UNKNOTTING INFORMATION FROM HEEGAARD FLOER HOMOLOGY

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Abstract. We use results of Cochran-Lickorish and Ozsváth-Szabó to obtain bounds on unknotting numbers. We determine the unknotting numbers of 9_{10}, 9_{13}, 9_{35}, 9_{38}, 10_{53}, 10_{101} and 10_{120}.

1. Introduction

Let K be a knot in S^3. Given any diagram D for K, a new knot may be obtained by changing one or more crossings of D. The unknotting number $u(K)$ is the minimum number of crossing changes required to obtain the unknot, where the minimum is taken over all diagrams for K.

Let $\Sigma(K)$ denote the double cover of S^3 branched along K. A theorem of Montesinos [4] tells us that for any knot K, $\Sigma(K)$ is given by Dehn surgery on some framed link with $u(K)$ components, with half-integral framing coefficients. In particular if $u(K) = 1$ then $\Sigma(K)$ is obtained by $\pm\det(K)/2$ Dehn surgery on a knot C in S^3. Ozsváth and Szabó have shown in [11] that the Heegaard Floer homology of a 3-manifold Y gives an obstruction to Y being given by half-integral surgery on a knot in S^3; they apply this to $\Sigma(K)$ to obtain an obstruction to K having unknotting number 1 .

Note that crossings in a knot diagram may be given a sign as in Figure 1 (independent of the choice of orientation of the knot). Let $\sigma(K)$ denote the signature of a knot K. It is shown in [2, Proposition 2.1] that if K' is obtained from K by changing a positive crossing, then

$$\sigma(K') \in \{\sigma(K), \sigma(K)+2\};$$

similarly if K' is obtained from K by changing a negative crossing then

$$\sigma(K') \in \{\sigma(K), \sigma(K)-2\}.$$

Now suppose that K may be unknotted by changing p positive and n negative crossings (in some diagram). Since the unknot has zero signature, it follows that a bound for n is given by

$$n \geq \sigma(K)/2, \tag{1}$$

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The following theorem, which uses a result of Cochran and Lickorish [2, Theorem 3.7], may be viewed as a refinement of Montesinos' theorem. It may be combined with results of Ozsváth and Szabó to get a useful obstruction to equality in (1).

Theorem 1. *Suppose that a knot K may be unknotted by changing p positive and n negative crossings, with $n = \sigma(K)/2$. Then $\Sigma(K)$ may be obtained by Dehn surgery on an oriented, framed $p+n$ component link $C_1, \dots, C_{p+n}$ in S^3 satisfying the following conditions.*

- *The framing on component C_i is $(2m_i - 1)/2$ for some $m_i \in \mathbb{N}$;*
- *let $a_{ij} = \mathrm{lk}(C_i, C_j)$ be the linking number of C_i and C_j. Then the inequalities*

$$|a_{ij}| < m_i \leq m_j$$

  are satisfied whenever $i < j$;
- *the corresponding four-dimensional 2-handlebody bounded by $\Sigma(K)$ (as in Lemma 2.2) is positive-definite.*

Moreover exactly n of the m_i are even.

Corollary 2. *The knots $9_{10}, 9_{13}, 9_{35}, 9_{38}, 10_{53}, 10_{101}, 10_{120}$ have unknotting number 3.*

For all but one of the knots in Corollary 2, the signature is 4 and the unknotting number computation follows from Theorem 1 and theorems of Ozsváth and Szabó. The exception is 9_{35}, whose signature is 2. The computation of $u(9_{35})$ uses Theorem 1 and also a result of Traczyk [12].

Corollary 2 completes the table of unknotting numbers for knots with 9 crossings or less.

Corollary 3. *Any unknotting sequence for 10_{145} contains at least two negative crossing changes.*

Acknowledgements.

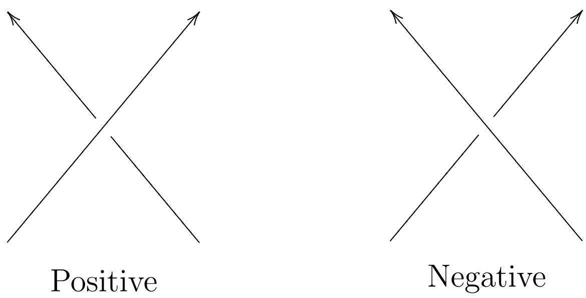

FIGURE 1. Signed crossings in a knot diagram.

2. Kirby-Rolfsen calculus

In this section we establish some preliminaries on Dehn surgery. For details on Dehn surgery and Kirby-Rolfsen calculus see [3].

A framed link L in S^3 with rational framing coefficients determines a three-manifold Y_L by Dehn surgery (remove a tubular neighbourhood of each component of L; the framing coefficient determines the gluing map to sew back a solid torus along the boundary). If the framing coefficients are integers one obtains a four-manifold W_L with boundary Y_L by attaching two-handles to B^4 along the components of L. Kirby-Rolfsen calculus describes when two framed links L, L' determine the same three-manifold Y_L.

Given a framed oriented link L with components $C_1, \dots, C_m$, let A denote the free abelian group with generators $c_1, \dots, c_m$. Define a symmetric bilinear form

$$Q : A \times A \to \mathbb{Q}$$

by

$$Q(c_i, c_j) = \begin{cases} \text{framing coefficient of } C_i & \text{if } i = j; \\ \text{linking number } \mathrm{lk}(C_i, C_j) & \text{if } i \neq j. \end{cases}$$

In other words, the matrix of Q in the basis $c_1, \dots, c_m$ is the *linking matrix* of L. (This is the intersection pairing on $H_2(W_L; \mathbb{Z})$ if the diagonal entries are integers.)

In the case that the framing coefficients on L are integers, any change of basis in A may be realised by a change in the link L. In particular the change of basis $c_i \mapsto c_i \pm c_j$ may be realised by a *handleslide*. Let λ_j denote a pushoff of C_j whose linking number with C_j equals the framing of C_j. A handleslide $C_i \mapsto C_i \pm C_j$ consists of replacing C_i by the oriented band sum of C_i with $\pm\lambda_j$. This gives a new link L' whose linking matrix is the matrix of Q in the basis $\{c_1, \dots, c_i' = c_i \pm c_j, \dots, c_m\}$ and with $Y_{L'} \cong Y_L$. It will be convenient to have the following generalisation of handlesliding to links with rational framings.

Proposition 2.1. *Let L be an oriented link in S^3 consisting of components $C_1, \dots, C_m$ with framings $\frac{p_1}{q_1}, \dots, \frac{p_m}{q_m}$, and let Q be the rational-valued bilinear pairing determined by the linking matrix of L. Then by replacing C_i in L it is possible to obtain a link L' whose linking matrix is the matrix of Q in the basis $\{c_1, \dots, c_i' = c_i \pm q_j c_j, \dots, c_m\}$ and with $Y_{L'} \cong Y_L$.*

Proof. For each $i = 1, \dots, m$ choose a continued fraction expansion

$$\frac{p_i}{q_i} = a^i_{l_i} - \cfrac{1}{a^i_{l_i - 1} - \ddots - \cfrac{1}{a^i_1}}.$$

(The numbers $a^i_{l_i}, \ldots, a^i_1$ arise from the Euclidean algorithm as follows:

$$\begin{aligned} r_{l_i} = p_i &= a^i_{l_i} q_i - r_{l_i-2} \\ r_{l_i-1} = q_i &= a^i_{l_i-1} r_{l_i-2} - r_{l_i-3} \\ &\vdots \\ r_2 &= a^i_2 r_1 - 1 \\ r_1 &= a^i_1 .) \end{aligned} \tag{2}$$

Use reverse "slam-dunks" to obtain an integral surgery description of Y_L: as shown in Figure 2, we add a chain of linked unknots linking each C_i, with framings $a^i_1, \ldots, a^i_{l_i-1}$, and replace the framing on C_i with $a^i_{l_i}$. (This is a standard procedure, see e.g. [3, §5.3].) Denote the resulting link by $L_{\mathbb{Z}}$, and let $Q_{\mathbb{Z}} : A_{\mathbb{Z}} \times A_{\mathbb{Z}} \to \mathbb{Z}$ denote the resulting bilinear form.

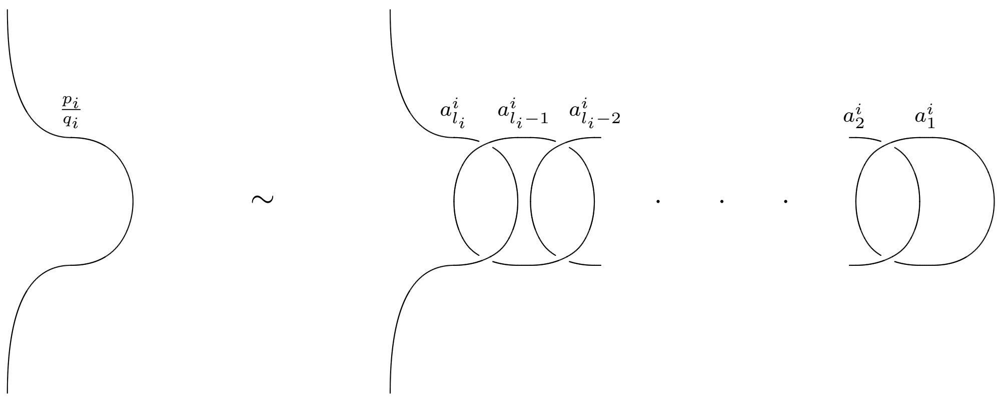

FIGURE 2. Converting Dehn surgery to integral surgery.

We now perform handleslides on this integer-framed link. Let $D_1, \ldots, D_{l_j-1}$ be the chain of unknots linking C_j as above, with $\mathrm{lk}(C_j, D_{l_j-1}) = 1$. Let $K_1 = C_i + D_1$, and note that

$$\mathrm{lk}(K_1, D_1) = a^j_1. \tag{3}$$

We now define K_k recursively for $2 \le k < l_j$. Choose any link diagram of $K_{k-1} \cup D_{k-1} \cup D_k$. By performing a handleslide over D_k for each crossing where K_{k-1} crosses over D_{k-1} we obtain a knot K_k which does not cross over D_{k-1} and therefore is separated from it by a two-sphere in S^3 (see Figure 3). The signed count of these handleslides is minus the linking number of K_{k-1} and D_{k-1}; thus we write

$$[K_k] = [K_{k-1}] - \mathrm{lk}(K_{k-1}, D_{k-1})[D_k],$$

where $[K_k]$ denotes the element of $A_{\mathbb{Z}}$ corresponding to the knot K_k. We may use this to compute linking numbers and the framing of K_k. In particular

$$\begin{aligned} \mathrm{lk}(K_k, D_k) &= \mathrm{lk}(K_{k-1}, D_k) - a^j_k \mathrm{lk}(K_{k-1}, D_{k-1}) \\ &= -\mathrm{lk}(K_{k-2}, D_{k-2}) - a^j_k \mathrm{lk}(K_{k-1}, D_{k-1}). \end{aligned} \tag{4}$$

Finally we let C'_i be obtained as above from K_{l_j-1} by sliding over C_j, with C'_i unlinked from each of $D_1, \dots, D_{l_j-1}$. As above we have

$$\begin{aligned} [C'_i] &= [K_{l_j-1}] - \mathrm{lk}(K_{l_j-1}, D_{l_j-1})[C_j], \\ \mathrm{lk}(C'_i, C_j) &= -\mathrm{lk}(K_{l_j-2}, D_{l_j-2}) - a^j_{l_j} \mathrm{lk}(K_{l_j-1}, D_{l_j-1}) \qquad (5) \end{aligned}$$

Comparing (3), (4), and (5) to (2) we see that

$$\begin{aligned} \mathrm{lk}(K_k, D_k) &= (-1)^{k+1} r^j_k \quad \text{for } k = 1, \dots, l_j - 2, \\ \mathrm{lk}(K_{l_j-1}, D_{l_j-1}) &= (-1)^{l_j} r^j_{l_j-1} = (-1)^{l_j} q_j, \\ \mathrm{lk}(C'_i, C_j) &= (-1)^{l_j} p_j. \end{aligned}$$

This yields

$$[C'_i] = [C_i] + \mathcal{D} + (-1)^{l_j+1} q_j [C_j],$$

where

$$\mathcal{D} = [D_1] + \sum_{k=2}^{l_j-1} (-1)^{k-1} r_{k-1} [D_k].$$

Note that by construction C'_i is separated by a two-sphere from each D_k and so $Q_{\mathbb{Z}}([C'_i], \mathcal{D}) = 0$. The framing of C'_i is given by

$$\begin{aligned} Q_{\mathbb{Z}}([C'_i], [C'_i]) &= Q_{\mathbb{Z}}([C_i] + \mathcal{D} \pm q_j [C_j], [C_i] + \mathcal{D} \pm q_j [C_j]) \\ &= Q_{\mathbb{Z}}([C_i] \pm q_j [C_j], [C_i] + \mathcal{D} \pm q_j [C_j]) \\ &= Q_{\mathbb{Z}}([C_i], [C_i]) \pm 2 q_j Q_{\mathbb{Z}}([C_i], [C_j]) + q_j^2 a^j_{l_j} - q_j r_{l_j-2} \\ &= a^i_{l_i} \pm 2 q_j \mathrm{lk}(C_i, C_j) + p_j q_j. \end{aligned}$$

Slam dunking to remove the chains of linking unknots from each of $C_1, \dots, C'_i, \dots, C_m$ gives the required link L' for the basis change $c'_i = c_i + (-1)^{l_j+1} q_j c_j$. To get the opposite sign construct C'_i as above but start with $K_1 = C_i - D_1$. □

The following lemma is an application of the standard procedure, referred to in the proof of Proposion 2.1 and illustrated in Figure 2, for converting a Dehn surgery description of a three-manifold to an integral surgery description.

Lemma 2.2. *Let $L = \{C_1, \dots, C_n\}$ be a framed link in S^3 with framing $(2m_i - 1)/2$ on C_i, and let Y be the three-manifold obtained by Dehn surgery on L. Then Y is equal to the boundary of the four-manifold W obtained by adding 2-handles to B^4 along either of the following $2n$-component framed links (as in Figure 4):*

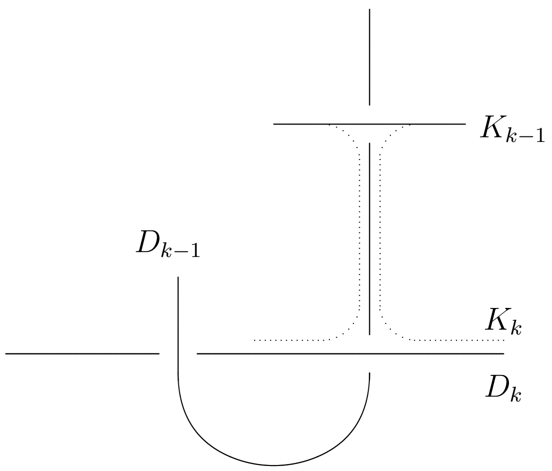

FIGURE 3. Handlesliding K_{k-1} over D_k yields K_k which is separated from D_{k-1} by a two-sphere.

(1) *the link consisting of the components* C_i *with framing* m_i *plus a small linking unknot with framing* 2*, for each* $i = 1, \ldots, n$*;*
(2) *the link consisting of* C_i *with framing* m_i*, plus a longitude* C_i' *with framing* m_i *and with the opposite orientation, with linking number* $\mathrm{lk}(C_i, C_i') = 1 - m_i$*, for each* $i = 1, \ldots, n$*.*

Proof. The fact that Y is the boundary of the four-manifold given by the framed link (1) follows from the continued fraction expansions $(2m_i - 1)/2 = m_i - \frac{1}{2}$. The equivalence between (1) and (2) follows by handlesliding: add C_i to C_i' to go from (2) to (1).

3. Proof of Theorem 1

We start with an algorithm for drawing a Kirby diagram of a four-manifold W bounded by $\Sigma(K)$. (For more details on Kirby diagrams of cyclic branched covers see [3, §6.3]; indeed what follows is a slight variation of the method in their Exercise 6.3.5(c).)

Let D be a diagram for a knot K which becomes a diagram for the unknot after changing some chosen set of p positive and n negative crossings. We think of $K \subset S^3$ as being in the boundary of B^4. Draw $(p + n)$ unlinked unknots beside D, each with

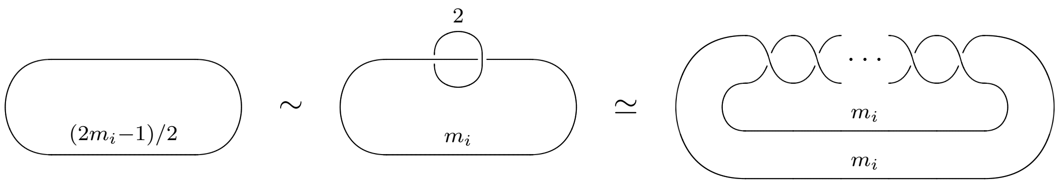

FIGURE 4. **Half integer surgery.** There are $2m_i - 2$ crossings in the diagram on the right.

framing $+1$. This is a Kirby diagram which represents K as a knot in the boundary of $X = B^4 \#(p+n)\mathbb{CP}^2$. As observed in [2], the knot K bounds a disk Δ in X. This may be seen from the diagram by sliding each of the chosen crossings in D over a $+1$-framed unknot as in Figure 5. Mark each of these changed crossings with a small arc α_i, $i = 1, \dots, (p+n)$, as shown in that figure.

The resulting diagram consists of:

- an unknot U which has been obtained from K by crossing changes;
- arcs $\alpha_1, \dots, \alpha_{p+n}$ (one per changed crossing);
- $+1$-framed unknots $\gamma_1, \dots, \gamma_{p+n}$.

Each γ_i bounds a disk D_i which retracts onto α_i and whose intersection with U consists of the endpoints of α_i.

Rearrange the diagram so that a point of U which is not the endpoint of an arc α_i is the point at infinity and U is a straight line; then Δ may be seen in this diagram as a half-plane with boundary U. (The arcs α_i may be knotted and linked in this diagram.) We may suppose all of the α_i are disjoint from the half-plane Δ, so that each γ_i intersects Δ in two points. Let $w(\gamma_i)$ denote the *writhe* of γ_i (i.e. the signed count of self-crossings). Similarly let $w(\alpha_i)$ denote the signed count of self-crossings of the arc α_i.

It is now straightforward to draw a diagram of the double cover W of X branched along Δ (take two copies of $S^3 - U$ cut open along Δ, and join the boundary half-planes in pairs). Each arc α_i lifts to a knot $\tilde{\alpha}_i$, and each D_i lifts to an annulus $\tilde{D}_i$ with core $\tilde{\alpha}_i$. The knot γ_i lifts to two knots C_i, C_i'; these are the boundary of the annulus $\tilde{D}_i$.

To compute the framings on these knots note that the blackboard framing of γ_i lifts to the blackboard framings on each of C_i, C_i'. The blackboard framing of γ_i is $w(\gamma_i)$. The blackboard framing of C_i and C_i' are each given by $w(\tilde{\alpha}_i) = 2w(\alpha_i)$. Thus the framing $+1$ on γ_i lifts to $2w(\alpha_i) - w(\gamma_i) + 1$; denote this number by m_i. The linking number $\mathrm{lk}(C_i, C_i')$ is $w(\gamma_i) - w(\tilde{\alpha}_i) = 1 - m_i$. (The number of twists in the annulus $\tilde{D}_i$ is $w(\gamma_i)$, while $-w(\tilde{\alpha}_i)$ counts crossings of C_i and C_i' due to knotting of $\tilde{D}_i$.)

We note that the resulting Kirby diagram for W matches that in Lemma 2.2 (2). That lemma then shows that $\Sigma(K) = \partial W$ is Dehn surgery on the framed link $L = C_1, \dots, C_{p+n}$ with framing $(2m_i - 1)/2$ on C_i. Thus we have recovered Montesinos' description of $\Sigma(K)$ as half-integral surgery on a link. But this four-dimensional description yields more information: the rank and signature of the intersection pairing of W are computed by Cochran and Lickorish in [2, Theorem 3.7] as follows:

$$\begin{aligned} b_2(W) &= 2(p+n) \\ \sigma(W) &= \sigma(K) + 2p. \end{aligned}$$

In particular if $\sigma(K) = 2n$ then W is positive-definite. Note that the inequalities $i < j \implies |a_{ij}| < m_i \leq m_j$ may then be satisfied after applying a finite number of rational handleslides as in Proposition 2.1 to the link L. (These rational handleslides are just a composition of handleslides on a Kirby diagram for W; in particular they preserve W as well as its boundary.)

Finally we must establish the claim that n of the numbers $m_1, \dots, m_{p+n}$ are even. Let $\mathcal{U}$ denote the set of crossings in the diagram D that we change to unknot K. We have given a description of $\Sigma(K)$ as half-integral surgery on a link L, with one component for each crossing in $\mathcal{U}$. Dehn surgery on a sublink of L gives the double branched cover of a knot which is obtained from D by changing a subset of the crossings in $\mathcal{U}$. In particular $(2m_i - 1)/2$ surgery on the knot C_i yields the double branched cover of the knot K' which is obtained by changing all but one of the crossings in $\mathcal{U}$.

The determinant $\det(K)$ of a knot K is equal to the order of $H^1(\Sigma(K); \mathbb{Z})$, which in turn is equal to the determinant of any simply connected positive-definite four-manifold W bounded by $\Sigma(K)$. Moreover the determinant and signature of K are shown in [5, Theorem 5.6] to satisfy

$$\det(K) \equiv \sigma(K) + 1 \pmod 4.$$

The fact that $\sigma(K) = 2n$ implies that every change of a negative crossing in $\mathcal{U}$ reduces the knot signature by 2, while every change of a positive crossing leaves the signature unchanged. It follows that if the knot K' is obtained by changing all but one crossing in $\mathcal{U}$, then K' has signature 2 if that crossing is negative and signature zero otherwise. The determinant of K' is $2m_i - 1$; thus m_i is even if and only if $\sigma(K') = 2$. Finally note that the parities of $m_1, \dots, m_{p+n}$ are not changed by rational handlesliding. $\square$

4. Heegaard Floer homology

In this section we recall some properties of the Heegaard Floer homology invariants of Ozsváth and Szabó. Details are to be found in their papers, in particular [9, 10, 11].

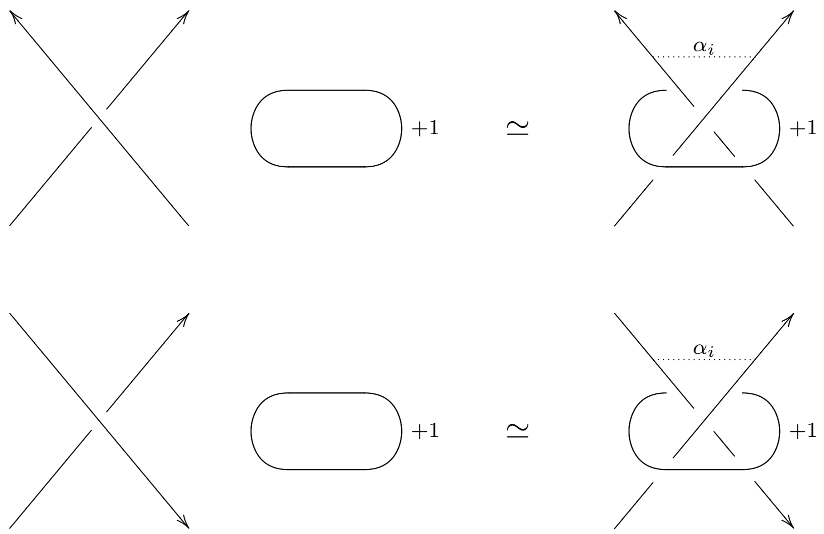

FIGURE 5. Changing crossings by sliding over a two-handle.

Let Y be an oriented rational homology sphere. Recall that the space $\mathrm{Spin}^c(Y)$ of spin^c structures on Y is isomorphic to $H^2(Y;\mathbb{Z})$; if $|H^2(Y;\mathbb{Z})|$ is odd then there is a canonical isomorphism which takes the unique spin structure to zero.

Fixing a spin^c structure $\mathfrak{s}$, the Heegaard Floer homology $HF^+(Y;\mathfrak{s})$ is a $\mathbb{Q}$-graded abelian group with an action by $\mathbb{Z}[U]$, where U lowers the grading by 2. The *correction term* invariant is a rational number $d(Y,\mathfrak{s})$; it is defined to be the lowest grading of a nonzero homogeneous element of $HF^+(Y;\mathfrak{s})$ which is in the image of U^n for all $n \in \mathbb{N}$. These have the property that $d(Y,\mathfrak{s}) = -d(-Y,\mathfrak{s})$, where $-Y$ denotes Y with the opposite orientation. We will describe below how these correction terms may be computed in certain cases.

Now let X be a positive-definite four-manifold with boundary Y. Then it is shown in [9] that for any spin^c structure $\mathfrak{s}$ on X,

$$c_1(\mathfrak{s})^2 - b_2(X) \geq 4d(Y,\mathfrak{s}|_Y), \tag{6}$$

$$\text{and} \quad c_1(\mathfrak{s})^2 - b_2(X) \equiv 4d(Y,\mathfrak{s}|_Y) \pmod 2. \tag{7}$$

This means that the correction terms of Y may be used to give an obstruction to Y bounding a four-manifold X with a given positive-definite intersection form. We will now elaborate on how this may be checked in practice.

Suppose for simplicity that X is simply-connected and that $|H^2(Y;\mathbb{Z})|$ is odd. Let n denote the second Betti number of X. Fix a basis for $H_2(X;\mathbb{Z})$ and thus an

isomorphism
$$H_2(X;\mathbb{Z}) \cong \mathbb{Z}^n.$$
Let Q be the matrix of the intersection pairing of X in this basis; thus Q is a symmetric positive-definite $n\times n$ integer matrix with $\det Q = |H^2(Y;\mathbb{Z})|$. The dual basis gives an isomorphism between the second cohomology $H^2(X;\mathbb{Z})$ and $\mathbb{Z}^n$. The set $\{c_1(\mathfrak{s}) \mid \mathfrak{s} \in \mathrm{Spin}^c(X)\} \subset H^2(X;\mathbb{Z})$ of first Chern classes of spin^c structures is equal to the set of characteristic covectors $Char(Q)$ for Q. These in turn are elements ξ of $\mathbb{Z}^n$ whose components ξ_i are congruent modulo 2 to the corresponding diagonal entries Q_{ii} of Q. The square of the first Chern class of a spin^c structure is computed using the pairing induced by Q on $H^2(X;\mathbb{Z})$; in our choice of basis this is given by $\xi^T Q^{-1}\xi$.

The long exact sequence of the pair (X,Y) yields the following short exact sequence:
$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{Q} \mathbb{Z}^n \longrightarrow H^2(Y;\mathbb{Z}) \longrightarrow 0.$$
Define a function
$$m_Q : \mathbb{Z}^n/Q(\mathbb{Z}^n) \to \mathbb{Q}$$
by
$$m_Q(g) = \min\left\{ \frac{\xi^T Q^{-1}\xi - n}{4} \;\middle|\; \xi \in Char(Q),\ [\xi] = g \right\}.$$
An easy argument shows that in computing m_Q it suffices to consider characteristic covectors $\xi = (\xi_1,\ldots,\xi_n)$ whose components are smaller in absolute value than the corresponding diagonal entries of Q:
$$-Q_{ii} \le \xi_i \le Q_{ii}.$$
(A more difficult argument in [10] shows that it suffices to restrict to
$$-Q_{ii} \le \xi_i \le Q_{ii} - 2.)$$
Thus it is straightforward, if tedious, to compute m_Q for a given positive-definite matrix Q.

The conditions (6) and (7) may now be expressed as follows: there exists a group isomorphism
$$\phi : \mathbb{Z}^n/Q(\mathbb{Z}^n) \to \mathrm{Spin}^c(Y)$$
with
$$m_Q(g) \ \ge\ d(Y,\phi(g)), \tag{8}$$
$$\text{and} \quad m_Q(g) \ \equiv\ d(Y,\phi(g)) \pmod 2 \tag{9}$$
for all $g \in \mathbb{Z}^n/Q(\mathbb{Z}^n)$.

The four-manifold X is said to be *sharp* if equality holds in (6). In this case the correction terms for Y can be computed using the function m_Q described above. Also, if a rational homology sphere Y bounds a negative-definite four-manifold X such that $-X$ is sharp, then the correction terms for Y can be computed using the formula $d(Y,\mathfrak{s}) = -d(-Y,\mathfrak{s})$. Note that if K is a knot in S^3 then the standard orientation on

S^3 induces an orientation on $\Sigma(K)$; letting $r(K)$ denote the reflection of K, we have $\Sigma(r(K)) \cong -\Sigma(K)$.

In particular let K be an alternating knot with double branched cover $\Sigma(K)$. Let G denote the Goeritz matrix computed from an alternating diagram for K (see e.g. [11]). This is a definite matrix; after reflecting K if necessary we may suppose it is positive-definite. It is shown in [11, Proposition 3.2] that G represents the intersection pairing of a sharp four-manifold bounded by $\Sigma(K)$. Thus the correction terms for $\Sigma(K)$ are given by m_G.

Also if K is a Montesinos knot then the double branched cover $\Sigma(K)$ is a Seifert fibred space which is given as the boundary of a plumbing of disk bundles over S^2. This plumbing is determined (nonuniquely) by the Montesinos invariants which specify K. After possibly reflecting K we may choose the plumbing so that its intersection pairing is represented by a positive-definite matrix P. It is shown in [10] that the plumbing is sharp, so that the correction terms for $\Sigma(K)$ are given by m_P. (See [6] for a detailed description of Montesinos knots and their branched double covers.)

Remark 4.1. *Checking the congruence condition (7) alone is equivalent to checking that the intersection pairing of X presents the linking pairing of Y; see* [8] *for a detailed discussion.*

5. Examples

Suppose that K is a knot for which the correction terms of $\Sigma(K)$ are known (for example if K is alternating or Montesinos they may be computed as in Section 4.) We wish to investigate whether it is possible to unknot K by changing p positive crossings and $n = \sigma(K)/2$ negative crossings. For ease of exposition we restrict to the case that $p+n = 2$. Then it follows from Theorem 1 that $\Sigma(K)$ is given by surgery on a two-component link C_1, C_2 with framing $(2m_i - 1)/2$ on C_i and $\operatorname{lk}(C_1, C_2) = a$. Moreover (after possibly changing the orientation of one component) these numbers satisfy

$$0 \leq a < m_1 \leq m_2,$$

and exactly n of m_1, m_2 are even. By Lemma 2.2 we see that $\Sigma(K)$ bounds a four-dimensional 2-handlebody W with intersection pairing

$$Q = \begin{pmatrix} m_1 & 1 & a & 0 \\ 1 & 2 & 0 & 0 \\ a & 0 & m_2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix};$$

Note also that Q is positive-definite. From the long exact sequence of the pair $(W, \Sigma(K))$ it follows that the determinant of Q is equal to the order of $H^2(\Sigma(K); \mathbb{Z})$, which in turn is equal to the determinant of K. There are finitely many choices of

(m_1, a, m_2) satisfying these conditions, which imply for example that

$$4a < \det K.$$

Thus there are finitely many possibilities for the matrix Q. For each one we compute the function m_Q as in Section 4. (We need to consider $4m_1m_2$ characteristic covectors ξ for Q in this computation.) If for all Q there does not exist a group isomorphism ϕ satisfying (8) and (9), we conclude that K cannot be unknotted by changing p positive crossings and $n = \sigma(K)/2$ negative crossings.

Proof of Corollary 2. For each knot in Corollary 2 we distinguish between K and its reflection $r(K)$ by specifying that K has positive signature.

We start with the knot $K = 9_{10}$ shown in Figure 6. This is the two-bridge knot $S(33, 23)$. It has signature 4, and it is easy to see that 3 crossing changes suffice to unknot it. Thus the unknotting number is either 2 or 3, and if it can be unknotted by changing two crossings then both are negative ($p = 0$ and $n = 2$).

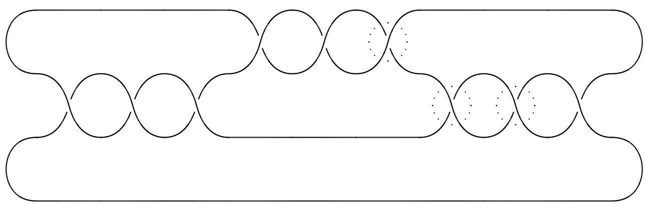

FIGURE 6. **The knot $9_{10} = S(33, 23)$.** Note that changing the circled crossings will give the unknot.

Two-bridge knots are both alternating and Montesinos, so the correction terms of $\Sigma(K)$ may be computed using either method described in Section 4, or alternatively using the recursion formula given in [9, Proposition 4.8]. (See also [6].) We find them to be:

$$A = \left\{ \begin{array}{ccccccccccc} -1, & -\frac{23}{33}, & \frac{7}{33}, & -\frac{3}{11}, & -\frac{5}{33}, & \frac{19}{33}, & -\frac{1}{11}, & -\frac{5}{33}, & \frac{13}{33}, & -\frac{5}{11}, & -\frac{23}{33}, \\ -\frac{1}{3}, & \frac{7}{11}, & \frac{7}{33}, & \frac{13}{33}, & \frac{13}{11}, & \frac{19}{33}, & \frac{19}{33}, & \frac{13}{11}, & \frac{13}{33}, & \frac{7}{33}, & \frac{7}{11}, \\ -\frac{1}{3}, & -\frac{23}{33}, & -\frac{5}{11}, & \frac{13}{33}, & -\frac{5}{33}, & -\frac{1}{11}, & \frac{19}{33}, & -\frac{5}{33}, & -\frac{3}{11}, & \frac{7}{33}, & -\frac{23}{33} \end{array} \right\}.$$

The order of this list corresponds to the cyclic group structure of $\mathrm{Spin}^c(\Sigma(K)) \cong H^2(\Sigma(K); \mathbb{Z})$, and the first element is the correction term of the spin structure.

The determinant of 9_{10} is 33. To find a matrix Q as above we need to find (m_1, a, m_2) with

$$(2m_1 - 1)(2m_2 - 1) - 4a^2 = 33,$$

$$0 \leq a < m_1 \leq m_2,$$

and m_1 and m_2 are even. There are two solutions: $(2, 0, 6)$ and $(4, 2, 4)$. Computing m_Q for each of the matrices

$$Q_1 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 4 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

yields the following lists:

$$B_1 = \left\{ \begin{array}{ccccccccccc} -1, & -\frac{5}{33}, & \frac{13}{33}, & \frac{7}{11}, & \frac{19}{33}, & \frac{7}{33}, & -\frac{5}{11}, & \frac{19}{33}, & \frac{43}{33}, & -\frac{3}{11}, & -\frac{5}{33}, \\ -\frac{1}{3}, & -\frac{9}{11}, & \frac{13}{33}, & \frac{43}{33}, & -\frac{1}{11}, & \frac{7}{33}, & \frac{7}{33}, & -\frac{1}{11}, & \frac{43}{33}, & \frac{13}{33}, & -\frac{9}{11}, \\ -\frac{1}{3}, & -\frac{5}{33}, & -\frac{3}{11}, & \frac{43}{33}, & \frac{19}{33}, & -\frac{5}{11}, & \frac{7}{33}, & \frac{19}{33}, & \frac{7}{11}, & \frac{13}{33}, & -\frac{5}{33} \end{array} \right\},$$

$$B_2 = \left\{ \begin{array}{ccccccccccc} -1, & -\frac{19}{33}, & \frac{23}{33}, & \frac{9}{11}, & -\frac{7}{33}, & -\frac{13}{33}, & \frac{3}{11}, & -\frac{7}{33}, & \frac{5}{33}, & -\frac{7}{11}, & -\frac{19}{33}, \\ \frac{1}{3}, & \frac{1}{11}, & \frac{23}{33}, & \frac{5}{33}, & \frac{5}{11}, & -\frac{13}{33}, & -\frac{13}{33}, & \frac{5}{11}, & \frac{5}{33}, & \frac{23}{33}, & \frac{1}{11}, \\ \frac{1}{3}, & -\frac{19}{33}, & -\frac{7}{11}, & \frac{5}{33}, & -\frac{7}{33}, & \frac{3}{11}, & -\frac{13}{33}, & -\frac{7}{33}, & \frac{9}{11}, & \frac{23}{33}, & -\frac{19}{33} \end{array} \right\}.$$

We claim that for both Q_1 and Q_2 it is impossible to find a group automorphism ϕ of $\mathbb{Z}/33$ for which (8) and (9) are satisfied. This is immediate in either case by considering the minimal elements. We have the entry $-9/11$ in B_1. By inspection there is no element in A which is less than or equal to $-9/11$, and differs from it by a multiple of 2. The same applies to $-7/11$ in B_2. We conclude that 9_{10} cannot be unknotted by two crossing changes and $u(9_{10}) = 3$.

Similar calculations show that $9_{13}, 9_{38}, 10_{53}, 10_{101}$ and 10_{120} cannot be unknotted with two crossing changes. All of these knots are alternating, have signature four and cyclic $H^2(\Sigma(K); \mathbb{Z})$. By inspection of their diagrams (see e. g. [1]), all can be unknotted with three crossing changes. For some details of the calculations for these knots, see Table 1.

Finally consider $K = 9_{35}$. This is the Montesinos knot $M(0; (3, 1), (3, 1), (3, 1))$. It has signature 2 and can be unknotted with 3 crossing changes. The opposite $-\Sigma(K)$ of its double branched cover $\Sigma(K)$ is the boundary of the plumbing of disk bundles over two-spheres specified by the graph in Figure 7. This has intersection pairing represented by the matrix

$$P = \begin{pmatrix} 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

We note that this presents $H^2(-\Sigma(K);\mathbb{Z})$ which is thus 2-cyclic; this shows (by Montesinos' theorem for example but by an inequality originally due to Wendt) that $u(K) \geq 2$. We can also use $-m_P$ to compute the correction terms of $\Sigma(K)$, which are

$$A = \begin{bmatrix} -\frac{1}{2} & \frac{19}{18} & -\frac{5}{18} & \frac{3}{2} & \frac{7}{18} & \frac{7}{18} & \frac{3}{2} & -\frac{5}{18} & \frac{19}{18} \\ \frac{1}{6} & -\frac{5}{18} & \frac{7}{18} & \frac{1}{6} & \frac{19}{18} & \frac{19}{18} & \frac{1}{6} & \frac{7}{18} & -\frac{5}{18} \\ \frac{1}{6} & -\frac{5}{18} & \frac{7}{18} & \frac{1}{6} & \frac{19}{18} & \frac{19}{18} & \frac{1}{6} & \frac{7}{18} & -\frac{5}{18} \end{bmatrix}.$$

Here the rectangular array shows the $\mathbb{Z}/3 \oplus \mathbb{Z}/9$ group structure; the top left entry is the correction term of the spin structure.

Suppose that 9_{35} may be unknotted by changing one positive and one negative crossing. Applying Theorem 1 we find that $\Sigma(K)$ is the boundary of a 2-handlebody with intersection pairing

$$Q = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

(This is the only matrix for which (m_1, a, m_2) satisfy the conclusions of Theorem 1 and which presents $\mathbb{Z}/3 \oplus \mathbb{Z}/9$.) Computing m_Q yields another array whose minimal entry is $-17/18$; we conclude that there is no automorphism ϕ of $\mathbb{Z}/3 \oplus \mathbb{Z}/9$ satisfying (8) and (9).

This is not enough to rule out the possibility that $u(9_{35}) = 2$; it does however show that if 9_{35} can be unknotted by two crossing changes, then they are both negative crossings. Using the value of the Jones polynomial at $e^{i\pi/3}$, Traczyk has shown in [12] that if 9_{35} can be unknotted by changing two crossings, then the crossings have different signs. We conclude that $u(9_{35}) = 3$. $\square$

Proof of Corollary 3. The knot $K = 10_{145}$ has signature 2. It is shown in [7] that the double branched cover $\Sigma(K)$ does not bound any positive-definite four-manifold. It follows from Theorem 1 that if K is unknotted by changing p positive and n negative crossing changes, then $n > 1$. $\square$

Knot	Correction term data	$\min\limits_{\mathfrak{s}\neq\mathfrak{s}_0}\{d(\Sigma(K);\mathfrak{s})\}$	(m_1,a,m_2)	$\min\limits_{g\neq 0}\{m_Q(g)\}$
9_{13}	$S(37,27)$	$-\frac{27}{37}$	$(10,9,10)$	$-\frac{33}{37}$
9_{38}	$G=\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & -2 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$	$-\frac{37}{57}$	$(2,0,10)$	$-\frac{51}{57}$
			$(6,4,6)$	$-\frac{45}{57}$
10_{53}	$M(0;(2,1),(3,2),(7,4))$	$-\frac{53}{73}$	$(4,1,6)$	$-\frac{59}{73}$
10_{101}	$G=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 \\ 0 & -1 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}$	$-\frac{59}{85}$	$(6,3,6)$	$-\frac{65}{85}$
			$(22,21,22)$	$-\frac{81}{85}$
10_{120}	$G=\begin{pmatrix} 4 & -2 & 0 & -1 \\ -2 & 4 & -1 & 0 \\ 0 & -1 & 4 & -2 \\ -1 & 0 & -2 & 4 \end{pmatrix}$	$-\frac{69}{105}$	$(2,0,18)$	$-\frac{99}{105}$
			$(4,0,8)$	$-\frac{91}{105}$
			$(6,2,6)$	$-\frac{83}{105}$
			$(10,8,10)$	$-\frac{93}{105}$

TABLE 1. **Data for knots in Corollary 2.** Correction term data consists of either a two-bridge or Montesinos description of the knot, or a Goeritz matrix; in each case these enable one to compute the correction terms of $\Sigma(K)$. The fourth column contains the surgery coefficients satisfying the conclusions of Theorem 1.

Remark 5.1. *The obstruction described in this section to equality in (1) does not use all of the information from Theorem 1. We have only used the information about the intersection pairing of the four-manifold W bounded by $\Sigma(K)$, and not the fact that*

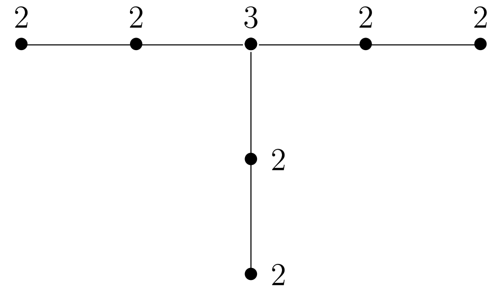

Figure 7. **Plumbing graph for $-\Sigma(9_{35})$.**

W is a surgery cobordism arising from a half-integral surgery. Comparing to Theorem 1.1 in [11], *we have generalised conditions (1) and (2) to the case of $u(K) > 1$ but not the symmetry condition (3). It is to be hoped that the symmetry condition may also be generalised in some way, and that this could lead to computation of some more unknotting numbers.*

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