CRITICAL POINTS OF FUNCTIONS, sl_2 REPRESENTATIONS, AND FUCHSIAN DIFFERENTIAL EQUATIONS WITH ONLY UNIVALUED SOLUTIONS

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Dedicated to V. I. Arnold on his 65th birthday

Abstract. Let a second order Fuchsian differential equation with only univalued solutions have finite singular points at $z_1, ..., z_n$ with exponents $(\rho_{1,1}, \rho_{2,1})$,...,$(\rho_{1,n}, \rho_{2,n})$. Let the exponents at infinity be $(\rho_{1,\infty}, \rho_{2,\infty})$. Then for fixed generic $z_1, ..., z_n$, the number of such Fuchsian equations is equal to the multiplicity of the irreducible sl_2 representation of dimension $|\rho_{2,\infty} - \rho_{1,\infty}|$ in the tensor product of irreducible sl_2 representations of dimensions $|\rho_{2,1} - \rho_{1,1}|$, ... , $|\rho_{2,n} - \rho_{1,n}|$. To show this we count the number of critical points of a suitable function which plays the crucial role in constructions of the hypergeometric solutions of the sl_2 KZ equation and of the Bethe vectors in the sl_2 Gaudin model. As a byproduct of this study we conclude that the set of Bethe vectors is a basis in the space of states for the sl_2 inhomogeneous Gaudin model.

1. Introduction

1.1. **Critical points and sl_2 representations.** Consider the Lie algebra sl_2 with standard generators e, f, h, $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$. Let L_a be the irreducible sl_2 module with highest weight $a \in \mathbb{C}$. The module L_a is generated by its singular vector v_a, $ev_a = 0$, $hv_a = av_a$. Vectors $v_a, fv_a, f^2v_a, \ldots$ form a basis of L_a. If a is a nonnegative integer, then $\dim L_a = a + 1$; otherwise L_a is infinite-dimensional.

If $m_1, \ldots, m_n$ are nonnegative integers, then the tensor product $L^{\otimes m} = L_{m_1} \otimes \cdots \otimes L_{m_n}$ is a direct sum of irreducible representations with highest weights $l(m) - 2k$, where

$$l(m) = m_1 + ... + m_n$$

and k is a nonnegative integer. Let $w(m,k)$ be the multiplicity of $L_{l(m)-2k}$ in $L^{\otimes m}$. We have

$$w(m,k) \geq 0 \text{ if } l(m) - 2k \geq 0; \quad w(m,k) = 0 \text{ if } l(m) - 2k < 0.$$

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Let $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ be a point with pairwise distinct coordinates. Let

$$\mathcal{A}=\mathcal{A}_{k,n}(z)=\bigcup_{i=1}^{k}\bigcup_{l=1}^{n}\{t\in\mathbb{C}^k \mid t_i=z_l\}\bigcup_{1\le i<j\le k}\{t\in\mathbb{C}^k \mid t_i=t_j\}$$

be a (discriminantal) arrangement of hyperplanes in $\mathbb{C}^k$, and $\mathcal{C}=\mathcal{C}_{k,n}(z)$ its complement. For $m=(m_1,\dots,m_n)\in\mathbb{C}^n$, consider the multivalued function $\Phi:\ \mathcal{C}\to\mathbb{C}$,

$$\Phi_{k,n}(t)=\Phi_{k,n}(t;z,m)=\prod_{i=1}^{k}\prod_{l=1}^{n}(t_i-z_l)^{-m_l}\prod_{1\le i<j\le k}(t_i-t_j)^2\,.$$

A point $t^0\in\mathcal{C}$ is called *a critical point of* Φ if

$$\frac{\partial\Phi}{\partial t_i}(t^0)=0,\quad i=1,\dots,k\,.$$

The symmetric group S^k acts on $\mathcal{C}$ permuting coordinates. Each orbit consists of $k!$ points. The action preserves the critical set of the function $\Phi_{k,n}(t)$.

Let $\lambda_1=\sum t_i,\ \ \lambda_2=\sum t_it_j,\ \ \dots,\ \ \lambda_k=t_1\cdots t_k$ be the standard symmetric functions of $t_1,...,t_k$. Denote $\mathbb{C}^k_\lambda$ the space with coordinates $\lambda_1,...,\lambda_k$.

Our first main result is

Theorem 1. *Let $m_1,\dots,m_n\in\mathbb{Z}_{\ge0}$ and $k\in\mathbb{Z}_{>0}$.*

- *If $l(m)+1-k>k$, then for generic z all critical points of $\Phi_{k,n}(t)$ are nondegenerate and the critical set consists of $w(m,k)$ orbits.*
- *If $l(m)+1-k=k$, then for any z the function $\Phi_{k,n}(t)$ does not have critical points.*
- *If $0\le l(m)+1-k<k$, then for generic z the function $\Phi_{k,n}(t)$ can have only nonisolated critical points. Written in symmetric coordinates $\lambda_1,...,\lambda_k$, the critical set consists of $w(m,l(m)+1-k)$ straight lines in the space $\mathbb{C}^k_\lambda$.*
- *If $l(m)+1-k<0$, then for any z the function $\Phi_{k,n}(t)$ does not have critical points.*

In this paper, the words "a point z is generic" mean that z does not belong to a suitable proper algebraic subset in $\mathbb{C}^n$.

Remark. Assume that $m_1,\dots,m_p\in\mathbb{Z}_{>0}$ and $m_{p+1}=...=m_n=0$. Then for any k, we have $\Phi_{k,n}(t_1,...,t_k;z_1,...,z_n,\ m_1,...,m_n)=\Phi_{k,p}(t_1,...,t_k;z_1,...,z_p,\ m_1,...,m_p)$. For generic $z_{p+1},...,z_n$, the two functions have the same number of isolated critical points and the same number of critical curves. We also have $w(\,(m_1,...,m_n),\,k\,)=w(\,(m_1,...,m_p),\,k\,)$. Thus to prove the Theorem it is enough to consider the case when all $m_1,...,m_n$ are positive integers.

Example. Let $n = 2$ and $z = (0, 1)$. We have

$$\Phi_{k,2}(t) = \prod_{i=1}^{k} t_i^{-m_1}(t_i - 1)^{-m_2} \prod_{1\le i<j\le k} (t_i - t_j)^2 .$$

The critical point system of the function $\Phi_{k,2}(t)$ being written with respect to symmetric coordinates $\lambda_1, \dots, \lambda_k$ is the following linear system,

$$(p+1)(-m_1 + p)\lambda_{k-p-1} = (k-p)(-m_1 - m_2 + k + p - 1)\lambda_{k-p} ,$$

where $p = 0, \dots, k-1$, and $\lambda_0 = 0$, see Lemma 1.3.4 in [V].

For $m_1, m_2 \in \mathbb{Z}_{>0}$, there are four possibilities.

(i) If $k \le m_1, m_2$, then the linear system has a single solution which defines $k!$ nondegenerate critical points of $\Phi_{k,2}(t)$. In this case the multiplicity of $L_{m_1+m_2-2k}$ in $L_{m_1} \otimes L_{m_2}$ is $w(m,k) = 1$.

(ii) If k is greater than exactly one of the numbers m_1, m_2, then the linear system still has a single solution, but the solution defines points lying in the arrangement $\mathcal{A}$. This means that $\Phi_{k,2}(t)$ does not have critical points. In this case $w(m,k) = w(m, l(m)+1-k) = 0$.

(iii) If $m_1, m_2 < k \le m_1 + m_2 + 1$, then the rank of the linear system is $k-1$. The solutions form a straight line in the space $\mathbb{C}^k_\lambda$ with coordinates $\lambda_1, \dots, \lambda_k$. The line defines a curve of critical points of the function $\Phi_{k,2}(t)$. In this case $w(m, l(m)+1-k) = 1$.

(iv) If $m_1 + m_2 + 1 < k$, then the system again has a single solution which defines points lying in the arrangement $\mathcal{A}$. The function $\Phi_{k,2}(t)$ does not have critical points.

For negative exponents $m_1, ..., m_n$ and real $z_1, ..., z_n$, the function $\Phi_{k,n}(t; z, m)$ has only nondegenerate critical points and the critical set consists of $\binom{k+n-2}{n-2}$ orbits [V]. If the exponents tend to positive integer values so that $l(m) - 2k$ remains nonnegative, some of critical points vanish at edges of the arrangement $\mathcal{A}_{k,n}(z)$. To prove the first part of Theorem 1 we count the number of vanishing critical points.

To show that critical points of $\Phi_{k,n}(t; z, m)$ form lines, if $l(m) - 2k$ is negative, we use the connection of critical points with Fuchsian equations having polynomial solutions.

On the number of critical points of a product of generic powers of arbitrary linear functions see [V, OT, Si]. In that case of generic exponents the critical points are isolated and nondegenerate and their number is equal to the absolute value of the Euler characteristic of the complement to the arrangement of zero hyperplanes of the linear functions. In contrast to generic exponents, the exponents of the function $\Phi_{k,n}(t; z, m)$ in Theorem 1 are highly resonant. It would be very interesting to find out how much of the phenomenon described in Theorem 1 can be generalized to more general arrangements.

1.2. **Critical points and Fuchsian equations with polynomial solutions.** Consider the differential equation

$$u''(x)+p(x)u'(x)+q(x)u(x)=0 \tag{1}$$

with meromorphic $p(x)$ and $q(x)$. A point $z_0\in\mathbb{C}$ is *an ordinary point* of the equation if the functions $p(x)$ and $q(x)$ are holomorphic at $x=z_0$. A non-ordinary point is called *singular*.

The point $z_0\in\mathbb{C}$ is *a regular singular point* of the equation if z_0 is a singular point, $p(x)$ has a pole at z_0 of order not greater than 1, and $q(x)$ has a pole at z_0 of order not greater than 2.

The equation has *an ordinary (resp., regular singular) point at infinity* if after the change $x=1/\xi$ the point $\xi=0$ is an ordinary (resp., regular singular) point of the transformed equation.

Let $x=z_0$ be a regular singular point,

$$p(x)=\sum_{l=0}^{\infty}p_l(x-z_0)^{l-1}\,,\quad q(x)=\sum_{l=0}^{\infty}q_l(x-z_0)^{l-2}$$

the Laurent series at z_0. If the function

$$u(x)=(x-z_0)^{\rho}\sum_{l=0}^{\infty}c_l(x-z_0)^l,\quad c_0=1, \tag{2}$$

is a solution of equation (1), then ρ must be a root of *the indicial equation*

$$\rho^2+(p_0-1)\rho+q_0=0\,.$$

The roots of the indicial equation are called *the exponents* of the equation at z_0.

If the difference $\rho_1-\rho_2$ of roots is not an integer, then the equation has solutions of the form (2) with $\rho=\rho_j$, $j=1,2$. If the difference $\rho_1-\rho_2$ is a nonnegative integer, then the equation has a solution u_1 of the form (2) with $\rho=\rho_1$. The second linearly independent solution u_2 is either of the form

$$u_2(x)=(x-z_0)^{\rho_2}\sum_{l=0}^{\infty}d_l(x-z_0)^l\,,\quad d_0=1\,,$$

or

$$u_2(x)=u_1(x)\ln(x-z_0)+(x-z_0)^{\rho_2}\sum_{l=0}^{\infty}d_l(x-z_0)^l\,.$$

A differential equation with only regular singular points is called *Fuchsian*. Let the singular points of a Fuchsian equation be $z_1,\dots,z_n$ and infinity. Let $\rho_{1,j}$ and $\rho_{2,j}$ be the exponents at z_j, $1\le j\le n$, and $\rho_{1,\infty}$, $\rho_{2,\infty}$ the exponents at infinity. Then

$$\rho_{1,\infty}+\rho_{2,\infty}+\sum_{j=1}^{n}(\rho_{1,j}+\rho_{2,j})=n-1\,.$$

Consider the equation

$$F(x)u''(x) + G(x)u'(x) + H(x)u(x) = 0\,, \tag{3}$$

where $F(x)$ is a polynomial of degree n, and $G(x)$, $H(x)$ are polynomials of degree not greater than $n-1$, $n-2$, respectively. If $F(x)$ has no multiple roots, then the equation is Fuchsian. Write

$$F(x) = \prod_{j=1}^{n}(x - z_j)\,, \qquad \frac{G(x)}{F(x)} = \sum_{j=1}^{n} \frac{-m_j}{x - z_j} \tag{4}$$

for suitable complex numbers m_j , z_j . Then 0 and m_j+1 are exponents at z_j of equation (3). If $-k$ is one of the exponents at ∞ , then the other is $k - l(m) - 1$.

Problem [S], Ch. 6.8. *Given polynomials $F(x)$, $G(x)$ as above,*

(i) *find a polynomial $H(x)$ of degree at most $n-2$ such that equation (3) has a polynomial solution of a preassigned degree k ;*
(ii) *find the number of solutions to Problem (i).*

The following result is classical.

Theorem 2. Cf. [S], Ch. 6.8.

- *Let $u(x)$ be a polynomial solution of (3) of degree k with roots $t_1^0, ..., t_k^0$ of multiplicity one. Then $t^0 = (t_1^0, \dots, t_k^0)$ is a critical point of the function $\Phi_{k,n}(t; z, m)$, where $z = (z_1, \dots, z_n)$ and $m = (m_1, \dots, m_n)$.*
- *Let t^0 be a critical point of the function $\Phi_{k,n}(t; z, m)$, then the polynomial $u(x)$ of degree k with roots $t_1^0, ..., t_k^0$ is a solution of (3) with $H(x) = (-F(x)u''(x) - G(x)u'(x))/u(x)$ being a polynomial of degree at most $n-2$.*

A critical point of the function $\Phi_{k,n}(t; z, m)$ defines a Fuchsian differential equation and its polynomial solution. The Fuchsian differential equation defined by a critical point t^0 will be called *associated* and denoted $E(t^0, z, m)$.

According to Theorem 2, the orbits of critical points of $\Phi_{k,n}(t; z, m)$ label solutions to Problem (i), and Problem (ii) turns out to be the question on the number of the orbits of critical points of $\Phi_{k,n}(t; z, m)$.

For fixed real $z_1, \dots, z_n$ and negative $m_1, \dots, m_n$, Problem (ii) was solved in the 19th century by Heine and Stieltjes. They showed that under these conditions the number of solutions is equal to $\binom{k+n-2}{n-2}$, see [S], Ch. 6.8.

1.3. **Fuchsian equations with only polynomial solutions.** If all solutions of the Fuchsian equation (3) are polynomials, then the numbers $m_1, ..., m_n$ in (4) are nonnegative integers. If k is the degree of the generic polynomial solution of that equation, then $k > l(m) + 1 - k$ and the equation also has polynomial solutions of degree $l(m) + 1 - k$.

Assume that all solutions of equation (3) are polynomials. If $m_j = 0$ for some j, then $x = z_j$ is a regular point of the equation. Indeed, the function $G(x)$ is clearly divisible by $x - z_j$. We also have

$$\frac{H(x)}{F(x)} = -\frac{u''(x)}{u(x)} - \frac{G(x)}{F(x)}\frac{u'(x)}{u(x)}$$

for any solution $u(x)$. Hence $H(x)$ is divisible by $x - z_j$.

Our second main result counts for generic $z_1, ..., z_n$ the number of Fuchsian equations with fixed positive integers $m_1, ..., m_n$, k having only polynomial solutions.

Theorem 3. *Let $z_1, \dots, z_n$ be pairwise distinct complex numbers, let $m_1, \dots, m_n$, $k \in \mathbb{Z}_{>0}$. Let $F(x)$ and $G(x)$ be determined by (4).*

- *If $k > l(m) + 1 - k \geq 0$, then for generic $z_1, ..., z_n$ there exist exactly $w(m, l(m) + 1 - k)$ polynomials $H(x)$ of degree not greater than $n - 2$ such that all solutions of equation (3) are polynomials with the degree of the generic solution equal to k.*
- *If $k \leq l(m) + 1 - k$ or $l(m) + 1 - k < 0$ then for any $z_1, ..., z_n$ there are no such polynomials $H(x)$.*

The assumption that $z_1, ..., z_n$ are generic is essential.

Example. Let $m = (1, 1, 1,)$ and $k = 3$. Then $l(m)+1-k = 1$ and $w(m, l(m)+1-k) = 2$. Let $z = (0, 1, c)$,

$$F(x) = x(x-1)(x-c)\,, \quad G(x) = \left(-\frac{1}{x} - \frac{1}{x-1} - \frac{1}{x-c}\right) \cdot F(x)\,.$$

Let $v(x) = x - \alpha$ be a solution of equation (3) with these $F(x)$ and $G(x)$ and a suitable $H(x)$. The number α is a critical point of the function $\Phi(x) = x^{-1}(x-1)^{-1}(x-c)^{-1}$, i.e. a root of the quadratic equation $3x^2 - 2(c+1)x + c = 0$.

If the discriminant of this equation $\Delta = c^2 - c + 1$ is non-zero, then there are two distinct roots and there are two linear polynomials

$$H(x) = -\frac{G(x)}{x - \alpha}$$

such that the corresponding equation (3) has only polynomial solutions with cubic generic solutions.

If the discriminant vanishes, i.e. the numbers $0, 1, c$ form an equilateral triangle, then there is only one critical point and only one differential equation.

Corollary of Theorem 3.

Let all solutions of a second order Fuchsian differential equation be univalued. Let the singular points be $z_1, ..., z_n$ and infinity. Let $\rho_{1,j}$ and $\rho_{2,j}$ be the exponents at z_j, and $\rho_{1,\infty}$, $\rho_{2,\infty}$ the exponents at infinity. Then for fixed generic $z_1, ..., z_n$, the number of such Fuchsian equations is equal to the multiplicity of the irreducible sl_2 representation of dimension $|\rho_{2,\infty} - \rho_{1,\infty}|$ in the tensor product of irreducible sl_2 representations of dimensions $|\rho_{2,1} - \rho_{1,1}|, \dots, |\rho_{2,n} - \rho_{1,n}|$.

Proof. Let

$$v''(x) \ + \ p(x)v'(x) \ + \ q(x)v(x) \ = \ 0$$

be such an equation. Assume that $\rho_{1,1} < \rho_{2,1}$, ..., $\rho_{1,n} < \rho_{2,n}$, $\rho_{1,\infty} < \rho_{2,\infty}$. Change the variable, $v(x) = u(x)\ (x-z_1)^{\rho_{1,1}} \cdots (x-z_n)^{\rho_{1,n}}$. The new equation with respect to $u(x)$ is a Fuchsian differential equation of type (3) with only polynomial solutions. Its exponents are $(0,\ \rho_{2,1}-\rho_{1,1})$, ... , $(0,\ \rho_{2,n}-\rho_{1,n})$, $(\rho_{1,\infty}+\sum_{j=1}^n \rho_{1,j},\ \rho_{2,\infty}+\sum_{j=1}^n \rho_{1,j})$. By Theorem 3 for fixed generic $z_1, ..., z_n$, the number of such equations is $w(m, l(m)+1-k)$ where $m = (\rho_{2,1}-\rho_{1,1}-1,\ \dots,\ \rho_{2,n}-\rho_{1,n}-1)$ and $k = -\rho_{1,\infty} - \sum_{j=1}^n \rho_{1,j}$. This gives the statement of the Corollary. □

1.4. **Two-dimensional spaces of polynomials with prescribed singularities.** Theorems 1 and 3 give the following corollary which can be considered as a statement from enumerative algebraic geometry, see for instance [GH]. Namely, consider a two-dimensional space V of polynomials of one variable x with complex coefficients. Let k_1 be the degree of generic polynomials in V and k_2 the degree of special polynomials in V, $k_1 > k_2$.

For two functions $f(x), g(x)$ let $W(f,g)(x) = f'(x)g(x) - f(x)g'(x)$ be the Wronskian. If f, g is a basis in V, then the Wronskian has degree k_1+k_2-1 and does not depend on the choice of the basis up to multiplication by a nonzero constant. The corresponding monic polynomial will be called the Wronskian of the space and denoted $W_V(x)$. Let

$$W_V(x) \ = \ \prod_{l=1}^{n} (x-z_l)^{m_l} \ .$$

We say that the vector space V is nondegenerate if for any complex number x_0 there is a polynomial $f(x)$ in V, such that $f(x_0)$ is not zero, and if the set of roots of a polynomial in V of degree k_2 does not intersect the set $z_1, ..., z_n$.

Problem *Assume that* $k_1 > k_2$ *and* $W_V(x) \ = \ \prod_{l=1}^n (x-z_l)^{m_l}$ *are fixed,* $m_1 + ... + m_n = k_1 + k_2 - 1$. *What is the number of nondegenerate vector spaces* V *with such characteristics ?*

Corollary of Theorems 1 and 3.

For generic $z_1, ..., z_n$ *the number of nondegenerate vector spaces* V *with such data is equal to the multiplicity of the representation* $L_{k_1-k_2-1}$ *in the tensor product* $L_{m_1} \otimes ... \otimes L_{m_n}$.

1.5. **Critical points and Bethe vectors.** For a positive integer a, let L_a be the irreducible sl_2 module with highest weight a. The Shapovalov form on L_a is the unique symmetric bilinear form S_a such that $S_a(v_a, v_a) = 1$ and $S_a(ex, y) = S_a(x, fy)$ for all $x, y \in L_a$.

Let $\Omega = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e$ be the Casimir operator.

For positive integers $m_1, ..., m_n$, define on $L^{\otimes m} = L_{m_1} \otimes \cdots \otimes L_{m_n}$ the Shapovalov form as $S = S_{m_1} \otimes \cdots \otimes S_{m_n}$.

For pairwise distinct complex numbers $z_1, ..., z_n$ and any $i = 1, ..., n$, introduce a linear operator $H_i(z) : L^{\otimes m} \to L^{\otimes m}$,

$$H_i(z) \;=\; \sum_{j,\, j\neq i} \frac{\Omega^{(i,j)}}{z_i - z_j}\,. \tag{5}$$

Here $\Omega^{(i,j)}$ is the operator acting as Ω in the i-th and j-th factors and as the identity in all other factors of the tensor product. The operators $H_i(z)$, $i = 1, ..., n$, commute and are called the Hamiltonians of the Gaudin model of an inhomogeneous magnetic chain [G].

The Bethe ansatz is a certain construction of eigenvectors for a system of commuting operators. The idea of the construction is to find a vector-valued function of a special form and determine its arguments in such a way that the value of this function is an eigenvector. The equations which determine the special values of arguments are called the Bethe equations. The corresponding eigenvectors are called the Bethe vectors. The main problem of the Bethe ansatz is to show that the construction gives a basis of eigenvectors. On the Bethe ansatz see for instance [F, TV].

One of the systems of commuting operators diagonalized by the Bethe ansatz is the system of Hamiltonians of the Gaudin model, see [G].

Let k be a positive integer. Let $J = (j_1, ..., j_n)$ be a vector with integer coordinates such that $j_1 + ... + j_n = k$ and for any l we have $0 \leq j_l \leq m_l$. Introduce a vector in the tensor product, $f_J v = f^{j_1} v_{m_1} \otimes ... \otimes f^{j_n} v_{m_n}$. Introduce a function

$$A_J(t_1, ..., t_k, z_1, ..., z_n) \;=\; \sum_{\sigma\in\Sigma(k;\, j_1,...,j_n)} \prod_{i+1}^{k} \frac{1}{t_i - z_{\sigma(i)}}\,,$$

the sum is over the set $\Sigma(k;\, j_1, ..., j_n)$ of maps σ from $\{1, ..., k\}$ to $\{1, ..., n\}$ such that for every l the cardinality of $\sigma^{-1}(l)$ is equal to j_l.

Theorem 4. [RV, V].

- *If t^0 is a nondegenerate critical point of $\Phi_{k,n}(t; z, m)$, then the vector*

$$v(t^0, z) \;=\; \sum_J A_J(t^0, z) f_J v$$

belongs to the subspace $\mathrm{Sing}(L^{\otimes m})_k = \{v \in L^{\otimes m}[l(m) - 2k] \mid ev = 0\}$ *of singular vectors of the weight $l(m) - 2k$ and is an eigenvector of operators $H_i(z)$, $i = 1, ..., n$.*
- *If t^0 is a nondegenerate critical point, then*

$$S(v(t^0, z), v(t^0, z)) = det_{1\,\leq i,j\leq k} \left(\frac{\partial^2}{\partial t_i \partial t_j}\, ln\, \Phi_{k,n}(t^0; z, m) \right).$$

- *If $l(m) - 2k \geq 0$, then for a generic z, the eigenvectors $v(t^0, z)$ generate the space* $\mathrm{Sing}(L^{\otimes m})_k$.

The vectors $v(t^0, z)$ are the Bethe ansatz vectors of the Gaudin model. The Bethe equations of the Gaudin model are the critical point equations for the function $\Phi_{k,n}(t; z, m)$.

Corollary of Theorems 1 and 4. *For a generic z, the set of the Bethe vectors is a basis in the space $\mathrm{Sing}(L^{\otimes m})_k$, that is each eigenvector is presented exactly once as a Bethe vector.*

Remarks on the Bethe ansatz for $sl(n+1)$ and critical points see in [MV].

On the connections between the Bethe ansatz and Fuchsian differential equations see [Sk1, Sk2].

1.6. **The function $\Phi_{k,n}(t; z, m)$ and hypergeometric solutions of the KZ equations.** The KZ equations [KZ] of the conformal field theory for a function $u(z_1, ..., z_n)$ with values in the tensor product $L^{\otimes m}$ is the system of equations

$$\kappa \frac{\partial u}{\partial z_i} = H_i(z)\, u\,, \qquad i = 1, ..., n\,.$$

Here $H_i(z)$ are the operators defined in (5). The number κ is a parameter of equations.

The KZ equations have hypergeometric solutions [SV],

$$u(z) = \sum_J \int_{\gamma(z)} \Phi_{k,n}(t; z, m)^{\frac{1}{\kappa}}\, A_J(t, z)\, dt_1 \wedge ... \wedge dt_k\ f_J v\,.$$

The hypergeometric solutions are labeled by suitable families of k-dimensional cycles $\gamma(z)$. Such a solution takes values in the subspace $\mathrm{Sing}(L^{\otimes m})_k$ of singular vectors.

Studying the semiclassical asymptotics of the hypergeometric solutions as κ tends to zero one gets the Bethe ansatz for the Gaudin model [RV, V].

We see that the function $\Phi_{k,n}(t; z, m)$ is the *"master function"* which governs solutions of KZ equations and Bethe vectors in the Gaudin model.

2. Isolated critical points

2.1. **Combinatorial remarks.** For an sl_2 module V and $l \in \mathbb{C}$, let $V[l] = \{v \in V \mid hv = lv\}$ be the weight subspace of weight l.

Let $m_1, \dots, m_a \in \mathbb{Z}_{>0}$ and $m_{a+1}, \dots, m_n \notin \mathbb{Z}_{>0}$. Consider the tensor product of irreducible sl_2 representations $L^{\otimes m} = L_{m_1} \otimes \cdots \otimes L_{m_a} \otimes L_{m_{a+1}} \otimes \cdots \otimes L_{m_n}$ and for a nonnegative integer k the difference of dimensions

$$d(m, k) = \dim L^{\otimes m}[l(m) - 2k] - \dim L^{\otimes m}[l(m) - 2k + 2].$$

Define the number

$$\sharp(k, n; m_1, \dots, m_a) = \sum_{q=0}^{a} (-1)^q \sum_{1 \le i_1 < \dots < i_q \le a} \binom{k + n - 2 - m_{i_1} - \dots - m_{i_q} - q}{n - 2}.$$

Remark. The formula for $\sharp(k,n;m_1,\dots,m_a)$ implies that if $k\le m_l$ for all $1\le l\le a$, then this number does not depend on $m_1,\dots,m_a$, and is equal to $\binom{k+n-2}{n-2}$.

Theorem 5. *We have* $d(m,k) = \sharp(k,n;m_1,\dots,m_a)$.

Proof. A basis of the weight subspace $L^{\otimes m}[l(m)-2k]$ is formed by the vectors $f_J v = f^{j_1}v_{m_1}\otimes ... \otimes f^{j_n}v_{m_n}$, where $j_1,\dots,j_n$ are integers such that $j_1+...+j_n=k$, and for any l we have $0\le j_l\le m_l$. The Inclusion-Exclusion Principle (see [B], Ch.5) says that the number of such J is equal to

$$\sum_{q=0}^{a}\ \sum_{1\le i_1<\dots<i_q\le a}(-1)^q\binom{k-m_{i_1}-\dots-m_{i_q}-q+n-1}{n-1}.$$

The Pascal triangle property,

$$\binom{a}{b}-\binom{a-1}{b}=\binom{a-1}{b-1},$$

implies the statement. □

Let $m=(m_1,\dots,m_n)\in\mathbb{R}^n$, $k\in\mathbb{Z}_{>0}$.

Definition. The pair $\{m,k\}$ is called *good* if $l(m)\ge 2k$ and m has the following form,

$$m=(m_1,\dots,m_a,m_{a+1},\dots,m_{a+b},m_{a+b+1},\dots,m_n),\quad 0\le a\le a+b\le n,$$

where

- $m_1,\dots,m_a$ are positive integers;
- $m_{a+1},\dots,m_{a+b}$ are positive numbers such that for any $1\le i\le j\le b$ the sum $m_{a+i}+\dots+m_{a+j}$ is not an integer;
- $m_{a+b+1},\dots,m_n$ are negative integers.

Example. If $m_1,\dots,m_n,k\in\mathbb{Z}_{>0}$, and $l(m)\ge 2k$, then the pair $\{m,k\}$ is good.

Remarks.

1. Let the pair $\{m,k\}$ be good and let $\mathrm{Sing}(L^{\otimes m})_k=\{v\in L^{\otimes m}[l(m)-2k]\mid ev=0\}$ be the subspace of singular vectors of the weight $l(m)-2k$. Then $\dim\mathrm{Sing}(L^{\otimes m})_k = d(m,k)$.

2. If $m_1,...,m_n,k$ are positive integers and $l(m)-2k\ge 0$, then $w(m,k)=\dim\mathrm{Sing}(L^{\otimes m})_k$.

Lemma 1. *Let p be a positive integer, $p<k$. If the pair $\{(m_1,\dots,m_{j-1},\ p-1,\ m_{j+1},\dots,m_n),k\}$ is good, then the pair $\{(m_1,\dots,m_{j-1},m_{j+1},\dots,m_n,-p-1),k-p\}$ is good.* □

2.2. **Main statements on isolated critical points.**

Theorem 6. *Let p be a positive integer, $p \leq k$. Assume that the function $\Phi_{k,n}(t;z,m)$ has an infinite sequence of critical points such that each of its first p coordinates tends to infinity and each of its remaining coordinates has a finite limit. Then $l(m) = 2k-p-1$.*

Corollary 1. *If $l(m)-2k > -2$, then the function $\Phi_{k,n}(t;z,m)$ has only isolated critical points.* □

Proof of Theorem 6. We write the system defining the critical points of $\Phi(t) = \Phi_{k,n}(t;z,m)$ in the form

$$(t_r - z_1)(\partial\Phi/\partial t_r)/\Phi = 0\,, \quad r = 1,\dots,k\,.$$

The r-th equation is

$$-m_1 - \sum_{l=2}^{n} \frac{m_l(t_r - z_1)}{t_r - z_l} + \sum_{\substack{1\leq j\leq k \\ j\neq r}} \frac{2(t_r - z_1)}{t_r - t_j} = 0\,,$$

and the sum of the first p equations is

$$-pm_1 - \sum_{r=1}^{p}\sum_{l=2}^{n} \frac{m_l(t_r - z_1)}{t_r - z_l} + 2\cdot\frac{p(p-1)}{2} + \sum_{r=1}^{p}\sum_{j=p+1}^{k} \frac{2(t_r - z_1)}{t_r - t_j} = 0\,.$$

Let $\{t^{(q)} = (t_1^{(q)},...,t_k^{(q)})\}$ be our sequence of critical points. Then

$$\frac{t_r^{(q)} - z_1}{t_r^{(q)} - z_l} \underset{q\to\infty}{\longrightarrow} 1\,, \quad \frac{t_r^{(q)} - z_1}{t_r^{(q)} - t_j^{(q)}} \underset{q\to\infty}{\longrightarrow} 1\,, \quad 1 \leq r \leq p\,, \quad 2 \leq l \leq n\,, \quad p+1 \leq j \leq k\,,$$

and this equation results in $-p(m_1 + \dots + m_n) + p(p-1) + 2p(k-p) = 0$. □

Theorem 7. *Let the pair $\{m,k\}$ be good and let a be a nonnegative integer such that $m_1,\dots,m_a \in \mathbb{Z}_{>0}$ and $m_{a+1},\dots,m_n \notin \mathbb{Z}_{>0}$. Then for a generic z in $\mathbb{C}^n$, all critical points of the function $\Phi_{k,n}(t;z,m)$ are nondegenerate and the critical set consists of $\sharp(k,n;m_1,\dots,m_a)$ orbits.*

Theorem 7 is proved in Sec. 2.8. Theorem 7 implies part 1 of Theorem 1.

2.3. **The bound from below.**

Theorem 8. *Let $\{m = (m_1,...,m_n),k\}$ be a good pair. Assume that the number a is such that $m_1,\dots,m_a \in \mathbb{Z}_{>0}$, $m_{a+1},\dots,m_n \notin \mathbb{Z}_{>0}$. Let s be a real number, $s \gg 1$, and $z^{(s)} = (s,s^2,\dots,s^n)$. Then the function $\Phi_{k,n}(t;z^{(s)},m)$ has at least $\sharp(k,n;m_1,\dots,m_a)$ orbits of nondegenerate critical points.*

This Theorem is a direct corollary of results in [RV], Sec. 9. For convenience, we sketch its proof here.

Definition. Let $m_1,m_2 \in \mathbb{R}$, $k \in \mathbb{Z}_{\geq 0}$. The triple $\{m_1,m_2;k\}$ is called *admissible* if the following two conditions are satisfied,

- $m_1 + m_2 - 2k \geq 0$,
- if for some $i \in \{1, 2\}$ we have $m_i \in \mathbb{Z}_{\geq 0}$, then $k \leq m_i$.

If the triple $\{m_1, m_2; k\}$ is admissible, then the function

$$\Phi_{k,2}(t) = \prod_{i=1}^{k} t_i^{-m_1}(t_i - 1)^{-m_2} \prod_{1\leq i<j\leq k} (t_i - t_j)^2$$

has exactly $k!$ critical points all of which are nondegenerate [V].

Definition. Let $\{m = (m_1, ..., m_n), k\}$ be a good pair. Let $I = (i_1, \ldots, i_n)$ be a sequence of nonnegative integers such that $i_1 = 0$ and $i_2 + \cdots + i_n = k$. The sequence I is called *an admissible sequence* for $\{m, k\}$ if all triples

$$\{m_1 + \cdots + m_{l-1} - 2(i_1 + \cdots + i_{l-1}),\, m_l;\, i_l\}\,,$$

for $l = 2, \ldots, k$ are admissible.

Proof of Theorem 8. For an admissible sequence $I = (i_1, \ldots, i_n)$, make a change of variables

$$t_j = s^l u_j \quad \text{if} \quad i_1 + \cdots + i_{l-1} < j \leq i_1 + \cdots + i_l\,, \quad l = 2, \ldots, n\,.$$

For $2 \leq l \leq n$, define the function

$$\begin{aligned}\Phi_{i_l,2} &= \Phi_{i_l,2}(u_{i_1+\cdots+i_{l-1}+1}, \ldots, u_{i_1+\cdots+i_l}) \\ &= \prod_{j=i_1+\cdots+i_{l-1}+1}^{i_1+\cdots+i_l} u_j^{-a_l}(u_j - 1)^{-m_l} \prod_{i_1+\cdots+i_{l-1}+1\leq i<j\leq i_1+\cdots+i_l} (u_i - u_j)^2\,,\end{aligned}$$

where $a_l = m_1 + \cdots + m_{l-1} - 2(i_1 + \cdots + i_{l-1})$.

Let $\Phi_I(u) = \Phi_{i_2,2} \cdots \Phi_{i_n,2}$. For any $l = 2, \ldots, n$, the function $\Phi_{i_l,2}$ has exactly one orbit of nondegenerate critical points according to Theorem 1.3.1 in [V]. Let $u_{(l)}$ be a critical point of $\Phi_{i_l,2}$, then $u_I = (u_{(2)}, \ldots, u_{(n)})$ is a nondegenerate critical point of the function $\Phi_I(u)$. In a neighborhood of u_I , the critical point system of the function $\Phi_{k,n}(t(u)) = \Phi_{k,n}(t(u); z^{(s)}, m)$ is a deformation of the critical point system of the function $\Phi_I(u)$ with deformation parameter s ,

$$\frac{\partial\Phi_{k,n}(t(u))/\partial u_j}{\Phi_{k,n}(t(u))} = \frac{\partial\Phi_I(u)/\partial u_j}{\Phi_I(u)} + O(s^{-1}) = 0\,, \quad j = 1, \ldots, k\,.$$

When $s \to \infty$, the function $\Phi_{k,n}(t(u))$ has a nondegenerate critical point $u_I(s)$ close to u_I , which defines a nondegenerate critical point $t_I(s)$ of the function $\Phi_{k,n}(t; z^{(s)}, m)$. Theorem 9.9 in [RV] and its corollaries imply that if I and I' are distinct admissible sequences, then the corresponding points $t_I(s)$ and $t_{I'}(s)$ cannot belong to the same orbit. To complete the proof it remains to note that the number of admissible sequences for $\{m, k\}$ is equal to the dimension of $\mathrm{Sing}(L^{\otimes m})_k$. This follows from the fact that the admissible sequences label a basis of iterated singular vectors in $\mathrm{Sing}(L^{\otimes m})_k$, see Sec. 8 in [RV]. □

2.4. **The maximal possible number of critical points.**

Theorem 9. *If the pair* $\{m,k\}$ *is good and if* $k \le m_i$ *for all* $m_i \in \mathbb{Z}_{>0}$, *then for a generic* z *in* $\mathbb{C}^n$, *the function* $\Phi_{k,n}(t;z,m)$ *has exactly* $\binom{k+n-2}{n-2}$ *orbits of critical points which all are nondegenerate.*

Corollary 2. *Theorem 7 is true for* $k=1$.

Proof of Theorem 9. If all numbers $z_1, ..., z_n$ are real and all numbers $m_1, ..., m_n$ are negative, then all critical points of the function $\Phi_{k,n}(t;z,m)$ are nondegenerate and the critical set consists of $\binom{k+n-2}{n-2}$ orbits [V]. Therefore for any z and m the total number of isolated orbits of critical points counted with multiplicities is not greater than $\binom{k+n-2}{n-2}$. Corollary 1 says that the function $\Phi_{k,n}(t;z,m)$ does not have non-isolated critical points. In order to finish the proof we apply Theorem 8. □

2.5. **Vanishing critical points.**

Set $m(\epsilon) = (m_1, \dots, m_{j-1}, k-1+\epsilon, m_{j+1}, \dots, m_n)$. Let K be the number of critical points of the function $\Phi_{k,n}(t;z,m(\epsilon))$ which tend to the vertex $\{t_1 = \dots = t_k = z_j\}$ when ϵ tends to zero.

Theorem 10. *If the pair* $\{(m_1, \dots, m_{j-1}, k-1, m_{j+1}, \dots, m_n), k\}$ *is good, then the number* K *is positive and divisible by* $k!$.

Theorem 10 is proved in Sec. 2.6.

Let p be a positive integer, $p < k$. For $m = (m_1, \dots, m_{j-1}, p-1, m_{j+1}, \dots, m_n)$, set $m(\epsilon) = (m_1, \dots, m_{j-1}, p-1+\epsilon, m_{j+1}, \dots, m_n)$, $m^{(p)} = (m_1, ..., m_{j-1}, -p-1, m_{j+1}, \dots, m_n)$.

Definition. The function

$$\Phi_{k-p,n}(t_{p+1}, ..., t_k; z, m^{(p)}) = \prod_{i=p+1}^{k} (t_i - z_j)^{p+1} \prod_{i=p+1}^{k} \prod_{l \ne j} (t_i - z_l)^{-m_l} \prod_{p+1 \le i < j \le k} (t_i - t_j)^2$$

is called the function *induced* by the function $\Phi_{k,n}(t;z,m(\epsilon))$ on the edge $\{t_1 = \dots = t_p = z_j\}$ as ϵ tends to zero.

Let ϵ tend to zero. Let $B = (b_{p+1}, \dots, b_k)$ be a nondegenerate critical point of the induced function $\Phi_{k-p,n}(t_{p+1}, ..., t_k; z, m^{(p)})$, and let K be the number of critical points of the function $\Phi_{k,n}(t;z,m(\epsilon))$ which tend to the point $\{t_1 = \dots = t_p = z_j, t_{p+1} = b_{p+1}, \dots, t_k = b_k\}$.

Theorem 11. *If* $\{(m_1, \dots, m_{j-1}, p-1, m_{j+1}, \dots, m_n), k\}$ *is a good pair, then* K *is positive and divisible by* $p!$.

Theorem 11 is proved in Sec. 2.7.

2.6. **Proof of Theorem 10.** After the translation $t_i \mapsto t_i - z_j$, $z_l \mapsto z_l - z_j$, and renumbering $z_1, \dots, z_n$, we can assume $z = (0, z_2, \dots, z_n)$ and $m(\epsilon) = (k-1+\epsilon, m_2, \dots, m_n)$. We estimate the number of critical points $t(\epsilon)$ of the function

$$\Phi_{k,n}(t; z, m(\epsilon)) = \prod_{i=1}^{k} \left[t_i^{-k+1-\epsilon} \prod_{l=2}^{n} (t_i - z_l)^{-m_l} \right] \prod_{1 \le i < j \le k} (t_i - t_j)^2$$

such that $t_r(\epsilon)$ tends to zero for $r = 1, \dots, k$ as ϵ tends to zero.

Blow-up the vertex $\{t_1 = \dots = t_k = 0\}$. In coordinates $u_1, \dots, u_k$, where

$$t_1 = u_1 u_k, \quad \dots, \quad t_{k-1} = u_{k-1} u_k, \quad t_k = u_k,$$

the function $\Phi_{k,n}(t; z, m(\epsilon))$ has the form

$$\begin{aligned} \tilde\Phi &= \Phi_{k,n}(u_1 u_k, \dots, u_{k-1} u_k, u_k; z, m(\epsilon)) = \\ &= \prod_{i=1}^{k-1} u_i^{-k+1-\epsilon} (u_i - 1)^2 \prod_{1 \le i < j \le k-1} (u_i - u_j)^2 \\ &\cdot \quad u_k^{-k\epsilon} \prod_{l=2}^{n} (u_k - z_l)^{-m_l} \prod_{i=1}^{k-1} \prod_{l=2}^{n} (u_i u_k - z_l)^{-m_l} . \end{aligned}$$

Consider this function as a function on the space $\mathbb{C}^{k+1}$ with coordinates $u_1, \dots, u_k, \epsilon$. Consider in $\mathbb{C}^{k+1}$ the set C of all critical points of this function with respect to coordinates $u_1, \dots, u_k$.

Lemma 2. *Near the divisor $\{ (u; \epsilon) \in \mathbb{C}^{k+1} \mid u_k = 0 \}$, the critical set C is the union of $(k-1)!$ nonsingular curves which intersect the divisor at $(k-1)!$ points $(A_\sigma, 0; 0)$, where A_σ runs through all permutations of $\{\alpha, \dots, \alpha^{k-1}\}$, $\alpha = \exp(2\pi i/k)$. The coordinate u_k is a local parameter at the intersection point on each of these curves.*

Proof. We write the system defining the critical points of $\tilde\Phi$ in the form

$$\frac{\partial \tilde\Phi / \partial u_q}{\tilde\Phi} = 0 , \; q = 1, \dots, k-1, \qquad u_k \cdot \frac{\partial \tilde\Phi / \partial u_k}{\tilde\Phi} = 0 ,$$

and get the following equations

$$\frac{-k+1-\epsilon}{u_q} + \frac{2}{u_q - 1} + \sum_{\substack{1 \le j \le k-1 \\ j \ne q}} \frac{2}{u_q - u_j} - \sum_{l=2}^{n} m_l \frac{u_k}{u_q u_k - z_l} = 0 ,$$

$$-k\epsilon - \sum_{l=2}^{n} m_l \frac{u_k}{u_k - z_l} - \sum_{i=1}^{k-1} \sum_{l=2}^{n} m_l \frac{u_i u_k}{u_i u_k - z_l} = 0 ,$$

where $q = 1, \dots, k-1$.

One can express ϵ in terms of $u_1, \dots, u_k$ from the last equation, and consider the equations

$$\frac{\partial\tilde{\Phi}/\partial u_q}{\tilde{\Phi}} = 0\,, \quad q = 1, \dots, k-1,$$

as a system of equations with respect to $u_1, \dots, u_{k-1}$ depending on the parameter u_k. For $u_k = 0$, this system turns into the critical point system of the function

$$\Phi_{k-1,2}(u) = \prod_{i=1}^{k-1} u_i^{-k+1}(u_i - 1)^2 \prod_{1\le i<j\le k-1} (u_i - u_j)^2\,.$$

Theorem 1.3.1 [V] implies that the function $\Phi_{k-1,2}$ has exactly $(k-1)!$ critical points, all of which are non-degenerate, and the coordinates of each of these critical points form the set of all roots of the equation $\xi^{k-1} + \xi^{k-2} + \dots + 1 = 0$. This gives the Lemma. □

Lemma 3. *For a given permutation A_σ, the number of critical points of the function $\tilde{\Phi}\,|_{\epsilon=\epsilon_0}$, which tend to $(A_\sigma, 0; 0)$ as ϵ_0 tends to zero, is positive and divisible by k.*

Proof. It is enough to prove the statement for $A = (\alpha, \alpha^2, \cdots, \alpha^{k-1})$. The function $\Phi_{k,n}(t)$ is invariant with respect to permutations of $\{t_1, \dots, t_k\}$. Therefore the critical set C is invariant with respect to the corresponding action of the symmetric group S^k on the space $\mathbb{C}^{k+1}$ with coordinates $u_1, \cdots, u_k$, ϵ. The connected component $C_A \subset C$ which contains $(A, 0; 0)$ is preserved by the map $\mathcal{P}$, the lifting of the cyclic permutation $t_1 \mapsto t_2 \mapsto \dots \mapsto t_k \mapsto t_1$; and the point $(A, 0; 0)$ is a fixed point of $\mathcal{P}$. According to Lemma 2, the coordinate u_k is a local parameter on the curve C_A,

$$C_A = \{\, (u; \epsilon) \mid u_j = \alpha^j + O(u_k),\ j = 1, \dots, k-1, \quad \epsilon = f(u_k) \,\},$$

where $f(u_k)$ is the germ of a suitable holomorphic function. This germ can not be identically zero, as in this case the function $\Phi_{k,n}(t; z, m)$ would have a curve of critical points,

$$\{\, t_j = \alpha^j t_k + O(t_k)\,, \quad j = 1, \dots, k-1 \,\},$$

but this is impossible by Corollary 1 because the pair $\{(m_1, \dots, m_{j-1}, k-1, m_{j+1}, \dots, m_n), k\}$ is good. The equation $\epsilon = f(u_k)$ has to be invariant with respect to the map $\mathcal{P}$ which does not change ϵ and maps u_k to $u_1 u_k = \alpha u_k + O(u_k^2)$. This means that the Taylor expansion of the germ $f(u_k)$ starts with a power of u_k divisible by k. □

Lemmas 2 and 3 imply Theorem 10. □

2.7. **Proof of Theorem 11.** We set

$$z = (0, z_2, \dots, z_n)\,, \quad m(\epsilon) = (p - 1 + \epsilon, m_2, \dots, m_n)\,,$$

and count the number of critical points $t(\epsilon)$ of the function

$$\Phi_{k,n}(t; z, m(\epsilon)) = \prod_{i=1}^{k} \left[t_i^{-p+1-\epsilon} \prod_{l=2}^{n} (t_i - z_l)^{-m_l} \right] \prod_{1\le i<j\le k} (t_i - t_j)^2$$

which satisfy

$$t_i(\epsilon) \to 0\,, \quad 1 \le i \le p, \quad t_j(\epsilon) \to b_j\,, \quad p+1 \le j \le k,$$

as ϵ tends to 0.

Blow-up the edge $\{t_1 = \cdots = t_p = 0\}$. In coordinates $u = (u_1, \dots, u_p)$, $t' = (t_{p+1}, \dots, t_k)$, where $t_1 = u_1 u_p,\ \dots,\ t_{p-1} = u_{p-1}u_p,\ t_p = u_p$, the function $\Phi_{k,n}(t; z, m(\epsilon))$ is

$$\begin{aligned}\tilde\Phi \ &= \ \Phi_{k,n}(u_1u_p, \dots, u_{p-1}u_p, u_p, t_{p+1}, \dots, t_k; z, m(\epsilon)) = \\ &\prod_{i=p+1}^{k}\left[t_i^{-p+1-\epsilon}\prod_{l=2}^{n}(t_i - z_l)^{-m_l}\right]\prod_{p+1\le i<j\le k}(t_i - t_j)^2 \\ &\cdot\prod_{i=1}^{p-1}\prod_{j=p+1}^{k}(t_j - u_iu_p)^2\prod_{j=p+1}^{k}(t_j - u_p)^2 \\ &\cdot\prod_{i=1}^{p-1}u_i^{-p+1-\epsilon}(u_i - 1)^2\prod_{1\le i<j\le p-1}(u_i - u_j)^2 \\ &\cdot u_p^{-p\epsilon}\prod_{l=2}^{n}(u_p - z_l)^{-m_l}\prod_{i=1}^{p-1}\prod_{l=2}^{n}(u_iu_p - z_l)^{-m_l}\,.\end{aligned}$$

We take the critical point system for $\tilde\Phi$ in the following form

$$\frac{\partial\tilde\Phi/\partial u_i}{\tilde\Phi} = 0\,, \qquad i = 1, \dots, p-1; \qquad (S_u)$$

$$\frac{\partial\tilde\Phi/\partial t_j}{\tilde\Phi} = 0\,, \qquad j = p+1, \dots, k; \qquad (S_{t'})$$

$$u_p \cdot \frac{\partial\tilde\Phi/\partial u_p}{\tilde\Phi} = 0\,. \qquad (S_p)$$

From equation (S_p), one can express ϵ in terms of u, t'. Therefore one can consider equations (S_u), $(S_{t'})$ as a system of equations with respect to $u_1, \dots, u_{p-1}$, $t_{p+1}, \dots, t_k$ depending on the parameter u_p. For $u_p = 0$, equations (S_u) turn into the critical point system of the function

$$\Phi_{p-1,2}(u) = \prod_{i=1}^{p-1}u_i^{-p+1}(u_i - 1)^2\prod_{1\le i<j\le p-1}(u_i - u_j)^2\,,$$

and equations $(S_{t'})$ turn into the critical point system of the induced function

$$\Phi_{k-p,n}(t'; z, m^{(p)}) = \prod_{i=p+1}^{k}\left[t_i^{p+1}\prod_{l=2}^{n}(t_i - z_l)^{-m_l}\right]\prod_{p+1\le i<j\le k}(t_i - t_j)^2\,.$$

Consider $\tilde{\Phi}$ as a function on the space $\mathbb{C}^{k+1}$ with coordinates u, t', ϵ. Consider in $\mathbb{C}^{k+1}$ the critical set of $\tilde{\Phi}$ with respect to u, t'. Similarly to Lemma 2 we get

Lemma 4. *The critical set near the plane* $\{ (u,t',\epsilon) \in \mathbb{C}^{k+1} \mid u_p = 0,\ t' = B \}$ *is the union of* $(p-1)!$ *nonsingular curves which intersect this plane at* $(p-1)!$ *points* $(A_\sigma, 0, B; 0)$, *where* A_σ *runs through all permutations of* $\{\alpha, ..., \alpha^{p-1}\}$, $\alpha = \exp(2\pi i/p)$. *The coordinate* u_p *is a local parameter at the intersection point on each these curves.* □

The function $\Phi_{k,n}(t)$ is invariant with respect to permutations of $\{t_1, \dots, t_p\}$, and hence the union of these $(p-1)!$ curves is invariant with respect to the corresponding action of the symmetric group S^p on the space $\mathbb{C}^{k+1}$ with coordinates u, t', ϵ.

Lemma 5. *For a given permutation* A_σ, *the number of critical points of the function* $\tilde{\Phi}|_{\epsilon=\epsilon_0}$ *which tend to* $(A_\sigma, 0, B; 0)$ *as* ϵ_0 *tends to zero is positive and divisible by* p.

Proof. We prove this statement for $A = (\alpha, \dots, \alpha^{p-1})$. The connected component of the critical set which contains the point $(A, 0, B; 0)$ is of the form

$$C_{A,B} = \{ u_i = \alpha^i + O(u_p),\ i = 1, \dots, p-1;\ \ t_j = b_j + O(u_p),\ j = p+1, \dots, k;\ \ \epsilon = f(u_p) \},$$

where $f(u_p)$ is the germ of a suitable holomorphic function. Similarly to Lemma 3, we conclude that f is a non-zero germ, that $C_{A,B}$ is invariant with respect to the map $\mathcal{P}$ which is the lifting of the permutation $t_1 \mapsto t_2 \mapsto \dots \mapsto t_p \mapsto t_1$, and that the Taylor expansion of the germ $f(u_p)$ starts with a power of u_p divisible by p. □

Lemmas 4 and 5 imply Theorem 11. □

2.8. **Proof of Theorem 7.** We prove the statement by a double induction with respect to k, the number of variables in $\Phi_{k,n}(t; z, m)$, and $a(m)$, the number of positive integers in m.

For $k = 1$ and any $a(m)$ the statement is true by Corollary 2. For any k and $a(m) = 0$, the statement holds by Theorem 9.

Assume that the Theorem is proved for $k < k_0$ and any $a(m)$ and for $k = k_0$ and $a(m) < a_0$. We prove the Theorem for $k = k_0$ and $a = a_0$.

Let $\{m = (m_1, \dots, m_n), k\}$ be a good pair. Assume that $m_1, \dots, m_a \in \mathbb{Z}_{>0}$ and $m_{a+1}, \dots, m_n \notin \mathbb{Z}_{>0}$. For $\epsilon \neq 0$ small enough, the pair $\{m(\epsilon) = (m_1, \dots, m_{a-1}, m_a + \epsilon, m_{a+1}, \dots, m_n), k\}$ is also good, and the number of positive integers in $m(\epsilon)$ is $a - 1$. Therefore according to the induction hypothesis, for a generic z the function $\Phi_{k,n}(t; z, m(\epsilon))$ has exactly $\sharp(k, n; m_1, \dots, m_{a-1})$ orbits of critical points which all are nondegenerate.

We study how the number of orbits of critical points of $\Phi_{k,n}(t; z, m(\epsilon))$ changes as $\epsilon \to 0$. According to Corollary 1, non-isolated critical points do not appear. For isolated critical points, there are three possibilities.

(1) If $m_a \geq k$, then the function $\Phi_{k,n}(t; z, m)$ has at most

$$\sharp(k, n; m_1, \dots, m_{a-1}) = \sharp(k, n; m_1, \dots, m_a)$$

orbits of critical points. Indeed, the number of orbits of isolated critical points cannot increase as $\epsilon \to 0$.

(2) If $m_a = k-1$, then according to Theorem 10 at least $k!$ critical points disappear at the vertex $t_1 = \cdots = t_k = z_a$ as $\epsilon \to 0$. Therefore the number of orbits of critical points of the function $\Phi_{k,n}(t;z,m)$ does not exceed

$$\sharp(k,n;m_1,\dots,m_{a-1}) - 1 = \sharp(k,n;m_1,\dots,m_{a-1},k-1) = \sharp(k,n;m_1,\dots,m_a)\,.$$

(3) If $m_a = p-1$ for some integer $1 < p \le k-1$, then critical points disappear at certain points of the edges of the form $\{t_{i_1} = \cdots = t_{i_p} = z_a\}$. These points are critical points of the functions induced by the function $\Phi_{k,n}(t;z,m(\epsilon))$ on the edges as $\epsilon \to 0$. The number of the edges is $\binom{k}{k-p}$. Any of the induced functions is a function of $k-p<k$ variables with the set of exponents

$$m^{(p)} = (m_1,\dots,m_{a-1},-p-1,m_{a+1},\dots,m_n)\,.$$

The pair $\{m^{(p)},k-p\}$ is good after renumbering the coordinates of the vector $m^{(p)}$, and $m^{(p)}$ contains $a-1$ positive integers. Hence according to the induction hypothesis, for a generic z any of the induced functions has exactly $(k-p)!\sharp(k-p,n;m_1,\dots,m_{a-1})$ critical points which all are nondegenerate. At each of these points, at least $p!$ critical points of the function $\Phi_{k,n}(t;z,m(\epsilon))$ disappear as $\epsilon \to 0$ by Theorem 11. Thus the total number of critical points which disappear as $\epsilon \to 0$ is at least

$$\binom{k}{k-p}(k-p)!\;p!\;\sharp(k-p,n;m_1,\dots,m_{a-1}) \;=\; k!\;\sharp(k-m_a-1,n;m_1,\dots,m_{a-1})\,.$$

Therefore $\Phi_{k,n}(t;z^{(s)},m)$ has at most

$$\sharp(k,n;m_1,\dots,m_{a-1}) - \sharp(k-m_a-1,n;m_1,\dots,m_{a-1}) = \sharp(k,n;m_1,\dots,m_a)$$

orbits of critical points.

Thus in all cases the number of orbits of critical points of the function $\Phi_{k,n}(t;z,m)$ is not greater than $\sharp(k,n;m_1,\dots,m_a)$. But Theorem 8 says that the number of orbits is at least $\sharp(k,n;m_1,\dots,m_a)$. This gives Theorem 7. □

3. Critical points and Fuchsian equations

3.1. **Critical points and Fuchsian equations with only polynomial solutions.** On Fuchsian equations see [R].

Lemma 6. *Let all solutions of the Fuchsian equation (3) be polynomials. Then generic solutions have no multiple roots.*

Proof. Let $v(x)$ be a solution. Assume that the order of $v(x)$ at some point $x = z_0$ is $r, \quad r \geq 2$. Then the order of $F(x)v''(x)$ at $x = z_0$ is at least $r-1$. Hence $F(z_0) = 0$. Therefore z_0 is one of the points $z_1, ..., z_n$ and the order of $v(x)$ at this point is $m_j + 1$. This means that $v(x)$ is not a generic solution. $\square$

Lemma 7. *Let $m = (m_1, ..., m_n) \in \mathbb{Z}^n_{>0}$, $k \in \mathbb{Z}_{>0}$. Let t^0 be a critical point of the function $\Phi_{k,n}(t; z, m)$. Then all solutions of the associated differential equation $E(t^0, z, m)$ are polynomials.*

Proof. Let $u(x) = (x - t^0_1) \cdots (x - t^0_k)$. For $j = 1, ..., n$, we have $u(z_j) \neq 0$. Hence all solutions are univalued at z_j. Therefore all solutions are univalued at infinity as well. Thus all solutions are polynomials. $\square$

Remarks.

1. If $l(m) + 1 - k > k$, then the generic solution of equation $E(t^0, z, m)$ has degree $l(m) + 1 - k$.

2. If $0 \leq l(m) + 1 - k < k$, then the generic solution of equation $E(t^0, z, m)$ has degree k, the equation also has solutions of degree $l(m) + 1 - k$.

3. If $l(m) + 1 - k = k$, then the two exponents at infinity are equal. Every Fuchsian differential equation with equal exponents has multivalued solutions. Hence the function $\Phi_{k,n}(t; z, m)$ does not have critical points. This is the second part of Theorem 1.

4. If $l(m) + 1 - k < 0$, then one of exponents at infinity is positive. Such a Fuchsian differential equation cannot have only polynomial solutions. Hence the function $\Phi_{k,n}(t; z, m)$ does not have critical points. This is the fourth part of Theorem 1.

5. Let $l(m) + 1 - k = 0$ and let equation (3) have only polynomial solutions with the degree of the generic solution equal to k. Then $H(x)$ is identically equal to zero and the solutions have the form

$$\int (x - z_1)^{m_1} \cdots (x - z_n)^{m_n} dx + \text{const}\,.$$

Hence the critical set of the function $\Phi_{k,n}(t; z, m)$, written in symmetric coordinates, forms a straight line. In this case $w(m, l(m) + 1 - k) = 1$. This statement gives part 3 of Theorem 1 for $l(m) + 1 - k = 0$.

Lemma 8. *Let $m = (m_1, ..., m_n) \in \mathbb{Z}^n_{>0}$, $k \in \mathbb{Z}_{>0}$, $l(m) - 2k \leq -2$. Let t^0 be a critical point of the function $\Phi_{k,n}(t; z, m)$. Then there exists a curve of critical points containing t^0. The curve being written in symmetric coordinates $\lambda_1 = \sum t_i$, ..., $\lambda_k = t_1 \cdots t_k$ is a straight line in $\mathbb{C}^k_\lambda$.*

Proof. Equation $E(t^0, z, m)$ has only polynomial solutions. Let $u_1(x) = (x - t^0_1) \cdots (x - t^0_k)$ and let $u_2(x)$ be a solution of degree $l(m)+1-k$. Then solutions $u_c(x) = u_1(x) + cu_2(x)$

correspond to a curve of critical points. The coefficients of $u_c(x)$ give a straight line in $\mathbb{C}^k_\lambda$. □

Lemma 9. *Let $m = (m_1, ..., m_n) \in \mathbb{Z}^n_{>0}$, $k \in \mathbb{Z}_{>0}$, $l(m) - 2k \le -2$. Then the straight lines in $\mathbb{C}^k_\lambda$ of critical points of $\Phi_{k,n}(t; z, m)$ do not intersect.*

Proof. If two critical points of $\Phi_{k,n}(t; z, m)$ belong to different lines, then the associated differential equations are different. Two differential equations of the form (3) with the same $F(x), G(x)$ and distinct $H(x)$ cannot have common nonzero solutions. □

If k is such that $l(m) - 2k \le -2$, then for $k' = l(m) + 1 - k$ we have $l(m) - 2k' \ge 0$.

Lemma 10. *Let $m = (m_1, ..., m_n) \in \mathbb{Z}^n_{>0}$, $k \in \mathbb{Z}_{>0}$, $l(m) - 2k \le -2$. Then the number of critical lines in $\mathbb{C}^k_\lambda$ of the function $\Phi_{k,n}(t; z, m)$ is not less than the number of orbits of critical points of the function $\Phi_{l(m)+1-k,\,n}(t; z, m)$.*

Proof. Let $t^0 \in \mathbb{C}^{l(m)+1-k}$ be a critical point of $\Phi_{l(m)+1-k,n}(t; z, m)$. Generic solutions of $E(t^0, z, m)$ are of degree k. They define a straight line in $\mathbb{C}^k_\lambda$ of critical points of $\Phi_{k,n}(t; z, m)$.

If two critical points of $\Phi_{l(m)+1-k,\,n}(t; z, m)$ belong to different orbits, then the associated differential equations are different. The corresponding straight lines do not intersect. □

Theorem 12. *Let $m = (m_1, ..., m_n) \in \mathbb{Z}^n_{>0}$, $k \in \mathbb{Z}_{>0}$, $0 < l(m) + 1 - k < k$. For a generic z in $\mathbb{C}^n$, let t^0 be a critical point of the function $\Phi_{k,n}(t; z, m)$. Let $u(x)$ be a solution of degree $l(m)+1-k$ of equation $E(t^0, z, m)$. Then roots of $u(x)$ form a critical point of the function $\Phi_{l(m)+1-k,\,n}(t; z, m)$.*

Corollary 3. *Under conditions of Theorem 12, for a generic z the number of critical lines of the function $\Phi_{k,n}(t; z, m)$ is equal to the number of orbits of critical points of the function $\Phi_{l(m)+1-k,\,n}(t; z, m)$.*

Corollary 3 implies Theorem 3 and part 3 of Theorem 1.

Theorem 12 is proved in Sections 3.2 and 3.3. To prove Theorem 12 one needs to show that if z is generic, then the polynomial $u(x)$ does not have multiple roots.

3.2. **Polynomial solutions with multiple roots.** Let $z_1, \dots, z_n$ be pairwise distinct complex numbers. Let $m_1, \dots, m_n$, $k \in \mathbb{Z}_{>0}$. Let $F(x)$ and $G(x)$ be determined by (4). Let $H(x)$ be a polynomial of degree not greater than $n-2$ such that the differential equation (3) has a polynomial solution $U(x)$ of degree k with a multiple root. Then the root is equal to one of the singular points z_j and the multiplicity of the root is equal to $m_j + 1$.

Assume that $z_a, z_{a+1}, ..., z_n$ are all multiple roots of the solution, where a is a suitable number. Then

(6) $$U(x) \,=\, (x-t_1^0)\cdots(x-t_b^0)\,\cdot\,(x-z_a)^{m_a+1}\cdots(x-z_n)^{m_n+1}$$

where $t_1^0, ..., t_b^0$ are roots of multiplicity 1 and $b+(m_a+1)+...+(m_n+1)=k$.

Lemma 11.

- *Under the above assumptions, $t^0=(t_1^0,...,t_b^0)$ is a critical point of the function $\Phi_{b,n}(t;z,\tilde m)$ where $\tilde m=(m_1,...,m_{a-1},-m_a-2,...,-m_n-2)$.*
- *If $t^0=(t_1^0,...,t_b^0)$ is a critical point of the function $\Phi_{b,n}(t;z,\tilde m)$, then there exists a unique polynomial $H(x)$ of degree not greater than $n-2$ such that the polynomial $U(x)$ given by (6) is a polynomial solution of equation (3).*

The differential equation of part 2 of Lemma 11 will be called *associated with the critical point t^0 and vectors m, $\tilde m$* and denoted $E(t^0,z,m,\tilde m)$.

Proof. The substitution $x=t_i^0$ into (3) gives

$$\frac{U''(t_i^0)}{U'(t_i^0)}=\sum_{l=1}^n\frac{m_l}{t_i^0-z_l}\,.$$

We have

$$U'(x)=U(x)\left(\sum_{i=1}^b\frac{1}{x-t_i^0}+\sum_{l=a}^n\frac{m_l+1}{x-z_l}\right)\,,$$

$$\begin{aligned}U''(x)\;&=\;U(x)\left(\sum_{i<j}\frac{2}{(x-t_i^0)(x-t_j^0)}+\sum_{i=1}^b\sum_{l=a}^n\frac{2(m_l+1)}{(x-t_i^0)(x-z_l)}\right.\\&+\;\left.\sum_{l<j}\frac{2(m_l+1)(m_j+1)}{(x-z_l)(x-z_j)}+\sum_{l=a}^n\frac{(m_l+1)\,m_l}{(x-z_l)^2}\right)\,.\end{aligned}$$

Thus

$$\frac{U''(t_i^0)}{U'(t_i^0)}=\sum_{j\neq i}\frac{2}{t_i^0-t_j^0}+\sum_{l=a}^n\frac{2(m_l+1)}{(t_i^0-z_l)}\,,$$

hence $(t_1^0,\dots,t_b^0)$ is a solution of the critical point system of the function $\Phi_{b,n}(t;z,\tilde m)$.

To prove the second statement we have to check that $H(x)=-[\,F(x)\,U''(x)+G(x)\,U'(x)\,]\,/\,U(x)$ is a polynomial. The requirement that the function $H(x)$ does not have poles at $x=t_1^0,\,...,\,t_b^0$ is equivalent to the fact that t^0 is a critical point of $\Phi_{b,n}(t;z,\tilde m)$. An easy direct calculation shows that $H(x)$ does not have poles at $x=z_a,\,...,\,z_n$. □

Lemma 12. *Let $m\in\mathbb{Z}^n_{>0}$, $k\in\mathbb{Z}_{>0}$. Let $\tilde m$ be as in Lemma 11. Assume that $l(m)-2k\geq 0$. Then the pair $\{\tilde m,b\}$ is good.* □

Lemma 13. *Let $m \in \mathbb{Z}^n_{>0}$ and $k \in \mathbb{Z}_{>0}$ be such that $k < l(m)+1-k$. Let $a \in \mathbb{Z}_{>0}$ be such that $a \le n$ and $k = (m_a+1)+...+(m_n+1)$. For pairwise distinct $z_1, ..., z_n$, consider the differential equation (3) where $F(x)$, $G(x)$ are defined by (4) and $H(x)$ is such that the differential equation has a solution $U(x) = (x-z_a)^{m_a+1}\cdots(x-z_n)^{m_n+1}$. Then for generic $z_1, ..., z_n$, the generic solution of this differential equation is multivalued.*

Proof. The substitution $u(x) = U(x)\, v(x)$ turns the equation into the differential equation

$$v''(x) + \left(\sum_{l=a}^{n} \frac{m_l+2}{x-z_l} - \sum_{l=1}^{a-1} \frac{m_l}{x-z_l} \right) v'(x) = 0 .$$

Its general solution is

$$v(x) = \int \frac{(x-z_1)^{m_1} \cdots (x-z_{a-1})^{m_{a-1}}}{(x-z_a)^{m_a+2} \cdots (x-z_n)^{m_n+2}} \; dx .$$

According to our assumptions, $m_1 + ... + m_{a-1} \ge (m_a+2) + ... + (m_n+2) - 1$. In this case the function $v(x)$ is multivalued for generic $z_1, ..., z_n$. To see this it is enough to notice that the residue of the integrand at infinity is not zero if $z_a = ... = z_n = 0$ and $z_1 = ... = z_{a-1} = 1$. □

Theorem 13. *Let $m \in \mathbb{Z}^n_{>0}$ and $k \in \mathbb{Z}_{>0}$ be such that $k < l(m)+1-k$. Let $a \in \mathbb{Z}_{>0}$ be such that $a \le n$ and $k > (m_a+1)+...+(m_n+1)$. Set $b = k-(m_a+1)-...-(m_n+1)$. For $s > 0$, set $z^{(s)} = (s\,,\ s^2\,,\ \dots\,,\ s^n)$. Let t^0 be any critical point of the function $\Phi_{b,n}(t, z^{(s)}, \tilde m)$ and let $E(t^0, z^{(s)}, m, \tilde m)$ be the associated differential equation. If $s \gg 1$, then the generic solution of $E(t^0, z^{(s)}, m, \tilde m)$ is a multivalued function.*

Theorem 13 implies Theorem 12.

3.3. **Proof of Theorem 13.** The pair $\{\tilde m, b\}$ is good by Lemma 12. The critical points of the function $\Phi_{b,n}(t, z^{(s)}, \tilde m)$ are labeled by admissible sequences $I = (i_1, ..., i_n)$, where $i_1 = 0$ and $i_2 + ... + i_n = b$, see Sec. 2.3.

Let $t_I(s) = (t_{I,1}(s), ..., t_{I,b}(s))$ be the critical point corresponding to a sequence I. According to the construction, as s tends to infinity for any j there exists the limit of $t_{I,j}(s)/s^n$. This limit is equal to zero if $j \le i_1 + ... + i_{n-1}$, and the limit is not equal to zero otherwise. Moreover, the limits of the last i_n coordinates form a critical point of the function $\Phi_{i_n,2}\,(\,t;\ (0,1),\ (m_{i_n,1}, m_{i_n,2})\,)$ where $m_{i_n,1} = l(m) - 2k + m_n + 2 + 2i_n$ and $m_{i_n,2} = -m_n - 2$. We denote $(T_1, ..., T_{i_n})$ the coordinates of that critical point.

Consider the polynomial (6) and make the change of variables $x = s^n y$, then

$$\begin{aligned} U(s^n y) &= s^{nk}(y - t^0_1(s)/s^n)\cdots(y - t^0_b(s)/s^n)\cdot(y - s^{a-n})^{m_a+1}\cdots(y - s^{n-n})^{m_n+1} \\ &= s^{nk}\, y^{k-i_n-m_n-1}\,(y-1)^{m_n+1}\,(y-T_1)\cdots(y-T_{i_n}) + \mathcal{O}(s^{nk-1}) . \end{aligned}$$

Denote

$$V(y) = y^{k-i_n-m_n-1}\,(y-1)^{m_n+1}\,(y-T_1)\cdots(y-T_{i_n}) .$$

Make the change of variables $x = s^n y$ in the differential equation $E(t_I(s), z^{(s)}, m, \tilde{m})$,

$$F(x)u''(x) + G(x)u'(x) + H(x)u(x) = 0\,.$$

We have

$$\begin{aligned}
F(s^n y) &= s^{n^2}(y - s^{1-n})\cdots(y - s^{n-n}) = s^{n^2} y^{n-1}(y-1) + \mathcal{O}(s^{n^2-1})\,,\\
G(s^n y) &= -\left(\sum_{l=1}^{n} \frac{m_l}{s^n y - s^l}\right) F(s^n y)\\
&= -s^{n(n-1)}\left(\frac{l(m) - m_n}{y} + \frac{m_n}{y-1}\right) y^{n-1}(y-1) + \mathcal{O}(s^{n(n-1)-1})\,,\\
H(s^n y) &= \frac{F(s^n y)\, U''(s^n y) + G(s^n y)\, U'(s^n y)}{U(s^n y)}\\
&= -s^{n(n-2)} \frac{f(y)\, V''(y) + g(y)\, V'(y)}{V(y)} + \mathcal{O}(s^{n(n-2)-1})\,.
\end{aligned}$$

Denote

$$\begin{aligned}
f(y) &= y^{n-1}(y-1),\\
g(y) &= -\left(\frac{l(m) - m_n}{y} + \frac{m_n}{y-1}\right) y^{n-1}(y-1)\,,\\
h(y) &= -\frac{f(y)\, V''(y) + g(y)\, V'(y)}{V(y)}\,.
\end{aligned}$$

As $s \to \infty$, the equation

(7) $$F(s^n y)\, u''(s^n y) + G(s^n y)\, u'(s^n y) + H(s^n y)\, u(s^n y) = 0$$

turns into the equation

(8) $$f(y)\, v''(y) + g(y)\, v'(y) + h(y)\, v(y) = 0\,,$$

and $V(y)$ is its solution. Rewrite equation (8) in the form

$$v''(y) + p(y)\, v'(y) + q(y)\, v(y) = 0\,.$$

We have

$$p(y) = -\frac{l(m) - m_n}{y} - \frac{m_n}{y-1}\,.$$

Lemma 14. *We have*

$$\begin{aligned} q(y) \;=\; & -\sum_{i<j}\frac{2}{(y-T_i)\,(y-T_j)}-\sum_{j=1}^{i_n}\frac{2\,(k-i_n-m_n-1)}{(y-T_j)\,y} \\ & -\sum_{j=1}^{i_n}\frac{2\,(m_n+1)}{(y-T_j)\,(y-1)}-\frac{2\,(k-i_n-m_n-1)\,(m_n+1)}{y\,(y-1)} \\ & -\frac{(k-i_n-m_n-1)\,(k-i_n-m_n-2)}{y^2}-\frac{(m_n+1)\,m_n}{(y-1)^2} \\ & +\left(\frac{l(m)-m_n}{y}+\frac{m_n}{y-1}\right)\left(\sum_{j=1}^{i_n}\frac{1}{y-T_j}+\frac{k-i_n-m_n-1}{y}+\frac{m_n+1}{y-1}\right). \end{aligned}$$

Proof. We have

$$V'(y)\;=\;V(y)\,\left(\sum_{j=1}^{i_n}\frac{1}{y-T_j}+\frac{k-i_n-m_n-1}{y}+\frac{m_n+1}{y-1}\right),$$

and

$$\begin{aligned} V''(y)\;=\;V(y)\,\Bigg(& \sum_{i<j}\frac{2}{(y-T_i)\,(y-T_j)}+\sum_{j=1}^{i_n}\frac{2\,(k-i_n-m_n-1)}{(y-T_j)\,y} \\ & +\sum_{j=1}^{i_n}\frac{2\,(m_n+1)}{(y-T_j)\,(y-1)}+\frac{2\,(k-i_n-m_n-1)\,(m_n+1)}{y\,(y-1)} \\ & +\frac{(k-i_n-m_n-1)\,(k-i_n-m_n-2)}{y^2}+\frac{(m_n+1)\,m_n}{(y-1)^2}\Bigg). \end{aligned}$$

Therefore

$$\begin{aligned} q(y) \;=\; & \frac{h(y)}{f(y)}=-\frac{f(y)V''(y)+g(y)V'(y)}{V(y)f(y)}=-\frac{V''(y)}{V(y)}-p(y)\frac{V'(y)}{V(y)} \\ =\; & -\sum_{i<j}\frac{2}{(y-T_i)(y-T_j)}-\sum_{j=1}^{i_n}\frac{2(k-i_n-m_n-1)}{(y-T_j)y} \\ & -\sum_{j=1}^{i_n}\frac{2(m_n+1)}{(y-T_j)(y-1)}-\frac{2(k-i_n-m_n-1)(m_n+1)}{y(y-1)} \\ & -\frac{(k-i_n-m_n-1)(k-i_n-m_n-2)}{y^2}-\frac{(m_n+1)m_n}{(y-1)^2} \\ & +\left(\frac{l(m)-m_n}{y}+\frac{m_n}{y-1}\right)\left(\sum_{j=1}^{i_n}\frac{1}{y-T_j}+\frac{k-i_n-m_n-1}{y}+\frac{m_n+1}{y-1}\right). \end{aligned}$$

Lemma 15. *Equation (8) is the Fuchsian differential equation with singular points* $0,\,1,\,\infty$ *and exponents* $(\,k-i_n-m_n-1,\ l(m)-k+i_n+2\,)$, $(\,0,\,m_n+1\,)$, $(\,-k,\,k-l(m)-1\,)$, *respectively.*

Proof. First, we prove that equation (8) is Fuchsian. To show this one needs to check that the function $q(y)$ can be written in the form

$$\sum \frac{A_i}{y-B_i} + \sum \frac{C_i}{(y-B_i)^2}\,,$$

where $B=(0,1,T_1,\dots,T_{i_n})$ and the numbers A_i satisfy the condition $\sum A_i = 0$. This statement clearly follows from the formula for $q(y)$ in Lemma 14. Thus $p(y)$ and $q(y)$ are of the required form, [R], Ch. 6.41, Theorem 25.

Now we check that any point T_j is an ordinary point of equation (8). The formula for $p(y)$ tells that $p(y)$ is holomorphic at T_j. To show that $q(y)$ is holomorphic at T_j, it is enough to verify that the limits q_0 and q_1 of the functions $(y-T_j)^2q(y)$ and $(y-T_j)q(y)$ as $y\to T_j$ vanish.

All summands in $q(y)$ contain $(y-T_j)$ in degree at most -1. Thus $q_0=0$.

We have

$$q_1 \;=\; -\;\frac{m_n+2}{T_j-1}\;+\;\frac{l(m)-2k+m_n+2+2i_n}{T_j}\;-\;\sum_{i\neq j}\frac{2}{T_j-T_i}\,.$$

Thus $q_1=0$, since it is exactly the j-th equation of the critical point equations for the function $\Phi_{i_n,2}(\,t;\ (0,1),\ (m_{i_n,1},m_{i_n,2})\,)$.

The exponents at the singular points 0, 1, ∞ are calculated using the indicial equation and formulas for $p(y)$, $q(y)$. □

Lemma 16. *The generic solution of equation (8) is multivalued.*

Proof. We write the P-symbol of equation (8), see [R], Ch. 6.45,

$$v = P\begin{pmatrix} 0 & 1 & \infty & \\ k-i_n-m_n-1 & 0 & -k & y \\ l(m)-k+i_n+2 & m_n+1 & k-l(m)-1 & \end{pmatrix}.$$

By a meromorphic change of variables, equation (8) can be reduced to the form

$$y(1-y)\,V''(y)\;+\;[\,c-(a+b+1)y\,]\,V'(y)\;-\;ab\,V(y)\;=\;0\,, \tag{9}$$

where

$$a=2k-l(m)-m_n-i_n-2\,,\qquad b=-i_n-m_n-1,\qquad c=2k-l(m)-m_n-2i_n-2\,,$$

see [R], Ch. 6.46, [H], Ch. 2.1.1.

Equation (9) is the Gauss hypergeometric equation. If at least one of the numbers a, b, $c-a$, $c-b$ is an integer, then formulas for two linearly independent solutions are listed in Ch. 2.1.2 in [H]. The corresponding table in [H], pp. 71-73, consists of 29 cases. Moreover, if generic solutions are multivalued, then this is stated in the table.

In equation (9), the numbers $a,\ b,\ c$ are negative integers with $b \geq a \geq c$. This is Case 24 with multivalued generic solutions. □

Let $W(y)$ be a multivalued solution of (8) with the initial values at some point z_0 being $W(z_0) = c_1\,,\ W'(z_0) = c_2$ for suitable numbers c_1, c_2. Analytical continuation of this solution along some closed curve Γ leads to a new value of $W(y)$ at z_0 which is different from the initial value.

Let $X(y,s)$ be the solution of equation (7) with the same initial values. Then the function $X(y,s)$, restricted to the curve Γ, tends to the function $W(y)$, restricted to the curve Γ, as $s \to \infty$. Thus $X(y,s)$ is a multivalued function for $s \gg 1$.

This proves Theorem 13. □

References

[B] R. Brualdi, "Introductory Combinatorics", North-Holland, 1977.

[F] L. Faddeev, Lectures on the Quantum Inverse Scattering Method. In: Integrable Systems, ed. by X.-G. Song, Nankai Lectures Math. Phys., World Scientific, Teaneck, N.J., 1990, 23-70.

[G] M. Gaudin, Diagonalization d'une class hamiltoniens de spin. *Journ. de Physique* **37**, no. 10 (1976), 1087 - 1098.

[GH] P. Griffiths and J. Harris, "Principles of Algebraic Geometry", J. Wiley and Sons, 1978.

[H] A. Erdélyi (ed.), "Higher transcendental functions", McGram-Hill book company, 1953.

[KZ] V. Knizhnik and A. Zamolodchikov, Current algebra and the Wess-Zumino model in two dimensions. *Nucl. Phys. B* **247** (1984), 83 - 103.

[MV] E. Mukhin and A. Varchenko, Remarks on Critical Points of Phase Functions and Norms of Bethe Vectors. In: Arrangements - Tokyo 1998. *Advanced Studies in Pure Mathematics* **27** (2000), 239 - 246.

[OT] P. Orlik and H. Terao, The Number of Critical Points of a Product of Powers of Linear Functions. *Invent. Math.* **120** (1995), no. 1, 1 - 14.

[R] E. Rainville, "Intermediate differential equations", The Macmillan Company, 1964.

[RV] N. Reshetikhin, A. Varchenko, Quasiclassical Asymptotics of Solutions to the KZ Equations. In: Geometry, Topology, and Physics for Raoul Bott, International Press, 1994, 293–322.

[Si] R. Silvotti, On a conjecture of Varchenko, *Invent. Math.* **126** (1996), no. 2, 235 - 248.

[Sk1] E. Sklyanin, Separation of variables in the Gaudin model. *J. Soviet Math.* **47** (1989), 2473 - 2488.

[Sk2] E. Sklyanin, The functional Bethe ansatz. In: Integrable and Superintegrable Systems, ed. by B. Kupershmidt, World Scientific, Singapore, 1990, 8 - 33.

[S] G. Szego, "Orthogonal polynomials", AMS, 1939.

[SV] V. Schechtman and A. Varchenko, Arrangements of hyperplanes and Lie algebra homology. *Invent. Math.* **106** (1991), 139 - 194.

[TV] V. Tarasov and A. Varchenko, Completeness of Bethe Vectors and Difference Equations with Regular Singular Points. *IMRN* (1995), No. 13, 637 - 669.

[V] A. Varchenko, Critical points of the product of powers of linear functions and families of bases of singular vectors. *Compositio Mathematica* **97** (1995), 385–401.