contains a repelling periodic point α of order p. There is a neighbourhood $W \subset V$ of α such that $f^p W \supset W$. On the other hand, the set of exceptional points is not attracting, and so there is a compact set $L \supset K$ not containing exceptional points such that $fL \supset L$. It follows from Montel's theorem that there is an l such that $f^{pl} W = \bigcup\limits_{k=1}^{p} f^{kl} W \supset L$. Then $f^m V \supset K$ for $m \geqslant pl$.

Corollary 1. *Let b be a non-exceptional point, and $\varepsilon > 0$. Then there is an integer N such that for $m \geqslant N$ the inverse image $f^{-m}\, b$ is an ε-net of $J(f)$.*

A continuous transform T of a compact set X is called *topologically mixing* if for any two neighbourhoods V, $W \subset X$ there is an N such that $T^{-m}\, W \cap V \neq \varnothing$ for $m \geqslant N$. It follows from category arguments that *a topologically mixing transform has a dense orbit $\{T^m a\}_{m=0}^{\infty}$.*

Corollary 2. *A rational endomorphism is topologically mixing on the Julia set.*

Corollary 3. *Let $f\colon \bar{\mathbf{C}} \circlearrowleft$ be a rational endomorphism, and $a \in \bar{\mathbf{C}}$. The following conditions are equivalent:* a) *$a \in J(f)$;* b) *the orbit $\{f^m a\}_{m=0}^{\infty}$ is not Lyapunov stable.*

2. ***Proposition*** 1.10. *Let $\{f_i^{-m}\}_{m,i}$ be a family of single-valued analytic branches of the inverse functions in a domain V. Then:* a) *the family $\{f_i^{-m}\}$ is normal;* b) *if $V \cap J(f) \neq \varnothing$, then $\| Df_i^{-m}(z) \| \to 0$ $(m \to \infty)$ uniformly on compact subsets of V. Here $\| Dg(z) \|$ is the spherical norm of the differential.*

Proof. a) We consider a cycle $\{\alpha_i\}_{i=0}^{p-1}$ of order greater than two and not contained in V (if necessary we make V smaller, which is possible, since normality is a local property). Then the α_i are exceptional values for the family $\{f_i^{-m}\}$ in V.

b) We suppose the contrary. Then, in view of a), for some sequence (m_k, i_k) we have $f_{i_k}^{-m_k} \to \varphi (m_k \to \infty)$ uniformly on compact subsets of V, where $\varphi \neq$ const. Then $f_{i_k}^{-m_k} V \supset \overline{W}$, where W is a neighbourhood of some point of $J(f)$. It follows that $f^{m_k} W \subset V$, which contradicts Theorem 1.15.

3. As we know, the attracting region of an attracting or neutral rational cycle contains a critical point. The following fact is a weakened form of this result for neutral irrational cycles. We denote by $\omega(z)$ the limit set of the orbit $\{f^m\, z\}$; let c_i be the critical points of an endomorphism f:

$$\omega_f = \bigcup_i \omega(c_i); \quad \omega_f' = \bigcup_{c_i \in J(f)} \omega(c_i).$$

Proposition 1.11. a) *Let α be a non-Siegel neutral irrational periodic point. Then $\alpha \in \omega(c)\setminus\{f^m c\}_{m=0}^{\infty}$ for some critical point $c \in J(f)$.*

b) *Let D be either a Siegel disc, or an Arnol'd–Herman ring. Then*

$$\partial D \subset \omega_f'.$$

Proof. a) We may assume that α is a fixed point. Suppose that there is a neighbourhood V of α not containing points $f^m c$ different from α, where c is a critical point and $m > 0$. We consider the component V_m of the inverse image $f^{-m}V$ that contains α. By Lemma 1.4 $f^m : V_m \to V$ is a univalent map. We consider the family of inverse maps f^{-m}: $V \to V_m$. We have $\|Df^{-m}(\alpha)\| = 1$, which contradicts Proposition 1.1. The inclusion $c \in J(f)$ follows from the description of the dynamics on $F(f)$ (Theorems 1.13 and 1.16).

b) Let $\beta \in \partial D\setminus\omega_f$. Then in a neighbourhood V of β we can define branches f^{-m} such that $f^{-m}\gamma \in D$, where $\gamma \in V \cap D$. We have $\| Df^{-m}(\gamma) \| \geqslant \geqslant c > 0$. This contradiction shows that $\partial D \subset \omega_f$. Now the description of the dynamics on $F(f)$ yields $\partial D \subset \omega_f'$.

Conjecture 1.5. a) *If α is a non-Siegel neutral fixed point, then the orbit of some critical point converges to α.*

b) *If D is either an invariant Siegel disc or an Arnol'd–Herman ring, then every component of the boundary ∂D contains a critical point of f.*

Herman [68] proved this conjecture for f: $z \mapsto z^d + w$ under the assumption that the multiplier of a neutral periodic point satisfies the estimate (1.4) in §1.11.

§1.15. The absence of wandering components of the Fatou set

In §§1.8–1.12 we have described the behaviour of the orbits on periodic components of the Fatou set $F(f)$. The following result by Sullivan shows that every orbit in $F(f)$ is absorbed by some cycle of periodic components. This result completes the description of the dynamics of a rational endomorphism on the set $F(f)$.

A set D is called *wandering* if $f^m D \cap f^n D = \varnothing$ for all natural numbers $m > n \geqslant 0$.

Theorem 1.16 (Sullivan [89], [90], Part I). *The Fatou set $F(f)$ of a rational endomorphism has no wandering components.*

This theorem is similar to the "finiteness theorem" of Ahlfors for Kleinian groups [30].

We need some information on prime ends [10]. Let D be a simply-connected domain. A *slit* is a simple curve in D whose ends lie on ∂D. We fix a point $z_0 \in D$. A sequence of slits $\{\gamma_n\}$ is called a *chain* if $\gamma_i \cap \gamma_j = \varnothing$ $(i \neq j)$, every slit γ_n separates z_0 in D from γ_{n+1}, and diam $\gamma_n \to 0$. Two chains are regarded as equivalent if one can choose infinite subsets of slits in them that together form a chain. The classes of equivalent chains are called *prime ends.* If a is a prime end represented by a chain $\{\gamma_n\}$, and $\gamma_n \to x$, then x is called a principal point of the prime end. Let $I(a)$ be the set of principal points of a. The basic theorem on the correspondence between boundaries under conformal maps says that *a conformal isomorphism of simply-connected domains extends to a one-to-one correspondence of the prime ends* (Carathéodory; see [10], Ch. II, §3). In particular, if we are given a conformal isomorphism $\mathbf{U} \to D$, then the prime ends of D are in one-to-one correspondence with the points of $\mathbf{T}$. This correspondence determines a topology of the set of prime ends.

Lemma 1.7. *The set of prime ends with common set of principal points is totally disconnected.*

Everything said above can be extended to the case of finitely-connected domains. We consider the group $G_{J(f)}$ of homeomorphisms of $J(f)$ that commute with f.

Lemma 1.8. *The group $G_{J(f)}$ is totally disconnected.*

Proof. Let Γ be a connected component of id of the group $G_{J(f)}$. Since the finite set $\mathrm{Per}_k(f)$ is invariant under $G_{J(f)}$, we see that all the points in $\mathrm{Per}_k(f)$ are fixed by Γ. Since the periodic points are dense in $J(f)$, we obtain $\Gamma = \mathrm{id}$.

Proof of Theorem 1.16. Let D_0 be a wandering component of $F(f)$, and $D_n = f^n D_0$ $(n \geqslant 0)$. We may assume that every D_n does not contain critical points of f, since the set of such points is finite. Then $f\colon D_n \to D_{n+1}$ is an unramified covering.

The first case. The domains D_n are finitely-connected and the maps $f\colon D_n \to D_{n+1}$ are univalent for $n \geqslant n_0$. We may assume that $n_0 = 0$. To make our exposition simpler, suppose that the domains D_n are simply-connected. We consider a conformal isomorphism $g: \mathbf{U} \to D_0$. We construct a family of diffeomorphisms $\psi_t\colon \bar{\mathbf{U}}\circlearrowleft$ depending continuously on $t \in \mathbf{R}^N$ (where $N > 4d+2$) and possessing the following property: all the maps $\psi_s^{-1} \circ \psi_t$ are not conformal on $\mathbf{T} = \partial\mathbf{U}$, that is, their restrictions to $\mathbf{T}$ differ from the transforms $z \mapsto \lambda\,(z - a)/(1 - \bar{a}z)\,(|\lambda| = 1,\ |a| < 1,\ |z| = 1)$. We consider in D_0 the family of conformal structures $\tilde{\mu}_t = (g \circ \psi_t)_* \sigma_{\mathbf{U}}$. We extend these structures to the large orbit of D_0 with the help of f. This can be done because D_0 is a wandering domain and all the maps $f\colon D_n \to D_{n+1}$ are univalent. To the remaining part of the sphere the structures obtained can be extended as standard. We obtain an N-parameter family of conformal structures μ_t on $\bar{\mathbf{C}}$. By the measurable Riemann theorem there is a continuous family of quasi-conformal homeomorphisms $\varphi_t\colon \bar{\mathbf{C}}\circlearrowleft$ such that $(\varphi_t)_* \mu_t = \sigma$. The transforms $f_t = \varphi_t \circ f \circ \varphi_t^{-1}$ are continuous and are locally conformal everywhere except for a finite number of points. By the theorem on removable singularities f_t is a rational function of degree d.

Thus, we have constructed a continuous map $t \mapsto f_t$ from $\mathbf{R}^N$ to the manifold $\mathfrak{R}_d$ of rational functions of degree d. Since $\dim \mathfrak{R}_d = 4d + 2 < N$, by a well-known topological theorem ([11], Chapter IV, §4) there is a non-empty connected set $M \subset \mathbf{R}^N$ such that $f_t = f_s$ for $t, s \in M$. We fix s and obtain the result that all the homeomorphisms $h_t = \varphi_s^{-1} \circ \varphi_t$ commute with f. By Lemma 1.8, $h_t = \mathrm{id}$ on $J(f)$ $(t \in M)$.

It is easy to see that D_0 is invariant under h_t $(t \in M)$. Consequently, h_t is a homeomorphism of the closure $\bar{D}_0$ and so it acts on the space of prime ends of D_0. It follows that the homeomorphisms $H_t = g^{-1} \circ h_t \circ g \colon \mathbf{U} \circlearrowleft$ extend to homeomorphisms of $\bar{\mathbf{U}}$. We show that $H_t | \mathbf{T} = \mathrm{id}$. Let a be a prime end of D_0. It is easy to see that the map $t \mapsto h_t(a)$ is continuous. In addition, $h_t(I(a)) = I(a)$, since $h_t | J(f) = \mathrm{id}$. It follows from Lemma 1.7 that $h_t(a) = a$ $(t \in M)$, as required. Thus, the map $\psi_s^{-1} \circ \psi_t$ on $\mathbf{T}$ coincides with $\psi_s^{-1} \circ H_t \circ \psi_t (t \in M)$. But the latter map is conformal in $\mathbf{U}$. We obtain a contradiction.

The second case. The domains D_n are finitely-connected and the maps $f\colon D_n \to D_{n+1}$ are non-univalent for infinitely many n. It follows from the Riemann-Hurwitz formula that all the domains are doubly-connected. We consider conformal maps $g_n \colon \mathrm{A}(r_n, \rho_n) \to D_n$ normalized so that $r_n < 1 < \rho_n$ and $g_{n+1}(1) = f(g_n(1))$. Then $g_{n+1}^{-1} \circ f \circ g_n : z \mapsto z^{l_n}$. We consider the simple closed curves $\Gamma_n = g_n \mathrm{T} \subset D_n$. The maps $f^n \colon \Gamma_0 \to \Gamma_n$ are d_n-sheeted coverings and $d_n \to \infty$. It follows from the normality of $\{f^n\}$ in D_0 that $| \Gamma_n | d_n = \int_{\Gamma_0} \| Df^n(z) \| \, ds \leqslant C$, where $|\Gamma_n|$ is the spherical length of Γ_n. Consequently, $|\Gamma_n| \to 0$. But then $\operatorname{diam} E_n \to 0$, where E_n is a component of the complement of Γ_n. In view of the fact that f is uniformly continuous on $\bar{\mathbf{C}}$ we have $fE_n = E_{n+1}$ $(n \geqslant N)$. Consequently, $\{f^n\}$ is normal in E_N. We obtain a contradiction.

The third case. All the domains D_n are infinitely-connected.

***Lemma* 1.9.** *Let $D_0 \xrightarrow[f_0]{} D_1 \xrightarrow[f_1]{} D_2 \to \dots$ be a sequence of analytic coverings of hyperbolic Riemann surfaces. Then there is a Riemann surface D_∞ and a sequence of analytic coverings $\pi_n \circ D_n \to D_\infty$ such that $\pi_n = \pi_{n+1} \circ f_n$. If the Riemann surfaces D_n are of infinite type, then D_∞ is also of infinite type*(1).

Applying this lemma to the sequence $D_0 \xrightarrow[f]{} D_1 \xrightarrow[f]{} \dots$, we consider an N-parameter family of conformal structures μ_t on the Riemann surface

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(1) We say that a Riemann surface V is of *infinite* (*topological*) *type* if its fundamental group is not finitely generated. This is equivalent to the fact that there is an infinite-parameter family on V of non-equivalent conformal structures.

D_∞ ($N > 4d+2$). We lift the μ_t to f-invariant conformal structures on the large orbit of D_0, and then extend them to the whole sphere by the standard structure. We obtain an N-parameter family of structures ω_t on $\bar{\mathbf{C}}$. Let φ_t: $\bar{\mathbf{C}}$ $\circlearrowleft$ and $(\varphi_t)_* \ \omega_t = \sigma$. Then $\varphi_t \circ f \circ \varphi_t^{-1}$ is an N-parameter family of rational functions of degree d. We obtain a contradiction.

§1.16. Rational endomorphisms satisfying Axiom A

1. This section is devoted to the most important and well-studied class of rational endomorphisms.

We endow the sphere with a smooth Riemann metric ω proportional to the spherical metric. We consider a closed f-invariant subset $X \subset \bar{\mathbf{C}}$. An endomorphism f is called *dilating on a set X* if there are constants $a > 0$, $\gamma > 1$ for which

$$\| Df^n(z) \| \geqslant a\gamma^n \quad (z \in X,\ n = 0, 1, \ldots),$$

where $\| \cdot \|$ denotes the norm in the metric ω. This definition is invariant under the choice of a Riemann metric in a neighbourhood of X. If $a = 1$, then ω is called a Lyapunov metric.

2. ***Theorem*** 1.17 (Fatou [60], §31). *The following properties of a rational endomorphism f are equivalent*:

(i) *f is dilating on the Julia set $J(f)$*;

(ii) *the orbits of its critical points converge to attracting or superattracting cycles.*

In addition, the orbits of all points of the Fatou set $F(f)$ converge to attracting or superattracting cycles.

Proof. (ii) ⇒ (i). We delete from the sphere the orbits $\{f^n c_i\}_{n=1}^\infty$ of critical points and invariant neighbourhoods of attracting cycles. The resulting domain S is hyperbolic and f^{-1}-invariant. Let $\pi : \mathbf{U} \to S$ be the universal covering. Then f^{-1} can be lifted to a single-valued function on $\mathbf{U}$. Applying the Schwarz lemma, we obtain $\|Df(z)\| > 1$ ($z \in S$) in the hyperbolic metric ρ_S. Consequently, $\|Df(z)\| \geqslant \gamma > 1$ ($z \in J(f)$) that is, the hyperbolic metric is Lyapunov.

(i) ⇒ (ii). All critical points of the endomorphism f lie in the Fatou set. Consequently, f has no Siegel discs or Arnol'd–Herman rings (Proposition 1.1). According to Corollary 2 of Proposition 1.7, f has no Leau domains either. It follows from the description of the dynamics on the Fatou set (Theorems 1.13, 1.16) that the orbits of all points of the Fatou set (and so, all critical points) converge to attracting and superattracting cycles.

By analogy with Smale's well-known definition for diffeomorphisms [26] we shall say that *an endomorphism f satisfies Axiom* A if properties (i) and (ii) of the theorem hold.

3. **Symbolic dynamics on the Julia set** (Fig. 8).
We denote by Σ_d^+ the space of one-sided sequences $(i_1 i_2 \ldots)$ of d symbols endowed with the weak topology. *The unilateral (topological) Bernoulli shift* is the transform β: $(i_1 i_2 \ldots) \mapsto (i_2 i_3 \ldots)$ of the space Σ_d^+. The unilateral Bernoulli shift provides an adequate model for the dynamics of a rational endomorphism on the Julia set [30]-[33], [64].

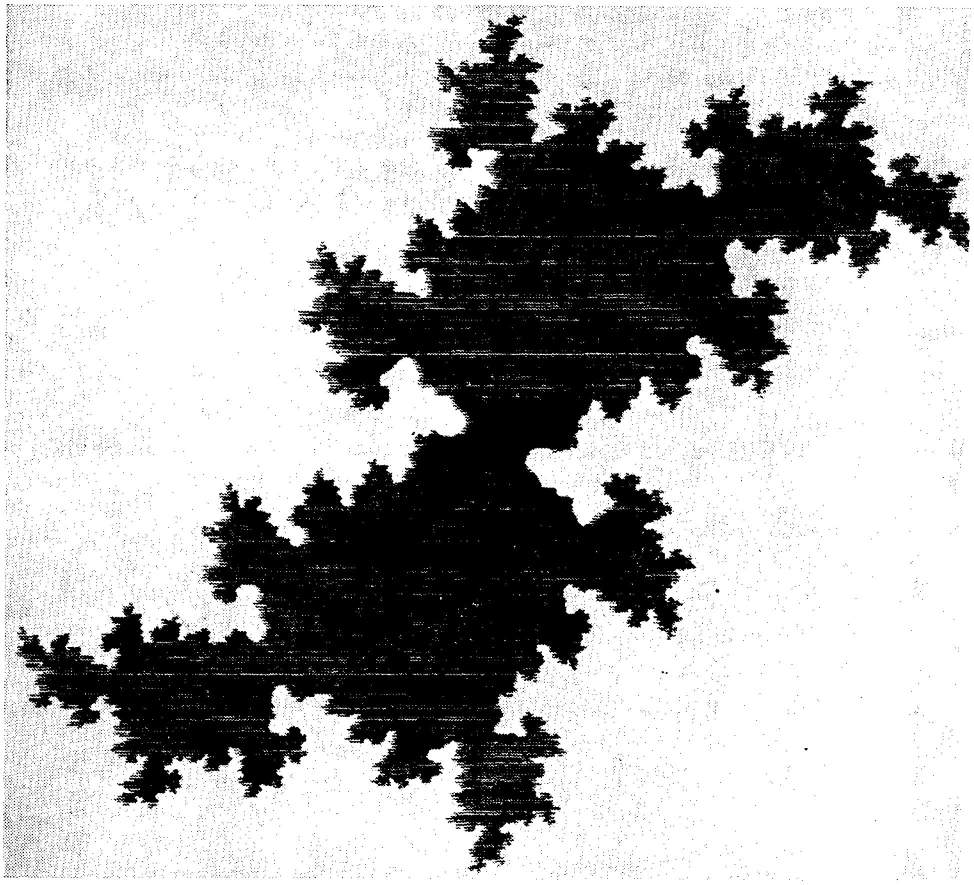

Fig. 8. The Julia set of the transform $z \to z^2 - 0.65i$ is a quasi-circle.

Suppose that an endomorphism f satisfies Axiom A. Then an appropriate point z can be joined to its inverse images $z_1, \ldots, z_d$ by smooth curves $l_1, \ldots, l_d$ so that, for all n, single-valued branches of f^{-n} are defined in a neighbourhood of $\bigcup_{i=1}^{d} l_i$. Then the inverse image $f^{-1}l_j$ consists of d curves l_{ij} starting at z_i, respectively. We denote the endpoint of l_{ij} by z_{ij}. Suppose that the points of the inverse image $f^{-(k-1)}z$ are numbered by d-nary sequences $(i_1 \ldots i_{k-1})$. The inverse image $f^{-(k-1)}l_{i_k}$ consists of d^{k-1} curves

$l_{i_1}\ldots{}_{i_{k-1}i_k}$ starting at $z_{i_1\ldots i_{k-1}}$. We denote the endpoint of $l_{i_1\ldots i_k}$ by $z_{i_1\ldots i_k}\in f^{-k}z$. It follows from Axiom A that $\rho(l_{i_1\ldots i_k}, J(f))\to 0$ $(k\to\infty)$, and the lengths of the curves $l_{i_1\ldots i_k}$ decay exponentially. Therefore, $\lim\limits_{k\to\infty} z_{i_1\ldots i_k} = \varphi(i_1 i_2\ldots)\in J(f)$ exists. We have defined a map $\varphi\colon \Sigma_d^+\to J(f)$. It follows from the exponential decay of the lengths of the curves $l_{i_1\ldots i_k}$ that φ is continuous, and from Corollary 1 of Theorem 1.16 that φ is onto. Obviously, $\varphi\circ\beta = f\circ\varphi$. We have thus proved the following result.

Proposition 1.12. *If an endomorphism f satisfies Axiom* A, *then* $f\colon J(f)\circlearrowleft$ *is a quotient of a unilateral Bernoulli shift.*

Using methods of the theory of Markov partitions, one can show that there is a number c such that for any $z\in J(f)$ we have $\mathbf{card}\varphi^{-1}z\leqslant c$ [33].

4. The case when $F(f)$ is connected.

In this case the symbolic dynamics is especially good. It was actually familiar to Fatou ([59], 252).

Theorem 1.18. *Suppose that an endomorphism f satisfies Axiom* A, *and the Fatou set* $F(f)$ *consists of a single component. Then the Julia set* $J(f)$ *is a Cantor set and* $f\colon J(f)\circlearrowleft$ *is topologically conjugate to the unilateral Bernoulli shift* $\beta\colon\Sigma_d^+\circlearrowleft$.

Proof. In our case all critical points c_i lie in the domain $D = F(f)$, and all their orbits converge to an attracting fixed point $\alpha\in D$. Suppose additionally that these orbits are disjoint(1). Then it is easy to construct an invariant simply-connected neighbourhood V of α with smooth boundary such that the first moment N when the orbit $\{f^n c_i\}_{n=1}^{\infty}$ falls into V does not depend on i. Let V_n be the connected component of $f^{-n}V$ containing α. Then V_{N-1} contains all critical values fc_i, but does not contain critical points c_i. We put $\Delta = \bar{\mathbf{C}}\setminus\bar{V}_{N-1}$. Then $f^{-1}\Delta$ consists of d components Δ_i such that $\bar\Delta_i\cap\bar\Delta_j = \emptyset$ $(i\neq j)$, $\Delta_i\subset\Delta$, and $f\colon\Delta_i\to\Delta$ is univalent. The inverse image $f^{-n}\Delta$ consists of d^n components $\Delta_{i_0\ldots i_{n-1}} = \Delta_{i_0}\cap f^{-1}\Delta_{i_1}\cap\ldots\cap f^{-(n-1)}\Delta_{i_{n-1}}$ with analogous properties. It follows from Axiom A that $\mathbf{diam}\,\Delta_{i_0\ldots i_{n-1}} < c\gamma^{-n}\to 0$ $(n\to\infty)$, and so $J(f) = \bigcap\limits_{n=0}^{\infty} f^{-n}\Delta$ is a Cantor set. We associate with $\underline{i} = (i_0 i_1\ldots)$ the point $x_{\underline{i}} = \bigcap\limits_{n=0}^{\infty}\Delta_{i_0 i_1\ldots i_{n-1}}$ and obtain the desired conjugation of the Bernoulli shift β and $f\colon J(f)\circlearrowleft$.

We have given above two examples (1.2 and 1.6) of a rational endomorphism such that $J(f)$ is a Cantor set. The first of them satisfies Axiom A, while the second does not. Fatou conjectured ([60], 84) that if $J(f)$ is a Cantor set, then all critical points lie in $F(f)$. This conjecture has been disproved by Brolin ([45], 136).

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(1) The particular case when the orbits of critical points intersect can be reduced to the general case by a perturbation of the endomorphism.

5. The case when $F(f)$ consists of two components.
A *quasi-circle* is by definition a Jordan curve that is a quasi-conformal image of the circle $\mathbf{T}$.

***Theorem* 1.19** ([59], [34]). *Suppose that an endomorphism f satisfies Axiom* A *and that $F(f)$ consists of two invariant components D_1, D_2. Then $J(f)$ is a quasi-circle and $f\colon J(f)\circlearrowright$ is topologically conjugate to the transform $z \mapsto z^d$ of the circle* $\mathbf{T}$.

Proof. Since the D_i are simply-connected (Proposition 1.4), it follows that $J(f)$ can be enclosed in an annulus A bounded by smooth curves $\gamma_i \subset D_i$, not containing critical points, and such that $\overline{f^{-1}A} \subset A$. We denote the interior of $f^{-1}\gamma_i \equiv \beta_i$ by $V_i \subset D_i$. We construct diffeomorphisms $h_i : \bar{V}_i \to \mathrm{U}_r$ $(0 < r < 1)$ such that $h_i(fz) = h_i(z)^d$ for $z \in \beta_i$. We now define a transform $g\colon \bar{\mathbf{C}}\circlearrowright$ as follows: $(z \in \overline{f^{-1}A})$ and $gz = h_i^{-1}(h_i(z)^d)$ $(z \in \bar{V}_i)$. We define on $\bar{\mathbf{C}}$ a g-invariant conformal structure μ. In the component D_i the structure μ is a g-invariant extension of the structure $(h_i^{-1})_*\sigma$ from V_i. The structure μ is standard on $J(f)$. It follows from the measurable Riemann theorem that g is quasi-conformally conjugate to the transform $z \mapsto z^d$ of the whole sphere.

6. We consider the transform $f\colon z \mapsto z^2 + \varepsilon$, where $|\,\varepsilon\,|$ is small. It satisfies the assumptions of Theorem 1.19. However, the situation is not as simple as it may appear at first sight: for $\varepsilon \neq 0$ the set $J(f)$, being a Jordan curve, has no tangent at any point. This follows from a general theorem of Fatou.

***Theorem* 1.20** ([61], 208–240). *Let α be either an attracting or a superattracting fixed point. Suppose that $\partial D(\alpha) \cap \omega_f = \emptyset$. Then we have the following alternatives*: a) *$\partial D(\alpha)$ is a circle*; b) *$\partial D(\alpha)$ has no tangent at any point.*

We clarify the nature of non-smoothness for the transform $f\colon z \mapsto z^2 + \varepsilon$ in the case when ε is not real. Then f has a repelling fixed point α with non-real multiplier $\lambda = 1 + \sqrt{1-4\varepsilon}$. Consequently, the curve γ in a neighbourhood of α looks like a logarithmic spiral and so has no tangent at α. But γ has a similar nature at all inverse images $f^{-n}\alpha$, which densely fill γ.

§1.17. Iterates of polynomials

1. The study of the dynamics of polynomials is simplified by the existence of a completely invariant component of the Fatou set, namely the attracting region of ∞, which we denote by $D(\infty)$. It has been shown earlier that $J(f)$ coincides with the boundary of $D(\infty)$, and every bounded component of $F(f)$ is simply-connected (Proposition 1.6). It follows from the Riemann–Hurwitz formula that $D(\infty)$ is simply-connected or infinitely-connected. The first case is equivalent to the fact that the Julia set $J(f)$ is connected.

Theorem 1.21 ([61], [65]). *The Julia set $J(f)$ of a polynomial f is connected if and only if the orbits of all finite critical points do not tend to ∞.*

Corollary. *If the Julia set is connected, then $f\colon D(\infty)\circlearrowleft$ is conformally conjugate to the transform $z \mapsto z^d$ of the disc* U.

Proof. a) If $D(\infty)$ is simply-connected, then according to the Riemann-Hurwitz formula there are $d-1$ critical points, taking account of multiplicities. But ∞ is itself a critical point of multiplicity $d-1$.

b) Let V_0 be a small invariant neighbourhood of ∞. Since $f^{-1}\infty = \{\infty\}$, it follows that $V_1 = f^{-1}V_0$ is a domain. If V_1 contains no finite critical points, then by the Riemann-Hurwitz formula V_1 is simply-connected. Similarly, all domains $V_n = f^{-n}V_0$ are simply-connected. Consequently, $D(\infty) = \bigcup_{n=0}^{\infty} V_n$ is also simply-connected.

2. (Fig. 9). In the case of a connected Julia set the question of its local connectedness is very interesting and non-trivial. This question is investigated with the help of the following classical criterion.

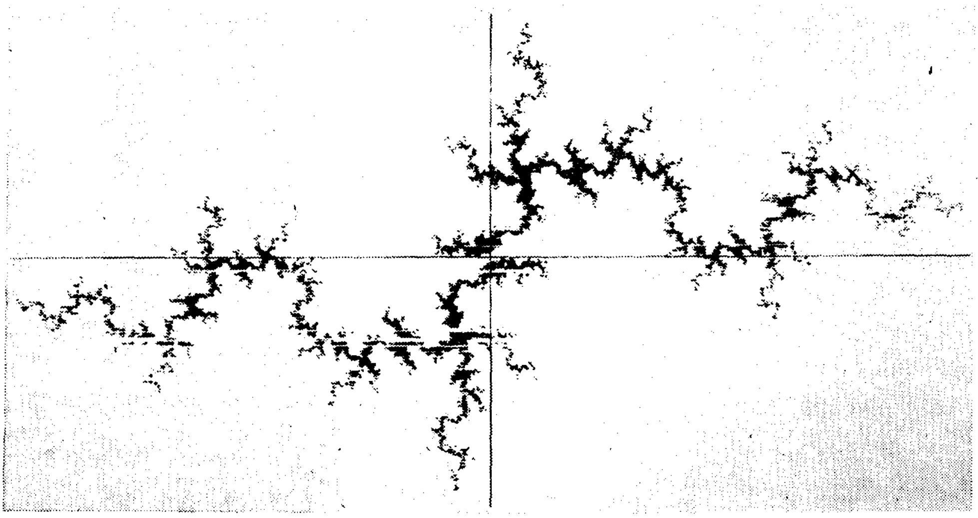

Fig. 9. The transform $z \to z^2 + c$ for $c = -(1.04+0.34i)$. Near this value of c a bifurcation from the connected Julia set to a Cantor set takes place.

Carathéodory's theorem [48]. *Let D be a simply-connected domain, and $\varphi\colon$ U $\to D$ a conformal map. The boundary ∂D is locally connected*(1) *if and only if φ is continuous up to the boundary of the disc* U.

The following example was announced in [43] as a result of Douady and Sullivan. It has been found independently by the author.

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(1) Therefore, ∂D is locally connected if and only if it is a Jordan curve, that is, a continuous image of the circle.

Example 1.9. $f_\lambda: z \mapsto \lambda z + z^2$. Suppose that $|\lambda| = 1$ and that f_λ is not conjugate to a rotation in a neighbourhood of the origin. Then $J(f_\lambda)$ is not locally connected.

For otherwise the conformal map $\varphi: \mathbf{U} \to D(\infty)$ is continuous up to the boundary. We put $X = \varphi^{-1}0 \subset \mathbf{T}$. Then the functions $\rho(z) = \operatorname{dist}(\varphi^{-1}z, X)$ and $|z|$ are equivalent in $D(\infty)$ (that is, $|z| < \delta \Leftrightarrow \rho(z) < \varepsilon$). Let f^{-1} be a branch of the inverse function for which $f^{-1}0 = 0$. Then $\rho(f^{-m} z) = = |\sqrt[2^m]{\varphi^{-1}z} - \sqrt[2^m]{\zeta}| \to 0\ (m \to \infty)$, where $\zeta \in X$, $\sqrt[2^m]{}$ is an appropriate branch of the root. Therefore, if $z \in D(\infty)$ and $|z| < \delta$, then $|f^{-m} z| < \varepsilon$. Thus, 0 is a Lyapunov stable position of equilibrium. We obtain a contradiction.

3. On the other hand, with the help of symbolic dynamics the following fact can be established:

Theorem 1.22 [61]. *Let f be a rational endomorphism, and D a simply-connected component of $F(f)$. Suppose that $f|\partial D$ is a dilating transform. Then ∂D is a Jordan curve.*

Corollary 1. *Under the assumptions of Theorem* 1.22 *all points of ∂D are attainable from D.*

Corollary 2. *Suppose that a polynomial f satisfies Axiom* A *and that $J(f)$ is connected. Then* a) *$J(f)$ is a Jordan curve*; b) *if D is a bounded connected component of $F(f)$, then ∂D is a simple Jordan curve.*

Under the assumptions of Theorem 1.22, ∂D in general is not a simple Jordan curve, as the example $D = D(\infty)$ for a polynomial shows. Another example is given in [33].

In conclusion we mention the following fact: *if, under the assumptions of Theorem* 1.22, *∂D is a simple Jordan curve, then ∂D is a quasi-circle.* It can be proved in the same way as Theorem 1.19 or with the help of the symbolic dynamics [34].

4. For polynomials Conjecture 1.3 has been proved by Douady and Hubbard:

Theorem 1.23 ([53], [57]). *A polynomial of degree d has at most $d-1$ non-repelling cycles on the plane.*

We introduce the following notion. Let V and V' be simply-connected domains with $V' \subset V$, and $g: V' \to V$ an analytic d-sheeted ramified covering. Then g is called a *polynomially similar map* of degree d. We denote by $K(g)$ the set $\{z: g^n z \in V'\ (n = 0, 1, 2, \dots)\}$.

Lemma 1.10. *Let g be a polynomially similar map of degree d. Then there is a polynomial h of degree d, a neighbourhood $V(g)$ of $K(g)$, and a neighbourhood $V(h)$ of the Julia set $J(f)$ such that $g|V(g)$ and $h|V(h)$ are quasi-conformally conjugate. The conjugating homeomorphism $\varphi: V(g) \to V(h)$ is conformal on $K(g)$.*

Proof. We consider a domain W such that $K(g) \subset W \subset V'$, W is diffeomorphic to the disc, and $\overline{g^{-1}W\setminus W}$ is homeomorphic to the annulus. Arguing as in Theorem 1.19, we extend $g|\overline{g^{-1}W}$ to a map $\tilde{g}:\bar{\mathbf{C}}\circlearrowleft$ which on $\bar{\mathbf{C}}\setminus\overline{g^{-1}W}$ is quasi-conformally conjugate to the map $z \mapsto z^d$ of $\mathbf{U}_{\underset{\sim}{r}}$ $(0 < r < 1)$. It follows from the measurable Riemann theorem that $\tilde{g}$ is quasi-conformally conjugate to a polynomial.

Proof of Theorem 1.23. Let N be the set of non-repelling cycles of a polynomial f of degree d. There is a polynomial h (of high degree) such that $h(z) = 0$ and $|(f+h)'(z)| < 1$ for any $z \in N$. Then for $\varepsilon \in (0, 1)$ the points $z \in N$ will be attracting periodic points for the polynomial $g_\varepsilon = f + \varepsilon h$. But for small ε, g_ε is a polynomially similar map of degree d in a neighbourhood of $\mathbf{C}\setminus D_f(\infty)$ and int $K(g_\varepsilon) \supset N$. The required estimate now follows from the preceding lemma and Theorem 1.4.

§1.18. Endomorphisms whose critical point orbits are absorbed by cycles

1. As we know (Proposition 1.2) the Julia set is either nowhere dense or coincides with the whole sphere. In all the examples considered above the first possibility has occurred. In particular, if f is a polynomial, then $J(f) = \bar{\mathbf{C}}$, since $\infty \in F(f)$. We now present a classical example of a rational endomorphism for which $J(f) = \bar{\mathbf{C}}$.

Example 1.10 (Lattes [71]). We consider the Weierstrass elliptic function $\wp_\tau$ with periods $\{1, \tau\}$ $(\operatorname{Im} \tau > 0)$. It satisfies the duplication formula $\wp_\tau(2z) = f_\tau(\wp_\tau(z))$, where f_τ is a rational function of degree 4 (see [19]). This formula can be interpreted as the commutativity of the following diagram:

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "z\mapsto 2z"] \arrow[d, "\pi_\tau"] \arrow[dd, bend right=60, "\wp_\tau"'] & \mathbf{C} \arrow[d, "\pi_\tau"'] \arrow[dd, bend left=60, "\wp_\tau"] \\
\mathbf{T}^2_\tau \arrow[r, "A_\tau"] \arrow[d] & \mathbf{T}^2_\tau \arrow[d] \\
\bar{\mathbf{C}} \arrow[r, "f_\tau"] & \bar{\mathbf{C}}
\end{tikzcd}$$

(π_τ being a quotient map over the lattice $\{m+n\tau:\ (m, n) \in \mathbf{Z}^2\}$).

The points $\pi_\tau\left(\frac{m+n\tau}{2^p-1}\right)$ are periodic for the transform A_τ. Since they are dense on the torus $\mathbf{T}^2_\tau$, the periodic points of f_τ are dense on $\bar{\mathbf{C}}$. Consequently, $J(f_\tau) = \bar{\mathbf{C}}$.

The function f_τ has three critical values, which coincide with the finite critical values of the Weierstrass function:

$$e_1(\tau) = \wp_\tau\left(\frac{1}{2}\right), \quad e_2(\tau) = \wp_\tau\left(\frac{\tau}{2}\right), \quad e_3(\tau) = \wp_\tau\left(\frac{\tau+1}{2}\right).$$

Moreover, f_τ: $e_i(\tau) \mapsto \infty$. It turns out that this property itself implies that $J(f) = \bar{\mathbf{C}}$.

2. ***Theorem*** 1.24. *Suppose that the orbits of all critical points of a rational endomorphism f are absorbed by cycles but are not cycles themselves. Then $J(f) = \bar{\mathbf{C}}$. Furthermore, all cycles of f are repelling.*

See [20][1] for a proof not using Sullivan's theorem on the absence of wandering components. We shall reap the fruits of the theory developed. Suppose that $J(f) \neq \bar{\mathbf{C}}$. Then the Fatou set $F(f)$ contains a periodic component D. If D is a Schröder or Leau domain, then it contains a critical point whose orbit is not absorbed by a cycle. If D is a Böttcher domain, then it contains a critical point whose orbit is a cycle. Finally, if D is a Siegel disc or Arnold–Herman ring, then $\partial D \subset \overline{\bigcup \omega(c_i)}$. Therefore, all cases contradict the assumptions of the theorem. The fact that all cycles of f are repelling follows from Proposition 1.11.

3. In the Lattes example the endomorphism f_τ is a quotient of the dilating endomorphism $z \mapsto 2z$ of $\mathbf{C}$. The fibres of the function $\mathcal{P}_\tau$ that accomplishes this factorization are orbits of the group Γ_τ, which is obtained from the group $\{z \mapsto z + m + n\tau\colon (m, n) \in \mathbf{Z}^2\}$ by adjoining the transform $z \mapsto -z$. Γ_τ preserves the Euclidean metric ρ. Consequently, the metric $\gamma_\tau = (\mathcal{P}_\tau)_* \rho$ with singularities at $e_i(\tau)$, ∞ is well-defined on $\bar{\mathbf{C}}$. This metric is good in the sense that f_τ becomes dilating in it: $\gamma_\tau(f_\tau x, f_\tau y) \geqslant t\gamma_\tau(x, y)$, where $t > 1$ (in spite of the fact that f_τ has critical points). Thurston has shown that a similar metric can be constructed for all rational endomorphisms whose orbits of critical points are absorbed by cycles but are not cycles themselves. The key point of Thurston's construction is the following classical fact:

Lemma 1.11. *Let us mark on $\bar{\mathbf{C}}$ a finite number of points x_i endowed with weights $m_i \in \mathbf{N}$, $m_i \geqslant 2$. Suppose that*

$$\sum_{i=1}^{n} \frac{1}{m_i} \leqslant n-2 \tag{1.5}$$

Then there is a compact Riemann surface S and a ramified covering $p : S \to \bar{\mathbf{C}}$ such that 1) *p is not ramified over $\mathbf{C} \setminus \{x_i\}_{i=1}^{n}$;* 2) *if $p(y) = x_i$, then y is a branch point of index m_i. If equality is attained in* (1.5), *then $S = \mathbf{T}^2$, the torus; otherwise S is a hyperbolic Riemann surface. The fibres of the covering p are orbits of a finite group of motions of S (in the metric ρ_S).*

We return now to a rational endomorphism f whose orbits of all critical points are absorbed by cycles but are not cycles themselves. We mark points of these orbits $\{f^n c_j\}_{n=1}^{n_j}$. The weight $m(x)$ is assigned to a point x as follows. We consider f^n as a ramified covering $\bar{\mathbf{C}} \circlearrowright$. Let $m_n(x)$ be the least common multiple of branch indices of all the points y for which $f^n y = x$.

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[1] Fatou possessed all the information needed for the proof of Theorem 1.24 ([60], 60–61), but he did not state it anywhere.

Then $m(x)$ is the least common multiple of the $m_n(x)$. It is easy to see that $m(x) < \infty$. We shall call the $m(x)$ the *Thurston numbers*. It is convenient to suppose that weights $m(x) = 1$ correspond to unmarked points.

Lemma 1.12. *The Thurston numbers m_i satisfy* (1.5). *Equality in* (1.5) *is equivalent to $m(fx) = m(x)\deg_x f$ for all x.*

Proof. Since $m(x)\deg_x f$ divides $m(fx)$, we have

$$\left(\frac{1}{m(fx)}-1\right)\deg_x f\leqslant\frac{1}{m(x)}-1-(\deg_x f-1).$$

Summing over all $x \in \bar{\mathbf{C}}$, we obtain

$$d\sum\left(\frac{1}{m_i}-1\right)\leqslant\sum\left(\frac{1}{m_i}-1\right)-(2d-2).$$

Hence, (1.5) follows. If $m(fx) = m(x)\deg_x f$ for all x, the above inequalities turn into equalities. The converse is also obvious.

Theorem 1.25 (Thurston [93]). *Suppose that the orbits of all critical points of a rational endomorphism f are absorbed by cycles but are not cycles themselves. Then there is a ramified covering $\mathcal{P}$: $V \to \bar{\mathbf{C}}$, where $V = \mathbf{C}$ or $\mathbf{U}$, and an analytic endomorphism g: $V\circlearrowleft$ such that the following diagram is commutative*:

(1.6)
$$\begin{tikzcd} V \arrow[d, "\mathcal{P}"'] & V \arrow[l, "g"'] \arrow[d, "\mathcal{P}"] \\ \bar{\mathbf{C}} \arrow[r, "f"'] & \bar{\mathbf{C}} \end{tikzcd}.$$

The fibres of the covering $\mathcal{P}$ are orbits of a certain group of motions Γ of the Riemann surface V.

Proof. We consider the ramified covering $p : S \to \bar{\mathbf{C}}$ from Lemma 1.11. Let $\pi : V \to S$ be the universal covering of the Riemann surface S. Since $m(x)\deg_x f$ divides $m(fx)$, the lifting g of the many-valued function f^{-1} to the Riemann surface V is a locally single-valued function. Since V is simply-connected, g is globally single-valued.

Proposition 1.13. *The following properties are equivalent*: a) $V = \mathbf{C}$; b) *equality is attained in* (1.5); c) *g is invertible. Moreover g: $z \mapsto kz + l$, where $|k| < 1$.*

Proof. a) $\Leftrightarrow$ b) by Lemma 1.11. By Lemma 1.12, b) $\Leftrightarrow m(fx) = m(x)\deg_x f$. The latter is equivalent to the fact that g^{-1} is locally (and so globally) single-valued. Therefore, b) $\Leftrightarrow$ c). Next, the invertible transform g of $\mathbf{C}$ is affine: $z \mapsto kz + l$. Finally, $\deg f = k^{-2}$, from which it follows that $|k| < 1$.

Corollary. *Under the assumptions of Theorem* 1.25 *there is a Riemann metric γ on $\bar{\mathbf{C}}$* (*having singularities at marked points*) *with respect to which f is dilating.*

Proof. Since the fibres of the ramified covering $\mathcal{P}$ in the diagram (1.6) are orbits of the group of motions Γ, the metric $\gamma = \mathcal{P}_* \rho_V$ is well-defined. If $V = \mathbf{U}$, then g is not invertible and consequently it is a uniform compression on the fundamental domain of Γ, since the latter is compact. If $V = \mathbf{C}$, then g: $z \mapsto kz + l$ $(|k| < 1)$ is also a uniform compression. In both cases f is a dilating transform in γ.

The second proof of Theorem 1.24 (due to Thurston). Since f is dilating in γ, all its trajectories are Lyapunov unstable.

4. Thurston has given a classification of transforms whose orbits of critical points are absorbed by cycles ([93], [56]). We treat only the simplest case when $V = \mathbf{C}$ (in the notation of Theorem 1.25). In this case the following commutative diagram holds:

$$\begin{tikzcd}
\mathbf{T}^2_\tau \arrow[r, "A"'] \arrow[d, "p"'] & \mathbf{T}^2_\tau \arrow[d, "p"] \\
\bar{\mathbf{C}} \arrow[r, "f"'] & \bar{\mathbf{C}}
\end{tikzcd},$$

where A is an affine endomorphism of the torus $\mathbf{T}^2_\tau$, and p is the quotient map over a finite group of motions G. We can assume that G has a fixed point (otherwise the torus can be replaced by a quotient torus). Therefore, the problem splits into two problems: 1) the description of groups of motions of the torus having a fixed point, and 2) the description of algebraic endomorphisms of the torus. The solution of the first problem is elementary: (i) on any torus there acts the group $\mathbf{Z}_2$, which is generated by the involution $z \mapsto -z$; (ii) on the torus $\mathbf{T}^2_i$ there acts the group $\mathbf{Z}_4$, which is generated by the rotation $z \mapsto iz$; (iii) on the torus $\mathbf{T}^2_{e^{i\pi/3}}$ there acts the group $\mathbf{Z}_6$, which is generated by the rotation $z \mapsto e^{i\pi/3}z$. Considering in the last case the subgroup $\mathbf{Z}_3$, we obtain case (iv). Let us emphasize that in (i) we have a one-parameter family of tori, while the cases (ii), (iii), (iv) are isolated.

On any torus $\mathbf{T}^2_\tau$ there is an algebriac endomorphism $A_{n,\tau}$ generated by the transform $z \mapsto nz$ of $\mathbf{C}$ $(n \in \mathbf{N},\ n \geqslant 2)$. Lowering it to $\bar{\mathbf{C}}$ in the case (i), we obtain a one-parameter family of rational endomorphisms (the Lattes example for $n = 2$). In the cases (ii), (iii), (iv) we obtain countably many more examples (up to a conformal conjugation). For some isolated values of τ the torus $\mathbf{T}^2_\tau$ admits algebraic endomorphisms different from $A_{n,\tau}$. They are generated by the *complex multiplications* $z \mapsto \alpha z$ $(\alpha \notin \mathbf{R})$. The enumeration of complex multiplications is a deep problem, which lies beyond the scope of the present survey (see [19]).

In each of the cases (i)–(iv) it is easy to write down explicitly an example of the corresponding rational transform. Namely, the transform f: $z \mapsto \left(\frac{z-2}{z}\right)^2$ corresponds to the endomorphism A: $z \mapsto \sqrt{2}iz$ of $\mathbf{T}^2_i$ (case (ii)).

In the examples of this section the property $J(f) = \bar{\mathbf{C}}$ was connected with the fact that the orbits of critical points are absorbed by cycles. In §2.4 we give essentially more complicated examples, in which $J(f) = \bar{\mathbf{C}}$.

§1.19. On the measure of the Julia set

Problem. *Is it true that if the Julia set $J(f)$ is nowhere dense, then* $\operatorname{mes} J(f) = 0$?

This is one of the crucial open questions in the theory of iterations of rational functions⁽¹⁾. The first remarks concerning this problem were made by Fatou ([59], 260). In 1965 Brolin [45] proved that $\operatorname{mes} J(f) = 0$ under the assumptions of Theorem 1.18. We shall give more general conditions under which $\operatorname{mes} J(f) = 0$. The main technical tool for us is the following theorem.

Koebe's distortion theorem (see [10], Ch. II, §4). *Let $\varphi : \mathrm{U}_r \to \bar{\mathbf{C}}$ be a univalent function, and $q \in (0, 1)$. Then there is a constant $K = K(q)$ not depending on φ and r such that*

$$\frac{\| D\varphi(x_1)\|}{\| D\varphi(x_2)\|} \leqslant K$$

for any $x_1, x_2 \in \mathrm{U}_{qr}$.

We denote by $B_\varepsilon(x) = \{y\colon \rho(x, y) < \varepsilon\}$ the disc in the spherical metric.

Proposition 1.14 [20]. *Suppose that $J(f)$ is nowhere dense. Then $\omega(z) \subset \omega_f'$ for almost all $z \in J(f)$.*

Proof. For brevity we shall write $J = J(f)$. We consider the set $\Lambda_\varepsilon = \{z \in J\colon \varlimsup_{m\to\infty} \rho(f^m z, \omega_f) > 2\varepsilon\}$. For $z \in \Lambda_\varepsilon$ we have $\rho(f^{m_k} z, \omega_f) > 2\varepsilon$ for some subsequence $\{m_k\}$. Therefore, there is a neighbou $D_{\delta,k}1$ of z such that f^{m_k} maps $D_{\delta,k}$ univalently onto $B_\delta(f^{m_k} z)$ $(\delta \leqslant 2\varepsilon)$. By Koebe's theorem

$$\frac{\operatorname{mes}(D_{\varepsilon,k} \setminus J)}{\operatorname{mes} D_{\varepsilon,k}} \geqslant C \frac{\operatorname{mes}(B_\delta(f^{m_k} z) \setminus J)}{\delta^2}, \tag{1.7}$$

where C does not depend on k. The expression on the right in (1.7) is separated from zero by a constant not depending on k. By Koebe's theorem $D_{\varepsilon,k}$ is an oval with a bounded distortion and so $\operatorname{diam} D_{\varepsilon,k} \to 0$ $(k \to \infty)$ (Theorem 1.15). Therefore, the lower density⁽²⁾ of J at z is less than 1. By Lebesgue's theorem on density points we have $\operatorname{mes} \Lambda_\varepsilon = 0$. This proves that $\omega(z) \subset \omega_f$ for almost all $z \in J$.

⁽¹⁾This question has an unsolved analogue in the theory of Kleinian groups: the well-known Ahlfors problem [36].

⁽²⁾That is,

$$\varliminf_{r\to 0} \frac{\operatorname{mes}(J \cap B_r(z))}{\operatorname{mes} B_r(z)}.$$

Next, it follows from the description of the dynamics on $F(f)$ that $\omega_f \cap J = \omega'_f \cup \alpha_i$, where the α_i are neutral rational cycles. Consequently, if $z \in J$ and $\omega(z) \subset \omega_f \backslash \omega'_f$, then the orbit of z converges to a neutral rational cycle. But the description of the Leau flower (§1.10) shows that in this case $z \in F(f)$.

Proposition 1.14 immediately yields the following result.

Theorem 1.26 ([20], [55], Part I). *Suppose that the orbits of all critical points converge to attracting, repelling, or neutral rational cycles (can be absorbed by them). Then we have the following alternatives:* a) $J(f) = \overline{\mathbf{C}}$; b) mes $J(f) = 0$.

Corollary. *If an endomorphism f satisfies Axiom* A, *then* mes $J(f) = 0$.

(This fact is standard in the general theory of dynamical systems [6].)

We note that analogous results are valid for the linear measure of $J(f)$ if it lies on a circle or a line. In particular, in Example 1.6 the linear measure of $J(f)$ is zero.

§1.20. The Newton iterative process

Let $p(z) = z^d + a_1 z^{d-1} + \dots + a_d$ be a complex polynomial, where $d > 1$. The Newton iterative process is one of the numerical methods for finding the roots of $p(z)$. The sequence of approximations z_m constructed by this method is an orbit of the rational endomorphism f: $z \mapsto z - p(z)/p'(z)$. Suppose that the roots $\{\alpha_i\}_{i=1}^{d}$ of $p(z)$ are simple. Then $\deg f = d$ and the α_i are superattracting fixed points of f. Moreover, f has a repelling fixed point ∞. The critical points of f are the α_i and the roots of $p''(z)$.

If the initial approximation z_0 is sufficiently close to α_i, then the Newton process converges to α_i superexponentially. We consider the question of the global convergence of the Newton process. It is clear at once that not every initial approximation is satisfactory from this point of view. For example, if $z_0 \in J(f)$, then obviously there is no convergence to a root. Therefore, it is natural to be interested in convergence almost everywhere. First we dwell on the elementary case of a quadratic polynomial.

Proposition 1.15. *Suppose that $p(z)$ has two simple roots α_1, α_2. Let L be the straight line perpendicular to the interval $[\alpha_1, \alpha_2]$ and passing through its midpoint, and P_1, P_2 the open half-planes into which L divides the plane, $\alpha_i \in P_i$. Then if $z_0 \in P_i$, the Newton process $\{z_m\}$ converges to α_i.*

Proof. The endomorphism f is conformally conjugate to $z \mapsto z^2$ with the help of the transform φ: $z \mapsto (z - \alpha_1)/(z - \alpha_2)$. Moreover, $L = \varphi^{-1}\mathbf{T}$.

Suppose now that $d \geqslant 3$ and $p(z)$ has real coefficients. The connected component $I(\alpha)$ of $\{x \in \mathbf{R}: f^n x \to \alpha\}$ containing α will be called the interval of immediate attraction of α.

***Lemma* 1.13.** *Let $\alpha \in \mathbf{R}$ be a simple root of a real polynomial $p(z)$. If the interval $I(\alpha)$ is bounded, then it contains a root of $p''(z)$.*

Proof. Let $I(\alpha) = (\beta, \gamma)$. Then $f\beta = \beta$ or $f\beta = \gamma$. But the first possibility is excluded, since finite fixed points of f are superattracting(1). Therefore, $f\beta = \gamma$ and similarly $f\gamma = \beta$. If $I(\alpha)$ contains no roots of $p''(z)$, then $f(x)$ monotonically decreases on (β, α) and increases on (α, γ) (or vice versa). Consequently, $f\gamma > f\alpha = \alpha > \beta$ (or $f\beta < \gamma$). We obtain a contradiction.

***Theorem* 1.27** [24]. *Suppose that the roots of $p(z)$ are simple and real. Then* a) *for almost all $z_0 \in \mathbf{C}$ the Newton process $\{z_m\}$ converges to one of the roots*; b) *for almost all $z_0 \in \mathbf{R}$ (with respect to the linear measure) the Newton process converges to one of the roots.*

Proof. a) Let $\alpha_1 < \alpha_2 < \ldots < \alpha_d$. Then the intervals $I(\alpha_k)$ $(2 \leqslant k \leqslant d-1)$ are bounded and by Lemma 1.13 contain at least one root of $p''(z)$. Since there are $d-2$ such intervals, all the roots of $p''(z)$ must be used. Therefore, f satisfies Axiom A. It remains to apply Theorems 1.17 and 1.26. The assertion b) can be proved in a similar way with the help of the theory of one dimensional (real) dynamical systems.

Example 1.11. We consider the Newton process $f\colon z \mapsto \frac{1}{d}\left[(d-1)z + \frac{a}{z^{d-1}}\right]$ for finding the roots of the equation(2) $z^d = a$. The only critical point of f different from the roots $\sqrt[d]{a}$ is the point $c = 0$. We have $f\colon c \mapsto \infty \circlearrowleft$. By Theorem 1.26 almost all orbits converge to roots. Recently many beautiful computer pictures of the Julia set of this process have been made [51].

Example 1.12. $p(z) = z^3 - z + b$. If $b = 1/\sqrt{2}$, then $\{0, 1/\sqrt{2}\}$ is a superattracting cycle of f. Consequently, an open set of orbits of the Newton process "cycles" (converges to this cycle). Perturbing b, we obtain a similar picture. The parameter b can be chosen so that f has an attracting cycle of arbitrarily large period and the process is perceived as chaotic(3).

CHAPTER II

HOLOMORPHIC FAMILIES OF RATIONAL ENDOMORPHISMS

The main subject of the present chapter is the description of quasiconformal deformations of rational endomorphisms with the help of suitable

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(1) The multiple roots of $p(z)$ are simply attracting.

(2) It is curious that Fatou considered this example for $d = 3$ ([60], 89) (without mentioning the convergence almost everywhere of orbits).

(3) A recent survey of Smale ([88], Ch. 2, § 1) contains a very similar exposition of these questions with additional references. In that survey a modified version of the Newton process is proposed, which leads to success with probability 1/6.

Teichmüller spaces (§2.5). Such an approach, traditional in the theory of Kleinian groups, has quite recently entered the theory of iterates of rational functions. It has not only made it possible to obtain strong concrete results, but has also shed light on the field in general. In §§2.1–2.2 we present a theorem on the structural stability of a rational function of general position. In §§2.3–2.4 we investigate bifurcations in the behaviour of the orbits of critical points. Although the first four sections are closely related to §2.5, they use more traditional techniques, in the first place, the theory of normal families (true, the functions depend on several variables or even on a point of an analytic set). §2.6 contains the most advanced progress in the Fatou problem mentioned in the introduction (a rational function of general position satisfies Smale's Axiom A). In the final section §2.7 we consider the quadratic family $f_w\colon z \mapsto z^2 + w$. The exposition is carried out from the point of view of the results of §2.5, which give a unified approach to two theorems of Douady and Hubbard which at first sight have little in common.

§2.1. The λ-lemma and *J*-stability

Let X be an analytic set (see [9]), and $\varphi_\alpha\colon X \to \overline{\mathbf{C}}$ a family of analytic maps. As in the case when X is a domain in $\overline{\mathbf{C}}$, the family $\{\varphi_\alpha\}$ is called *normal* if it is precompact in the compact-open topology. Moreover, the main test for normality, namely Montel's theorem on three exceptional values or functions, remains valid (see [69]).

The rational endomorphisms of degree d form a $2d+1$-dimensional complex analytic manifold $\mathfrak{R}_d$. Let $\mathfrak{M} \subset \mathfrak{R}_d$ be an arbitrary complex submanifold. One of the most important special cases is $\mathfrak{M} = \mathcal{P}_d$, the space of polynomials of degree d.

A rational endomorphism $f_0 \in \mathfrak{M}$ is called *J-stable* (in the family $\mathfrak{M}$) if for all $f \in \mathfrak{M}$ sufficiently close to f_0 the transforms $f_0\colon J(f_0)\circlearrowleft$ and $f\colon J(f)\circlearrowleft$ are topologically conjugate, and the conjugating homeomorphism $h_f\colon J(f_0) \to J(f)$ depends continuously on f (the space of maps $J(f_0) \to \overline{\mathbf{C}}$ is endowed with the uniform topology).

Theorem 2.1. *The set of J-stable endomorphisms is open and dense in* $\mathfrak{M}$.

Thus, *in any holomorphic family* $\mathfrak{M}$ *a rational endomorphism in general position is J-stable.* The main tool in the proof of Theorem 2.1 is the possibility of extending the conjugating homeomorphism from a dense set to its closure.

Lemma 2.1 (the λ-lemma). *Let* $\varphi_z\colon \mathbf{C}^k \to \overline{\mathbf{C}}$ *be a family of analytic functions in a domain* $W \subset \mathbf{C}^k$ *that depend parametrically on a point* z *of a set* $X \subset \overline{\mathbf{C}}$; $\varphi_z(w_0) \equiv z$. *Suppose that for* $z_1 \neq z_2$ $(z_i \in X)$ *we have* $\varphi_{z_1}(w) \not\equiv \varphi_{z_2}(w)$ *for all* w. *Then there is a family of quasi-conformal homeomorphisms* h_w $(w \in W)$ *of* X *onto the image such that* $h_w(z) = \varphi_z(w)$ *for* $z \in X$. *Moreover,* $h_w(z)$ *is analytic in* w *for every* $z \in \overline{X}$.

The quasi-conformality of a map defined not in a domain is understood in the sense of Pesin (see [5], Ch. I, §4).

Proof. The family of functions $\mathfrak{A} = \{\varphi_z\}_{z\in X}$ is normal. For if we delete from it three arbitrary functions φ_i, then the remaining family satisfies the assumptions of the generalized Montel theorem with exceptional functions φ_i. Therefore, the family $\overline{\mathfrak{A}}$ is compact. Furthermore, if φ and ψ are different functions in $\overline{\mathfrak{A}}$, then $\varphi(w) \neq \psi(w)$ for any $w \in W$. For suppose the contrary. There are sequences $\varphi_i \to \varphi$ and $\psi_i \to \psi$, where $\varphi_i, \psi_i \in \mathfrak{A}$. By the Hurwitz theorem we have $\varphi_i(w_i) = \psi_i(w_i)$ for some $w_i \in W$ and sufficiently large i. By hypothesis, $\varphi_i = \psi_i$ and so $\varphi = \psi$. We obtain a contradiction.

We consider the map π_w: $\overline{\mathfrak{A}} \to \overline{\mathbf{C}}$, $\varphi \mapsto \varphi(w)$. By what we have proved, π_w is one-to-one. Since $\overline{\mathfrak{A}}$ is compact, π_w is a homeomorphism onto its image. We now put $h_w = \pi_w \circ \pi_{w_0}^{-1}$. All the properties of h_w are obvious, except for quasi-conformality. We denote by $[z_1 z_2 z_3 z_4] = \frac{z_3 - z_1}{z_4 - z_1} \cdot \frac{z_4 - z_2}{z_3 - z_2}$ the cross-ratio of a quadruple of points. It is sufficient to show that if $C^{-1} \leqslant |\,[z_1 z_2 z_3 z_4]\,| \leqslant C$, then

$$K^{-1} \leqslant |\,[h_w(z_1),\ h_w(z_2),\ h_w(z_3),\ h_w(z_4)]\,| \leqslant K = K(w,\ C).$$

But the last cross-ratio is a holomorphic function of w that does not take the values 0, 1, ∞. Now the required assertion follows from Schottki's theorem (see [10], Ch. 8, §2), taking account of the fact that $h_{w_0}(z_i) = z_i$. The λ-lemma is now proved.

We now consider a many-valued analytic function α_p: $\mathfrak{M} \to \overline{\mathbf{C}}$ satisfying the algebraic equation $f^p z = z$. As we mentioned in §1.10, this function has only algebraic singularities. We denote by N_p the set of its singularities. It is a proper analytic subset of $\mathfrak{M}$. We put $N = \overline{\bigcup_{1 \leqslant p < \infty} N_p}$, $\Sigma = \mathfrak{M} \setminus N$. We denote by $\lambda_p(f)$ the analytic function on $\mathfrak{M}$ defined by $\lambda_p(f) = \frac{df^p}{du}(\alpha_p(f))$, where u is a local parameter on $\overline{\mathbf{C}}$ in a neighbourhood of $\alpha_p(f)$ ($\lambda_p(f)$ is the multiplier of the periodic point $\alpha_p(f)$ or some power of it). It follows from the implicit function theorem that if $f \in N_p$, then $\lambda_p(f) = 1$ for some branch of λ_p.

Proof of Theorem 2.1. Let us show that if $f_0 \in \Sigma$, then f is J-stable. We consider a simply-connected neighbourhood $U \subset \Sigma$ of f_0. Then all branches $\alpha_{p,i}$ of α_p are single-valued in U. Furthermore, if $\alpha_{p,i}(f) = \alpha_{q,j}(f)$, then $\alpha_{p,i} \equiv \alpha_{q,j}$. For otherwise f is a singular point of α_{pq}. Therefore, $\{\alpha_{p,i}\}_{p,i}$ satisfies the assumptions of the λ-lemma. The homeomorphism h_f constructed in the λ-lemma conjugates f_0: $\overline{\mathrm{Per}\, f_0}\circlearrowleft$ and f: $\overline{\mathrm{Per}\, f}\circlearrowleft$, where Per f is the set of periodic points of f. Since $\overline{\mathrm{Per}\, f}$ is the union of $J(f)$ and a finite number of isolated points, it follows that h_f transforms $J(f_0)$ into $J(f)$.

Let us show that Σ is dense in $\mathfrak{M}$. We consider those values of p for which some branch of λ_p is identically equal to 1. There are at most $2d-2$ such values of p (corollary of Theorem 1.8). Let p_0 be the largest of them. Since the set $\bigcup\limits_{1\leqslant p\leqslant p_0} N_p$ is nowhere dense in $\mathfrak{M}$, we can assume that all the functions considered below lie outside it.

We denote by $s(f)$ the number of attracting cycles of f. Let $f_0 \in N$, $\varepsilon > 0$. Then there is an $\tilde{f} \in N_p$ such that dist $(f_0, \tilde{f}) < \varepsilon$. We have $\lambda_{p,i}(\tilde{f}) = 1$ for a suitable branch of λ_p, and $\lambda_{p,i} \not\equiv 1$ by the agreement adopted above. Therefore, there is a point f_1 such that $|\lambda(f_1)| < 1$, dist $(\tilde{f}, f_1) < \varepsilon$. Since attracting cycles are stable under perturbations, it follows that $s(f_1) > s(f_0)$ for sufficiently small ε. If $f_1 \in N$, the process can be repeated and the number of attracting cycles increases. By the corollary of Theorem 1.4 the process breaks off no later than at the $(2d-2)$-th step. As a result we obtain a function $f \in \Sigma$ close to f_0. The theorem is now proved.

Remark. It is easy to show conversely that if f_0 is J-stable, then $f_0 \in \Sigma$.

The results of this section were obtained independently by the author [21], [23] (except for the quasi-conformality of the conjugating homeomorphism) and Mañé, Sad, and Sullivan [77].

§2.2. Structural stability is a generic property

A rational endomorphism $f_0 \in \mathfrak{M}$ is called *structurally stable* (in $\mathfrak{M}$) if for every $f \in \mathfrak{M}$, close enough to f_0, the transforms $f_0\colon \bar{\mathbf{C}}\circlearrowleft$ and $f\colon \bar{\mathbf{C}}\circlearrowleft$ are topologically conjugate, and the conjugating homeomorphism depends continuously on f.

Theorem 2.2. *The set of structurally stable endomorphisms is open and dense in* $\mathfrak{M}$. *The conjugating homeomorphisms can be chosen to be quasi-conformal.*

Proof. We consider a connected component W of the set Σ of J-stable endomorphisms; $f_0 \in W$. Let $h_f : J(f_0) \to J(f)$ be a homeomorphism that J-conjugates f_0 and $f \in W$. The problem is to extend h_f to the components of the normality set $F(f)$ (under the condition that f_0 and f are in general position in W). This extension has been carried out by Mañé, Sad, and Sullivan ([77], [90], Part III)⁽¹⁾. Moreover, $h_f(z)$ depends analytically on f, and the λ-lemma automatically ensures that these extensions are joined together continuously and that h_f is quasi-conformal.

We restrict ourselves to a description of the construction in the case when all periodic components of $N(f_0)$ are Schröder domains.

⁽¹⁾For an extension to Siegel discs and Arnol'd-Herman rings a stronger version of the λ-lemma was needed, which was proved by Sullivan and Thurston [92] (see also [42]).

To simplify the notation we shall assume that $f_0 \in W$ has a unique Schröder domain connected with an attracting fixed point $\alpha(f_0)$. A function f sufficiently close to f_0 has an attracting fixed point $\alpha(f)$. The critical points $c_i(f)$ can be numbered so that they depend continuously on f (each point being counted with its multiplicity). We suppose that the first r critical points of f_0 lie in the attracting domain of $\alpha(f_0)$, while the others lie in the Julia set $J(f_0)$. Then it follows from J-stability that the same properties hold for any close function f. We also suppose that for some $m, l \geqslant 0, i, j \in [1, r]$

$$(2.1) \qquad f^m c_i(f) = f^l c_j(f),$$

where this relation breaks down under a perturbation of f. We denote the set of such f by Λ. Suppose that the preceding considerations are valid in a neighbourhood W_0 of f_0. Let us show that $\Lambda \cap W_0$ is closed and nowhere dense in W_0. We denote by Z the set of functions $f \in W_0$ for which the multiplier $\lambda(f)$ of the fixed point $\alpha(f)$ vanishes. Z is a proper analytic subset of W_0. Therefore, it is sufficient to show that Λ is closed and nowhere dense in a neighbourhood W_1 of $f_1 \in W_0 \backslash Z$.

Let $\overline{W}_1 \subset W_0 \backslash Z$. Then there is an $\varepsilon > 0$ such that any transform $f \in W_1$ univalently maps $\{z\colon |z - \alpha(f)| < \varepsilon\}$ into itself. On the other hand, there is a k such that $|f^m c_i(f) - \alpha(f)| < \varepsilon$ for $m \geqslant k$ ($f \in W_1$, $i = 1, \ldots, r$). Consequently, if $f \in W_1 \cap \Lambda$, then f satisfies some equality (2.1) with $l = k$.

We now consider the set X of $f \in W$ such that $f^k(c_i(f)) = \alpha(f)$ for some i, where this equality breaks down under a small perturbation of f. For the same reasons as above, it is sufficient to show that Λ is closed and nowhere dense in a neighbourhood W_2 such that $\overline{W}_2 \subset W_1 \backslash X$. But

$$\inf \{|f^k(c_i(f)) - \alpha(f)|\colon f \in W_2,\ 1 \leqslant i \leqslant r\} > 0,$$

and $f^m(c_i(f)) \to \alpha(f)$ ($m \to \infty$) uniformly in W_2. Therefore, the equations (2.1) for $l = k$ and large m have no solutions in W_2. It follows that $\Lambda \cap W_2$ is a proper analytic subset of W_2.

We now show that an endomorphism $f \in W_0 \backslash \Lambda$ is structurally stable. If $f \in W_0 \backslash \Lambda$, then the multiplier $\lambda(f)$ of the fixed point $\alpha(f)$ is non-zero. We denote by $\varphi_f\colon z \mapsto z + \beta(f)z^2 + \ldots$ the normalized Koenigs function for f. It is univalent in a neighbourhood V_f of $\alpha(f)$ and satisfies there the Schröder equation $\varphi_f(fz) = \lambda(f)\varphi_f(z)$. Each construction of the Koenigs function shows that $\varphi_f(z)$ is analytic in all variables. We diminish the neighbourhood $\bigcup_{f \in W_0} V_f$ (without changing the notation) so that $\varphi_f(V_f) = U_\varepsilon = \{z\colon |z| < \varepsilon\}$ and the orbits $\{f^m c_i(f)\}_{m=0}^{\infty}$ are disjoint with ∂V_f. Let $a_i(f)$ be the first point of $\{f^m c_i(f)\}_{m=0}^{\infty}$ that falls into V_f ($i = 1, \ldots, r$). Then $a_i(f)$ depends analytically on f and we have the following alternatives: $a_i(f) \equiv a_j(f)$ or $a_i(f) \neq a_j(f)$ for any $f \in W_0 \backslash \Lambda$. We now put $b_i(f) = \varphi_f(a_i(f))$ ($i = 1, \ldots, r$; $f \in W_0 \backslash \Lambda$).

It is easy to construct (in some neighbourhood $\Omega \subset W_0$) a family of quasi-conformal homeomorphisms $g_f\colon \mathbf{U}_\varepsilon \circlearrowright$ such that (i) $g_f(\lambda(f_0)z) = \lambda(f)g_f(z)$; (ii) $g_f : b_i(f_0) \to b_i(f)$; (iii) g_f depends analytically on f for any z; (iv) $g_{f_0}(z) \equiv z$. We now put $h_f = \varphi_f^{-1} \circ g_f \circ \varphi_{f_0}$. Then $h_f\colon V_{f_0} \to V_f$ is a quasi-conformal homeomorphism conjugating f_0 and f locally and depending analytically on f.

We extend h_f to the whole attracting domain $\Delta(f_0)$ of the fixed point $\alpha(f_0)$. Let $z \in f_0^{-1}V_{f_0} \setminus V_{f_0}$ and $f_0 z \neq a_i(f_0)$. It follows from the implicit function theorem that there is an analytic function $\varphi_z : \Omega \to \overline{\mathbf{C}}$ satisfying $f(\varphi_z(f)) = h_f(f_0 z)$, $\varphi_z(f_0) = z$. The family of these functions extends h_f to $f_0^{-1}V_{f_0}$ with some punctures. Iterating, we extend h_f to the attracting domain $\Delta(f_0)$ with punctured inverse images of the $a_i(f_0)$ of all orders. An application of the λ-lemma completes the proof of Theorem 2.2.

A remarkable example of J-stable endomorphisms (in the whole manifold $\mathfrak{R}_d$) is given by the endomorphisms satisfying Axiom A. Their structural stability follows both from general arguments of the theory of smooth dynamical systems (see [26], [32]) and from our considerations (see, for example, Theorem 2.4 below).

Conjecture 2.1 (Fatou ([60], 73)). *In the family $\mathfrak{R}_d$ of all rational endomorphisms of degree d, J-stability is equivalent to Axiom* A.

§2.3. The behaviour of orbits of critical points

1. As we have repeatedly seen, the global dynamics of a rational endomorphism depends essentially on the behaviour of the orbits of critical points. Therefore, it is useful, leaving aside the other orbits for the time being, to investigate the dependence of these orbits on parameters. This approach has been used independently by Levin [17] (in the context of one-parameter families of polynomials) and by the author [21]-[23].

Suppose first that some critical point $c(f)$ is a single-valued analytic function of f. Then the orbit $\{f^m c(f)\}_{m=0}^{\infty}$ of this point defines a sequence of holomorphic functions of f, which needs to be studied. However, critical points in general are many-valued functions having singularities. Therefore, for a full description of the situation it is necessary to pass to the analytic set $\mathrm{Cr} = \{(f, c) \in \mathfrak{M} \times \overline{\mathbf{C}}\colon\ Df(c) = 0\}$. We define on it a sequence of analytic functions $F_m : \mathrm{Cr} \to \overline{\mathbf{C}}$ as follows: $F_m(f, c) = f^m c$ $(m = 0, 1, 2, \ldots)$. This sequence contains complete information on the dependence of the orbits of critical points on f. Montel's theorem for families of functions on an analytic set (see [69]) is a tool for investigating it.

We denote by Reg the set of points $(f, c) \in \mathrm{Cr}$ in whose neighbourhoods $\{F_m\}_{m=0}^{\infty}$ is a normal family, and by Irr the complementary set $\mathrm{Cr} \setminus \mathrm{Reg}$. If $(f_0, c_0) \in \mathrm{Reg}$ and $(f, c) \in \mathrm{Cr}$ is close to (f_0, c_0), then the orbits $\{f_0^m c_0\}_{m=0}^{\infty}$ and $\{f^m c\}_{m=0}^{\infty}$ are uniformly close.

Proposition 2.1 [17]. a) *The set* Reg *is open and dense in* Cr; b) *the set* Irr *is either perfect or empty.*

In fact, Irr locally has a very complicated structure: if U is a neighbourhood of a point $x \in \text{Irr}$, then Irr separates U into countably many components. It follows, of course, that x is not isolated. Moreover, if x is a smoothness point of Cr, then $U \cap \text{Irr}$ is not contained in any smooth submanifold. Below we describe in more detail the structure of Irr for the quadratic family f_w: $z \mapsto z^2 + w$ (§2.7).

We denote by Fin the set of $(f, c) \in \text{Cr}$ such that the orbit of a critical point c is absorbed by some repelling cycle, but this property is unstable under a perturbation of (f, c). The next assertion follows easily from Montel's theorem.

Proposition 2.2. Fin *is a dense subset of* Irr.

2. A subset of a locally compact metric space X is called *massive* if it contains a dense subset of type G_δ (the intersection of a countable number of open sets). Next, a property depending on a point $x \in X$ will be called *typical* if it is satisfied for a massive subset of X. Our proofs of typicalness are based on the following remark: *the set of zeros of a non-negative upper semi-continuous function* $\rho : X \to \mathbf{R}$ *is of type* G_δ.

Theorem 2.3 ([21]-[23]). *For a typical point* $(f, c) \in \text{Irr}$ *the orbit* $\{f^m c\}_{m=0}^{\infty}$ *is contained in the Julia set* $J(f)$ *and is dense there.*

Proof. We denote by $\rho_p(f, c)$ the spherical distance from the orbit $\{f^m c\}_{m=0}^{\infty}$ to the farthest repelling periodic point of f of order p. Let us show that $\rho_p(x) = 0$ on a massive subset of Irr. The set Neu_p of points $x = (f, c) \in \text{Irr}$ such that f has a neutral cycle of order p that can be made attracting under a perturbation of f is nowhere dense in Irr (in view of the complicated local structure of Irr described above). Therefore, it is sufficient to check that in some neighbourhood of any point $x_0 = (f_0, c_0) \in \text{Irr} \setminus \text{Neu}_p$ the property $\rho_p(x) = 0$ is typical. Let $\alpha_{p,1}(f_0), \ldots, \alpha_{p,l}(f_0)$ be repelling periodic points of f_0 of order p, and $z = \alpha_{p,i}(f)$ a parametrization of the curve $f^p z = z$ in a neighbourhood U of $(x_0, \alpha_{p,i}(f_0)) \in \mathfrak{M} \times \bar{\mathbf{C}}$ $(i = 1, \ldots, l)$. If $U \cap \text{Neu}_p = \emptyset$, then $\alpha_{p,i}(f)$ is the complete set of repelling periodic points of f of order p. The function $\rho_{p,i}^{(m)}(f, c) = \text{dist}\ (f^m c, \alpha_{p,i}(f))$ is continuous in U. Therefore, $\rho_{p,i} = \inf\limits_{1 \leqslant m < \infty} \rho_{p,i}^{(m)}$ is upper semi-continuous, and so the set $L_{p,i}$ of its zeros is of type G_δ. On the other hand, $L_{p,i}$ contains points $(f, c) \in U$ such that $f^m c = \alpha_{p,i}(f)$ for some m. But the set of such points (f, c) is dense in $\text{Irr} \cap U$ (a minor refinement of Proposition 2.2). Consequently, $L_{p,i}$ is massive in $\text{Irr} \cap U$. Thus, the property $\rho_p(x) = 0$ is satisfied on the massive set $\bigcap\limits_{i=1}^{l} L_{p,i}$. Therefore, we have verified that the property $\rho_p(x) = 0$ is typical in Irr. But then the set of zeros of the function $\rho = \sup\limits_{1 \leqslant p < \infty} \rho_p$ is also massive in Irr. It remains to observe that the equality $\rho(x) = 0$ means that all repelling cycles of f are contained in $\overline{\{f^m c\}_{m=0}^{\infty}}$, that is, $J(f) \subset \overline{\{f^m c\}_{m=0}^{\infty}}$.

To prove that the opposite inclusion is typical we consider the function $r_p(f, c)$, the distance from c to the closest repelling periodic point of f of order p. r_p is upper semi-continuous, since repelling cycles are stable under perturbations. Therefore, the function $\operatorname{dist}(c, J(f)) = \inf\limits_{1 \leqslant p < \infty} r_p(f, c)$ is also upper semi-continuous, and so it vanishes on a set L of type G_δ On the other hand, Fin $\subset L$. By Proposition 2.2 L is dense in Irr. The theorem is now proved.

3. In conclusion we mention a result that relates J-stability with the stability (with respect to a parameter) of orbits of critical points.

Theorem 2.4 ([21]-[23]). *For a function $f \in \mathfrak{M}$ to be J-stable it is necessary and sufficient that $(f, c_i) \in$ Reg for all critical points c_i of f.*

§2.4. The family f_w: $z \mapsto 1 + wz^{-2}$

1. Our aim is to show that there are many functions in the family f_w for which $J(f_w) = \overline{\mathbf{C}}$, the orbits of critical points of these functions having a behaviour essentially different from that in the examples of §1.18. All transforms f_w $(w \in \mathbf{C}^*)$ have two critical points 0 and ∞. However, these points lie in one large orbit, since f_w: $0 \mapsto \infty$ for all $w \in \mathbf{C}^*$. Therefore, it is sufficient for us to follow the orbit of one of the critical points. The choice of the family f_w is connected precisely with this circumstance.

The analytic set Cr defined in the preceding section is parametrized by two punctured planes $\mathrm{Cr}_1 = \{(f_w, 0): w \in \mathbf{C}^*\}$ and $\mathrm{Cr}_2 = \{(f_w, \infty): w \in \mathbf{C}^*\}$. Since f_w: $0 \mapsto \infty$, the properties $(f_w, 0) \in$ Irr and $(f_w, \infty) \in$ Irr are equivalent. By Theorem 2.4 these properties hold if and only if f_w is J-unstable. The corresponding values of w will be called unstable, and we preserve the notation Λ for the set of such w.

The set Λ is non-empty. For example, for $w_0 = -4/27$ the function f_{w_0} has fixed point $\alpha_0 = 2/3$ with multiplier 1. For $w = -4/27 + \varepsilon$ $(\varepsilon > 0)$ the point α_0 gives rise to a pair of fixed points, one repelling and one attracting. Therefore, f_{w_0} is J-unstable $(f_{w_0} \in N_1$ in the notation of §2.1). Thus, Λ is a non-empty nowhere dense perfect subset of $\mathbf{C}^*$. We shall not describe its structure in detail. This can be done as in the case of the family $z \mapsto z^2 + w$, which we discuss in detail in §2.7.

Proposition 2.3. *For a typical unstable value of a parameter $w \in \Lambda$ we have:* a) *the Julia set $J(f_w)$ coincides with the whole sphere;* b) *the orbit $\{f_w^m c\}_{m=0}^{\infty}$ of the critical point $c = \infty$ is dense on the sphere.*

Proof. We consider the set $X = \{w \in \Lambda: J(f_w) = \overline{\mathbf{C}}\}$. Let us show that X has type G_δ (see Proposition 1.8 and Theorem 2.3). For let $\{a_i\}_{i=1}^{\infty}$ be a countable dense set on $\overline{\mathbf{C}}$, and $r_{p,i}(w)$ the distance from a_i to the closest repelling periodic point of order p. Since $r_{p,i}$ is upper semi-continuous, the same is true for $r_i(w) \equiv \operatorname{dist}(a_i, J(f_w)) = \inf\limits_{p,i} r_{p,i}(w)$. Therefore, r_i vanishes on a set of type G_δ. But X coincides with the set of common zeros of the r_i. Let us now show that X is dense in Λ. For by Theorem 1.24 if the orbit $\{f_w^m \infty\}_{m=0}^{\infty}$ is absorbed by a repelling cycle, then $J(f_w) = \overline{\mathbf{C}}$. In other words, Fin $\subset X$. By Proposition 2.2 Fin is dense in Λ and a) is proved. Now b) follows directly from Theorem 2.3.

Sullivan's theorem shows that for $w \in \Lambda$ one of the following possibilities holds: (i) f_w has a neutral cycle; (ii) f_w has a cycle of Arnol'd-Herman rings; (iii) $J(f_w) = \overline{\mathbf{C}}$(1). The possibility (i) is realized on a countable union of piecewise analytic curves, and so on a subset of Λ of the first category. From this point of view Proposition 2.3 shows that the possibility (ii) can also be realized only on a subset of the first category. It is interesting to investigate the structure of the set of values of the parameter $w \in \mathbf{C}$ for which the possibility (ii) is realized (and whether there are such values).

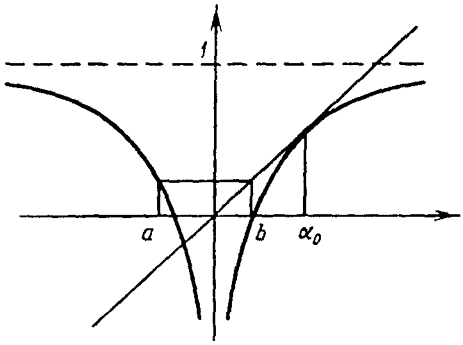

Fig. 10. The transform $z \mapsto 1 - \dfrac{4}{27z^2}$

2. We now illustrate on our example $f_w\colon z \mapsto 1 + wz^{-2}$ how complicated the bifurcations of the oribt of a critical point may be in a neighbourhood of an unstable value of the parameter. We take for such a value our known value $w_0 = -4/27$ for which $f_0 \equiv f_{w_0}$ has neutral point $\alpha_0 = 2/3$. There are two points $a < 0 < b$ such that $f_0[0, b] = [-\infty, 0]$, $f_0[a, 0] = [-\infty, b]$ (see Fig. 10). For $x \in [a, 0) \cup (0, b]$ we have $|f'(x)| \geqslant \gamma > 1$. We consider the invariant set

$$Q_0 = \{x\colon f^m x \in [a, b] \quad (m = 0, 1, 2, \ldots)\}.$$

As in Example 1.2 it can be shown that Q_0 is a Cantor set. With each point $x \in Q_0$ we can associate a sequence $Hx = (\varepsilon_0, \varepsilon_1, \ldots)$ of ± 1 as follows: $\varepsilon_m = \operatorname{sign}(f^m x)$. The map H topologically conjugates $f_0\colon Q_0 \circlearrowleft$ and the one-

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(1)The possibilities (i) and (iii) are not alternatives: if f_w has a neutral non-Siegel cycle, then $J(f_w) = \overline{\mathbf{C}}$. The possibilities (i) and (ii) are apparently alternatives, but a proof of this is not known.

sided topological Markov chain (TMC) σ_A: $\Sigma_A^+ \circlearrowleft$ with transition matrix[1] $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. It follows that the periodic points of the transform f_0: $Q_0 \circlearrowleft$ are dense in Q_0.

The Cantor set Q_0 is an invariant repelling set for f_0. We use the general theory of structural stability of hyperbolic sets of smooth dynamical systems. For invertible systems it is presented, for example, in the appendix to the book [26], while our irreversible situation can be investigated in a similar way with some simplifications. It follows from this theory that if w is close to w_0, then f_w has an invariant repelling set Q_w such that f_0: $Q_0 \circlearrowleft$ and f_w: $Q_w \circlearrowleft$ are topologically conjugate. In addition, the conjugating homeomorphism $h_w : Q_0 \to Q_w$ depends continuously on w.

We shall say that *the f-orbit of a point z copies the g-orbit of* ζ if the natural map of the orbits $g^m \zeta \mapsto f^m z$ $(m = 0, 1, 2, ...)$ extends to a homeomorphism of their closures. If for some $l \geqslant 0$ the orbit of $f^l z$ copies the orbit of ζ, we shall say that *the orbit of z copies asymptotically the orbit of* ζ.

Proposition 2.4. *For any point* $y \in \Sigma_A^+$ *arbitrarily close to* $w_0 = -4/27$ *there is a value of the parameter w such that the orbit* $\{f_w^m c\}_{m=0}^{\infty}$ *of the critical point* $c = \infty$ *copies asymptotically the orbit* $\{\sigma_A^m y\}_{m=0}^{\infty}$ *of the TMC with matrix* $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. We consider a point $z = H^{-1}y \in Q_0$. If z is a periodic point of f_0, then $h_w(z)$ depends analytically on w. Since the periodic points are dense in Q_0, $h_w(z)$ depends analytically on w for any $z \in Q_0$. Now it follows from Montel's theorem that there is a w close to w_0 and a natural number l for which $f_w^l c = h_w(z)$. This gives the required assertion.

Proposition 2.4 gives a huge variety in the asymptotic behaviour of the orbit of a critical point, which is a most important topological invariant. In particular, in any neighbourhood of w_0 we obtain a continual supply of different topological types of endomorphisms f_w. Of course, in our considerations the specific features of the family f_w: $z \mapsto 1 + wz^{-2}$ are illusory. But for this family we can claim additionally that all the functions constructed in Proposition 2.4 have the property $J(f_w) = \overline{\mathbf{C}}$ (this follows, for example, from the complete description of the dynamics on the Fatou set $F(f)$).

The results of the present section have been obtained by the author [21]-[23]. Another continual set of topologically non-conjugate endomorphisms for which $J(f) = \overline{\mathbf{C}}$ has been constructed by Herman [67].

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[1] We can become acquainted with the definition and properties of TMC in the survey of Bowen [6].

§2.5. Classes of quasi-conformal conjugacy and Teichmüller spaces

1. We denote by qc(f) the set of rational endomorphisms quasi-conformally conjugate to f modulo the action of the group $\mathrm{PSL}_2(\mathbf{C})$. In this section we describe a parametrization of qc(f) due to Sullivan with the help of a suitable Teichmüller space.

The reader can become acquainted with the theory of Teichmüller spaces in [1], [2]. For convenience we give the basic definition in a form slightly different from that in those books. Let S be a Riemann surface of finite topological type (that is, the fundamental group $\pi_1(S)$ is finitely generated). With the help of the uniformization theorem S can be embedded in a compact Riemann surface with boundary $\overline{S} = S \cup \partial S$, where ∂S is called *the ideal boundary* of S (see [1], Ch. II). Any quasi-conformal homeomorphism $S_1 \to S_2$ extends to a homeomorphism $\overline{S}_1 \to \overline{S}_2$.

Suppose that some isolated points are marked on S. We denote by Z_S the space of conformal structures on S. Two structures $\mu, \nu \in Z_S$ are called *isotopic* if there is a continuous family of quasi-conformal homeomorphisms $\varphi_t\colon \overline{S}\circlearrowleft$ identical on ∂S and at the marked points such that $\varphi_0 = \mathrm{id}$, $(\varphi_1)_*\mu = \nu$. *The Teichmüller space T_S of a surface S with marked points* is the space of classes of isotopic conformal structures. We shall denote the element of T_S corresponding to a conformal structure μ by $\overline{\mu}$. The space T_S can be endowed with the structure of a complete metric space (the corresponding metric is named after Teichmüller) and with the structure of a complex analytic manifold (infinite-dimensional in general).

Suppose that a group Γ of conformal transforms (not preserving the marked points) acts on S. Then *the equivariant Teichmüller space* can be defined naturally (it must be required in the above definition that the conformal structures are invariant and the isotopies are equivariant, that is, the homeomorphisms φ_t commute with Γ). In this situation we preserve the notation Z_S and T_S, bearing in mind that the Riemann surface S remembers the action of Γ (and equally the marked points). The appearence of equivariant Teichmüller spaces is a curious nuance in the theory of deformations of rational functions, which apparently has no prototypes.

2. In §§1.8–1.12 with each cycle of components of the Fatou set we associate a Riemann surface with marked points and, perhaps, with the action of the rotation group $\mathbf{T}$. Let $S(f)$ be the union of these surfaces, and $T_{S(f)}$ the corresponding equivariant Teichmüller space.

We dwell in more detail on the spaces T_S connected with cycles of the components of the Fatou set.

(i) The torus T^2 with $k \geqslant 1$ marked points.
We realize it as the quotient of $\mathbf{C}$ over the group $\Delta = \{me_1 + ne_2 : m, n \in \mathbf{Z}\}$, where $\mathrm{Im}\,(e_2/e_1) > 0$. Suppose that the marked points correspond to $d_0, \ldots, d_{k-1} \in \mathbf{C}$. The conformal structure μ on the torus is realized as a Δ-invariant conformal structure μ' on the plane. By the measurable Riemann theorem there is a quasi-conformal homeomorphism $h\colon \mathbf{C}\circlearrowleft$ normalized by the conditions $h(e_1) = 1$,

$h(d_0) = 0$ for which $h_*\mu' = \sigma$. We put $\tau = h(e_2)$, $b_i = h(d_i)$ $(i = 1, ..., k-1)$. We consider the map $\eta\colon T_S \to \mathcal{P}_S \subset \mathbf{C}^k$, which associates with a class $\tilde{\mu}$ of isotopic structures the point $(\tau, b_1, ..., b_{k-1})$. It is easy to verify that η is well-defined and is an unramified covering over its image $\mathcal{P}_S$. Therefore, η induces an analytic structure on T_S. Furthermore, $(\tau, b_1, ..., b_{k-1})$ are local parameters (moduli) on T_S and $\dim_{\mathbf{C}} T_S = k$. In particular, we always have $\dim_{\mathbf{C}} T_S \geqslant 1$. In the case $k = 1$ the space T_S is identified with the upper half-plane $\mathbf{H}$ (with the help of the map η).

(ii) **The punctured plane $\mathbf{C}^*$ with $k \geqslant 1$ marked points.**
In this case $\dim T_S = k-1$. For $k = 1$ the space T_S is a single point.

(iii) **The annulus $\mathrm{A} = \mathrm{A}(1, r)$ with $k \geqslant 0$ marked points $d_1, ..., d_k$ and the action of the rotation group $\mathbf{T}$.**
To determine moduli of the equivariant Teichmüller space T_S we mark two more points $d_0 \in \mathbf{T}$, $d_{k+1} \in \mathbf{T}_r$ on the components of the boundary of $\mathbf{A}$. Let μ be a $\mathbf{T}$-invariant conformal structure on $\mathbf{A}$. Then there is a quasi-conformal homeomorphism $h : \mathbf{A} \to \mathbf{A}(1, R)$ commuting with the rotations such that $h_*\mu = \sigma$, $h(d_0) = 1$. The value of h at one point of the concentric circle T_ρ determines $h|\mathrm{T}_\rho$. Therefore, it is sufficient to keep one marked point $d_{i_0} = d_0, d_{i_1}, ..., d_{i_{l+1}} = d_{k+1}$ $(l \geqslant 0)$ on each concentric circle. Then $a_n = h(d_{i_n})$ $(n = 1, ..., l+1)$ are the moduli of T_S. Thus, $\dim_{\mathbf{C}} T_S = l+1 \geqslant 1$.

(iv) **The punctured disc U^* with $k \geqslant 0$ marked points and the action of the rotation group.**
This case is completely analogous to the preceding one, but here $\dim_{\mathbf{C}} T_S = l$, where l is the number of concentric circles with marked points. For $k = 0$ the space T_S is a single point.

3. We now consider the Julia set $J(f)$. If $\operatorname{mes} J(f) > 0$, then it is meaningful to consider measurable conformal structures on $J(f)$. A structure μ is given by a Beltrami differential $k(z) \exp 2i\theta(z)d\bar{z}/dz$ on $J(f)$ (see §1.7). If μ is non-standard (that is, $k(z) \not\equiv 0$), it determines *a measurable field of directions* $\theta(z)$ on $\{z\colon k(z) \neq 0\}$, *the support* of $\theta(z)$ (of course, the support is defined up to sets of zero measure). To an invariant conformal structure there corresponds the invariant field of directions $\theta(fz) = \theta(z) + \arg f'(z) + \pi k$. Let $T_{J(f)}$ *be the space of invariant conformal structures whose support is contained in* $J(f)$.

We consider the partition Erg of the Julia set $J(f)$ into the *ergodic classes* of f. It is defined within the framework of the theory of measurable partitions [27] as the measurable hull of the partition into large orbits.

***Lemma* 2.2.** *On the Julia set there are finitely many ergodic classes X_j of positive measure and invariant measurable fields of directions $\theta_j(z)$ whose supports coincide with X_j $(j = 1, ..., k)$ such that any invariant conformal structure on $J(f)$ is given by a Beltrami differential* $\sum_{j=1}^{k} \lambda_j \exp(2i\theta_j(z))d\bar{z}/dz$, *where $\lambda_j \in \mathrm{U}$. Therefore, $T_{J(f)}$ is diffeomorphic to U^k.*

Proof. The space $T_{J(f)}$ is finite-dimensional. For otherwise we would obtain an infinite-parameter deformation of f (our familiar construction, taking account of Lemma 1.8).

Let $V \subset J(f)$ be the support of some invariant field of directions $\theta(z)$. If the quotient space V/Erg is infinite-dimensional, then the space of invariant measurable functions $\lambda(z)$ on V is infinite-dimensional. But each such function generates an invariant Beltrami differential $\lambda(z) \exp(2i\theta(z))d\bar{z}/dz$.

The contradiction shows that V consists of finitely many ergodic classes of positive measure.

We consider all components X_j of positive measure that admit invariant fields of directions $\theta_j(z)$ $(1 \leqslant j \leqslant k)$. They generate a k-dimensional family of invariant Beltrami differentials $\sum_{j=1}^{k} \lambda_j \exp(2i\theta_j(z))\, d\bar{z}/dz$. Consequently, $k < \infty$. Finally, by what we proved above, the support of any structure $\mu \in T_{J(f)}$ is contained in $\bigcup_{j=1}^{k} X_j$, so μ is given by the Beltrami differential mentioned above.

Corollary. *On the Julia set there are no wandering sets X of positive measure on which all iterates $f^n | X$ are one-to-one.*

4. *The Teichmüller space of a function f* is by definition $T(f) = T_{S(f)} \times T_{J(f)}$. We consider the group $G(f)$ of quasi-conformal homeomorphisms of the sphere commuting with f. The quotient group of $G(f)$ modulo the connected component of the unit is called *the modular group* Mod (f) *of the function f.* The group $G(f)$ acts naturally on the space $Z(f)$ of f-invariant conformal structures: $\mu \mapsto h_* \mu$ $(h \in G(f),\ \mu \in Z(f))$. By Lemma 1.8 this action induces an action of Mod f on $T(f)$. It can be shown that the transforms of the modular group are isometric diffeomorphisms of $T(f)$. Let $I(f)$ be the group of conformal transforms $z \mapsto (az + b)/(cz + d)$ of the sphere that commute with f, and $\tilde{g} \in \mathrm{qc}(f)$ the class of endomorphisms conformally conjugate to g.

***Theorem* 2.5.** *The group* Mod(f) *acts on $T(f)$ in a properly discontinuous way. The orbit space $T(f)/$*Mod(f) *is identified naturally with* qc(f). *The isotropy group of a point $x \in T(f)$ is isomorphic to the group $I(g)$, where* $\tilde{g} \in$ qc(f) *corresponds to the orbit of x.*

Proof. a) **A construction of the projection $\pi_f : T(f) \to \mathrm{qc}(f)$.** Let μ be a T-invariant conformal structure on the Riemann surface $S(f)$, and τ an invariant conformal structure on the Julia set $J(f)$. The structure μ is lifted to an f-invariant conformal structure $\hat{\mu}$ on the normality set $F(f)$. The structures $\hat{\mu}$ and τ determine an f-invariant conformal structure $\nu = (\hat{\mu}, \tau)$ on the whole sphere $\bar{\mathbf{C}}$. By the measurable Riemann theorem there is a quasi-conformal homeomorphism $h_{\mu,\tau}: \bar{\mathbf{C}} \circlearrowleft$ such that $(h_{\mu,\tau})_* \nu = \sigma_{\bar{\mathbf{C}}}$. Then $f_{\mu,\tau} = h_{\mu,\tau} \circ f \circ h_{\mu,\tau}^{-1}$ is a rational endomorphism of the Riemann sphere. Since $h_{\mu,\tau}$ is defined up to a composition with a conformal transform of the sphere, $f_{\mu,\tau}$ is defined up to conformal conjugacy. Therefore, we associate with each equivariant conformal structure (μ, τ) on $S(f) \cup J(f)$ an element $\tilde{f}_{\mu,\tau}$ in qc(f).

Let us show that to equivariantly isotopic structures $\varkappa$ and μ there correspond conformally conjugate transforms $f_{\varkappa,\tau}$ and $f_{\mu,\tau}$. In addition, we can assume that the structures $\varkappa = \sigma_{S(f)}$ and $\tau = \sigma_{J(f)}$ are standard, $f_{\varkappa,\tau} = f$. By Lemma 2.3, proved below, the structures $\varkappa$ and μ can be joined by an equivariant isotopy H_t: $S(f)\circlearrowright$ piecewise analytic in t. Then the family H_t is lifted to a piecewise analytic family $\hat{H}_t$ of quasi-conformal homeomorphisms of the sphere commuting with f, $\hat{H}_0 = \mathrm{id}$. The lifting construction is described in §§2.1–2.2, but we now deal with the trivial family of endomorphisms consisting of a single endomorphism $f_t \equiv f$. In particular, we have $\hat{H}_1^{-1} \circ f \circ \hat{H}_1 = f$. It follows that $(\hat{H}_1)_* \sigma_{F(f)} = \hat{\mu}$ on $F(f)$ (since $(H_1)_* \sigma_{S(F)} = \mu$ on $S(f)$). Moreover, $\hat{H}_t = \mathrm{id}$ on the Julia set. Hence, $(\hat{H}_1)_* \sigma_{\bar{\mathbf{C}}} = \nu$. It follows from the uniqueness in the measurable Riemann theorem that $\varphi = h_{\mu,\tau} \circ \hat{H}_1$ is a conformal transform of the sphere. But φ conjugates f and $f_{\mu,\tau}$. Therefore, the map $\pi_f : T(f) \to \mathrm{qc}(f)$ is well-defined.

b) **The map π_f is onto.** For if $g = h \circ f \circ h^{-1}$, then the structure $h_*^{-1} \sigma_{\bar{\mathbf{C}}}$ is f-invariant and so induces a T-invariant conformal structure μ on $S(f)$ and an invariant conformal structure τ on $J(f)$. It follows from the uniqueness in the measurable Riemann theorem that $\tilde{g} = \tilde{f}_{\mu,\tau}$.

c) **The fibres of π_f are orbits of the modular group Mod(f)**, which can be verified directly.

d) **A description of the isotropy group of a point $x \in T(f)$.** We can assume that $x = (\bar{\sigma}_{S(f)}, \sigma_{J(f)})$. We consider the group $I(f)$ of conformal transforms of the sphere commuting with f. The natural homeomorphism $j : I(f) \to \mathrm{Mod}(f)$ is an embedding. For if $h \in G(f)$ is isotropic to id, then $h|J(f) = \mathrm{id}$ (Lemma 1.8). Consequently, if in addition h is conformal, then $h = \mathrm{id}$. Therefore, Ker j is trivial. We identify the group $I(f)$ with its image in Mod(f).

Obviously, $I(f)$ is contained in the isotropy group of x. Conversely, suppose that $h_* x = x$ ($h \in G(f)$). This means that the structures $\sigma_{S(f)}$ and $H_* \sigma_{S(f)}$ are isotopic, where the homeomorphism H: $S(f)\circlearrowright$ is induced by the homeomorphism h: $\bar{\mathbf{C}}\circlearrowright$. Again applying Lemma 2.3, we consider a piecewise analytic isotopy Φ_t: $S(f)\circlearrowright$ joining $\sigma_{S(f)}$ and $H_* \sigma_{S(f)}$. As above we lift the family Φ_t to the family φ_t of quasi-conformal homeomorphisms of the sphere commuting with f such that $\varphi_0 = \mathrm{id}$. Then $(\varphi_1)_* \sigma_{\bar{\mathbf{C}}} = h_* \sigma_{\bar{\mathbf{C}}}$, since $(\Phi_1)_* \sigma_{S(f)} = H_* \sigma_{S(f)}$. Therefore, the transform $\varphi_1^{-1} \circ h$ is conformal. We now consider the isotopy $\varphi_t^{-1} \circ h$. It joins h with a conformal homeomorphism. This completes the proof of the fact that the isotropy group of x is isomorphic to $I(f)$.

e) **The modular group Mod(f) acts on the Teichmüller space $T(f)$ in a properly discontinuous way.** It is sufficient to verify this for some subgroup of finite index. We denote by $\mathrm{Per}_q(f)$ the set of periodic points of f with period q. It is invariant under the group $G(f)$. We consider a normal subgroup $G_q(f) \subset G(f)$ of homeomorphisms $h \in G(f)$ such that $h | \mathrm{Per}_q(f) = \mathrm{id}$. We choose the period q so that $|\mathrm{Per}_q(f)| \geqslant 3$. Then there are no non-trivial conformal transforms in $G(f)$. The image of $G(f)$ in $\mathrm{Mod}(f)$ under the natural homeomorphism is our desired subgroup of finite index. We denote it by $\mathrm{Mod}_q(f)$. By d) $\mathrm{Mod}_q(f)$ is embedded into the group of isometric diffeomorphisms of $T(f)$ endowed with the Teichmüller metric. We denote the image under this embedding by $\Gamma_q(f)$.

Let Δ be the group of all isometric diffeomorphisms of $T(f)$. It is a Lie group (see [15], Ch. 1, §4). Let us show that $\Gamma_q(f)$ is closed in Δ. Let $\gamma_i \in \Gamma_q(f)$, $\gamma_i \to \gamma \in \Delta$; $x = (\tilde{\sigma}_{S(f)}, \sigma_{I(f)}) \in T(f)$. Then there are conformal structures μ_i and μ on $S(f) \cup J(f)$ representing the points $\gamma_i x$ and γx of $T(f)$ such that $\mu_i \to \mu$. We consider liftings $\hat{\mu}_i$, $\hat{\mu}$ of μ_i, μ to the sphere $\bar{\mathbf{C}}$. Using the construction of lifting analytical deformations, which we have applied repeatedly in the proof, we find quasi-conformal homeomorphisms $h_i \in G_q(f)$ such that $(h_i)_* \sigma_{\bar{\mathbf{C}}} = \hat{\mu}_i$. Furthermore, by the measurable Riemann theorem there is a quasi-conformal homeomorphism $h\colon \bar{\mathbf{C}} \circlearrowleft$ with three fixed points in $\mathrm{Per}_q(f)$ such that $h_* \sigma_{\bar{\mathbf{C}}} = \hat{\mu}$. Since $h_i | \mathrm{Per}_q(f) = \mathrm{id}$, it follows that $h_i \to h$ (the continuous dependence in the measurable Riemann theorem). Therefore, $h \in G_q(f)$ and an element $\gamma \in \Delta$ corresponds to h under the natural homeomorphism $G_q(f) \to \Delta$. This shows that $\gamma \in \Gamma_q(f)$.

Therefore, $\Gamma_q(f)$ is a Lie subgroup of Δ. Being totally disconnected, $\Gamma_q(f)$ is discrete. But any discrete group of isometric diffeomorphisms is properly discontinuous ([15], Ch. I, §4). Theorem 2.5 is now proved modulo the following lemma.

***Lemma* 2.3.** *Let S be a Riemann surface of one of the types listed in subsection 2 (with marked points and perhaps with an action of the rotation group), and let $\varkappa$ and μ be $\mathbf{T}$-invariant conformal structures on S. If $\varkappa$ and μ are equivariantly isotopic, then they can be joined by a piecewise analytic (in the parameter) equivariant isotopy.*

Proof. It is sufficient to show that if the structures $\varkappa$ and μ are isotopic and close enough, then they can be joined by an analytic isotopy. As usual, we can assume that $\varkappa = \sigma_S$. We consider the universal covering space $\hat{S}$ of S. It is the plane in the case $S = \mathbf{T}^2$ or $S = \mathbf{C}^*$, and the upper half-plane in the case $S = \mathbf{U}^*$. When $S = \mathbf{A}$, the annulus, it is convenient to realize $\hat{S}$ as a horizontal strip. In all cases except the first, the covering $\hat{S} \to S$ has the form: $z \mapsto \exp 2\pi i z$. We mark on $\hat{S}$ the inverse images of the points marked on S. Finally, the action of the rotation group on $\mathbf{U}^*$ or $\mathbf{A}$ generates the action of $\mathbf{R}$ on the half-plane or the strip, $z \mapsto z + t$ ($z \in \hat{S}$, $t \in \mathbf{R}$).

We lift μ to a conformal structure $\hat{\mu}$ on $\hat{S}$. Then there is a quasi-conformal homeomorphism $\mathbf{id} + \psi\colon \hat{S} \circlearrowleft$ close to id, identical on $\partial\hat{S}$ and on the marked points, such that $(\mathbf{id} + \psi)_* \sigma_{\hat{S}} = \hat{\mu}$. In the case of the half-plane or the strip $\psi(z+t) = \psi(z)$ $(t \in \mathbf{R})$. If ψ is sufficiently small, then the maps $h_w = \mathrm{id} + w\psi$ $(|w| < 2)$ are quasi-conformal homeomorphisms of $\hat{S}$. Obviously, the h_w are identical on $\partial\hat{S}$ and on the marked points and are equivariant. Lowering h_w to S, we obtain the required isotopy.

Theorem 2.5 will be used repeatedly in what follows. We now state two direct consequences of it.

Corollary 1. *The total number of cycles of the Schröder domains and Arnol'd–Herman rings does not exceed* $2d-2$.

Proof. Each of these cycles brings into $\mathrm{qc}(f)$ at least one complex parameter.

Corollary 2. *The space* $\mathrm{qc}(f)$ *is connected.*

Remark. Theorem 1.16 on the absence of wandering domains is well interpreted from the point of view of the theory developed above. For example, the absence of simply-connected wandering domains is connected with the fact that the universal Teichmüller space $T_{\mathbf{U}}$ (where the disc $\mathbf{U}$ is not endowed with the rotation group) is infinite-dimensional.

§2.6. A-domains of the parameter space

Let W be a connected component of the set Σ of J-stable functions. W will be called an A-*domain* if the endomorphisms $f \in W$ satisfy Axiom A. For this it is sufficient that some f_0 in W satisfies Axiom A. The Fatou conjecture stated in §2.2 claims in this language that in the manifold $\mathfrak{R}_d$ of all rational functions of degree d any component W of Σ is an A-domain. The following result is the most advanced achievement in this direction.

Theorem 2.6 (Mañé, Sad, Sullivan [76], [77]). *Suppose that an endomorphism f is J-stable in $\mathfrak{R}_d$ and has no measurable invariant fields of directions on the Julia set $J(f)$. Then f satisfies Axiom* A.

Proof. By Theorem 2.2, perturbing f, we can make it structurally stable. We shall assume that this is so. Then f cannot have Böttcher, Leau, or Siegel domains that are destroyed by a small perturbation. It is shown in [76] that the Arnol'd–Herman rings are also unstable. Therefore, each periodic component of $F(f)$ is a Schröder domain. Moreover, there are no invariant fields of directions on $J(f)$. Therefore, $\dim_{\mathbf{C}} T(f)$ is equal to the number k of critical points in $F(f)$ (in a general position the orbits of the critical points are disjoint). By Theorem 2.5, $\dim_{\mathbf{C}}(\mathrm{qc}(f)) = k$. On the other hand, $\dim_{\mathbf{C}}(\mathrm{qc}(f)) = 2d-2$, since f is structurally stable. Thus, $k = 2d-2$, that is, all critical points lie in $F(f)$. Therefore, the orbits of all critical points converge to attractive cycles, which is equivalent to Axiom A.

Corollary. *If* mes $J(f) = 0$ *for a rational endomorphism in a general position, then Conjecture* 2.1 *of Fatou is valid.*

In conclusion we state an important result in the direction of topological classification of endomorphisms satisfying Axiom A.

Theorem 2.7 (Sullivan [90], Part III). *If endomorphisms f and g satisfying Axiom* A *are topologically conjugate, then they are quasi-conformally conjugate.*

The next assertion follows from Corollary 2 of Theorem 2.5.

Corollary. *If endomorphisms f and g satisfying Axiom* A *are topologically conjugate, then they lie in the same* A*-domain.*

§2.7. The Mandelbrot set

1. We consider the simplest one-parameter family of polynomials $f_w\colon z \mapsto z^2 + w$ $(w \in \mathbf{C})$. For real w the f_w have invariant circle $\bar{\mathbf{R}}$ and generate interesting and non-trivial dynamics on it, which has been intensively studied for the last 10 years (see [35], [49]). The bifurcation diagram for complex values of the parameter has been considered in detail by Mandelbrot [74] and independently by Levin [18] (see also [46]). Use of the quasi-conformal technique has enabled Douady and Hubbard to establish some remarkable properties of this diagram. In the same way, the problem of the growth of topological entropy of $f_w\colon \mathbf{R}\circlearrowleft$ as $w \in \mathbf{R}$ decreases has been solved (Douady and Hubbard, Milnor, Sullivan, Thurston).

The transforms $f_w\colon z \mapsto z^2 + w$ are pairwise not conformally conjugate, and each quadratic polynomial is conformally conjugate to an f_w. Therefore, the family f_w is the quotient space of the space $\mathcal{F}_2$ of quadratic polynomials modulo the action of the affine group $z \mapsto az + b$ by conjugations.

We consider the orbit $0 \mapsto w \mapsto w^2 + w \mapsto w^4 + 2w^3 + w^2 + w \mapsto \dots$ of a critical point of f_w. Here there arises the sequence of polynomials $F_m(w) = f_w^m(0)$, $\deg F_m = 2^{m-1}$. By Theorem 1.18, if $|F_m(w)| \to \infty$, then the Julia set $J(f_w)$ is a Cantor set. Otherwise $J(f_w)$ is connected. The set $M = \{w \in \mathbf{C}\colon J(f_w)$ is connected$\}$ is called *the Mandelbrot set.* Therefore, $\mathbf{C}\setminus M = \{w\colon |F_m(w)| \to \infty \ (m \to \infty)\} = \{w\colon J(f_w)$ is a Cantor set$\}$.

Proposition 2.5. a) *The Mandelbrot set is compact*; b) *its complement* $\mathbf{C}\setminus M$ *is connected*; c) *each connected component of its interior* M^0 *is simply-connected*; d) *the set of J-unstable endomorphisms* f_w *coincides with the boundary* ∂M *of the Mandelbrot set.*

Proof. We put $\Delta_m = \{w\colon |F_m(w)| > 2\}$. It is easy to show that $\mathbf{C}\setminus M = \bigcup_{m=0}^{\infty} \Delta_m$. Each set Δ_m is a domain, since $F_m\colon \bar{\mathbf{C}}\circlearrowleft$ is a ramified covering and $F_m^{-1}(\infty) = \{\infty\}$ (or from the maximum modulus principle).

Consequently, $\mathbf{C}\backslash M$ is a domain containing $\{w: |w| > 2\}$, which proves a) and b). c) follows from the following general fact of spherical topology: if V is a domain on the sphere, then the connected components of $(\overline{\mathbf{C}}\backslash V)^0$ are simply-connected (in our case $V = \overline{\mathbf{C}}\backslash M$).

Let us prove d). Let W be a connected component of the interior of M^0. Since $|F_m(w)| \leqslant 2$ on M, the family $\{F_m(w)\}_{m=0}^{\infty}$ is normal in W. By Theorem 2.4, $W \subset \Sigma$. In $\mathbf{C}\backslash M$ the sequence $\{F_m(w)\}$ tends to ∞ uniformly on compact sets. For the same reasons, $\mathbf{C} \diagdown M \subset \Sigma$. Therefore, $M^0 \cup (\mathbf{C} \diagdown M) \subset \Sigma$. Conversely, let $w \in \partial M$. Then $|F_m(w)| \leqslant 2$ $(m = 0, 1, 2, ...)$, but $|F_m(w_i)| \to \infty$ $(m \to \infty)$ for some sequence $w_i \to w$. Again applying Theorem 2.4 we see that f_w is J-unstable.

2. For the quadratic family f_w: $z \mapsto z^2 + w$, as in the general case, it is not known whether each connected component W of M^0 is an A-domain. Let W be an A-domain. This means that for $w \in W$ the function f_w has an attracting or superattracting cycle $\alpha(w) = \{\alpha_k(w)\}_{k=0}^{p-1}$. It follows from the implicit function theorem and the fact that W is simply-connected that the functions $z = \alpha_k(w)$ are single-valued branches in W of an algebraic function given by the equation $f_w^p(z) = z$. Consequently, the multiplier

$\lambda(w) = 2^p \prod\limits_{k=0}^{p-1} \alpha_k(w)$ is also a single-valued branch in W of an algebraic

function. When reaching the boundary ∂W the cycle $\alpha(w)$ becomes neutral, and so $|\lambda(w)| = 1$ on ∂W. Therefore, $\lambda : W \to \mathbf{U}$ is a ramified covering. In particular, λ necessarily has zeros in W. They are those values of w for which $\alpha(w)$ becomes superattracting. The following theorem shows that in each A-domain there is exactly one such value of the parameter.

Theorem 2.8 (Douady and Hubbard [53]-[55]). *The multiplier* $\lambda : W \to \mathbf{U}$ *is a univalent conformal map of the* A*-domain* W *onto the disc* $\mathbf{U}$.

Proof. We consider the domain W^* obtained from W by puncturing the zeros of λ. Let $w_0 \in W^*$, $f_0 \equiv f_{w_0}$, $\lambda_0 \equiv \lambda(w_0)$. By Theorem 2.2, W^* coincides with $\mathrm{qc}(f_0)$ and so there is a natural projection $\pi_0 : T(f_0) \to W^*$, whose fibres are the orbits of $\mathrm{Mod}(f_0)$ (Theorem 2.5$^{(1)}$).

The Riemann surface $S(f_0)$ is the union of the torus S_0 with one marked point and the punctured disc $\mathbf{U}^*$ with the action of the rotation group. The latter does not influence the space $T(f_0)$ and we can forget about it. The Julia set has zero measure (Corollary of Theorem 1.26) and so does not influence $T(f_0)$ either. Therefore, $T(f_0) = T_{S_0}$ is the upper half-plane $\mathbf{H}$.

Let $|\lambda_0| = r_0$. With the help of the Koenigs function (see § 1.8) we realize the torus S_0 as the quotient space of the punctured plane $\mathbf{C}^*$ modulo

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(1) The functions f_w $(w \neq 0)$ do not commute with the non-identical linear-fractional transforms of the sphere, since such transforms leave invariant the points $0, w, \infty$. Therefore, $\mathrm{Mod}(f_0)$ acts freely on $T(f_0)$ and the projection π_0 is an unramified covering.

the cyclic group$\{\zeta \mapsto \lambda_0^n\zeta\colon n \in \mathbf{Z}\}$. In this representation we define the following homeomorphism (*the Dehn twist map*): $\varphi\colon S_0 \circlearrowleft$, $r \exp i\theta \mapsto$ $\mapsto r \exp i\left(\theta + 2\pi \frac{r - r_0}{1 - r_0}\right)$ (φ is identical on the interior boundary of the fundamental annulus $\mathbf{A}[r_0, 1]$ and twists the exterior boundary through an angle 2π). Let $\gamma\colon T(f_0)\circlearrowleft$ be the transform of the Teichmüller space induced by φ (in the half-plane model γ reduces to the form $\tau \mapsto \tau + 1$). Let $w_i = \pi_0(\tau_i)$ $(i = 1, 2)$. Clearly, $\lambda(w_1) = \lambda(w_2)$ if and only if τ_1 and τ_2 lie in the same orbit of the cyclic group $\Gamma = \{\gamma^n\}_{n \in \mathbf{Z}}$. Therefore, to prove that λ is univalent it needs to be shown that $\mathrm{Mod}(f_0) = \Gamma$ (it is clear in advance only that $\mathrm{Mod}(f_0) \subset \Gamma$). In other words, there must be a quasi-conformal homeomorphism $h\colon \bar{\mathbf{C}}\circlearrowleft$ that commutes with f_0 and induces on S_0 a transform isotopic to the Dehn twist φ.

To construct such a homeomorphism we map conformally the Schröder domain D_0 of f_0 onto the unit disc $\psi : D_0 \to \mathbf{U}$. Under an appropriate normalization we have $g_0 = \psi \circ f^p \circ \psi^{-1}\colon z \mapsto z\frac{z - \lambda_0}{1 - \bar{\lambda}_0 z}$, where p is the order of the attracting fixed point[1] of f_0. The space of large orbits of the transform $g_0\colon \mathbf{U}\circlearrowleft$ is isomorphic to the torus S_0. Using the measurable Riemann theorem we construct a quasi-conformal homeomorphism $H\colon \mathbf{U}\circlearrowleft$ having the following properties: 1) the map $\bar{H}$ induced on S_0 is isotopic to the Dehn twist φ; 2) the transform $g_1 = H \circ g_0 \circ H^{-1}$ also has the form $z \mapsto z\frac{z - \lambda_1}{1 - \bar{\lambda}_1 z}$. It follows from property 1) that g_0 and g_1 have equal multipliers at the origin, that is, $\lambda_0 = \lambda_1$. Therefore, $g_0 = g_1$ and so H commutes with g_0. By Theorem 1.19, $g_0|\mathbf{T}$ is topologically conjugate to $z \mapsto z^2$. It can be shown that there are no non-identical homeomorphisms $\mathbf{T}\circlearrowleft$ that preserve orientation and commute with $z \mapsto z^2$. Therefore, $h|\mathbf{T} = \mathrm{id}$.

Next, the boundary of the Schröder domain D_0 is a simple Jordan curve (Corollary 2 of Theorem 1.22) and so the conformal map ψ extends to a homeomorphism $\bar{D}_0 \to \mathbf{U}$. We consider a quasi-conformal homeomorphism $h = \psi^{-1} \circ H \circ \psi\colon D_0\circlearrowleft$. It commutes with f_0 and induces on S_0 a transform isotopic to the Dehn twist φ, and $h|\partial D_0 = \mathrm{id}$. Such a homeomorphism extends easily to a quasi-conformal homeomorphism $h\colon \bar{\mathbf{C}}\circlearrowleft$ of the whole sphere commuting with f_0 (this extension satisfies $h|\overline{D(\infty)} = \mathrm{id}$). The theorem is now proved.

We shall say that *an* A-*domain* W *has period* (*order*) p if for $w \in W$ the attracting cycle $\alpha(w)$ has period (order) p.

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(1)A two-sheeted ramified covering $g\colon \mathbf{U}\circlearrowleft$ having a fixed point $\beta \in \mathbf{U}$ reduces by a conformal conjugation to the form $z \mapsto z\,\frac{z - \lambda}{1 - \bar{\lambda} z}$, where $\lambda = g'(\beta)$. Therefore, g is uniquely (up to a conformal conjugation) restored from the multiplier λ.

Corollary. *The number of* A*-domains with period p is equal to* 2^{p-1}.

Proof. By Theorem 2.8 each A-domain W contains one root w_0 of $F_p(w)$. Since $\deg F_p(w) = 2^{p-1}$, we have to show that the roots of $F_p(w)$ are simple. Again by Theorem 2.8, w_0 is a simple zero of $\lambda(w)$. We have $\lambda(w) = 2^q \prod_{k=0}^{q} \alpha_k(w)$, where q is the order of the A-domain W. Therefore, w_0 is a simple zero of one of the functions $\alpha_k(w)$, say $\alpha_0(w)$ to be definite. The function $z = \alpha_0(w)$ satisfies the equation $\varphi(w, z) \equiv f_w^p(z) - z = 0$, in which $\frac{\partial\varphi}{\partial z}(w_0, 0) \neq 0$. But the solution of this equation has at $w = w_0$ a root of the same order as the function $\varphi(w, 0) = F_p(w)$. Therefore, w_0 is a simple root of $F_p(w)$.

Remark. If we denote by l_q the number of A-domains of order q, then $\sum_{q \mid p} l_q = 2^{p-1}$. From this l_q can be found with the help of the Möbius inversion formula $l_q = \sum_{s \mid q} \mu(q/s)2^{s-1}$, where μ is the Möbius function in number theory.

Theorem 2.9 (Douady and Hubbard [53]-[55]). *The Mandelbrot set is connected.*

Proof. An equivalent statement is: the domain $\mathbf{C}\backslash M$ is doubly-connected. This can be proved in the same way as it is proved in Theorem 2.8 that W^* is doubly-connected, but the role of the Schröder domain is played here by the Böttcher domain $D_0(\infty)$ containing the critical point 0. Let $w_0 \in \mathbf{C}\backslash M$, $f_0 \equiv f_{w_0}$. Then $\mathbf{C}\backslash M = \mathrm{qc}(f_0)$. Therefore, $\mathbf{C}\backslash M = T(f_0)/\mathrm{Mod}(f_0)$. Now we have to check that $T(f_0)$ is isomorphic to $\mathbf{H}$, and $\mathrm{Mod}(f_0)$ is cyclic and generated by a parabolic element $\gamma \in \mathrm{PSL}_2(\mathbf{R})$. We leave the verification to the reader.

Remark. We consider the Böttcher function φ_w for f_w in a neighbourhood of ∞, $\varphi(\infty) = \infty$. It does not extend to the whole Böttcher domain $D_w(\infty)$ but extends to an invariant domain B_w bounded by a curve of figure-eight type with self-crossing at the origin, $w \in B_w$. Then the transform $w \longmapsto \varphi_w(w)$ univalently maps $\mathbf{C}\backslash M$ onto $\mathbf{C}\backslash \bar{\mathrm{U}}$ [54]. This assertion is a precise analogue of Theorem 2.8.

Problem. a) *Is it true that the Mandelbrot set is locally connected?*$^{(1)}$ b) *Is it true that its boundary has positive measure?*

3. We describe in more detail the structure of the Mandelbrot set (Fig. 11). We consider the A-domain $W_1 = \{w\colon f_w$ has an attracting fixed point $\}$. It is bounded by the cardioid γ_1 with cusp $w_1 = 1/4$. There is a unique point $w_2 = -3/4$ on γ_1 for which $\lambda^{(1)}(w_1) = -1$. At w_2 duplication bifurcation

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(1) This problem is discussed in detail by Douady and Hubbard [55].

takes place: the attracting fixed point $\alpha^{(1)}(w)$ "gives birth to" an attracting cycle $\alpha^{(2)}(w)$ of the second order⁽¹⁾. Therefore, an A-domain W_2 of the second order adjoins the A-domain W_1. ∂W_2 also has a unique point $w_2 = -5/2$ at which $\lambda^{(2)}(w_2) = -1$. At this point a bifurcation of the birth of a cycle of order 4 takes place: a component W_3 of order 4 clings to the component W_2, and so on⁽²⁾. The components W_n accumulate to a limiting point w_*, which corresponds to the famous Feigenbaum map (see [35], [49]).

Fig. 11. The Mandelbrot set

Besides duplication bifurcations, on the boundary of any A-domain triplication bifurcations, quadriplication bifurcations (and so on) occur. The bifurcation of the s times enlargement of the period occurs at points

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(1) In fact at $w = w_1$ a confluence of the attracting fixed point $\alpha^{(1)}(w)$ and the repelling cycle $\alpha^{(2)}(w)$ takes place. Then they split, but the cycle $\alpha^{(2)}(w)$ becomes attracting.

(2) The sequence of duplication bifurcations first appeared in the paper of Myrberg [80].

$w \in \partial W$ for which $\lambda(w) = \exp 2\pi ir/s$, where r and s are coprime. Since λ maps ∂W homeomorphically onto $\mathbf{T}$, the bifurcation points are dense on ∂W. Therefore, a tree of A-domains grows from W_1. A simple computation (using Theorem 2.8, of course) shows that a countable number of such trees go into the Mandelbrot set.

With the help of a computer one can see the microscopic structure of details of the Mandelbrot set. The colour pictures of it are extremely beautiful (see [12]).

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Added in proof: Conjectures 1.1, 1.3, 1.4 have recently been proved by M. Shishikura. M. Herman has found a counterexample to Conjecture 1.5 b).