UNKNOTTING INFORMATION FROM HEEGAARD FLOER HOMOLOGY

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Abstract. We use Heegaard Floer homology to obtain bounds on unknotting numbers. This is a generalisation of Ozsváth-Szabó's obstruction to unknotting number one. We determine the unknotting numbers of 9_{10}, 9_{13}, 9_{35}, 9_{38}, 10_{53}, 10_{101} and 10_{120}; this completes the table of unknotting numbers for prime knots with crossing number nine or less. Our obstruction uses a Kirby calculus description of a four-manifold W bounded by the branched double cover of the knot, and a theorem of Cochran and Lickorish which computes the signature of W.

1. Introduction

Let K be a knot in S^3. Given any diagram D for K, a new knot may be obtained by changing one or more crossings of D. The unknotting number $u(K)$ is the minimum number of crossing changes required to obtain the unknot, where the minimum is taken over all diagrams for K.

Let $\Sigma(K)$ denote the double cover of S^3 branched along K. A theorem of Montesinos [6] tells us that for any knot K, $\Sigma(K)$ is given by Dehn surgery on some framed link in S^3 with $u(K)$ components, with half-integral framing coefficients. In particular if $u(K) = 1$ then $\Sigma(K)$ is obtained by $\pm\delta/2$ Dehn surgery on a knot C, where δ is the determinant of K. Ozsváth and Szabó have shown in [13] that the Heegaard Floer homology of a 3-manifold Y gives an obstruction to Y being given by half-integral surgery on a knot in S^3; they apply this to $\Sigma(K)$ to obtain an obstruction to K having unknotting number one.

Note that crossings in a knot diagram may be given a sign as in Figure 1 (independent of the choice of orientation of the knot). Let $\sigma(K)$ denote the signature of a knot K. It is shown in [3, Proposition 2.1] that if K' is obtained from K by changing a positive crossing, then

$$\sigma(K') \in \{\sigma(K), \sigma(K) + 2\};$$

similarly if K' is obtained from K by changing a negative crossing then

$$\sigma(K') \in \{\sigma(K), \sigma(K) - 2\}.$$

Now suppose that K may be unknotted by changing p positive and n negative crossings (in some diagram). Since the unknot has zero signature, it follows that a bound

Date: May 1, 2005.

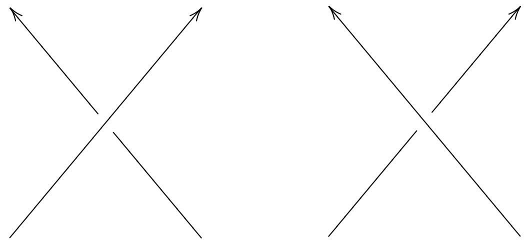

FIGURE 1. Signed crossings in a knot diagram.

for n is given by

$$n \geq \sigma(K)/2. \tag{1}$$

The main result of this paper is an obstruction to equality in (1). This is easiest to state for the case of an alternating knot; the obstruction is then a condition on the positive-definite Goeritz matrix obtained from an alternating projection of K. (We will recall the definition of the Goeritz matrix in Section 4.) We also restrict for now to knots which can be unknotted with two crossing changes.

A positive-definite integral matrix Q of rank r presents a finite group Γ_Q via the short exact sequence

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^r \longrightarrow \Gamma_Q \longrightarrow 0.$$

A *characteristic covector* for Q is an element of $\mathbb{Z}^r$ which is congruent modulo 2 to the diagonal of Q, i.e., an element of

$$\mathrm{Char}(Q) = \{\xi \in \mathbb{Z}^r \mid \xi_i \equiv Q_{ii} \pmod 2\}.$$

Define a function

$$m_Q : \Gamma_Q \to \mathbb{Q}$$

by

$$m_Q(g) = \min\left\{ \frac{\xi^T Q^{-1}\xi - r}{4} \;\middle|\; \xi \in \mathrm{Char}(Q),\ [\xi] = g \right\}.$$

(The minimum exists since Q is positive-definite.)

Theorem 1. *Let K be an alternating knot which may be unknotted by changing p positive and n negative crossings, where $n = \sigma(K)/2$ and $p + n = 2$. Let G be the positive-definite Goeritz matrix obtained from an alternating diagram for K. Then there exists a positive-definite matrix*

$$Q = \begin{pmatrix} m_1 & 1 & a & 0 \\ 1 & 2 & 0 & 0 \\ a & 0 & m_2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix},$$

with

$$\det Q = \det K,$$
$$0 \leq a < m_1 \leq m_2 \quad \textit{(and hence } a < \det K/4\textit{)},$$

and exactly n *of* $\{m_1, m_2\}$ *are even; and a group isomorphism*

$$\phi : \Gamma_Q \to \Gamma_G$$

with

$$\begin{array}{rrcl} & m_Q(g) & \geq & m_G(g), \\ \textit{and} & m_Q(g) & \equiv & m_G(g) \pmod 2 \end{array}$$

for all $g \in \Gamma_Q$*.*

Applying Theorem 1 to the alternating knots which were listed in [1] as having unknotting number 2 or 3 yields the following:

Corollary 2. *The knots* $9_{10}, 9_{13}, 9_{35}, 9_{38}, 10_{53}, 10_{101}, 10_{120}$ *have unknotting number 3.*

For all but one of the knots in Corollary 2, the signature is 4 and the unknotting number computation follows directly from Theorem 1. The exception is 9_{35}, whose signature is 2. The computation of $u(9_{35})$ uses Theorem 1 and also a result of Traczyk [14].

Corollary 2 completes the table of unknotting numbers for prime knots with 9 crossings or less.

Recall that for an oriented framed link $C_1, \dots, C_r$ in S^3, the linking matrix is the symmetric matrix (a_{ij}) with each diagonal entry a_{ii} given by the framing on C_i, and off-diagonal entries a_{ij} given by the linking numbers $\mathrm{lk}(C_i, C_j)$. The following theorem uses a result of Cochran and Lickorish [3, Theorem 3.7], and may be viewed as a refinement of Montesinos' theorem.

Theorem 3. *Suppose that a knot* K *may be unknotted by changing* p *positive and* n *negative crossings, with* $n = \sigma(K)/2$*. Then the branched double cover* $\Sigma(K)$ *may be obtained by Dehn surgery on an oriented, framed* $p+n$ *component link* $C_1, \dots, C_{p+n}$ *in* S^3 *with linking matrix* $\frac{1}{2}Q$*, where* Q *is a positive-definite integral matrix which is congruent to the identity modulo 2, and exactly* n *of the diagonal entries of* Q *are congruent to 3 modulo 4.*

By handlesliding and changing orientations one may replace the linking matrix with $\frac{1}{2}PQP^T$*, for any* $P \in GL(p+n, \mathbb{Z})$ *which is congruent to the identity modulo 2. This preserves the congruences modulo 4 on the diagonal.*

It is shown in [9] that the double branched cover of the Montesinos knot 10_{145} does not bound any positive-definite four-manifold. This knot has signature two. Combining this with Theorem 3, or indeed with the above-mentioned result of Cochran and Lickorish, yields the following:

Corollary 4. *If 10_{145} is unknotted by changing p positive crossings and n negative crossings, then $n \geq 2$.*

Given a matrix Q in $M(r, \mathbb{Z})$ which is conjugate modulo 2 to the identity, associate a matrix $\tilde{Q} \in M(2r, \mathbb{Z})$ by replacing each entry by a 2×2-block as follows:

$$\begin{aligned} &\text{odd entries:} \qquad 2m-1 \quad \mapsto \begin{bmatrix} m & 1 \\ 1 & 2 \end{bmatrix} \\ &\text{even entries:} \qquad 2a \quad \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{2}$$

Thus for example if $r = 2$,

$$Q = \begin{pmatrix} 2m_1 - 1 & 2a \\ 2a & 2m_2 - 1 \end{pmatrix} \mapsto \tilde{Q} = \begin{pmatrix} m_1 & 1 & a & 0 \\ 1 & 2 & 0 & 0 \\ a & 0 & m_2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

For a rational homology three-sphere Y, the *correction terms* of Ozsváth and Szabó are a set of rational numbers $\{d(Y, \mathfrak{s}) \mid \mathfrak{s} \in \mathrm{Spin}^c(Y)\}$ which provide constraints on which four-manifolds Y may bound. We recall these constraints in Section 4; combining these with Theorem 3 yields the following unknotting obstruction, of which Theorem 1 is a special case.

Theorem 5. *Let K be a knot in S^3 which may be unknotted by changing p positive and n negative crossings, where $n = \sigma(K)/2$. Let $Q_1, \ldots, Q_k$ be a complete set of representatives of the finite quotient*

$$\frac{\{Q \in M(p+n, \mathbb{Z}) \mid Q \text{ is positive-definite, } \det Q = \det K,\ Q \equiv I \pmod 2\}}{\{P \in GL(p+n, \mathbb{Z}) \mid P \equiv I \pmod 2\}},$$

and let $\tilde{Q}_1, \ldots, \tilde{Q}_k$ be the corresponding elements of $M(2(p+n), \mathbb{Z})$. Then for some Q_i which has exactly n diagonal entries conjugate to 3 modulo 4, there exists a group isomorphism

$$\phi : \Gamma_{\tilde{Q}_i} \to \mathrm{Spin}^c(\Sigma(K))$$

with

$$\begin{aligned} m_{\tilde{Q}_i}(g) &\geq d(\Sigma(K), \phi(g)), \\ \textit{and} \quad m_{\tilde{Q}_i}(g) &\equiv d(\Sigma(K), \phi(g)) \pmod 2 \end{aligned}$$

for all $g \in \Gamma_{\tilde{Q}_i}$.

Acknowledgements. The problem of generalising the obstruction in [13] to higher unknotting numbers was suggested to me by Peter Ozsváth. I am grateful to Peter Ozsváth, Ravi Ramakrishna, and Sašo Strle for helpful discussions. Some Maple programs used in verifying Corollary 2 were written jointly with Sašo Strle.

2. Kirby-Rolfsen calculus

In this section we establish some preliminaries on Dehn surgery. For details on Dehn surgery and Kirby-Rolfsen calculus see [5].

A framed link L in S^3 with rational framing coefficients determines a three-manifold Y_L by Dehn surgery (remove a tubular neighbourhood of each component of L; the framing coefficient determines the gluing map to sew back a solid torus along the boundary). If the framing coefficients are integers one obtains a four-manifold W_L with boundary Y_L by attaching two-handles to B^4 along the components of L. Kirby-Rolfsen calculus describes when two framed links L, L' determine the same three-manifold Y_L.

Given a framed oriented link L with components $C_1, \ldots, C_m$, let A denote the free abelian group with generators $c_1, \ldots, c_m$. Define a symmetric bilinear form

$$Q : A \times A \to \mathbb{Q}$$

by

$$Q(c_i, c_j) = \begin{cases} \text{framing coefficient of } C_i & \text{if } i = j; \\ \text{linking number } \mathrm{lk}(C_i, C_j) & \text{if } i \neq j. \end{cases}$$

In other words, the matrix of Q in the basis $c_1, \ldots, c_m$ is the linking matrix of L. (This is the intersection pairing on $H_2(W_L; \mathbb{Z})$ if the diagonal entries are integers.)

In the case that the framing coefficients on L are integers, any change of basis in A may be realised by a change in the link L. In particular the change of basis $c_i \mapsto c_i \pm c_j$ may be realised by a *handleslide*. Let λ_j denote a pushoff of C_j whose linking number with C_j equals the framing of C_j. A handleslide $C_i \mapsto C_i \pm C_j$ consists of replacing C_i by the oriented band sum of C_i with $\pm\lambda_j$. This gives a new link L' whose linking matrix is the matrix of Q in the basis $c_1, \ldots, c_i' = c_i \pm c_j, \ldots, c_m$ and with $Y_{L'} \cong Y_L$, $W_{L'} \cong W_L$. It will be convenient to have the following generalisation of handlesliding to links with rational framings.

Proposition 2.1. *Let L be an oriented link in S^3 consisting of components $C_1, \ldots, C_m$ with framings $\frac{p_1}{q_1}, \ldots, \frac{p_m}{q_m}$, and let Q be the rational-valued bilinear pairing determined by the linking matrix of L. Then by replacing C_i in L it is possible to obtain a link L' whose linking matrix is the matrix of Q in the basis $c_1, \ldots, c_i' = c_i \pm q_j c_j, \ldots, c_m$ and with $Y_{L'} \cong Y_L$.*

Proof. For each $j = 1, \ldots, m$ choose a continued fraction expansion

$$\frac{p_j}{q_j} = a^j_{l_j} - \cfrac{1}{a^j_{l_j - 1} - \ddots - \cfrac{1}{a^j_1}}.$$

(The numbers $a^j_{l_j}, \dots, a^j_1$ arise from the Euclidean algorithm as follows:

$$\begin{aligned} r_{l_j} = p_j &= a^j_{l_j} q_j - r_{l_j-2} \\ r_{l_j-1} = q_j &= a^j_{l_j-1} r_{l_j-2} - r_{l_j-3} \\ &\vdots \\ r_2 &= a^j_2 r_1 - 1 \\ r_1 &= a^j_1.) \end{aligned} \tag{3}$$

Use reverse "slam-dunks" to obtain an integral surgery description of Y_L: as shown in Figure 2, we add a chain of linked unknots linking each C_j, with framings $a^j_1, \dots, a^j_{l_j-1}$, and replace the framing on C_j with $a^j_{l_j}$. (This is a standard procedure, see e.g. [5, §5.3].) Denote the resulting link by $L_\mathbb{Z}$, and let $Q_\mathbb{Z} : A_\mathbb{Z} \times A_\mathbb{Z} \to \mathbb{Z}$ denote the resulting bilinear form.

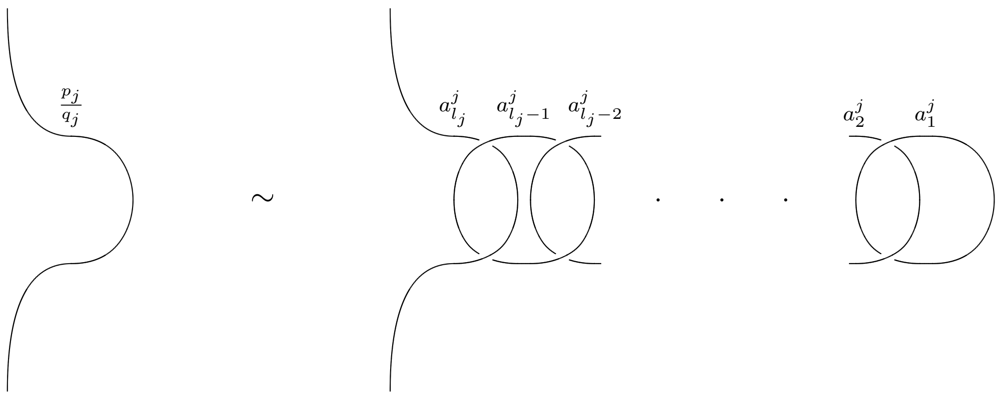

FIGURE 2. Converting Dehn surgery to integral surgery.

We now perform handleslides on this integer-framed link. Let $U_1, \dots, U_{l_j-1}$ be the chain of unknots linking C_j as above, oriented so that $\mathrm{lk}(C_j, U_{l_j-1}) = \mathrm{lk}(U_k, U_{k-1}) = -1$, for $2 \le k < l_j$. Let $K_1 = C_i + U_1$, and note that

$$\begin{aligned} \mathrm{lk}(K_1, U_1) &= a^j_1, \\ \mathrm{lk}(K_1, U_2) &= -1. \end{aligned} \tag{4}$$

We now define K_k recursively for $2 \le k < l_j$. Choose any link diagram of $K_{k-1} \cup U_{k-1} \cup U_k$. By performing a handleslide over U_k for each crossing where K_{k-1} crosses over U_{k-1} we obtain a knot K_k which does not cross over U_{k-1} and therefore is separated from it by a two-sphere in S^3 (see Figure 3). The signed count of these handleslides is equal to the linking number of K_{k-1} and U_{k-1}; thus we write

$$[K_k] = [K_{k-1}] + \mathrm{lk}(K_{k-1}, U_{k-1})[U_k],$$

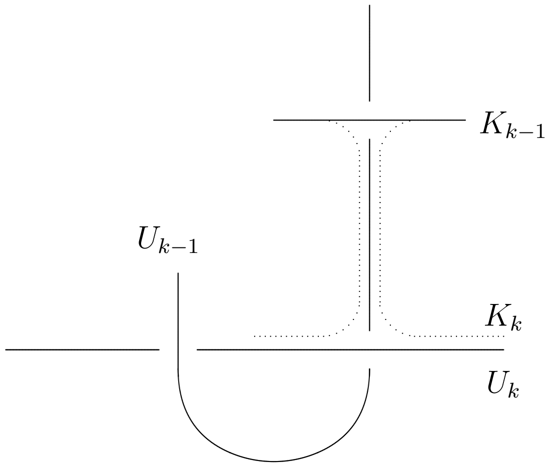

FIGURE 3. Handlesliding K_{k-1} over U_k yields K_k which is separated from U_{k-1} by a two-sphere.

where $[K_k]$ denotes the element of $A_{\mathbb{Z}}$ corresponding to the knot K_k. We may use this to compute linking numbers and the framing of K_k. In particular

$$\mathrm{lk}(K_2, U_2) = -1 + a_2^j \,\mathrm{lk}(K_1, U_1), \tag{5}$$

and for $2 < k < l_j$,

$$\begin{aligned} \mathrm{lk}(K_k, U_k) &= \mathrm{lk}(K_{k-1}, U_k) + a_k^j \,\mathrm{lk}(K_{k-1}, U_{k-1}) \\ &= -\mathrm{lk}(K_{k-2}, U_{k-2}) + a_k^j \,\mathrm{lk}(K_{k-1}, U_{k-1}). \end{aligned} \tag{6}$$

Finally we let C_i' be obtained as above from K_{l_j-1} by sliding over C_j, with C_i' unlinked from each of $U_1, \ldots, U_{l_j-1}$. We then have

$$\begin{aligned} [C_i'] &= [K_{l_j-1}] + \mathrm{lk}(K_{l_j-1}, U_{l_j-1})[C_j], \\ \mathrm{lk}(C_i', C_j) &= -\mathrm{lk}(K_{l_j-2}, U_{l_j-2}) + a_{l_j}^j \mathrm{lk}(K_{l_j-1}, U_{l_j-1}) + \mathrm{lk}(C_i, C_j). \end{aligned} \tag{7}$$

Comparing (4), (5), (6), and (7) to (3) we see that

$$\begin{aligned} \mathrm{lk}(K_k, U_k) &= r_k^j \quad \text{for } k = 1, \ldots, l_j - 2, \\ \mathrm{lk}(K_{l_j-1}, U_{l_j-1}) &= r_{l_j-1}^j = q_j, \\ \mathrm{lk}(C_i', C_j) &= p_j + \mathrm{lk}(C_i, C_j). \end{aligned}$$

This yields

$$[C_i'] = [C_i] + \mathcal{U} + q_j[C_j],$$

where

$$\mathcal{U} = [U_1] + \sum_{k=2}^{l_j-1} r_{k-1}[U_k].$$

Note that by construction C_i' is separated by a two-sphere from each U_k and so $Q_{\mathbb{Z}}([C_i'],\mathcal{U}) = 0$. The framing of C_i' is given by

$$\begin{aligned} Q_{\mathbb{Z}}([C_i'],[C_i']) &= Q_{\mathbb{Z}}([C_i] + \mathcal{U} + q_j[C_j], [C_i] + \mathcal{U} + q_j[C_j]) \\ &= Q_{\mathbb{Z}}([C_i] + q_j[C_j], [C_i] + \mathcal{U} + q_j[C_j]) \\ &= Q_{\mathbb{Z}}([C_i],[C_i]) + 2q_jQ_{\mathbb{Z}}([C_i],[C_j]) + q_j^2 a^j_{l_j} - q_j r_{l_j-2} \\ &= a^i_{l_i} + 2q_j \mathrm{lk}(C_i, C_j) + p_j q_j. \end{aligned}$$

Slam dunking to remove the chains of linking unknots from each of $C_1, \ldots, C_i', \ldots, C_m$ gives the required link L' for the basis change $c_i' = c_i + q_jc_j$. To get the opposite sign construct C_i' as above but start with $K_1 = C_i - U_1$. □

The following lemma is an application of the standard procedure, referred to in the proof of Proposion 2.1 and illustrated in Figure 2, for converting a Dehn surgery description of a three-manifold to an integral surgery description.

Lemma 2.2. *Let $L = \{C_1, \ldots, C_r\}$ be a framed link in S^3 with framing $(2m_i - 1)/2$ on C_i, and let Y be the three-manifold obtained by Dehn surgery on L. Then Y is equal to the boundary of the four-manifold W obtained by adding 2-handles to B^4 along either of the following $2n$-component framed links (as in Figure 4):*

(i) *the link consisting of the components C_i with framing m_i plus a small linking unknot with framing 2, for each $i = 1, \ldots, r$;*
(ii) *the link consisting of C_i with framing m_i, plus a longitude C_i' with framing m_i and with the opposite orientation, with linking number $\mathrm{lk}(C_i, C_i') = 1 - m_i$, for each $i = 1, \ldots, r$.*

Proof. The fact that Y is the boundary of the four-manifold given by the framed link (i) follows from the continued fraction expansions $(2m_i - 1)/2 = m_i - \frac{1}{2}$. The equivalence between (i) and (ii) follows by handlesliding: add C_i to C_i' to go from (ii) to (i). □

Recall that to each matrix $Q \in M(r, \mathbb{Z})$ which is congruent to the identity modulo 2, we associate the matrix $\tilde{Q} \in M(2r, \mathbb{Z})$ as in (2). If a 3-manifold Y is given by Dehn surgery on a link with linking matrix $\frac{1}{2}Q$, then by Lemma 2.2, Y is the boundary of a simply-connected four-manifold with intersection pairing $\tilde{Q}$. Also note that $\det Q = \det \tilde{Q}$, and Q is positive-definite if and only if $\tilde{Q}$ is positive-definite: let

$$\Delta_k(Q) = \det(Q_{ij})_{i,j\leq k}.$$

Then

$$\begin{aligned}\Delta_{2k}(\tilde{Q}) &= \Delta_k(Q),\\ \Delta_{2k-1}(\tilde{Q}) &= (\Delta_{2k-2}(\tilde{Q}) + \Delta_{2k}(\tilde{Q}))/2.\end{aligned}$$

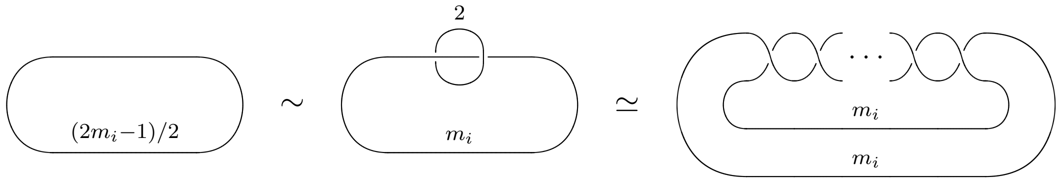

FIGURE 4. **Half-integer surgery.** There are $2m_i - 2$ crossings in the diagram on the right.

3. Proof of Theorem 3

We start with an algorithm for drawing a Kirby diagram of a four-manifold W bounded by $\Sigma(K)$. (For more details on Kirby diagrams of cyclic branched covers see [5, §6.3]; indeed what follows is a variation of the method in their Exercise 6.3.5(c).)

Let D be a diagram for a knot K which becomes a diagram for the unknot after changing some chosen set of p positive and n negative crossings. We think of $K \subset S^3$ as being in the boundary of B^4. Draw $(p+n)$ unlinked unknots beside D, each with framing $+1$. This is a Kirby diagram which represents K as a knot in the boundary of the "blown up" four-ball $X = B^4 \# (p+n)\mathbb{CP}^2$. As observed in [3], the knot K bounds a disk Δ in X. This may be seen from the diagram by sliding each of the chosen crossings in D over a $+1$-framed unknot as in Figure 5. Mark each of these changed crossings with a small arc α_i, $i = 1, \ldots, (p+n)$, as shown in that figure.

The resulting diagram consists of:

- an unknot U which has been obtained from K by crossing changes;
- arcs $\alpha_1, \ldots, \alpha_{p+n}$ (one per changed crossing);
- $+1$-framed unknots $\gamma_1, \ldots, \gamma_{p+n}$.

Each γ_i bounds a disk D_i which retracts onto α_i and whose intersection with U consists of the endpoints of α_i.

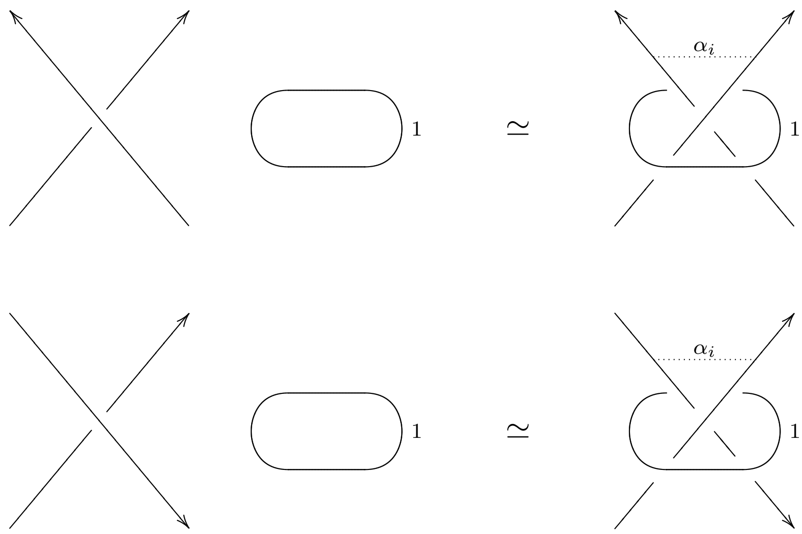

Figure 5. Changing crossings by sliding over a two-handle.

It is also observed in [3] that $H_1(X - \Delta; \mathbb{Z}/2) \cong \mathbb{Z}/2$, with generator given by the meridian of K. (To see this note from Figure 5 that the linking number of U with each of the $+1$-framed unknots is even. Now use the Mayer-Vietoris sequence for the decomposition of X into $X - \Delta$ and a neighbourhood of Δ, with $\mathbb{Z}/2$ coefficients.) Thus there exists a unique double cover W of X branched along Δ; this is a four-manifold with boundary $\Sigma(K)$.

Rearrange the diagram so that a point of U which is not the endpoint of an arc α_i is the point at infinity and U is a straight line; then Δ may be seen in this diagram as a half-plane with boundary U. (For a simple example see Figure 6. Note in general the arcs α_i may be knotted and linked.) We may suppose all of the α_i are disjoint from the half-plane Δ, so that each γ_i intersects Δ in two points. There are now four crossings involving the the line U and each knot γ_i; by twisting γ_i if necessary arrange that the sequence of undercrossings and overcrossings between U and each γ_i is nonalternating, in the order that they occur along U (as in Figure 6). Let $w(\gamma_i)$ denote the *writhe* of γ_i (i.e. the signed count of self-crossings). Similarly let $w(\alpha_i)$ denote the signed count of self-crossings of the arc α_i.

It is now straightforward to draw a Kirby diagram of W, as in Figure 6. (Take two copies of $S^3 - U$ cut open along Δ, and join the boundary half-planes in pairs). Each arc α_i lifts to a knot $\tilde{\alpha}_i$, and each D_i lifts to an annulus $\tilde{D}_i$ with core $\tilde{\alpha}_i$. The knot γ_i lifts to two knots C_i, C_i'; these are the boundary of the annulus $\tilde{D}_i$.

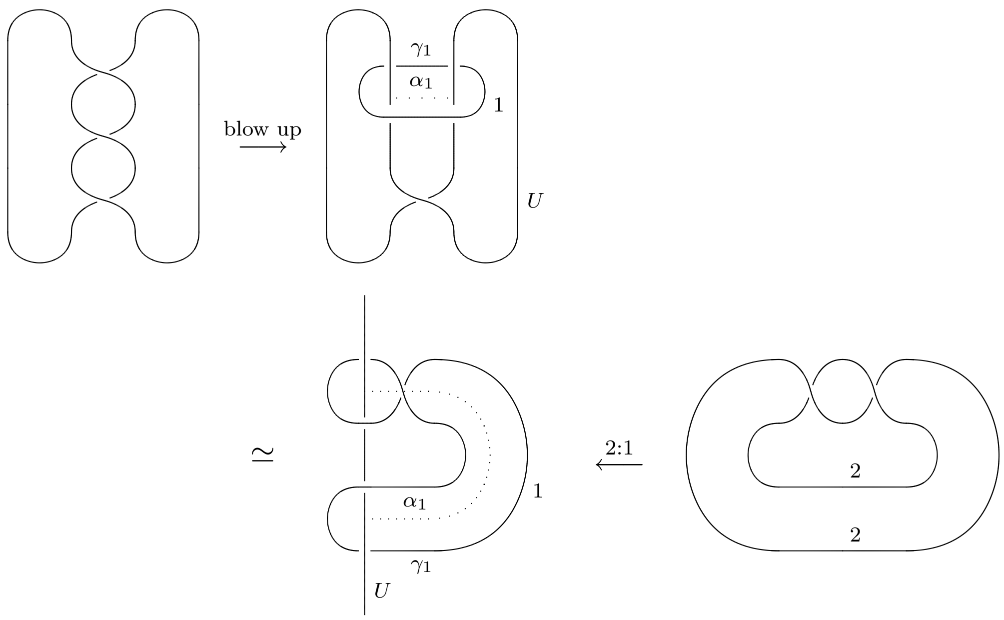

FIGURE 6. **A four-manifold bounded by the double branched cover of the left-handed trefoil.** For this example $w(\alpha_1) = 0$, $w(\gamma_1) = -1$.

To compute the framings on these knots note that the blackboard framing of γ_i lifts to the blackboard framing on each of C_i, C_i'. The blackboard framing of γ_i is $w(\gamma_i)$. The blackboard framing of C_i and C_i' are each given by $w(\tilde{\alpha}_i) = 2w(\alpha_i)$. Thus the framing $+1$ on γ_i lifts to $w(\tilde{\alpha}_i) - w(\gamma_i) + 1$; denote this number by m_i. The linking number $\mathrm{lk}(C_i, C_i')$ is $w(\gamma_i) - w(\tilde{\alpha}_i) = 1 - m_i$. (The number of twists in the annulus $\tilde{D}_i$ is $w(\gamma_i)$, while $-w(\tilde{\alpha}_i)$ counts crossings of C_i and C_i' due to knotting of $\tilde{D}_i$.)

We note that the resulting Kirby diagram for W matches that in Lemma 2.2 (ii). That lemma then shows that $\Sigma(K) = \partial W$ is Dehn surgery on the framed link $L = C_1, \dots, C_{p+n}$ with framing $(2m_i - 1)/2$ on C_i. Thus we have recovered Montesinos' description of $\Sigma(K)$ as Dehn surgery on a link with linking matrix $\frac{1}{2}Q$, where Q is congruent to the identity modulo 2. But this four-dimensional description yields more information: the rank and signature of the intersection pairing $\tilde{Q}$ of W are computed by Cochran and Lickorish in [3, Theorem 3.7] as follows:

$$\begin{aligned} b_2(W) &= 2(p+n) \\ \sigma(W) &= \sigma(K) + 2p. \end{aligned}$$

In particular if $\sigma(K) = 2n$ then W (and hence Q, by the observation after Lemma 2.2) is positive-definite.

We now establish the claim that n of the diagonal entries $2m_1 - 1, \ldots, 2m_{p+n} - 1$ of Q are congruent to 3 modulo 4. Let $\mathcal{U}$ denote the set of crossings in the diagram D that we change to unknot K. We have given a description of $\Sigma(K)$ as half-integral surgery on a link L, with one component for each crossing in $\mathcal{U}$. Dehn surgery on a sublink of L gives the double branched cover of a knot which is obtained from D by changing a subset of the crossings in $\mathcal{U}$. In particular $(2m_i - 1)/2$ surgery on the knot C_i yields the double branched cover of the knot K' which is obtained by changing all but one of the crossings in $\mathcal{U}$.

The determinant $\det(K)$ of a knot K is equal to the order of $H^1(\Sigma(K); \mathbb{Z})$, which in turn is equal to the determinant of the intersection pairing of any simply connected four-manifold W bounded by $\Sigma(K)$. Moreover the determinant and signature of K are shown in [7, Theorem 5.6] to satisfy

$$\det(K) \equiv \sigma(K) + 1 \pmod 4.$$

The fact that $\sigma(K) = 2n$ implies that every change of a negative crossing in $\mathcal{U}$ reduces the knot signature by 2, while every change of a positive crossing leaves the signature unchanged. It follows that if the knot K' is obtained by changing all but one crossing in $\mathcal{U}$, then K' has signature 2 if that crossing is negative and signature zero otherwise. The determinant of K' is $2m_i - 1$, which is thus congruent to 3 modulo 4 if and only if $\sigma(K') = 2$.

Finally note that we may reorient any of the link components $C_1, \ldots, C_{p+n}$ without changing the resulting Dehn surgery. Also by rational handlesliding as in Proposition 2.1 we may change the linking matrix by "adding" $\pm 2C_j$ to C_i for any i, j. These operations preserve the congruence classes modulo 4 of the diagonal. The last claim in the statement of Theorem 3 now follows from the following lemma.

Lemma 3.1. *Any matrix $P \in GL(r, \mathbb{Z})$ which is congruent to the identity modulo 2 may be obtained from the identity by a sequence of row operations, each of which is either multiplying a row by -1 or adding an even multiple of one row to another.*

Proof. Let $\mathbf{b} = (b_1, \ldots, b_r)$ be an element of $\mathbb{Z}^r$ with $\gcd(b_1, \ldots, b_r) = 1$. Assume $b_i \geq 0$ for all i, and that b_1 is odd but the other components $b_2, \ldots, b_r$ are even. Let b_j be the least positive component. By subtracting even multiples of b_j and then possibly changing sign, we may replace every other component b_i by b'_i, with $0 \leq b'_i \leq b_j$. By the gcd condition, the least positive b'_i is less than b_j unless $b_j = j = 1$. By iterating this procedure we see that $\mathbf{b}$ may be reduced to $(1, 0, \ldots, 0)$.

Now suppose $P \in GL(r, \mathbb{Z})$ is congruent to I modulo 2, and let $\mathbf{b}$ be the first column of Q. The argument just given shows that Q may be replaced by a matrix with $(1, 0, \ldots, 0)$ in the first column using the specified row operations. Then replacing the second column with $(*, 1, 0, \ldots, 0)$ by row operations on the last $r - 1$ rows, and so on, we see that we may reduce P to I in this manner. $\square$

4. Heegaard Floer homology

In this section we recall some properties of the Heegaard Floer homology invariants of Ozsváth and Szabó. Details are to be found in their papers, in particular [11, 12, 13].

Let Y be an oriented rational homology sphere. Recall that the space $\mathrm{Spin}^c(Y)$ of spin^c structures on Y is isomorphic to $H^2(Y;\mathbb{Z})$. If $|H^2(Y;\mathbb{Z})|$ is odd then there is a canonical isomorphism which takes the unique spin structure to zero; this gives $\mathrm{Spin}^c(Y)$ a group structure.

Fixing a spin^c structure $\mathfrak{s}$, the Heegaard Floer homology $HF^+(Y;\mathfrak{s})$ is a $\mathbb{Q}$-graded abelian group with an action by $\mathbb{Z}[U]$, where U lowers the grading by 2. The *correction term* invariant is a rational number $d(Y,\mathfrak{s})$; it is defined to be the lowest grading of a nonzero homogeneous element of $HF^+(Y;\mathfrak{s})$ which is in the image of U^n for all $n \in \mathbb{N}$. These have the property that $d(Y,\mathfrak{s}) = -d(-Y,\mathfrak{s})$, where $-Y$ denotes Y with the opposite orientation. We will describe below how these correction terms may be computed in certain cases.

Now let X be a positive-definite four-manifold with boundary Y. Then it is shown in [11] that for any spin^c structure $\mathfrak{s}$ on X,

$$c_1(\mathfrak{s})^2 - b_2(X) \geq 4d(Y,\mathfrak{s}|_Y), \tag{8}$$

$$\text{and} \quad c_1(\mathfrak{s})^2 - b_2(X) \equiv 4d(Y,\mathfrak{s}|_Y) \pmod 2. \tag{9}$$

This means that the correction terms of Y may be used to give an obstruction to Y bounding a four-manifold X with a given positive-definite intersection form. We will now elaborate on how this may be checked in practice.

Suppose for simplicity that X is simply-connected and that $|H^2(Y;\mathbb{Z})|$ is odd. Let r denote the second Betti number of X. Fix a basis for $H_2(X;\mathbb{Z})$ and thus an isomorphism

$$H_2(X;\mathbb{Z}) \cong \mathbb{Z}^r.$$

Let Q be the matrix of the intersection pairing of X in this basis; thus Q is a symmetric positive-definite $r\times r$ integer matrix with $\det Q = |H^2(Y;\mathbb{Z})|$. The dual basis gives an isomorphism between the second cohomology $H^2(X;\mathbb{Z})$ and $\mathbb{Z}^r$. The set $\{c_1(\mathfrak{s}) \mid \mathfrak{s} \in \mathrm{Spin}^c(X)\} \subset H^2(X;\mathbb{Z})$ of first Chern classes of spin^c structures is equal to the set of characteristic covectors $Char(Q)$ for Q. These in turn are elements ξ of $\mathbb{Z}^r$ whose components ξ_i are congruent modulo 2 to the corresponding diagonal entries Q_{ii} of Q. The square of the first Chern class of a spin^c structure is computed using the pairing induced by Q on $H^2(X;\mathbb{Z})$; in our choice of basis this is given by $\xi^T Q^{-1}\xi$.

The long exact sequence of the pair (X,Y) yields the following short exact sequence:

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^r \longrightarrow H^2(Y;\mathbb{Z}) \longrightarrow 0.$$

As in the introduction, define a function

$$m_Q : \mathbb{Z}^r/Q(\mathbb{Z}^r) \to \mathbb{Q}$$

by

$$m_Q(g) = \min\left\{ \frac{\xi^T Q^{-1}\xi - r}{4} \;\middle|\; \xi \in Char(Q),\ [\xi] = g \right\}.$$

An easy argument shows that in computing m_Q it suffices to consider characteristic covectors $\xi = (\xi_1, \ldots, \xi_r)$ whose components are smaller in absolute value than the corresponding diagonal entries of Q:

$$-Q_{ii} \leq \xi_i \leq Q_{ii}.$$

(A more difficult argument in [12] shows that it suffices to restrict to

$$-Q_{ii} \leq \xi_i \leq Q_{ii} - 2.)$$

Thus it is straightforward, if tedious, to compute m_Q for a given positive-definite matrix Q.

The conditions (8) and (9) may now be expressed as follows:

Theorem 4.1 (Ozsváth-Szabó). *Let Y be a rational homology three-sphere which is the boundary of a simply-connected positive-definite four-manifold X, with $|H^2(Y;\mathbb{Z})|$ odd. If the intersection pairing of X is represented in a basis by the matrix Q then there exists a group isomorphism*

$$\phi : \mathbb{Z}^r/Q(\mathbb{Z}^r) \to \mathrm{Spin}^c(Y)$$

with

$$m_Q(g) \geq d(Y, \phi(g)), \tag{10}$$

$$\text{and} \quad m_Q(g) \equiv d(Y, \phi(g)) \pmod 2 \tag{11}$$

for all $g \in \mathbb{Z}^r/Q(\mathbb{Z}^r)$.

The four-manifold X is said to be *sharp* if equality holds in (10). In this case the correction terms for Y can be computed using the function m_Q described above. Also, if a rational homology sphere Y bounds a negative-definite four-manifold X such that $-X$ is sharp, then the correction terms for Y can be computed using the formula $d(Y, \mathfrak{s}) = -d(-Y, \mathfrak{s})$. Note that if K is a knot in S^3 then the standard orientation on S^3 induces an orientation on $\Sigma(K)$; letting $r(K)$ denote the reflection of K, we have $\Sigma(r(K)) \cong -\Sigma(K)$.

In particular let K be an alternating knot with double branched cover $\Sigma(K)$. Let G denote the positive-definite Goeritz matrix computed from an alternating diagram for K as follows. Colour the knot diagram in chessboard fashion according to the convention shown in Figure 7. (Note that this is the opposite convention to that used in [13], since they use the negative-definite Goeritz matrix.) Let $v_1, \ldots, v_{k+1}$ denote the vertices of the white graph. Then G is the $k \times k$ symmetric matrix (g_{ij}) with entries

$$g_{ij} = \begin{cases} \text{the number of edges containing } v_i & \text{if } i = j \\ \text{minus the number of edges joining } v_i \text{ and } v_j & \text{if } i \neq j \end{cases}$$

for $i, j = 1, \ldots, k$. It is shown in [13, Proposition 3.2] that G represents the intersection pairing of a sharp four-manifold bounded by $\Sigma(K)$. Thus the correction terms for $\Sigma(K)$ are given by m_G (for any choice of alternating diagram and any ordering of the white regions). Also it follows from [4] that with this colouring convention, the signature of K is given by

$$\sigma(K) = k - \mu,$$

where μ is the number of positive crossings in the alternating diagram used to compute G.

Also if K is a Montesinos knot then the double branched cover $\Sigma(K)$ is a Seifert fibred space which is given as the boundary of a plumbing of disk bundles over S^2. This plumbing is determined (nonuniquely) by the Montesinos invariants which specify K. After possibly reflecting K we may choose the plumbing so that its intersection pairing is represented by a positive-definite matrix P. It is shown in [12] that the plumbing is sharp, so that the correction terms for $\Sigma(K)$ are given by m_P. (See [8] for a detailed description of Montesinos knots and their branched double covers.)

Remark 4.2. *Checking the congruence condition (9) alone is equivalent to checking that the intersection pairing of X presents the linking pairing of Y; see* [10] *for a detailed discussion.*

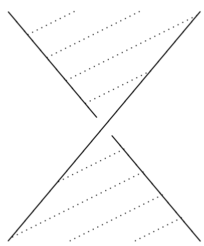

Figure 7. Colouring convention for alternating knot diagrams.

5. Obstruction to unknotting

In this section we prove Theorems 1 and 5.

Let $\mathcal{Q}(r, \delta)$ denote the set of positive-definite symmetric integer matrices of rank r and determinant δ, on which $GL(r, \mathbb{Z})$ acts by $P \cdot Q = PQP^T$ with finite quotient (see e.g. [2]). Let $\mathcal{Q}(r, \delta)_2 \subset \mathcal{Q}(r, \delta)$ (resp. $GL(r, \mathbb{Z})_2 \subset GL(r, \mathbb{Z})$) denote the subset (resp. subgroup) consisting of matrices which are congruent to the identity modulo 2. Then the subset $\mathcal{Q}(r, \delta)_2/GL(r, \mathbb{Z})$ is clearly finite, and thus so is $\mathcal{Q}(r, \delta)_2/GL(r, \mathbb{Z})_2$ since $GL(r, \mathbb{Z})_2$ is a finite index subgroup of $GL(r, \mathbb{Z})$.

Proof of Theorem 5. By Theorem 3, the unknotting hypothesis implies that $\Sigma(K)$ is given by Dehn surgery on a link in S^3 with linking matrix $\frac{1}{2}Q_i$ for some i, where n of the diagonal entries of Q_i are congruent to 3 modulo 4. By Lemma 2.2, $\Sigma(K)$ bounds the 2-handlebody W specified by an integer-framed link with positive-definite linking matrix $\tilde{Q}_i$, which then represents the intersection pairing of W. The conclusion now follows from Theorem 4.1. □

Proof of Theorem 1. Theorem 1 follows from Theorem 5 since a finite set of representatives of $\mathcal{Q}(2,\delta)_2/GL(2,\mathbb{Z})_2$ is given by the set of matrices

$$\left\{ Q = \begin{pmatrix} 2m_1 - 1 & 2a \\ 2a & 2m_2 - 1 \end{pmatrix} \;\middle|\; \det Q = \delta,\, 0 \leq a < m_1 \leq m_2 \right\},$$

and since the correction terms $d(\Sigma(K), \mathfrak{s})$ may be computed using a positive-definite Goeritz matrix G when K is alternating. □

Remark 5.1. *Theorems 1 and 5 do not use all of the information from Theorem 3. We have only used the information about the intersection pairing of the four-manifold W bounded by $\Sigma(K)$, and not the fact that W is a surgery cobordism arising from a half-integral surgery. Comparing to Theorem 1.1 in* [13]*, we have generalised conditions (1) and (2) to the case of $u(K) > 1$ but not the symmetry condition (3). It is to be hoped that the symmetry condition may also be generalised in some way, and that this could lead to computation of some more unknotting numbers.*

6. Examples

Proof of Corollary 2. For each knot in Corollary 2 we distinguish between K and its reflection $r(K)$ by specifying that K has positive signature.

We start with the knot $K = 9_{10}$ shown in Figure 8. This is the two-bridge knot $S(33, 23)$. It has signature 4, and it is easy to see that 3 crossing changes suffice to unknot it. Thus the unknotting number is either 2 or 3, and if it can be unknotted by changing two crossings then both are negative ($p = 0$ and $n = 2$).

With the white regions labelled as shown in the figure, the Goeritz matrix is

$$G = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}.$$

Using m_G, we find the correction terms of $\Sigma(K)$ to be:

$$A = \left\{ \begin{array}{ccccccccccc} -1, & -\frac{23}{33}, & \frac{7}{33}, & -\frac{3}{11}, & -\frac{5}{33}, & \frac{19}{33}, & -\frac{1}{11}, & -\frac{5}{33}, & \frac{13}{33}, & -\frac{5}{11}, & -\frac{23}{33}, \\ -\frac{1}{3}, & \frac{7}{11}, & \frac{7}{33}, & \frac{13}{33}, & \frac{13}{11}, & \frac{19}{33}, & \frac{19}{33}, & \frac{13}{11}, & \frac{13}{33}, & \frac{7}{33}, & \frac{7}{11}, \\ -\frac{1}{3}, & -\frac{23}{33}, & -\frac{5}{11}, & \frac{13}{33}, & -\frac{5}{33}, & -\frac{1}{11}, & \frac{19}{33}, & -\frac{5}{33}, & -\frac{3}{11}, & \frac{7}{33}, & -\frac{23}{33} \end{array} \right\}.$$

The order of this list corresponds to the cyclic group structure of $\mathrm{Spin}^c(\Sigma(K)) \cong H^2(\Sigma(K);\mathbb{Z})$, and the first element is the correction term of the spin structure.

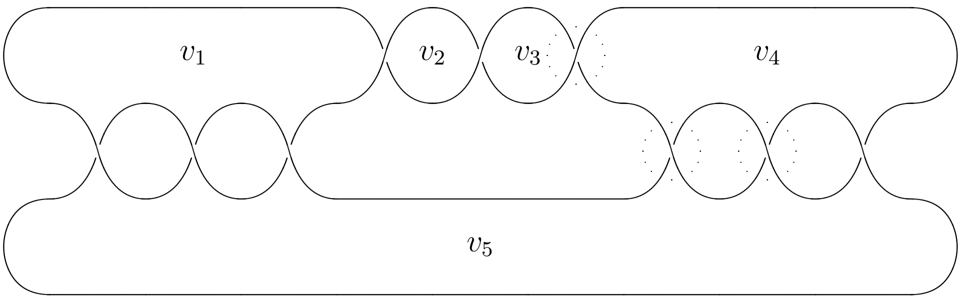

FIGURE 8. **The knot** $9_{10} = S(33, 23)$**.** Note that changing the circled crossings will give the unknot. The labels $v_1, \ldots, v_5$ correspond to vertices of the white graph.

The determinant of 9_{10} is 33. To find a matrix Q as in Theorem 1 we need to find (m_1, a, m_2) with

$$(2m_1 - 1)(2m_2 - 1) - 4a^2 = 33,$$

$$0 \leq a < m_1 \leq m_2,$$

and m_1 and m_2 are even. There are two solutions: $(2, 0, 6)$ and $(4, 2, 4)$. Computing m_Q for each of the matrices

$$Q_1 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 4 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

yields the following lists:

$$B_1 = \left\{ \begin{array}{ccccccccccc} -1, & -\frac{5}{33}, & \frac{13}{33}, & \frac{7}{11}, & \frac{19}{33}, & \frac{7}{33}, & -\frac{5}{11}, & \frac{19}{33}, & \frac{43}{33}, & -\frac{3}{11}, & -\frac{5}{33}, \\ -\frac{1}{3}, & -\frac{9}{11}, & \frac{13}{33}, & \frac{43}{33}, & -\frac{1}{11}, & \frac{7}{33}, & \frac{7}{33}, & -\frac{1}{11}, & \frac{43}{33}, & \frac{13}{33}, & -\frac{9}{11}, \\ -\frac{1}{3}, & -\frac{5}{33}, & -\frac{3}{11}, & \frac{43}{33}, & \frac{19}{33}, & -\frac{5}{11}, & \frac{7}{33}, & \frac{19}{33}, & \frac{7}{11}, & \frac{13}{33}, & -\frac{5}{33} \end{array} \right\},$$

$$B_2 = \left\{ \begin{array}{ccccccccccc} -1, & -\frac{19}{33}, & \frac{23}{33}, & \frac{9}{11}, & -\frac{7}{33}, & -\frac{13}{33}, & \frac{3}{11}, & -\frac{7}{33}, & \frac{5}{33}, & -\frac{7}{11}, & -\frac{19}{33}, \\ \frac{1}{3}, & \frac{1}{11}, & \frac{23}{33}, & \frac{5}{33}, & \frac{5}{11}, & -\frac{13}{33}, & -\frac{13}{33}, & \frac{5}{11}, & \frac{5}{33}, & \frac{23}{33}, & \frac{1}{11}, \\ \frac{1}{3}, & -\frac{19}{33}, & -\frac{7}{11}, & \frac{5}{33}, & -\frac{7}{33}, & \frac{3}{11}, & -\frac{13}{33}, & -\frac{7}{33}, & \frac{9}{11}, & \frac{23}{33}, & -\frac{19}{33} \end{array} \right\}.$$

We claim that for both Q_1 and Q_2 it is impossible to find a group automorphism ϕ of $\mathbb{Z}/33$ satisfying the required inequality and congruence conditions. This is immediate in either case by considering the minimal elements (excluding -1 which appears in all 3 lists). We have the entry $-9/11$ in B_1. By inspection there is no element in A which is less than or equal to $-9/11$, and differs from it by a multiple of 2. The same applies to $-7/11$ in B_2. We conclude that 9_{10} cannot be unknotted by two crossing changes and $u(9_{10}) = 3$.

Similar calculations show that $9_{13}, 9_{38}, 10_{53}, 10_{101}$ and 10_{120} cannot be unknotted with two crossing changes. All of these knots are alternating, have signature four and cyclic $H^2(\Sigma(K); \mathbb{Z})$. By inspection of their diagrams (see e.g. [1]), all can be unknotted with three crossing changes. For some details of the calculations for these knots, see Table 1. Note that we use the knot diagrams from [1] to compute the Goeritz matrices for these knots, after possibly reflecting to ensure positive signature.

Finally consider $K = 9_{35}$, pictured in Figure 9. It has signature 2 and can be unknotted with 3 crossing changes. The Goeritz matrix from the figure is

$$G = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}.$$

We note that this presents $H^2(\Sigma(K); \mathbb{Z})$ which is thus 2-cyclic; this shows (by Montesinos' theorem for example but by an inequality originally due to Wendt) that $u(K) \geq 2$. We can use m_G to compute the correction terms of $\Sigma(K)$, which are

$$A = \begin{bmatrix} -\frac{1}{2} & \frac{19}{18} & -\frac{5}{18} & \frac{3}{2} & \frac{7}{18} & \frac{7}{18} & \frac{3}{2} & -\frac{5}{18} & \frac{19}{18} \\ \frac{1}{6} & -\frac{5}{18} & \frac{7}{18} & \frac{1}{6} & \frac{19}{18} & \frac{19}{18} & \frac{1}{6} & \frac{7}{18} & -\frac{5}{18} \\ \frac{1}{6} & -\frac{5}{18} & \frac{7}{18} & \frac{1}{6} & \frac{19}{18} & \frac{19}{18} & \frac{1}{6} & \frac{7}{18} & -\frac{5}{18} \end{bmatrix}.$$

Here the rectangular array shows the $\mathbb{Z}/3 \oplus \mathbb{Z}/9$ group structure; the top left entry is the correction term of the spin structure.

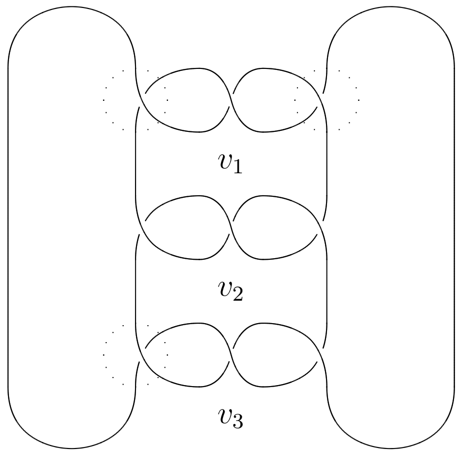

FIGURE 9. **The Montesinos knot** $9_{35} = M(0; (3,1), (3,1), (3,1))$**.**

Suppose that 9_{35} may be unknotted by changing one positive and one negative crossing. The only matrix which satisfies the conditions of Theorem 1 and which presents $\mathbb{Z}/3 \oplus \mathbb{Z}/9$ is

$$Q = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Computing m_Q yields another array whose minimal entry is $-17/18$; we conclude that there is no automorphism ϕ of $\mathbb{Z}/3 \oplus \mathbb{Z}/9$ satisfying the conclusion of Theorem 1.

This is not enough to rule out the possibility that $u(9_{35}) = 2$; it does however show that if 9_{35} can be unknotted by two crossing changes, then they are both negative crossings. Using the value of the Jones polynomial at $e^{i\pi/3}$, Traczyk has shown in [14] that if 9_{35} can be unknotted by changing two crossings, then the crossings have different signs. We conclude that $u(9_{35}) = 3$. $\square$

Knot	Goeritz matrix	$\min\limits_{g\neq 0}\{m_G(g)\}$	(m_1, a, m_2)	$\min\limits_{g\neq 0}\{m_Q(g)\}$
9_{13}	$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}$	$-\frac{27}{37}$	$(10, 9, 10)$	$-\frac{33}{37}$
9_{38}	$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & -2 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$	$-\frac{37}{57}$	$(2, 0, 10)$	$-\frac{51}{57}$
			$(6, 4, 6)$	$-\frac{45}{57}$
10_{53}	$\begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 \\ 0 & -1 & 4 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$	$-\frac{53}{73}$	$(4, 1, 6)$	$-\frac{59}{73}$
10_{101}	$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 \\ 0 & -1 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}$	$-\frac{59}{85}$	$(6, 3, 6)$	$-\frac{65}{85}$
			$(22, 21, 22)$	$-\frac{81}{85}$
10_{120}	$\begin{pmatrix} 4 & -2 & 0 & -1 \\ -2 & 4 & -1 & 0 \\ 0 & -1 & 4 & -2 \\ -1 & 0 & -2 & 4 \end{pmatrix}$	$-\frac{69}{105}$	$(2, 0, 18)$	$-\frac{99}{105}$
			$(4, 0, 8)$	$-\frac{91}{105}$
			$(6, 2, 6)$	$-\frac{83}{105}$
			$(10, 8, 10)$	$-\frac{93}{105}$

TABLE 1. **Data for knots in Corollary 2.** The fourth column contains possible coefficients of the matrix Q in Theorem 1.

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