DENSITY OF RESONANCES FOR SCHOTTKY GROUPS

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1. Introduction

In this talk I should like to present an estimate on the number of zeros of the Selberg zeta function for a class of Schottky groups. The method applies in greater generality and this talk should be considered as an announcement of the results of [4]. Here, I give a full proof of a special case.

Our motivation comes from the study of the distribution of quantum resonances – see [18] for a general introduction. Since the work of Sjöstrand [15] on geometric upper bounds for the number of resonances, it has been expected that for chaotic scattering systems the density of resonances near the real axis can be approximately given by a power law with the power equal to half of the dimension of the trapped set (see (1.1) below). Upper bounds in geometric situations have been obtained in [16] and [17].

Recent numerical studies in the semi-classical and several convex obstacles settings, [6],[7] and [8] respectively, show that the density of resonances satisfies a lower bound related to the dimension of the trapped set. In complicated situations which were studied numerically, the dimension is a delicate concept and it may be that different notions of dimension have to be used for upper and lower bounds – this point has been emphasized in [8].

In the case of *convex co-compact hyperbolic quotients*, $X = \Gamma\backslash\mathbb{H}^2$, studied in [17] the situation is particularly simple as the quantum resonances coincide with the zeros of the zeta-function – see [11]. The notion of the dimension of the trapped set is also clear as it is given by $2(1+\delta)$. Here $\delta = \dim \Lambda(\Gamma)$ is the dimension of the limit set of Γ, that is the set of accoumulation points of the elements of Γ (they are all hyperbolic), $\Lambda(\Gamma) \subset \partial\mathbb{H}^2$.

Hence we expect that

$$\sum_{|\operatorname{Im} s| \leq r \,,\, \operatorname{Re} s > -C} m_\Gamma(s) \sim r^{1+\delta} \,, \tag{1.1}$$

where $m_\Gamma(s)$ is the multiplicity of the zero of the zeta function of Γ at s.

Referring for definitions of Schottky groups and zeta functions to Sections 2 and 3 respectively we have

Theorem. *Suppose that Γ is a convex co-compact Schottky group and that $Z_\Gamma(s)$ is its Selberg zeta function. Then for any $C_0 > 0$ there exists C_1 such that for $|\operatorname{Re} s| < C_0$*

$$|Z_\Gamma(s)| \leq C_1 \exp(C_1 |s|^\delta) \,, \quad \delta = \dim \Lambda(\Gamma) \,. \tag{1.2}$$

The proof of this result is quite simple once we apply L^2-techniques to the study of the determinants of the Ruelle transfer operators and choose our spaces carefully.

If we use the convergence of the product representation (3.2) of the zeta function for $\operatorname{Re} s$ large and apply Jensen's theorem we obtain the following

Corollary. *Let $m_\Gamma(s)$ be the multiplicity of a zero of Z_Γ at s. Then*

$$\sum \{ m_\Gamma(s) \ : \ r \leq |\operatorname{Im} s| \leq r+1 \,, \quad \operatorname{Re} s > -C_0 \} \leq C_1 r^\delta \,, \tag{1.3}$$

where $\delta = \dim \Lambda(\Gamma)$.

This is stronger than the result obtained in [17] where the upper bound of the type (1.1) was given. In fact, the upper bound (1.3) is what we would obtain had we had a Weyl law of the form $r^{1+\delta}$ with a remainder $\mathcal{O}(r^\delta)$. That local upper bounds of this type are expected despite the absence of a Weyl law has been known since [12].

2. Schottky groups

For simplicity we will discuss only the first non-trivial case: a convex co-compact Schottky group on two generators – as will be seen in the argument increasing the number of generators will only add to notational complexity.

Thus let D_j^0, $j = 0, 1, 2$, be three open dics with disjoint closures intersecting $\partial \mathbb{H}^2 = \mathbb{R}$ (we will take the upper half plane model) perpendicularly. Let R_j be the reflections with respect to D_j's and R be the reflection with respect to $\mathbb{R}$ (in the disc model it would be the reflection with respect to the boundary of the disc). We put

$$\rho_j = R_j \circ R \,, \quad f_{ij} = \rho_i^{-1} \restriction_{D_j^0} \ . \tag{2.1}$$

Clearly, f_{ij} are holomorphic and contracting

$$|f'_{ij}(z)| < \alpha < 1 \,. \tag{2.2}$$

We now take Γ to be the group generated by $\rho_i \rho_j$. Then Γ is a discrete subgroup of $SL_2(\mathbb{R})$ and it is a free group on two generators. An example coming from [9] is shown (in the disc model) in Fig.2.

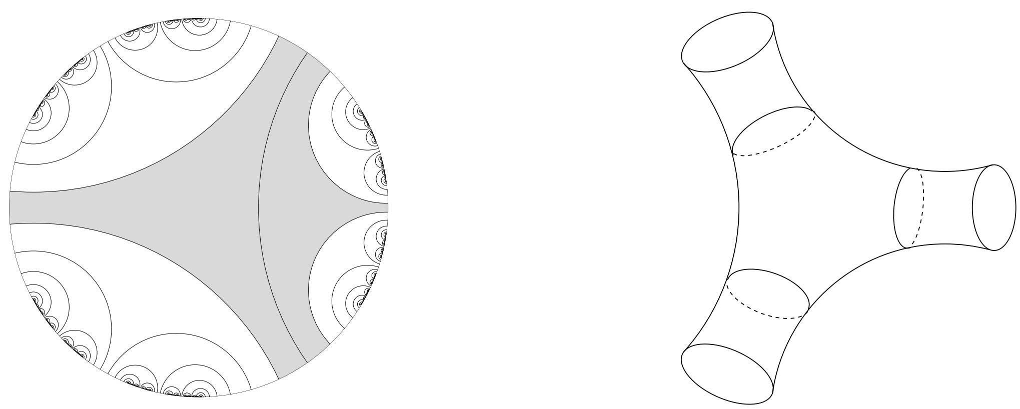

Figure 1. Tesselation by the Schottky group, Γ_θ, $\theta = 110^\circ$, generated by inversions in three symmetrically placed circles each cutting the unit circle in an 110° angle, with the fundamental domain of its subgroup of direct isometries and the associated Riemann surface. The dimension of the limit set is $\delta = 0.70055063\ldots$.

3. Properties of the Selberg Zeta-function

For Γ, a discrete subgroup of $SL_2(\mathbb{R})$, the Selberg zeta function is defined as follows

$$Z_\Gamma(s) = \exp\left(- \sum_{\{\gamma\}} \sum_{n=1}^{\infty} \frac{1}{n} e^{-sn\ell(\gamma)} (1 - e^{-n\ell(\gamma)})^{-1} \right) . \tag{3.1}$$

Here

$$\{\gamma\} = \text{ the conjugacy class of a primitive hyperbolic element } \gamma.$$

Such an element is called primitive if it is *not* a non-trivial power of another element. Using the expansion of $\log(1-z)$ we see that

$$Z_\Gamma(s) = \prod_{\{\gamma\}} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell(\gamma)}\right) . \tag{3.2}$$

We will now modify slightly the discussion in [13]. Let

$$\begin{gathered} D = D_0 \cup D_1 \cup D_2 \,, \quad D_j \text{ an open neighbourhood of } D_j^0 \cap \Lambda(\Gamma) \\ \overline{f_{ij}(D_j)} \subset D_i \,. \end{gathered} \tag{3.3}$$

With notational simplifications in the upper half-plane model† we follow [13] and define a special Ruelle transfer operator:

$$\begin{gathered} \mathcal{L}(s)u(z) = \sum_{i \neq j} [f_{ij}'(z)]^s u(f_{ij}(z)) \,, \quad z \in D_j \,, \\ u \in H^2(D) \,, \quad H^2(D) = \{ u \text{ holomorphic in } D \ : \ \iint_D |u(z)|^2 dm(z) < \infty \} \,. \end{gathered} \tag{3.4}$$

The only difference lies in choosing L^2 spaces of holomorphic functions instead of Banach spaces. However we still obtain the analogue of a (special case of a) result of Ruelle [14] and Fried [2]:

Proposition 1. *Suppose that $\mathcal{L}(s) : H^2(D) \to H^2(D)$ is defined by (3.4). Then for all $s \in \mathbb{C}$ $\mathcal{L}(s)$ is a trace class operator and*

$$|\det(I - \mathcal{L}(s))| \leq \exp(C|s|^2) \,. \tag{3.5}$$

Proof. The proof is based on estimates of the characteristic values, $\mu_l(\mathcal{L}(s))$. We will show that there exists $C > 0$ such that

$$\mu_l(\mathcal{L}(s)) \leq e^{C|s| - l/C} \,. \tag{3.6}$$

To see how that is obtained and how it implies (3.5) let us first recall some basic properties of characteristic values of a compact operator $A : H_1 \to H_2$ where H_j's are Hilbert spaces. We define

$$\|A\| = \mu_0(A) \geq \mu_1(A) \geq \cdots \geq \mu_\ell(A) \to 0 \,,$$

to be the eigenvalues of $(A^*A)^{\frac{1}{2}} : H_1 \to H_1$, or equivalently of $(AA^*)^{\frac{1}{2}} : H_2 \to H_2$. The min-max principle shows that

$$\mu_\ell(A) = \min_{\substack{V \subset H_1 \\ \operatorname{codim} V = \ell}} \ \max_{\substack{v \in V \\ \|v\|_{H_1} = 1}} \|Av\|_{H_2} \,. \tag{3.7}$$

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†In that case f_{ij}' is positive on the real axis so we have a well defined power $[f_{ij}(z)]^s$. In the disc model we extend the function $|f_{ij}(z)|^s$ holomorphically from the boundary of the disc.

The following rough estimate will be enough for us here: suppose that $\{x_j\}_{j=0}^{\infty}$ is an orthonormal basis of H_1, then

$$\mu_\ell(A) \leq \sum_{j=\ell}^{\infty} \|Ax_j\|_{H_2} . \tag{3.8}$$

To see this we will use $V_\ell = \text{span}\, \{x_j\}_{j=\ell}^{\infty}$ in (3.7): for $v \in V_\ell$ we have, by the Cauchy-Schwartz inequality, and the obvious $\ell^2 \subset \ell^1$ inequality,

$$\|Av\|_{H_2}^2 = \left\| \sum_{j=\ell}^{\infty} \langle v, x_j \rangle_{H_1} A x_j \right\| \leq \|v\|_{H_1}^2 \left(\sum_{j=\ell}^{\infty} \|Ax_j\|_{H_2} \right)^2 ,$$

from which (3.7) gives (3.8).

We will also need some real results about characteristic values The first is the *Weyl inequality* (see [3], and also [15, Appendix A]). It says that if $H_1 = H_2$ and $\lambda_j(A)$ are the eigenvalues of A,

$$|\lambda_0(A)| \geq |\lambda_1(A)| \geq \cdots \geq |\lambda_\ell(A)| \to 0 ,$$

then for any N,

$$\prod_{\ell=0}^{N} (1 + |\lambda_\ell(A)|) \leq \prod_{\ell=0}^{N} (1 + |\mu_\ell(A)|) .$$

In particular if the operator A is of trace class, that is if, $\sum_\ell \mu_\ell(A) < \infty$, then the determinant

$$\det(I + A) \stackrel{\text{def}}{=} \prod_{\ell=0}^{\infty} (1 + \lambda_\ell(A)) ,$$

is well defined and

$$|\det(I + A)| \leq \prod_{\ell=0}^{\infty} (1 + \mu_\ell(A)) . \tag{3.9}$$

We also need to recall the following standard inequality about characteristic values (see [3]):

$$\mu_{\ell_1+\ell_2}(A + B) = \mu_{\ell_1}(A) + \mu_{\ell_2}(B) \tag{3.10}$$

We finish the review, as we started, with an obvious equality: suppose that $A_j : H_{1j} \to H_{2j}$ and we form $\bigoplus_{j=1}^J A_j : \bigoplus_{j=1}^J H_{1j} \to \bigoplus_{j=1}^J H_{2j}$, as usual, $\bigoplus_{j=1}^J A_j (v_1 \oplus \cdots \oplus v_J) = A_1 v_1 \oplus \cdots \oplus A_J v_J$. Then

$$\sum_{\ell=0}^{\infty} \mu_\ell \left(\bigoplus_{j=1}^{J} A_j \right) = \sum_{j=1}^{J} \sum_{\ell=0}^{\infty} \mu_\ell(A_j) . \tag{3.11}$$

With these preliminary facts taken care of, we see that (3.6) implies (3.5). In fact, (3.9) shows that

$$\det(I - \mathcal{L}(s)) \leq \prod_{\ell=0}^{\infty} (1 + e^{C|s| - \ell/C}) \leq e^{C^3 |s|^2} .$$

Hence it remains to establish (3.6). For that we will write

$$H^2(D) = \bigoplus_{j=0}^{2} H^2(D_j^0) ,$$

and introduce

$$\mathcal{L}_{ij}(s) : H^2(D_i^0) \to H^2(D_j^0) , \quad \mathcal{L}_{ij}(s) u(z) \stackrel{\text{def}}{=} [f_{ij}'(z)]^s u(f_{ij}(z)) , \quad z \in D_j^0 , \quad i \neq j$$

From (3.10) and a version of (3.11) we then have

$$\mu_\ell(\mathcal{L}(s)) \leq \max_{\substack{0\leq i,j\leq 2\\ i\neq j}} 2\mu_{[\ell/6]}(\mathcal{L}_{ij}(s))\,.$$

To estimate $\mu_k(\mathcal{L}_{ij}(s))$ we use (3.8) with a basis of $H^2(D_i^0)$ given by

$$x_k = \sqrt{2k+1}\,r_i^{-1}((z-a_i)/r_i)^k\,, \quad D_i^0 = D(a_i, r_i)\,.$$

Since $\overline{f_{ij}(D_j^0)} \subset D_j^0$,

$$\|((f_{ij}(z)-a_i)/r_i)^k\|_{H^2(D_i^0)} \leq C\alpha^k\,,$$

for some $0<\alpha<1$. Since $[f_{ij}(z)]^s \leq e^{C|s|}$, we obtain

$$\mu_\ell(\mathcal{L}_{ij}(s)) \leq C\sum_{k\geq\ell}\|\mathcal{L}_{ij}(s)(x_k)\| \leq C\sum_{k\geq\ell} e^{C|s|}\alpha^k \leq Ce^{C|s|}\frac{\alpha^\ell}{1-\alpha} \leq C_1 e^{C|s|-\ell/C_1}\,,$$

for some C_1, which completes the proof of (3.6). □

Remark. The simple proof above is inspired by the work on the distribution of resonances in Euclidean scattering where the Fredholm determinant method and the use of Weyl inequalities were introduced by Melrose [10] and developed further by many authors – see [15],[18], and references given there. That was done at about the same time as David Fried (across the Charles River from Melrose) was applying the Grothendieck-Fredholm theory to multidimensional zeta-functions [2]. In both situation the enemy is the exponential growth for complex energies s, which is eliminated thanks to analyticity properties of the kernel of the operator.

The next proposition is an easy modification of [13, (3)]:

Proposition 2. *Let $\mathcal{L}(s)$ be defined by* (3.4) *and Γ be generated by $\rho_i\rho_j$ where ρ_j's and f_{ij}'s are as in* (2.1). *Then*

$$Z_\Gamma(s) = \det(I-\mathcal{L}(s))\,.$$

Proof. For the reader's convenience we reproduce the well known arguments of [14],[13] in the L^2 setting and for the Schottky group. For s fixed and $z\in\mathbb{C}$

$$h(z) \stackrel{\text{def}}{=} \det(I - z\mathcal{L}(s))$$

is, in view of (3.6) and (3.9) an entire function of order 0. For $|z|$ sufficiently small $\log(I-z\mathcal{L}(s))$ is well defined and we have

$$\det(I-z\mathcal{L}(s)) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\operatorname{tr}(\mathcal{L}(s))^n\right)\,. \tag{3.12}$$

To analyse the traces we introduce an *expanding map* T:

$$T\ :\ \bigcup_{i\neq j} f_{ij}(D_j) \to \bigcup_j D_j\,,\quad T\restriction_{f_{ij}(D_j)} = \rho_i\restriction_{f_{ij}(D_j)} = f_{ij}^{-1}\restriction_{f_{ij}(D_j)}\ .$$

In terms of T, $\mathcal{L}(s)$ takes the usual Ruelle-Perron-Frobenius form:

$$\mathcal{L}(s)u(z) = \sum_{Tw=z}[T'(w)]^{-s}u(w)\,.$$

The correspondence between the closed geodesic (or, equivalently, conjugacy classes of hyperbolic elements) and the periodic orbits of T is particularly simple for Schottky groups and we recall it in the form given in [13, Proposition 3.4] (where it is given in a more complicated setting of co-compact groups):

Closed geodesics on $\Gamma\backslash\mathbb{H}^2$, γ of length $l(\gamma)$, and word lenght $|\gamma|$ are in one to one correspondence with periodic orbits $\{x, Tx, \cdots, T^{n-1}x\}$ such that $(T^n)'(x) = \exp \ell(\gamma)$, and $n = |\gamma|$. For prime closed geodesics we have the same correspondence with primitive periodic orbits of T.

It is not needed for us to recall the precise definition of the *word length*. Roughly speaking it is the number of generators of Γ needed to write down γ.

The Schwartz kernel of $\mathcal{L}(s)^n$ can be written using the Bergman kernel for D_j's and the evaluation of the trace † gives

$$\operatorname{tr}\mathcal{L}(s)^n = \sum_{T^n x = x} \frac{[(T^n)'(x)]^{-s}}{1-[(T^n)'(x)]^{-1}} .$$

Returning to (3.12) and using the identification with closed geodesics quoted above, we obtain for $\operatorname{Re} s$ sufficiently large,

$$\begin{aligned}
\det(I - z\mathcal{L}(s)) &= \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{[(T^n)'(x)]^{-s}}{1-[(T^n)'(x)]^{-1}} \right) \\
&= \exp\left(-\sum_{n=1}^{\infty} \sum_{\substack{x,\cdots T^{n-1}x \\ T^n x = x,\, n \text{least}}} \sum_{k=1}^{\infty} \frac{z^{nk}}{k} \frac{[(T^n)'(x)]^{-s}}{1-[(T^n)'(x)]^{-1}} \right) \\
&= \exp\left(-\sum_{k=1}^{\infty} \sum_{\{\gamma\}} \frac{z^{|\gamma| k}}{k} \frac{e^{-s\ell(\gamma)}}{1-e^{-\ell(\gamma)}} \right) \\
&= \exp\left(-\sum_{k=1}^{\infty} \sum_{\{\gamma\}} \frac{z^{|\gamma| k}}{k} e^{-s\ell(\gamma)} \sum_{m=0}^{\infty} e^{-m\ell(\gamma)} \right) \\
&= \prod_{\{\gamma\}} \prod_{k=0}^{\infty} \left(1 - z^{|\gamma|} e^{-(s+k)\ell(\gamma)} \right) ,
\end{aligned}$$

which in view of (3.2) proves the proposition once we put $z = 1$. □

It is important that in the proof we did not use any properties of the open sets D_j other than the ones given in (3.3).

Finally, we remark that in view of the lower bounds on the number of zeros of Z_Γ obtained in [5] we see from Proposition 2 that the upper bound (3.5) is optimal for any Γ.

4. Estimates in terms of the dimension of $\Lambda(\Gamma)$.

For the proof of Theorem stated in Sect.1 we will choose D_j's in the definition of $\mathcal{L}(s)$ in way dependent on the size of s. Let $h = 1/|s|$. The self-similarity structure of $\Lambda(\Gamma)$ will show that we can choose $D_j = D_j(h)$ to be a union of $\mathcal{O}(h^{-\delta})$ disjoint discs of radii $\sim h$ and $d(\partial D_j, \Lambda(\Gamma)) \sim h$. The argument used in the proof of Proposition 1 will then give (1.2).

We start with

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†To see how it works we consider a simple case of f holomorphic in the unit dics, $f(0) = 0$, and $|f(z)| < |z|$ for $z \neq 0$. The pull back by f has the kernel (as a map on $H^2(D(0,1))$) given by $\pi^{-1}(1 - f(z)\bar{\zeta})$. The trace is then $\pi^{-1} \iint_{D(0,1)} (1 - f(z)\bar{z}) dm(z) = (1 - |f'(0)|)^{-1}$.

Proposition 3. *Let $\Lambda(\Gamma) \subset \mathbb{R}$ be the limit set of Γ in the disc model. Then there exists a constant $K = K(\Gamma)$ such that for $\delta < \delta_0$ the connected components of $\Lambda(\Gamma) + [-\delta, \delta]$ have length at most $K\delta$.*

Proof. The discussion of "cookie-cutter sets" in [1] and in particular [1, Corollary 4.4] can be applied to $\Lambda(\Gamma)$ showing that it is a *quasi-self-similar set*. More precisely there exist $c > 0$ and $r_0 > 0$ such that for any $x_0 \in \Lambda(\Gamma)$ and $r < r_0$ there exists a map $g : [x_0 - r, x_0 + r] \to \mathbb{R}$ with the properties

$$g(\Lambda(\Gamma) \cap [x_0 - r, x_0 + r]) \subset \Lambda(\Gamma)$$
$$cr^{-1}|x - y| \le |g(x) - g(y)| \le c^{-1}r^{-1}|x - y|\,, \quad x, y \in [x_0 - r, x_0 + r]$$

Hence the proposition follows by a scaling argument. □

Proof of Theorem. As outlined in the beginning of the section we put $h = 1/|s|$, where $|s|$ is large but $|\operatorname{Re} s|$ is uniformly bounded. We then define

$$I_j(h) \stackrel{\text{def}}{=} (\Lambda(\Gamma) \cap D_j^0 + [-h, h]) = \bigcup_{p=1}^{P_j(h)} [x_p^j - r_p^j, x_p^j + r_p^j]\,, \quad x_{p+1}^j - r_{p+1}^j > x_p^j + r_p^j\,,$$

that is $[x_p^j - r^j - p, x_p^j + r_p^j]$'s are the connected components of $I_j(h)$. Proposition 3 shows that $r_p^j < Kh$.

The open set $D(h)$ is defined as

$$D(h) = \bigcup_{j=0}^{2} D_j(h)\,, \quad D_j(h) = \bigcup_{p=1}^{P_j} D_{jp}(h)\,, \quad D_{jp}(h) = D(x_p^j, r_p^j)\,,$$

and since $f_{ij} : \Lambda(\Gamma) \cap D_j^0 \to \Lambda(\Gamma) \cap D_i^0$ we see that the condition (3.3) holds: for each D_{jp} there exists a $p' = p(i, j, p)$ for which

$$d(\partial D_{ip'}(h), f_{ij}(D_{jp})) > (1 - \beta)h\,,$$

for a fixed constant $0 < \beta < 1$. From this we also see that $P_j(h) = P(h)$ is independent of $j = 0, 1, 2$.

The now classical results of Patterson and Sullivan on the dimension of the limit set show that $P(h) = \mathcal{O}(h^{-\delta})$: what we are using here is the fact that the Hausdorff measure of $\Lambda(\Gamma)$ is finite.

We can now apply the same procedure as in the proof of Proposition 1. What we have gained is a bound on the weight: since $|\operatorname{Re} s| \le C$ and f_{ij}' is real on the real axis

$$|[f_{ij}'(z)]^s| \le C \exp(|s||\arg f_{ij}'(z)|) \le C \exp(C_1|s||\operatorname{Im} z|) \le C_2\,, \quad z \in D_j(h)\,.$$

We write $\mathcal{L}(s)$ as a sum of six operators $\mathcal{L}_{ij}(s)$ each of which is a direct sum of $P(h)$ operators. The discs and contractions are uniform after rescaling by h and hence the characteristic values of each of these operators satisfy the bound $\mu_l \le C\gamma^l$, $0 < \gamma < 1$. Using (3.9) and (3.11) we obtain the bound

$$\log|\det(I - \mathcal{L}(s))| \le CP(h) = \mathcal{O}(h^{-\delta})\,,$$

and this is (1.2). □

Proof of Corollary. The definition of $Z_\Gamma(s)$ (3.1) shows that for $\operatorname{Re} s > C_1$ we have $|Z_\Gamma(s)| > 1/2$. The Jensen formula then shows that the left hand side of (1.3) is bounded by

$$\sum\{m_\Gamma(s) \ : \ |s - ir - C_1| \le C_2\} \le 2 \max_{\substack{|s| \le r + C_3 \\ |\operatorname{Re} s| \le C_0}} \log|Z_\Gamma(s)| + C_4\,,$$

and (1.3) follows from (1.2). □

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