ERGODIC PROPERTIES OF TRANSFORMATIONS OF AN INTERVAL

A. M. Blokh and M. Yu. Lyubich

UDC 517.9

We consider the class $\mathfrak{S}$ of piecewise monotone transformations $f: [0, 1] \to [0, 1]$ having the following properties:

1) inside the intervals of monotonicity, a) $f \in C^3$, b) f has no critical points, and c) f has a negative Schwartzian

$$Sf = f'''/f' - 1{,}5\cdot(f''/f')^2 < 0,$$

2) in the neighborhood of extrema c_i, $|f'(x)| \asymp |x - c_i|^{n_i}$, where $n_i > 0$.

Let λ be Lebesgue measure on $[0, 1]$, let $\omega(x)$ be the limit set of the trajectory $\{f^n x\}_{n=0}^{\infty}$, and let $\mathrm{rl}(A) = \{x: \omega(x) \subset A\}$ be the region of attraction of the set $A \subset [0, 1]$.

We call a closed invariant set $A \subset [0, 1]$ such that 1) $\lambda(\mathrm{rl}(A)) > 0$; 2) $\lambda(\mathrm{rl}(A) \setminus \mathrm{rl}(A')) > 0$ for every closed invariant subset $A' \subset A$ an <u>attractor in the sense of Milnor</u> or a <u>metric attractor</u> [1]. We call an attractor <u>indecomposable</u> if it is not the union of two smaller attractors.

In [2] and [3] it is shown that almost every f-trajectory approaches some indecomposable attractor A, and one of the following three possibilities holds: 1) A is a limit cycle; 2) A is a cycle of a periodic interval; 3) $A = \omega(c) \ni c$, where c is a critical point.

A transformation $f: X \to X$ of a space with quasiinvariant measure λ is said to be <u>ergodic</u> if there exists no completely invariant subset $Y \subset X$ (i.e., $f^{-1}Y = Y$) such that $\lambda(Y) > 0$, $\lambda(X \setminus Y) > 0$.

<u>THEOREM 1</u>. Let A be an indecomposable attractor of the transformation $f \in \mathfrak{S}$ which is not a limit cycle. Then $f/\mathrm{rl}(A)$ is ergodic.

For unimodal $f \in \mathfrak{S}$ having transitive periodic intervals, this result is established in [2] (for the proof, see Ukr. Mat. Zh., <u>41</u> (1989)).

<u>COROLLARY 1</u>. The indecomposable attractors of a transformation $f \in \mathfrak{S}$ are minimal. Almost every trajectory of $f \in \mathfrak{S}$ approaches some minimal attractor.

A set X is said to be wandering if $f^n X \cap X = \emptyset$ ($n = 1, 2, \ldots$), and it is said to be strongly wandering if $f^n X \cap f^m X = \emptyset$ ($n > m \geq 0$). We put $B_f = [0, 1] \setminus \cup\, \mathrm{rl}(Z_i)$, where the Z_i are all possible limit cycles of f. The set B_f does not contain strongly wandering intervals (M. Y. Lyubich (1987); this result was obtained for unimodal $f \in \mathfrak{S}$ by Guckenheimer [4]).* Theorem 1 implies a measurable analogue of this proposition (cf. Sullivan [5], Theorem 2):

<u>COROLLARY 2</u>. There exists no strongly wandering set $X \in B_f$ of positive measure for which f^n/X is injective ($n \geq 0$).

Let d be the number of critical points in B_f.

<u>COROLLARY 3</u>. A transformation $f \in \mathfrak{S}$ has no more than d absolutely continuous invariant ergodic measures.

A transformation $f: X \to X$ of a space of quasiinvariant measure is said to be <u>conservative</u> if f has no wandering sets of positive measure.

*We note that, as is shown by the authors (1987), a theorem concerning the absence of strongly wandering intervals holds also for C^3-smooth transformations with nonsingular critical points (in the unimodal case, this was proved by de Melo and van Strien (1986)).

Republican Information-Computing Center of Ministry of Public Health of RSFSR. All-Union Scientific-Research Institute for Design of Medical Laboratory Equipment. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 23, No. 1, pp. 59-60, January-March, 1989. Original article submitted November 26, 1987.

 0016-2663/89/2301-0048$12.50

THEOREM 2. Let A be an attractor of the transformation $f \in \mathfrak{S}$. Then f/A is conservative.

We state a fundamental lemma from which Theorems 1 and 2 follow immediately. For this, we define a local involution $\tau: x \to x'$ in the neighborhood of extrema by means of the following property: $f(x) = f(x')$.

LEMMA. Let c be some extremum, and let $X \subset \{x: \omega(x) \ni c\}$ be a measurable invariant subset, $\lambda(X) > 0$. Then: 1) c is an accumulation point of the set $X \cup \tau(X)$; 2) the set X has positive upper density at every point $x \in \omega(c)$.

The following result strengthens Corollary 2.

THEOREM 3. If $f \in \mathfrak{S}$ and A is an attractor, then there exist no strongly wandering sets $X \subset \mathrm{rl}\,(A)$, $\lambda(X) > 0$ (here A does not contain limit cycles or solenoids).

It is possible to define a topological attractor analogously to the metric attractor: instead of positiveness of measure, one requires that the corresponding sets be of second Baire category. A complete description of topological attractors T for a transformation $f \in \mathfrak{S}$ (and also for smooth transformations with nonsingular critical points) follows from the absence of wandering intervals and from results in [6] and [7]. In fact, one of three possibilities holds: 1) T is a limit cycle; 2) T is a cycle of a periodic interval; 3) $T = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{p_k} f^k I_n$, where $I_1 \supset I_2 \supset \ldots$ is a sequence of periodic intervals of order $p_n \to \infty$, and $\mathrm{int}\, T = \emptyset$ (such an attractor is said to be a solenoid).

In the real case, metric attractors clearly coincide with topological attractors. This important fact follows from the following two hypotheses.

HYPOTHESIS 1. Let f \ [0, 1] be topologically transitive. Then $\omega(x) = [0, 1]$ for almost all x.

Remark. We note that the property "$\omega(x) = [0, 1]$ for almost all x" is equivalent to f being conservative [3]. We note also that from the above results it follows that, for topologically transitive f, either $\omega(x) = [0, 1]$ for almost all x or there exist a finite number of minimal attractors $A_k = \omega(c_k) \ni c_k$ $(k = 1, 2, \ldots)$ and $\omega(x) = A_{k(x)}$ for almost all x. In addition, the entire interval [0, 1] is the only topological attractor (since topological transitivity implies that $\omega(x) = [0, 1]$ for a Baire massive set of points x).

HYPOTHESIS 2. If R is a topological repeller, then $\lambda(R) = 0$.

In conclusion, we deal with the question of the measure of a solenoid. If S is a dyadic solenoid of the unimodal transformation $f \in \mathfrak{S}$, then $\lambda(S) = 0$ ([8]). We have obtained an analogous result for arbitrary (not only dyadic) solenoids:

THEOREM 4. Let S be a solenoid of the transformation $f \in \mathfrak{S}$. Then $\lambda(S) = 0$.

Remark Added in Proof. All of our results can be generalized to the smooth polynomial case.

LITERATURE CITED

1. J. Milnor, Commun. Math. Phys., 99, No. 2, 177-196 (1985).
2. A. M. Blokh and M. Yu. Lyubich, Funkts. Anal. Prilozhen., 21, No. 2, 70-71 (1987).
3. A. M. Blokh and M. Yu. Lyubich, Teor. Funktsii Funktsional. Anal. Prilozhen., No. 49, Kharkov (1988), pp. 5-16.
4. J. Guckenheimer, Commun. Math. Phys., 70, No. 2, 133-160 (1979).
5. D. Sullivan, Ann. Math., 122, No. 3, 401-418 (1985).
6. A. M. Blokh, Usp. Mat. Nauk, 38, No. 5, pp. 179-180 (1983).
7. A. M. Blokh, Teor. Funktsii Funktsional. Anal. Prilozhen., No. 46, Kharkov (1986), pp. 8-18.
8. J. Guckenheimer, Commun. Math. Phys., 110, No. 4, 655-659 (1987).