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0.0 Preface

These notes provide an introduction to pseudo-differential operators from an invariant point of view. Our modest goal is to discuss and elaborate on a subject contained in Hörmander's article "Fourier Integral Operators I", emphasizing the geometric description of the distributions which play a natural role in the theory, namely the Schwartz kernels of the operators under consideration. These distributions have singularities with some stability relative to the action of vector fields which are tangent to the diagonal in the product manifold. Lifting them to distributions on the normal bundle and applying Fourier transform along the fibers, one is led immediately to the notion of symbol and the corresponding symbol calculus, which for so many years has proven to be a powerful tool in the analysis of equations of various kind.

One area where the power of these operators can be appreciated is in the study of elliptic equations. I have always been fascinated by the naturality with which they appear in the Atiyah-Singer index theorem and their success in the theory of elliptic boundary value problems, both subjects of extreme importance in the last few years. And I believe that their invariant presentation is the most natural way for geometers to understand and quickly get used to them. It shows painlessly why (for many purposes) it is sufficient to use integration by parts and Fourier transform to come to grips with the subject.

The Schwartz kernel of a pseudo-differential operators is an important example of a conormal distribution, that is to say, distributions whose singularities are nicely placed along a closed embedded submanifold of the base, and stable when acted on by vector fields tangent to it. After some preliminaries, we start the real course of this work by developing the notion of conormal distributions of type ρ, δ with $\rho > \delta$. Given our main motivation, it will suffice to keep in mind only the case $\rho = 1$ and $\delta = 0$, although the general situation is just a little more complicated. After that, the rest of the work follows a natural flow and we discuss in detail important examples of elliptic differential operators frequently seen in differential geometry, whose inverses are pseudo-differential operators of type $(1, 0)$ (see (6.3.5), (6.3.6) and (6.3.7)). Among the many results we learned in [Ho1] and [KN] (some of which are reproduced here) we prove that elliptic pseudo-differential operators define Fredholm maps when acting on sections of bundles of a conveniently fixed Sobolev order. As a consequence, the analytic part of the Atiyah-Singer index theorem follows: any such operator has an index (the difference of the dimension of its kernel and cokernel).

I learned the geometric version of this subject from lectures by Richard Melrose at MIT. The courses in microlocal analysis that I taught at SUNY during 1987 kept me interested in this project. The students and I benefited from the interaction and I felt that providing them with written details would be helpful to improve their understanding of the subject. I originally intended to include also a discussion of Fourier integral operators here, but decided against it as the notes became too long and ambitious in scope, goal which we did not set for ourselves at the beginning.

I have made no attempt to trace the results to the original references. I apologize if I offended anybody and encourage the avid reader to look at the references provided in [Ho3] covering this and other topics.

Special thanks are due to many friends with whom I discussed this material at one time or another. Unfortunately, I can only mention a few: to view analysis with geometric spectacles is something for which I will always be indebted to Richard Melrose. Steven Andrea and Gerardo Mendoza initiated me into the subject. Kevin Payne read a great deal of the manuscript and provided helpful criticism. Luisa F. Ruiz was always around helping me with the computers and other things. That all errors that remain are only mine, as it is sometimes said, goes without saying.

January 1990.

Chapter 1

Distributions

In this chapter we recall notions which are well understood nowadays with the purpose of making these notes self-contained. In the process, we establish the notation to be used all throughout this work.

1.1 Smooth and rapidly decreasing functions

Let $n \geq 1$ and let Ω be an open subset of R^n. We denote by ${}_c(\Omega)$ the set of complex valued functions which are differentiable with continuity arbitrarily many times and have compact support in Ω. The support of $f \in_c (\Omega)$ will be denoted by $\operatorname{supp} f$. Given a compact subset K of Ω, we define

$$D_K(\Omega) = \{f \in_c (\Omega) : \ \operatorname{supp} f \subset K\}$$

and topologize it using the seminorms

$$p_{K,m}(f) = \sup_{|\alpha| \leq m, x \in K} |D^\alpha f(x)| \ , \ \ m < \infty \ . \tag{1.1.1}$$

Here (and in the sequel) $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $D^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}$, $D_{x_j} = \dfrac{1}{i} \dfrac{\partial}{\partial x_j}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

With this topology, $D_K(\Omega)$ is a locally convex linear topological space. Moreover if K_1 and K_2 are compact subsets of Ω with $K_1 \subset K_2$, then the topology of $D_{K_1}(\Omega)$ is identical with the relative topology of $D_{K_1}(\Omega)$ as a subset of $D_{K_2}(\Omega)$. Notice also that each $D_K(\Omega)$ is a linear subspace of ${}_c(\Omega)$ with

$$ {}_c(\Omega) = \cup_K D_K(\Omega) \ . $$

A convex, balanced and absorbing subset U of ${}_c(\Omega)$ will be declared open if and only if for all compact subset K of Ω, $U \cap D_K(\Omega)$ is an open subset of $D_K(\Omega)$ containing the zero vector. In other words, we provide ${}_c(\Omega)$ with the inductive limit topology of the $D_K(\Omega)$'s. The space of compactly supported functions topologized in this way will be denoted by $D(\Omega)$. The space of distributions in Ω is defined as the topological dual of $D(\Omega)$:

$$ D'(\Omega) = (D(\Omega))^* \ . \tag{1.1.2} $$

$D'(\Omega)$ will be provided with the weak topology.

For practical purposes, one should explore the definition above and define distributions using criteria which are easier to deal with. In order to do that, we explore further the topology of $D(\Omega)$.

Let F be a topological vector space provided with a countable family of norms $\| \ \|_i$. We can assume that $\| \ \|_{i+1} \geq \| \ \|_i$. Otherwise set $\| \ \|_i' = \sum_{j \leq i} \| \ \|_j$. Then

$$ d(f,g) = \sum_{i=0}^{\infty} 2^{-i} \frac{\|f-g\|_i}{1+\|f-g\|_i} $$

is a translation invariant metric on F. The space F is said to be Fréchet if it is complete in this metric. As an example, consider

$$ C_b^k(\Omega) = \{ f : \Omega \to \mathbf{C} : \ \ D^\alpha f \text{ is continuous and } \sup |D^\alpha f| < \infty \ \forall \ \alpha \text{ s.t. } |\alpha| \leq k \} \ . $$

With the norm given by $\|f\|_k = \sup_{|\alpha| \leq k} |D^\alpha f|$, $C_b^k(\Omega)$ becomes a Banach space. This assertion is a consequence of the Arzela-Ascoli theorem. Then

$$ C_b^\infty(\Omega) = \cap_{k=0}^\infty C_b^k(\Omega) $$

is a Fréchet space. Let d^∞ be the distance function in $C_b^\infty(\Omega)$. Recall that a sequence $\{K_j\}$ of compact subsets of Ω is said to be exhaustive if $K_{j-1} \subset \overset{\circ}{K}_j$ for all j and $\Omega = \cup_j K_j$.

Proposition 1.1.3 *Let $\{f_n\}$ be a sequence in $D(\Omega)$ which converges to f in the topology of $D(\Omega)$. Then there exists a compact set $K \subset \Omega$ such that* $\operatorname{supp} f_n \subseteq K$ *for all n, and for any α, $D^\alpha f_n$ converges uniformly to $D^\alpha f$ on K. Conversely, these conditions imply that $f_n \to f$ in $D(\Omega)$.*

Proof. It will be enough to assume that $f = 0$. Suppose the result is false. Then there exists a exhaustive sequence $\{K_j\}$ of compact subsets of Ω such that for some subsequence f_{n_j} we have supp $f_{n_j} \subset K_j$ and $f_{n_j}(x_j) \neq 0$ for some point $x_j \in K_j - \overset{\circ}{K}_{j-1}$. Indeed, $\{K_j\}$ can be chosen to be a subsequence of $\{\tilde{K}_j\}$ where $\tilde{K}_j = \{x \in \Omega : \ d(x, R^n - \Omega) \geq 1/j\} \cap \{x : \|x\| \leq j\}$. We shall find an open set $\mathcal{O}$ contradicting the hypothesis that $f_n \to 0$ in $D(\Omega)$.

Set $\varepsilon_j = d^\infty(f_{n_j}, 0) > 0$. Without loss of generality, we can assume these real numbers form a decreasing sequence. Consider

$$\mathcal{O} = \cup_{j=0}^{\infty}\{g \in D_{K_j}(\Omega) : \ d^\infty(g, 0) < \varepsilon_j/2\} .$$

This set is an open neighborhood of the zero vector in $D(\Omega)$. For if K is a compact subset of Ω, given an element $h \in \mathcal{O}_K = D_K(\Omega) \cap \mathcal{O}$, then for some j, $K \subset K_j$ and $h = \overline{h}$ with $\overline{h} \in D_{K_j}(\Omega)$ and $d^\infty(\overline{h}, 0) < \varepsilon_j/2$. Thus, if $h' \in D_K(\Omega)$ and $d^\infty(h', 0) < \varepsilon_j/2 - d^\infty(\overline{h}, 0)$, then $h + h' \in \mathcal{O}_K$. Clearly, $f_{n_j} \notin \mathcal{O}$, reaching the desired conclusion.

The converse is straightforward. ∎

A convergent sequence $\{f_n\}$ in $D(\Omega)$ is a particular example of a bounded subset of $D(\Omega)$. In general, $\mathcal{B} \subset D(\Omega)$ is bounded iff there exists a compact set K such that

$$\text{supp}\, \varphi \subseteq K \ \forall \ \varphi \in \mathcal{B} \ \text{ and } \ \sup_{x \in K, \varphi \in \mathcal{B}} |D^\alpha \varphi| < \infty \ \forall \ \alpha \ . \tag{1.1.4}$$

The proof of (1.1.4) involves the same arguments as the one in the previous proposition. From this we obtain the following characterization of distributions.

Theorem 1.1.5 *A map $T : D(\Omega) \to \mathbf{C}$ is an element of $D'(\Omega)$ iff T is linear and for any compact subset $K \subset \Omega$, there exist m and C such that*

$$|T(\varphi)| \leq C p_{K,m}(\varphi) \ \forall \ \varphi \in D_K(\Omega) \ .$$

Proof. The necessity is obvious since $D(\Omega)$ is the inductive limit of the spaces $D_K(\Omega)$ which implies that the composition $D_K(\Omega) \overset{i_K}{\to} D(\Omega) \overset{T}{\to} \mathbf{C}$ is continuous. Given (1.1.4), the converse follows from the fact that T must be bounded on bounded sets. ∎

Example 1.1.6 Let $f \in C^0(\Omega)$, the space of continuous functions. Then

$$T_f(\varphi) = \int_\Omega f(x)\varphi(x)dx$$

defines an element T_f of $D^{'}(\Omega)$. In this way, we identify $C^0(\Omega)$ with a subset of $D^{'}(\Omega)$. Similarly, we identify $L^2(\Omega)$ with a subset of $D^{'}(\Omega)$. Notice that in both of these cases we make use of the Lebesgue measure of Ω, structure which is naturally available unlike the more general case of distributions on manifolds to be discussed below.

Example 1.1.7 Consider the functional

$$\delta_x(\varphi) = \varphi(x) , \quad x \in \Omega .$$

Then $\delta_x \in D^{'}(\Omega)$. This is the so called Dirac delta function at x. It is not of the form T_f (as in the previous example) for any L^2-function f.

Let us now consider the space (Ω) of smooth functions in Ω topologized with the semi-norms in (1.1.1). Clearly the inclusion map $i : D(\Omega) \to (\Omega)$ is continuous and has dense image. Following the usual principle of functional analysis in search of interesting spaces, we dualize the inclusion above and obtain a map

$$D^{'}_c(\Omega) \stackrel{i^*}{\to} D^{'}(\Omega) , \tag{1.1.8}$$

where $D^{'}_c(\Omega) = ((\Omega))^*$.

Definition 1.1.9 *$u \in D^{'}(\Omega)$ is said to vanish on U iff the composition $u \circ i_U : D(U) \to \mathbf{C}$ is identically zero. Here, $i_U : D(U) \to D(\Omega)$ is the inclusion map. The support of u, supp u, is the complement of the union of all the open sets where u vanishes.*

Proposition 1.1.10 *The image of* (1.1.8) *is the set of distributions in $D^{'}(\Omega)$ with compact support.*

Proof. Let $v = i^*u$ for some $u \in D^{'}_c(\Omega)$ and assume that v does not have compact support. Then there is a sequence $\{x_j\}$, without accumulation point in Ω, entirely contained in supp u. Choose a sequence of disjoint open sets $\{U_j\}$ with $x_j \in U_j$ and such that for any compact set K there is an index j_K with $U_j \cap K = \emptyset$ if $j \geq j_K$. Since $x_j \in U_j$, we can find $f_j \in_c (U_j)$ such that $v(f_j) = 1$. Since the sum converges in (Ω) we define $f = \sum_j i_* f_j$. Then, $u(f) = \sum u(i_* f_j) = \sum i^* u(f_j) = \sum v(f_j) = \infty$, contradicting the fact that $u \in D^{'}_c(\Omega)$.

This shows that the image of (1.1.8) is contained in the set of distributions with compact support. The converse is obvious. ∎

Abusing notation, we denote by $D_c^{'}(\Omega)$ the set of distributions in $D^{'}(\Omega)$ with compact support.

The derivative D_{x_j} of a distribution u is defined by the identity

$$(D_{x_j}u)(\varphi) = u(-D_{x_j}\varphi) \ .$$

Show that the map

$$D_{x_j} : D^{'}(\Omega) \longrightarrow D^{'}(\Omega)$$

is continuous and that supp $D_{x_j}u \subset$ supp u. Generalize this result to partial differential operators $P = \sum a_\alpha(x) D_x^\alpha$ with coefficients ${}_\alpha(x) \in (\Omega)$.

A distribution $u \in D^{'}(\mathbf{R}^n)$ is said to be positively homogeneous of degree z iff

$$m_t^* u = t^z u \ \ \forall\, t \in \mathbf{R}^+ \ ,$$

where $m_t : \mathbf{R} \longrightarrow \mathbf{R}$ is the multiplication operator $m_t x = tx$.

1. Show that for complex numbers z such that Re $z > -1$ the function

$$x_+^z = \begin{cases} x^z = e^{z \log x} \ , & x > 0 \\ 0 \ , & x < 0 \end{cases}$$

is locally integrable and it therefore defines a distribution $x_+^z \in D^{'}(bfR)$.

2. Prove that if Re $z > 0$,

$$\frac{d}{dx} x_+^z = z x_+^z \ .$$

Use this property to extend the definition of x_+^z as a homogeneous distribution of degree z with values in the space of meromorphic functions in $\mathbf{C}$ having at most simple poles at $z = -1, -2, \ldots$. The distribution so defined is positively homogeneous of degree z.

3. Show that for any integer $k \geq 1$,

$$\lim_{z \to -k} (z + k) x_+^z = \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{d}{dx}\right)^{k-1} \delta_0 \ .$$

4. For any positive integer k, define

$$x_+^{-k} = \lim_{z \to -k} (x_+^z - \frac{(-1)^{k-1}}{(z+k)(k-1)!} \left(\frac{d}{dx}\right)^{k-1} \delta_0) .$$

Show that x_+^{-k} is not a homogeneous distribution.

5. For any z not a negative integer, define x_-^z by $x_-^z = T^* x_+^z$ where $T(x) = -x$. For a positive integer k, define x_-^{-k} as above replacing the role of x_+^z by x_-^z. Show that the distribution $x_+^{-1} - x_-^{-1}$ is the so called principal value

$$\text{p.v.} \frac{1}{x}(\varphi) = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx .$$

Finally let us consider the space of smooth functions in $\mathbf{R}^n$ for which

$$\|f\|_{M,N} = \sup_{|\alpha| \le N,\, x \in \mathbf{R}^n} (1 + \|x\|^2)^{\frac{M}{2}} |D_x^\alpha f(x)|$$

is bounded for all non-negative integers M and N. These functions decrease at ∞ faster than any polynomial. With the topology defined by these norms, we obtain the space of rapidly decreasing functions $\mathcal{S}(\mathbf{R}^n)$, a Fréchet space. It is rather clear that $D(\mathbf{R}^n) \subset \mathcal{S}(\mathbf{R}^n)$ continuously. Hence, the space of tempered distributions,

$$\mathcal{S}'(\mathbf{R}^n) = (\mathcal{S}(\mathbf{R}^n))^*$$

is included in $D'(\mathbf{R}^n)$.

Find an element of $D'(\mathbf{R}^n)$ which is not a tempered distribution.

1.2 Convolution and Fourier transform

Given f and g in $\mathcal{S}(\mathbf{R}^n)$ we define their convolution to be the rapidly decreasing function

$$(f * g)(x) = \int f(x-y) g(y) dy = \int f(y) g(x-y) dy . \tag{1.2.1}$$

This is an associative and commutative multiplication on $\mathcal{S}(\mathbf{R}^n)$ without identity. Moreover, if we identify f with an element of $D'(\mathbf{R}^n)$ and suppose that $g \in_c (\mathbf{R}^n)$, we can write

$$(f * g)(x) = f(\tau_x g) \tag{1.2.2}$$

where τ_x is the operator $g(\,.\,) \to g(x-\,.\,)$. This allows us to extend the convolution operation to the space $D^{'}(\mathbf{R}^n) \otimes_c (\mathbf{R}^n)$. By duality, it can be extended to $D^{'}(\mathbf{R}^n) \otimes D^{'}_c(\mathbf{R}^n)$. In order to see this, we must prove the following:

Proposition 1.2.3 *Let $f \in D^{'}(\mathbf{R}^n)$ and $g \in_c (\mathbf{R}^n)$. Then $f * g$ is smooth and $D^\alpha(f*g) = D^\alpha f * g = f * D^\alpha g$. If $f \in D^{'}_c(\mathbf{R}^n)$, then supp $f*g \subset$ supp f+supp g.*

Proof. Let e_k be the unit vector in the direction of the positive k-th axis. Then for $g \in_c (\mathbf{R}^n)$ and fixed x, the function $g_h(y) = \frac{1}{ih}(g(x+he_k-y) - g(x-y))$ converges to $D_{x_k} g(x-y)$ in the topology of $D(\mathbf{R}^n)$ (see proposition 1.1.3). Hence

$$D_{x_k}(f*g)(x) = \lim_{h\to 0} f(g_h)$$
$$= f(\tau_x D_{x_k} g) = (f * D_{x_k} g)(x)\ .$$

Since $D_{x_k} g(x-y) = -D_{y_k} g(x-y)$, it follows that

$$(f * D_{x_k} g)(x) = (D_{x_k} f * g)(x)\ ,$$

proving the desired formula when $|\alpha| = 1$. The general expression follows by induction.

Having this result available, to prove the smoothness of $f*g$ it will suffice to show that $f*g$ is continuous. But if $x_j \to x$ then $g(x_j - y) \to g(x-y)$ in $D_c(\mathbf{R}^n)$ as a function of y. Hence

$$\lim_{j\to\infty} (f*g)(x_j) = \lim_{j\to\infty} f(\tau_{x_j} g) = f(\tau_x g) = (f*g)(x)\ .$$

Finally, observe that $f(\tau_x g)$ is zero unless supp $f \cap$supp $\tau_x g \neq \emptyset$. This condition means that for some $y \in$ supp f, $x - y \in$ supp g. That is to say,

$$x = y + z\,,\quad y \in \operatorname{supp} f,\ z \in \operatorname{supp} g\ ,$$

showing the desired result for the support of the convolution. ∎

Using proposition 1.2.3 we can show that $D(\mathbf{R}^n)$ is dense in $D^{'}(\mathbf{R}^n)$. In fact, that statement follows easily from

Proposition 1.2.4 *Let $\varphi \in_c (\mathbf{R}^n)$, φ a non-negative real valued function such that $\int \varphi = 1$ and supp $\varphi \subset \{x :\ \|x\| \leq 1\}$. Consider the sequence $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then, if $u \in D^{'}(\mathbf{R}^n)$, $u_\varepsilon = u * \varphi_\varepsilon$ is smooth and converges to u in the topology of $D^{'}(\mathbf{R}^n)$.*

The proof of this proposition is left as an exercise for the reader.

If f and g are distributions with at least one of them compactly supported, then $f * g$ can be defined by

$$D(\mathbf{R}^n) \ni \varphi \longrightarrow g(\check{f} * \varphi) ,$$

where $\check{f}$ is the pull-back of f under the diffeomorphism $x \to -x$. It follows from proposition 1.2.3 that the function $\check{f} * \varphi$ is smooth and that the right hand-side of the expression above is well-defined if either f or g has compact support. Moreover, the map defined in the right side is continuous in the topology of $D(\mathbf{R}^n)$: for if $\varphi_n \to 0$, $\check{f} * \varphi_n \to 0$ if f has compact support, or it does so on the support of g; in either case we conclude that $g(\check{f} * \varphi_n) \to 0$. In this form, the convolution operation is extended to distributions when at least one of the distributions involved has compact support. In the distributional sense, the properties of proposition 1.2.3 carry over at this level, and when defined, the convolution is associative.

Consider the Fourier transform

$$\begin{array}{c} \mathcal{S}(\mathbf{R}^n) \stackrel{\mathcal{F}}{\longrightarrow} \mathcal{S}(\mathbf{R}^n) \\ f(x) \longrightarrow \hat{f}(\xi) = \int e^{-i} f(x) dx \end{array} . \tag{1.2.5}$$

It is a well-defined map. Indeed, it is trivial to check that $\widehat{D_x^\alpha f}(\xi) = \xi^\alpha \hat{f}(\xi)$ and that $\widehat{x^\alpha f}(\xi) = (-1)^\alpha D_\xi^\alpha \hat{f}(\xi)$, implying that $\hat{f} \in \mathcal{S}(\mathbf{R}^n)$.

Theorem 1.2.6 *The map* (1.2.5) *is an isomorphism and* $\widehat{f * g} = \hat{f} \cdot \hat{g}$.

Proof. Let

$$\mathcal{G} f(x) = \frac{1}{(2\pi)^n} \int e^{i} f(\xi) d\xi .$$

We shall prove that $(\mathcal{G} \circ \mathcal{F}) f(x) = f(x)$. We have

$$(\mathcal{G} \circ \mathcal{F}) f(x) = \frac{1}{(2\pi)^n} \int e^{i} \hat{f}(\xi) d\xi = \frac{1}{(2\pi)^n} \int \int e^{i\langle x-y,\xi\rangle} f(y) dy d\xi .$$

Since the last integral does not converge absolutely, we cannot interchange the order of integration. We can only do so after regularizing it. Just observe that

$$\begin{aligned} \frac{1}{(2\pi)^n} \int \int e^{i\langle x-y,\xi\rangle} f(y) dy d\xi \ , &= \lim_{t \to 0} \frac{1}{(2\pi)^n} \int \int e^{i\langle x-y,\xi\rangle - t\|\xi\|^2} f(y) dy d\xi \\ &= \lim_{t \to 0} (f * g_t)(x) , \end{aligned}$$

where $g_t(y) = \dfrac{1}{(2\pi)^n}\int e^{i-t\|\xi\|^2}d\xi$. Completing the square we have:

$$\begin{aligned} g_t(y) &= \frac{1}{(2\pi)^n}e^{-\frac{\|y\|^2}{4t}}\int e^{-t\|\xi-\frac{i}{2t}y\|^2}d\xi = \frac{1}{(2\pi)^n}e^{-\frac{\|y\|^2}{4t}}\prod_{j=1}^{n}\int e^{-t(\xi_j-\frac{i}{2t}y_j)^2}d\xi_j \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{\|y\|^2}{4t}} , \end{aligned}$$

the latter obtained by shifting the contour of integration to the complex line Im $z = i\dfrac{y_j}{4t}$. As it can be seen, $f * g_t$ is nothing but the convolution of f with the fundamental solution of the heat equation g_t. Notice that the later is such that $g_t \geq 0$ and $\int g_t(y)dy = 1$. Hence, if we choose δ such that $|f(y) - f(x)| < \varepsilon$ for $\|x - y\| < \delta$ and we set $M = \sup|f(x)|$, then

$$\begin{aligned} |(f * g_t(x) - f(x))| = &\ |\int (f(y) - f(x))g_t(x-y)dy| \\ \leq &\ \int_{\|x-y\|<\delta}|f(y) - f(x)|g_t(x-y)dy + \int_{\|x-y\|\geq\delta}|f(y) - f(x)|g_t(x-y)dy \\ \leq &\ \varepsilon\int g_t(x-y)dy + 2M\int_{\|x-y\|\geq\delta}g_t(x-y)dy . \end{aligned}$$

But $\int_{\|x-y\|\geq\delta} g_t(x-y)dy = \pi^{-\frac{n}{2}}\int_{\|\zeta\|\geq\frac{\delta}{2\sqrt{t}}} e^{-\|\zeta\|^2}d\zeta$, which shows that the second integrand in the right hand-side of the expression above is less than ε for t small. Thus, $|(f * g_t(x) - f(x))| < 2\varepsilon$ for small t, with ε arbitrary. The result follows.

Finally, let $h = f * g$. Since the integral converges absolutely we have:

$$\begin{aligned} \hat{h}(\xi) = \int e^{-i}(f * g)(x)dx &= \int\int e^{-i}f(x-y)g(y)dydx \\ &= \int\int e^{-i\langle z+y,\xi\rangle}f(z)g(y)dzdy \\ &= \hat{f}(\xi)\hat{g}(\xi) , \end{aligned}$$

concluding the proof. ∎

Prove the result above as follows (see [Gi]):

1. Show that $\mathcal{G}\hat{f}(0) = 0$ if f is such that $f(0) = 0$.

2. For a general function f use the decomposition $f = f(0)f_0 + (f - f(0)f_0)$ where $f_0 = e^{-\|x\|^2}$. Show that $\mathcal{G}\hat{f}(0) = f(0)$ by computing explicitly the value of $\int \hat{f}_0(\xi)d\xi$.

3. For $x_0 \in R^n$ set $g(x_0) = f(x + x_0)$. Using 1 and 2 show that $f(x_0) = (\mathcal{G}\hat{f})(x_0)$.

By duality one defines the Fourier transform of tempered distributions, i.e., if $u \in \mathcal{S}'(\mathbf{R}^n)$, then $\mathcal{F}u(f) = u(\mathcal{F}f)$ for all $f \in \mathcal{S}(\mathbf{R}^n)$. The map

$$\mathcal{S}'(\mathbf{R}^n) \xrightarrow{\mathcal{F}} \mathcal{S}'(\mathbf{R}^n) \tag{1.2.7}$$

is an isomorphism.

If $u \in D_c'(\mathbf{R}^n) = ((\mathbf{R}^n))^*$, its Fourier transform can be defined using the duality pairing between $D_c'(\mathbf{R}^n)$ and $(\mathbf{R}^n)$. Indeed, e^{-i} is an element of $(\mathbf{R}^n)$, and

$$\hat{u}(\xi) = \mathcal{F}u(\xi) = u(e^{-i}) \ . \tag{1.2.8}$$

This function is smooth and has a holomorphic extension to $\mathbf{C}^n$. The resulting extension is by definition the Fourier-Laplace transform of u. Observe that if $u_1 \in D_c'(\mathbf{R}^n)$ and $u_2 \in \mathcal{S}'(\mathbf{R}^n)$, the product $\hat{u_1} \cdot \hat{u_2}$ is well-defined, and by the isomorphism (1.2.8) it is the Fourier transform of a tempered distribution which we call $u_1 * u_2$.

The following theorem is the crucial tool in studying the notion of singularity of distributions. The statement as well as proof follows closely that in [Ho2].

Theorem 1.2.9 *(Paley-Wiener) An entire analytic function $U(\xi)$ is the Fourier-Laplace transform of a distribution with support in the ball $B_A = \{x : \|x\| \leq A\}$ if, and only if, for some constants C and N we have*

$$|U(\xi)| \leq C(1 + \|\xi\|)^N e^{A|\mathrm{Im}\,\xi|} \ .$$

U is the Fourier-Laplace transform of a function u ub $_c(B_A)$ if, and only if, for every integer N there exists a constant C_N such that

$$|U(\xi)| \leq C_N(1 + \|\xi\|)^{-N} e^{A|\mathrm{Im}\,\xi|} \ .$$

Proof. Suppose $U(\xi)$ is the Fourier-Laplace transform of a distribution with support in B_A. By theorem 1.1.5 we know that for some constants C and N,

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq N} \sup |D^\alpha \varphi| \ , \quad \varphi \in_c (B_A) \ .$$

Let $\psi \in (\mathbf{R})$ such that $\psi \equiv 1$ on $(-\infty, 1/2)$ and $\psi \equiv 0$ on $(1, \infty)$. Then,

$$\varphi_\xi(x) = e^{-i}\psi(\|\xi\|(\|x\| - A))$$

is in $_c(\mathbf{R}^n)$ and coincides with e^{-i} is a neighborhood of B_A. Since u has support in B_A, we have:

$$|\hat{u}(\xi)| = |u(\varphi_\xi)| \leq C \sum_{|\alpha|\leq N} \sup|D^\alpha \varphi_\xi| \ .$$

But $\varphi_\xi \neq 0$ when

$$\|\xi\|(\|x\| - A) < 1 \Longleftrightarrow \|x\| < \frac{1}{\|\xi\|} + A \ .$$

Therefore $|e^{-i}| \leq e^{A|\mathrm{Im}\xi|+1}$ on the support of φ_ξ. Hence, estimating the supremum of the derivatives of φ_ξ on the support of u, we obtain

$$|\hat{u}(\xi)| \leq Ce^{A|\mathrm{Im}\xi|}(1 + \|\xi\|)^N \ ,$$

proving that the first condition is necessary.

The necessity of the second condition is quite obvious since

$$\xi^\beta D^\alpha \hat{\varphi}(\xi) = \int e^{-i} D^\beta((-x)^\alpha \varphi(x))dx \ .$$

Using this with $\alpha = 0$ we obtain that

$$\|\xi\|^\beta |\hat{u}(\xi)| \leq C_\beta e^{A|\mathrm{Im}\xi|}$$

for any β, from which the desired result follows.

That the second condition is sufficient is proven as follows: set

$$u(x) = (2\pi)^{-n} \int U(\xi)e^i d\xi \ ,$$

which makes sense because U is a tempered distribution. Then $\hat{u} = U$. Furthermore, $u \in (\mathbf{R}^n)$. We only need to prove that $\operatorname{supp} u \subset B_A$.

But the second condition allows us th shift the integration in $u(x) = (2\pi)^{-n}\int U(\xi)e^i d\xi$ into the complex domain $u(x) = (2\pi)^{-n}\int U(\xi + i\eta)e^{i\langle x,\xi+i\eta\rangle}d\xi$. Here η is an arbitrary vector in $\mathbf{R}^n$. Thus, using the condition with $N = n + 1$ we have

$$|u(x)| \leq C_N e^{A\|\eta\|-\langle x,\eta\rangle}((2\pi)^{-n}\int \frac{d\xi}{(1 + \|\xi\|)^{n+1}} \leq C'_N e^{A\|\eta\|-\langle x,\eta\rangle} \ .$$

Choose $\eta = tx$ and let $t \to +\infty$. It follows that $u(x) = 0$ if $\|x\| > A$.

To prove the sufficiency of the first condition, we first note that $U \in \mathcal{S}'(\mathbf{R}^n)$ and so, by (1.2.8), $U = \hat{u}$ for some $u \in \mathcal{S}'(\mathbf{R}^n)$. Let $\varphi \in_c (\mathbf{R}^n)$ such that $\varphi \geq 0$, supp $\varphi \subset \{x : \ \|x\| \leq 1\}$ and $\int \varphi(x)dx = 1$. As previously done, we set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then if $u_\varepsilon = u * \varphi_\varepsilon$ we have that $\hat{u}_\varepsilon = \hat{u}\hat{\varphi}_\varepsilon$. But u_ε is an element of $_c(B_{A+\varepsilon})$ and it therefore satisfies the second condition with A replaced by $A+\varepsilon$. Letting $\varepsilon \to 0$ we obtain that supp $u \subset B_A$. ∎

According to Paley-Wiener theorem, if $v \in D_c'(\mathbf{R}^n)$ is smooth, then $\hat{v}(\xi)$ is a rapidly decreasing function. In the general case, one could study those rays ξ along which the estimates

$$|\hat{v}(\xi)| \leq C_N(1+\|\xi\|)^{-N} \ \ \forall \, N \tag{1.2.10}$$

fail as a measure of the singularity of the distribution v. In other words, the cone $\sum(v)$ consisting of those rays where (1.2.11) is not valid captures the singular behavior of the distribution v. If $u \in D'(\Omega)$, for $x \in \Omega$ we set

$$\Sigma_x = \cap_\phi \sum(\phi u) \,, \quad \phi \in_c (\Omega) \ \ \phi(x) \neq 0 \,.$$

Definition 1.2.11 *If $u \in D'(\Omega)$ the wave front set of u is defined as*

$$WF\, u = \{(x,\xi) \in \Omega \times \mathbf{R}^n - 0 \, : \, \xi \in \Sigma_x\} \,.$$

This coordinatized version of $WF\,u$ will be proven to be invariant in the following chapters and we therefore shall not emphasize that at this moment. Let us just add that $WF\,u$ should be considered as a closed-conic subset of $T^*\Omega - 0$, the complement of the zero section of the cotangent bundle of Ω.

Example 1.2.12 For the Dirac delta function we have $WF\,\delta_{x_0} = \{(x_0,\xi) : \ \xi \neq 0\}$.

Compute $WF\,H$ where H is the Heaviside function.

Show that if P is a differential operator with smooth coefficients, $WF\,Pu \subset WF\,u$.

1.3 Structure theorems

Let $u \in D'(\Omega)$. If there exists n such that for all compact sets $K \subset \Omega$, $|u(\varphi)| \leq C_K p_{K,n}(\varphi)$ for all $\varphi \in D_K(\Omega)$, then u is said to be of finite order and its order is the smallest of all the n's with the property above.

Theorem 1.3.1 *Let $u \in D^{'}(\Omega)$, and let U be an open set with compact closure in Ω. Then, there exists a continuous function f in U and a differential operator P such that $u = Pf$.*

Proof. Consider an open set V with compact closure such that $\overline{U} \subset V$ and let $\varphi \in C_c^\infty(V)$, $\varphi \equiv 1$ in a neighborhood of U. Extend φ by zero outside V, and set $u_1 = \varphi u$ as an element of $D^{'}(\mathbf{R}^n)$. Then $u_1 \mid_U = u \mid_U$. From theorem 1.1.5 it follows that u_1 has finite order. Since it also has compact support, there are constants C and N such that

$$|u_1(\varphi)| \leq C \mathrm{sup}_{|\alpha| \leq N, x \in \overline{V}} |D^\alpha \varphi(x)| \, , \quad \varphi \in C_c^\infty(\mathbf{R}^n) \; . \tag{1.3.2}$$

From the Hahn-Banach theorem we obtain an extension of u_1 as a linear functional on the space of N-times differentiable functions on $\mathbf{R}^n$, $C^N(\mathbf{R}^n)$, provided with the semi-norm on the right hand-side of (1.3.2), taken over a compact neighborhood of supp u_1. Set

$$g(x) = \begin{cases} \dfrac{1}{n(N+1)!} (x_1 x_2 \ldots x_n)^{N+1} & , x_1, \ldots, x_n \geq 0 \\ 0 \quad \text{otherwise} & \end{cases} ,$$

and define $v = u_1 * g$. Then v is continuous since $\tau_x g \to \tau_{x_0} g$ in $C^N(\mathbf{R}^n)$ when $x \to x_0$. Moreover, for $\varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned} \partial_{x_1}^{N+2} \ldots \partial_{x_n}^{N+2} v(\varphi) =& (-1)^{n(N+2)} v(\partial_{x_1}^{N+2} \ldots \partial_{x_n}^{N+2} \varphi) \\ =& (-1)^{n(N+2)} u_1(\tau_x g)(\partial_{x_1}^{N+2} \ldots \partial_{x_n}^{N+2} \varphi) \\ =& (-1)^{n(N+2)} u_1 \left(\int g(x-y) \partial_{x_1}^{N+2} \ldots \partial_{x_n}^{N+2} \varphi(x) dx \right) \\ =& u_1(\varphi) \quad , \end{aligned}$$

completing the proof. ∎

Consider a distribution $K \in D^{'}(\Omega_1 \times \Omega_2)$ with Ω_1 and Ω_2 open sets in $\mathbf{R}^{n_1}$ and $\mathbf{R}^{n_2}$, respectively. Such a kernel defines a map

$$\begin{aligned} T_K \; : \; D(\Omega_1) \longrightarrow D^{'}(\Omega_2) \\ \varphi_1 \longrightarrow (T_K \varphi_1)(\varphi_2) = K(\varphi_1 \otimes \varphi_2) \end{aligned} ,$$

where $\varphi_1 \otimes \varphi_2$ is the element of $D(\Omega_1 \times \Omega_2)$ defined by $(\varphi_1 \otimes \varphi_2)(x, y) = \varphi_1(x)\varphi_2(y)$. Observe that if K_1 and K_2 are two kernels for which $T_{K_1} = T_{K_2}$, then $K_1 = K_2$. Indeed, if $D(\Omega_1) \otimes D(\Omega_2)$ stands for the span of the set of functions $\varphi_1 \otimes \varphi_2$ with $\varphi_i \in D(\Omega_i)$, this claim is an easy consequence of the following lemma, whose proof we omit:

Lemma 1.3.3 *The inclusion map*

$$D(\Omega_1) \otimes D(\Omega_2) \longrightarrow D(\Omega_1 \times \Omega_2)$$
$$\varphi_1 \otimes \varphi_2 \longrightarrow \varphi_1 \otimes \varphi_2$$

has dense range. ∎

More interesting (and difficult) is the converse of the situation analyzed above. Any continuous linear map T from $D(\Omega_1)$ to $D^{'}(\Omega_2)$ can be represented uniquely as T_K, for some $K \in D^{'}(\Omega_1 \times \Omega_2)$. This result is known as the Schwartz kernel theorem.

Theorem 1.3.4 *Let $T : D(\Omega_1) \longrightarrow D^{'}(\Omega_2)$ be a linear continuous map. Then there exists a unique distribution $K \in D^{'}(\Omega_1 \times \Omega_2)$ such that $T(\varphi_1)(\varphi_2) = T_K(\varphi_1)(\varphi_2) = K(\varphi_1 \otimes \varphi_2)$ for all $\varphi_1 \in D(\Omega_1),\ \varphi_2 \in D(\Omega_2)$.*

Proof. If K and $K^{'}$ are distributions representing T, then $K - K^{'}$ is a kernel representing the zero map from $D(\Omega_1)$ to $D^{'}(\Omega_2)$. That means that $K - K^{'}$ vanishes on $D(\Omega_1) \otimes D(\Omega_2)$. Using lemma 1.3.3, it must vanish on $D(\Omega_1 \times \Omega_2)$. Thus $K = K^{'}$ and therefore, the kernel representing T, if it exists, is unique.

For the existence of K observe that T induces a bilinear map

$$\begin{array}{c} D(\Omega_1) \otimes D(\Omega_2) \longrightarrow \mathrm{C} \\ \sum \varphi_j \otimes \psi_j \longrightarrow \sum T(\varphi_j)(\psi_j) \end{array} . \tag{1.3.5}$$

Thus if $\Phi = \sum \varphi_j \otimes \psi_j \in D(\Omega_1 \times \Omega_2)$, K can be defined as

$$K(\Phi) = \sum T\varphi_j(\psi_j) \ . \tag{1.3.6}$$

We need to prove that the K so defined is continuous in the topology of $D(\Omega_1) \otimes D(\Omega_2)$ induced by $D(\Omega_1 \times \Omega_2)$. From that and lemma 1.3.3, this K will define an element of $D^{'}(\Omega_1 \times \Omega_2)$, which represents T in virtue of (1.3.6). Let C_1 and C_2 be compact subsets of Ω_1 and Ω_2, respectively. We want to find constants C and N such that

$$| \ K(\Phi) \ | \leq C p_{C_1 \times C_2, N}(\Phi) \tag{1.3.7}$$

for any $\Phi = \sum \varphi_j \otimes \psi_j$ with $\operatorname{supp} \Phi \subset C_1 \times C_2$. Let E_1 and E_2 be compact neighborhoods of C_1 and C_2. Then we can assume that $\operatorname{supp} \varphi_j \subset E_1$ and $\operatorname{supp} \psi_j \subset E_2$.

Consider $f_i \in C_c^\infty(\Omega_i)$, $f_i \equiv 1$ on E_i, $i = 1, 2$. We have:

$$T\varphi_j = \frac{1}{(2\pi)^{n_1}} \int T(f_1(\cdot)e^{i(\cdot,\eta)})\hat{\varphi}_1(\eta)d\eta \ ,$$

$$\psi_j = \frac{1}{(2\pi)^{n_2}} \int f_2(\cdot)e^{i(\cdot,\xi)}\hat{\varphi}_2(\xi)d\xi \ ,$$

and

$$\begin{aligned} T(\varphi_j)(\psi_j) &= \frac{1}{(2\pi)^{n_1+n_2}} \int T(f_1(\cdot)e^{i(\cdot,\eta)})\hat{\varphi}_1(\eta)d\eta \ \left(\int f_2(\cdot)e^{i(\cdot,\xi)}\hat{\psi}_j(\xi)d\xi\right) \\ &= \frac{1}{(2\pi)^{n_1+n_2}} \int\int T(f_1(\cdot)e^{i(\cdot,\eta)}) \left(f_2(\cdot)e^{i(\cdot,\xi)}\right) \hat{\varphi}_j(\eta)\hat{\psi}_j(\xi)d\eta d\xi \\ &= \int\int F(\eta,\xi)\hat{\varphi}_j(\eta)\hat{\psi}_j(\xi)d\eta d\xi \ , \end{aligned} \tag{1.3.8}$$

where $F(\eta,\xi) = \dfrac{1}{(2\pi)^{n_1+n_2}} T(f_1(\cdot)e^{i(\cdot,\eta)})(f_2(\cdot)e^{i(\cdot,\xi)})$.

The map (1.3.5) is separately continuous. If V_1 and V_2 are compact subsets of Ω_1 and Ω_2, there are constants A_1, A_2, N and M, such that

$$| \ T(\varphi)(\psi) \ | \leq A_1 p_{V_2,N}(\psi) \ \ \forall \, \psi \in D_{V_2}(\Omega_2) \ , \varphi \in D_{V_1}(\Omega_1) \text{ fixed}$$

and

$$| \ T(\varphi)(\psi) \ | \leq A_2 p_{V_1,M}(\varphi) \ \ \forall \, \varphi \in D_{V_1}(\Omega_1) \ \ , \psi \in D_{V_2}(\Omega_2) \text{ fixed} \ .$$

The first of the estimates above comes from the fact that $T(\varphi) \in D'(\Omega_2)$, while the second comes from continuity of T. But $D_{V_1}(\Omega_1)$ and $D_{V_2}(\Omega_2)$ are Fréchet spaces. From the Baire category theorem (see exercise 1.3.11 below), (1.3.5) is jointly continuous and

$$| \ T(\varphi)(\psi) \ | \leq A p_{V_1,N}(\varphi) p_{V_2,M}(\psi) \ .$$

In particular, we can assume that $\operatorname{supp} f_i \subset V_i$ and apply this estimates to the functions $f_1(x)e^{i(x,\eta)}$ and $f_2(y)e^{i(y,\xi)}$. We conclude that for some N and M,

$$| \ F(\eta,\xi) \ | = | \ T(f_1(\cdot)e^{i(\cdot,\eta)})(f_2(\cdot)e^{i(\cdot,\xi)}) \ | \leq C(1 + \|\eta\|)^N(1 + \|\xi\|)^M \ ,$$

showing that $F(\eta,\xi)$ has polynomial growth. Define a linear map on $C_c^\infty(\Omega_1 \times \Omega_2)$ by

$$\tilde{K}(\Phi) = \int\int F(\eta,\xi)\hat{\Phi}(\eta,\xi)d\eta d\xi \ . \tag{1.3.9}$$

Choose an integer q such that $N + M - q \; < \; -(n_1 + n_2)$. If supp $\Phi \subset V$ where V is a compact subset of $\Omega_1 \times \Omega_2$, one easily obtain the estimate

$$| \hat{\Phi}(\eta, \xi) | \leq \; C \; (1 + \|\xi\| + \|\eta\|)^{-q} \; p_{V,q}(\Phi) \; .$$

Hence

$$\begin{aligned} | \tilde{K}(\Phi) | \leq & C p_{V,q}(\Phi) \int\int (1 + \|\xi\| + \|\eta\|)^{-q} (1 + \|\eta\|)^N (1 + \|\xi\|)^M d\eta d\xi \\ \leq & C p_{V,q}(\Phi) \; . \end{aligned} \tag{1.3.10}$$

Thus, (1.3.9) defines distribution which, by (1.3.8), agrees with $\sum T(\varphi_j)(\psi_j)$ when $\Phi = \sum \varphi_j \otimes \psi_j$ and supp $\Phi \subset C_1 \times C_2$. Estimates (1.3.7) comes from (1.3.10) when $V = C_1 \times C_2$ and $N = q$. This finishes the proof. ∎

Assume F, F' are Fréchet spaces and let $B : F \times F' \longrightarrow C$ be a separately continuous bilinear map. Show it is jointly continuous.

Hint: For each $k \in N$ define

$$S_k = \{ v \in F : | \; B(w, v) \; | \leq k \|w\|_k, \; \forall \; w \in F \} \; .$$

Prove that each S_k is closed and that $F = \cup_k S_k$. Then use Baire's category theorem.

Consider a differential operator $P = \sum a_\alpha(x) D_x^\alpha$ as in exercise 1.1.11. Show that its Schwartz kernel K_P is

$$K_P(x, y) = \sum a_\alpha(x) (D_x^\alpha \delta)(x - y) \; .$$

Conversely, show that any distribution like this is the Schwartz kernel of a differential operator.

1.4 Densities and density valued distributions

Contrary to the case of an open set in $\mathbf{R}^n$ where there is a naturally defined notion of integration, the case of general manifold X is a little bit more complicated. A measure defines a continuous linear functional on $C_c^\infty(X)$, but there is no natural way to associate with a continuous function an element in $[C_c^\infty(X)]^*$. This problem can be solved making additional assumptions on X. Rather, we shall define a complex line bundle over X which will make possible to integrate with a desirable degree of generality.

Let us start looking at the linear case. Consider a vector space V of dimension n with basis $\{v_1, \ldots, v_n\}$, and let $\bigwedge^n V^*$ be the vector space of exterior n-forms on V. It is well known that $\bigwedge^n V^*$ is a one-dimensional vector space with basis $\omega = v_1^* \wedge \ldots \wedge v_n^*$. Here $\{v_1^*, \ldots, v_n^*\}$ is the basis of V^* dual to $\{v_1, \ldots, v_n\}$.

Definition 1.4.1 *An α-density d on V is a map $d : \bigwedge^n V^* - \{0\} \longrightarrow C$ such that for all $s \in R - 0$ and for all $w \in \bigwedge^n V^* - \{0\}$, we have $d(sw) =\mid s \mid^\alpha d(w)$.*

Given an α-density d on V, it is completely determined by its value over ω. Indeed, any element $w \in \bigwedge^n V^*$ can be written as $w = c\omega$ for some constant c and therefore, $d(w) =\mid c \mid^\alpha d(\omega)$. The set of all α-densities on V will be denoted by $\Omega^\alpha V$. With the obvious operations, it has the structure of a complex vector space. Its dimension is one. For, let d_α be that α-density for which $d_\alpha(\omega) = 1$, and consider any other α-density d such that $d(\omega) = c$. Then, $d = cd_\alpha$ since both, left and right hand-side take the same value over ω. If $\{e_1, \ldots, e_n\}$ is another basis of V, then there exists a matrix $A = (a_{ij})$ such that $e_i = \sum a_{ij} v_j$. Hence,

$$\begin{aligned} d(e_1^* \wedge \ldots e_n^*) =\mid \det A \mid^\alpha d(\omega) =&\mid \det A \mid^\alpha cd_\alpha(\omega) \\ =& cd_\alpha(e_1^* \wedge \ldots e_n^*)\ , \end{aligned} \tag{1.4.2}$$

and the result is independent of the basis chosen.

Consider a smooth paracompact manifold X of dimension n. For each $x \in X$, T_xX is a vector space, and the set $\Omega^\alpha T_xX$ a complex vector space of dimension one. As x varies, this makes up a complex line bundle over X, $\Omega^\alpha X$, with transition functions as in (1.4.2). If (U, φ) is a local chart in X with $\varphi(p) = (x_1(p), \ldots, x_n(p))$, pointwise we can define an α-density on U that takes thevalue 1 over $dx_1 \wedge \ldots \wedge dx_n$, a local generator for $\bigwedge^n T^*X \mid_U$. It is customary to denote it by $\mid dx_1 \wedge \ldots \wedge dx_n \mid^\alpha$. Hence, $\mid dx_1 \wedge \ldots \wedge dx_n \mid^\alpha$ will provide a basis for $\Omega^\alpha X \mid_U$. If (U', φ') is another chart with $\varphi'(p) = (y_1(p), \ldots, y_n(p))$ and $U \cap U' \neq \emptyset$, then

$$\mid dx_1 \wedge \ldots dx_n \mid^\alpha_{U\cap U'} =\mid \det \partial x_i/\partial y_j \mid^\alpha \mid dy_1 \wedge \ldots \wedge dy_n \mid^\alpha_{U\cap U'}\ . \tag{1.4.3}$$

From the construction above, we can see that there is a globally defined smooth non-zero section of $\Omega^\alpha X$, showing that in fact, $\Omega^\alpha X$ is trivial, although not naturally so. A section s

of $\Omega^\alpha X$ which for any local chart (U, φ) can be written as

$$s = s(x) \mid dx_1 \wedge \ldots \wedge dx_n \mid^\alpha ,$$

with $s(x)$ a C^k-function, will be called a C^k-section of $\Omega^\alpha X$. The space of C^k-sections of $\Omega^\alpha X$ will be denoted by $C^k(X; \Omega^\alpha X)$. The space of C^k-sections with compact support will be denoted by $C^k_c(X; \Omega^\alpha X)$. If E is any vector bundle sitting over X, using tensor product with the corresponding space of sections, we define $C^k(X; E \otimes \Omega^\alpha X)$ and $C^k_c(X; E \otimes \Omega^\alpha X)$, respectively.

Proposition 1.4.4 *The family of bundles $\{\Omega^\alpha X\}$ satisfies the following properties:*

1. $\Omega^0 X \cong X \times C$.
2. $\Omega^\alpha X \otimes \Omega^\beta X \cong \Omega^{\alpha+\beta} X$.
3. $(\Omega^\alpha X)^* \cong \Omega^{-\alpha} X$.

Proof. 1. An element of $\Omega^0 T_x X$ is a constant function. The result follows at once.
2. The space $\Omega^\alpha T_x X \otimes \Omega^\beta T_x X$ is naturally isomorphic to the space of finite linear combinations of products of elements in the first factor with elements in the second factor. It is clear that if $d \in \Omega^\alpha T_x X$ and $d' \in \Omega^\beta T_x X$, then $dd' \in \Omega^{\alpha+\beta} T_x X$. The result follows.
3. Given $d' \in \Omega^{-\alpha} X$, the map

$$\Omega^\alpha X \xrightarrow{h_{d'}} \Omega^0 X$$

$$d \longrightarrow dd'$$

defines an element of the bundle $\mathrm{Hom}\,(\Omega^\alpha X, \varepsilon^1)$ where ε^1 is the trivial line bundle. The map sending d' into $h_{d'}$ is clearly bijective. ∎

If $\varphi : X \to Y$ is a diffeomorphism of manifolds, then φ induces an isomorphism $\varphi^* : \Omega^\alpha Y \to \Omega^\alpha X$. More importantly, consider a continuous one density d with compact support, i.e., $d \in C^0_c(X; \Omega X)$. Assume that the support of d is contained in the domain of a local chart (U, φ) with $\varphi(p) = (x_1(p), \ldots, x_n(p))$. If $d = d(x) \mid dx_1 \wedge \ldots \wedge dx_n \mid$, we define

$$\int_K d = \int_{\varphi(K)} d(\varphi^{-1}(x)) dx_1 \wedge \ldots \wedge dx_n , \qquad (1.4.5)$$

where K is any compact set contained in U. In (1.4.5), the right hand side is the usual Lebesgue integral over a compact set of R^n. Due to (1.4.3) and the change of variable formula for integrals, the expression (1.4.5) is well-defined. The global definition of the integral of a one density with certain growth outside a compact set can be achieved by making use of a partition of unity subordinated to a locally finite covering of X by local charts. In this way, if $\mathcal{M}(X;\Omega^\alpha X)$ is the line bundle of measurable α-densities, one defines

$$L^1(X;\Omega X) = \{d \in \mathcal{M}(X;\Omega X) \;:\; \|d\| = \lim_{K\subseteq X, K\nearrow X} \int_K \mid d \mid < \infty\} \,. \tag{1.4.6}$$

If E is a bundle over X, using tensor product we define $L^1(X; E\otimes \Omega X)$.

If d and $d^{'}$ are smooth densities of order α and $1-\alpha$, respectively, with $d^{'}$ compactly supported, the tensor product $dd^{'}$ is smooth one density with compact support, and therefore, integrable. Topologizing the space $C_c^\infty(X;E^*\otimes\Omega^{1-\alpha}X)$ in the usual way, we define the distributions with values in $E\otimes\Omega^\alpha X$ as

$$D^{'}(X;E\otimes\Omega^\alpha X) = [C_c^\infty(X;E^*\otimes\Omega^{1-\alpha}X)]^{'} \,. \tag{1.4.7}$$

Clearly, all the spaces C^k are embedded into $D^{'}(X;E\otimes\Omega^\alpha X)$. The space $C_c^\infty(X;E^*\otimes\Omega^\alpha X)$ is dense in $D^{'}(X;E\otimes\Omega^\alpha X)$. Note that $D^{'}(X;E)$ corresponds to the dual of $C_c^\infty(X;E^*\otimes\Omega X)$. When the bundle E is trivial of rank one, we shall drop it from the notation writting simply $D^{'}(X;\Omega^\alpha X)$ instead of $D^{'}(X;E\otimes\Omega^\alpha X)$.

The set $L^1(X;\Omega X)$ is a Banach space. With its help, we can define the Hilbert space of measurable half-densities on X. Indeed, if $d\in\mathcal{M}(X;\Omega^{\frac{1}{2}}X)$, we say that $d\in L^2(X;\Omega^{\frac{1}{2}}X)$ if and only if $\mid d\mid^2\in L^1(X;\Omega X)$. The bilinear map

$$\begin{array}{c} C_c^\infty(X;\Omega^{\frac{1}{2}}X)\otimes C_c^\infty(X;\Omega^{\frac{1}{2}}X) \longrightarrow C \\ d\otimes d^{'} \longrightarrow \int d\cdot \bar{d^{'}} \end{array} \,,$$

defines a pre-Hilbert structure on $C_c^\infty(X;\Omega^{\frac{1}{2}}X)$ and the completion of this space in the norm defined by this structure coincides with $L^2(X;\Omega^{\frac{1}{2}}X)$. Once again, tensoring with the space of sections of E, a bundle over X, we define $L^2(X;E\otimes\Omega^{\frac{1}{2}}X)$, which sits inside $D^{'}(X;E\otimes\Omega^{\frac{1}{2}}X)$.

Let $\varphi\,:\,X\longrightarrow Y$ be a submersion. Show that the pull-back map φ^* defined on continuous functions by $\varphi^*f = f\circ\varphi$ can be extended by continuity to a map $\varphi^*\,:\,D^{'}(Y)\longrightarrow D^{'}(X)$. **Hint**: Show that the push-forward is a map from $C_c^\infty(X;\Omega X)$ into $C_c^\infty(Y;\Omega Y)$. The desired result follows by duality.

Let X and Y be smooth manifolds, and consider a continuous linear operator

$$T\ : C_c^\infty(Y) \longrightarrow D^{'}(X)\ . \tag{1.4.8}$$

Prove the Schwartz kernel theorem stating that there exists a unique distribution $K_T \in D^{'}(X \times Y; \Pi_Y^* \Omega Y)$ such that

$$T(\varphi)(\phi) = K(\varphi \otimes \phi)\ ,$$

for all $\varphi \in C_c^\infty(Y)$ and $\phi \in C_c^\infty(X; \Omega X)$. Here $\Pi_Y\ : X \times Y \longrightarrow Y$ is the natural projection.

Show that the map in (1.4.10) extends to a map from $D_c^{'}(Y)$ to $C^\infty(X)$ if, and only if, $K_T \in C^\infty(X \times Y; \Pi_Y^* \Omega Y)$.

1.5 Fourier transform

We would like to extend the notion of Fourier transform previously defined, keeping as much as possible the coordinate free approach used in the previous section. The construction here will be of crucial importance in chapter 4, where we shall prove the symbol isomorphism theorem between conormal distributions to a submanifold S and symbols on N^*S.

Let V be a vector space of dimension n, and consider a family $\{V_1, \ldots, V_n\}$ of constant vector fields that spans the tangent space of V at each point. Let $\|\ \|$ be some norm defined on V. The space of rapidly decreasing functions can be defined as

$$\mathcal{S}(V) = \{u \in (V) : \mid V_1^{\alpha_1} \ldots V_n^{\alpha_n} u(x) \mid \leq C_N (1 + \|x\|)^{-N}\ \forall\ \alpha_1, \ldots, \alpha_n, N\}\ . \tag{1.5.1}$$

Since all the norms on a finite dimensional vector space are equivalent and a new set of constant vector fields amounts simply for a change of basis, the definition above is independent of the choice of $\{V_1, \ldots, V_n\}$ and$\|\ \|$. The space of rapidly decreasing α-densities $; \Omega^\alpha V)$ is defined as $; \Omega^\alpha V) =) \otimes \Omega^\alpha V$. Here, $\Omega^\alpha V$ is considered as a bundle over V, canonically isomorphic to $V \times \Omega^\alpha V$, and the last $\Omega^\alpha V$ on the left side is the vector space of α-densities on V.

Let denote the dual pairing between $x \in V$ and $\xi \in V^*$, and define the Fourier transform on densities by

$$\begin{array}{c} ; \Omega V) \xrightarrow{\mathcal{F}} \mathcal{S}(V^*) \\ f(x)dx \longrightarrow \hat{f}(\xi) = \int e^{-i} f(x) dx \end{array}\ . \tag{1.5.2}$$

Proposition 1.5.3 *The map* (1.5.2) *is well defined, that is to say,* $\hat{f}(\xi) \in \mathcal{S}(V^*)$.

Proof. It is clear that $\hat{f}(\xi)$ is a smooth complex-valued function on V^*. It remains to check that it is rapidly decreasing. This follows from the fact that if α is any multi-index,

$$\xi^\alpha e^{-i} = \xi^{\alpha_1} \dots \xi^{\alpha_n} e^{-i} = (\sum a_{1j} V_j)^{\alpha_1}) \dots (\sum a_{nj} V_j)^{\alpha_n}) e^{-i} \ ,$$

for some constants a_{ij}, $1 \leq i, j \leq n$. The result follows integrating by parts. ∎

It is easy to see that $\Omega^\alpha V$ is isomorphic to $\Omega^{-\alpha} V^*$. Indeed, if $\{v_1, \dots, v_n\}$ is basis of V, $v_1 \wedge \dots \wedge v_n$ is a basis of $\bigwedge^n V^*$. Hence, given $d \in \Omega^\alpha V$, we define h_d by

$$h_d(v_1 \wedge \dots \wedge v_n) = d(v_1^* \wedge \dots \wedge v_n^*) \ ,$$

where $\{v_1^*, \dots, v_n^*\}$ is the basis of V^* dual to $\{v_1, \dots, v_n\}$. If the j-th vector is replaced by sv_j then

$$\begin{aligned} h_d(s(v_1 \wedge \dots \wedge v_n)) &= h_d(v_1 \wedge \dots \wedge sv_j \wedge \dots \wedge v_n) \\ &= d(v_1^* \wedge \dots \wedge \frac{v_j^*}{s} \wedge \dots \wedge v_n^*) \\ &= \mid s \mid^{-\alpha} d(v_1^* \wedge \dots \wedge v_n^*) \\ &= \mid s \mid^{-\alpha} h_d(v_1 \wedge \dots \wedge v_n) \ . \end{aligned}$$

Thus, h_d is an element of $\Omega^{-\alpha} V^*$. The correspondence $d \longrightarrow h_d$ is an isomorphism. Hence $\Omega^\alpha V \cong \Omega^{-\alpha} V^* \cong (\Omega^\alpha V^*)^*$ (the last congruency obtained using proposition 1.4.4). Therefore, tensoring the domain and range of (1.5.2) with $\Omega^{-\alpha} V$ and using the identification above, we extend the Fourier transform to a map

$$; \Omega^{1-\alpha} V) \xrightarrow{\mathcal{F}} \mathcal{S}(V^*; \Omega^\alpha V^*) \ .$$

Note the preferred role played by 1/2-densities. If $\mathcal{S}'(V; \Omega^\alpha V)$ denotes the dual of $\mathcal{S}(V; \Omega^{1-\alpha} V)$, i.e., the space of tempered α-densities distributions, by duality we get

$$\mathcal{S}'(V; \Omega^\alpha V) \xrightarrow{\mathcal{F}} \mathcal{S}'(V^*; \Omega^{1-\alpha} V^*) \ . \tag{1.5.4}$$

Theorem 1.5.5 *The map* (1.5.4) *is an isomorphism.*

Proof. It is clearly well-defined and to show it produces an isomorphism, it will be enough to show that (1.5.2) is an isomorphism. But that is nothing more than theorem 1.2.6 with the presence of a density factor (see also (1.2.8)). ∎

Finally, let E be a smooth vector bundle over X with projection Π, and assume for simplicity that X is compact. We consider the space of rapidly decreasing functions along the fibers:

$$\mathcal{S}(E) = \{f \in (E) : \ f\,|_{\Pi^{-1}(x)} \in \mathcal{S}(\Pi^{-1}(x)) \ \forall\, x \in X\}\ .$$

By letting the Fourier transform act along the fibers, we obtain an isomorphism

$$\mathcal{S}(E; \Omega^{1-\alpha}_{fiber}E) \xrightarrow{\mathcal{F}} \mathcal{S}(E^*; \Omega^{\alpha}_{fiber}E^*)\ , \tag{1.5.6}$$

and by duality an isomorphism

$$\mathcal{S}'(E; \Omega^{\alpha}_{fiber}E) \xrightarrow{\mathcal{F}} \mathcal{S}'(E^*; \Omega^{1-\alpha}_{fiber}E^*)\ .$$

Both of these shall be strongly used for the case of the conormal bundle to Y, a given closed submanifold of a paracompact manifold X.

If the base manifold is not compact, we shall denote by $\mathcal{S}_c(E)$ the intersection $\mathcal{S}(E) \cap \{f \in (E) : \ \text{supp}\, f \subset\subset X\}$. There are isomorphisms

$$\begin{array}{c} \mathcal{S}_c(E; \Omega^{\alpha}_{fiber}E) \cong \mathcal{S}_c(E) \otimes \Omega^{\alpha}_{fiber}E \\ \Pi^*\Omega^{\alpha}X \otimes \Omega^{\alpha}_{fiber}E \cong \Omega^{\alpha}E \end{array}\ . \tag{1.5.7}$$

The map (1.5.6) induces a map on $\mathcal{S}_c(E)$. If we tensor domain and range with $\Pi^*\Omega^{1/2}X$, using (1.5.7) we conclude that

$$\mathcal{S}_c(E; \Omega^{1/2}E) \xrightarrow{\mathcal{F}} \mathcal{S}_c(E^*; \Omega^{1/2}E^*) \tag{1.5.8}$$

is an isomorphism. We summarize this result in the following

Theorem 1.5.9 *The Fourier transform* (1.5.8) *is an isomorphism.* ∎

Chapter 2

Sobolev Spaces

In this chapter we introduce subspaces of $D'(\Omega)$ which allow us to measure the singularities of distributions. These spaces are generally defined for open domains with smooth boundary, but we present them in here in a way that applies also to domains with certain type of singularities at the boundary. In particular, our definitions applied to polyhedral domains in the plane, or to domains with edges in higher dimensions.

2.1 Sobolev spaces: definitions

Any element f of $L^2(\mathbf{R}^n)$ defines a distribution by the expression $\varphi \longrightarrow \int \varphi(x)f(x)dx$. The question one immediately asks is how far from an L^2-function a given distribution is. A criterion to answer it will provide a rule to determine how singular a distribution can be.

With this purpose in mind we introduce the following spaces. Let Ω be an open subset of $\mathbf{R}^n$:

1. If $k \in N = \{0, 1, 2, \ldots\}$,

$$u \in H^k_c(\Omega) \Longleftrightarrow D^\alpha_x u \in L^2(\Omega) \cap D'_c(\Omega)\ \forall\ |\alpha| \leq k\ .$$

2. When s is positive but not an integer, $0 < s < 1$, we interpolate

$$u \in H^s_c(\Omega) \Longleftrightarrow u \in L^2(\Omega) \cap D'_c(\Omega) \text{ and } \frac{|\ u(x) - u(y)\ |}{\|x - y\|^{\frac{n}{2}+s}} \in L^2(\Omega_x \times \Omega_y)\ .$$

3. If $s \geq 0$,

$$H_c^s(\Omega) = \{u \in D_c^{'}(\Omega) \; : \; D_x^\alpha u \in H_c^r(\Omega) \text{ if } \mid \alpha \mid < [s], \; r = r - [s]\} \; .$$

4. If $s < 0$,

$$H_c^s(\Omega) = \{u \in D_c^{'}(\Omega) \; : \; \exists u_\alpha \in H_c^r(\Omega) \text{ such that } u = \sum_{|\alpha| \leq -[s]} D_x^\alpha u_\alpha \text{ for } r = -[s] + s\} \; .$$

It is clear that for any s, the space $H_c^s(\Omega)$ is contained in $D_c^{'}(\Omega)$. We present below a different description of $H_c^s(\Omega)$ in terms of the Fourier transform.

Let $u \in H_c^s(\Omega)$ for $0 < s < 1$. Then

$$\frac{\mid u(x) - u(y) \mid}{\|x - y\|^{\frac{n}{2} + s}} \in L^2(\Omega_x \times \Omega_y) \; .$$

We have:

$$\int\int \frac{\mid u(x) - u(y) \mid^2}{\|x - y\|^{n+2s}} dxdy = \int\int \frac{\mid u(x) - u(x - z) \mid^2}{\|z\|^{n+2s}} dxdz \; .$$

As a function of (x, z), the last integrand is square integrable. Thus, for almost every z, it is square integrable as a function of x. Extending itby zero outside its support and using Plancherel's theorem, we obtain

$$\begin{aligned} \int\int \frac{\mid u(x) - u(x - z) \mid^2}{\|z\|^{n+2s}} dxdz &= (2\pi)^{-\frac{n}{2}} \int\int \mid \hat{u}(\xi) \mid^2 \frac{\mid (1 - e^{iz \cdot \xi}) \mid^2}{\|z\|^{n+2s}} dzd\xi \\ &= \int\int \mid \hat{u}(\xi) \mid^2 F(\xi) d\xi \; , \end{aligned}$$

where $F(\xi) = (2\pi)^{-\frac{n}{2}} \int \mid (1 - e^{iz \cdot \xi}) \mid^2 \|z\|^{-n-2s} dz$.

Observe that $F(t\xi) = t^{2s} F(\xi)$, i.e., $F(\xi)$ is homogeneous of degree $2s$. Then $F(\xi) = \|\xi\|^{2s} F(\hat{\xi})$ where $\hat{\xi} = \xi / \|\xi\|$. Hence, $u \in H_c^s(\Omega)$ if, and only if, $u \in L_c^2(\Omega)$ and

$$\|\xi\|^s \hat{u}(\xi) \in L_c^2(\Omega) \; .$$

These two conditions imply that

$$(1 + \|\xi\|^2)^{\frac{s}{2}} \hat{u}(\xi) \in L^2(\mathbf{R}^n) \; .$$

Generalizing this to any s, and conclude that

$$H_c^s(\Omega) = \{u \in D^{'}(\mathbf{R}^n) : \; \operatorname{supp} u \subset \Omega \text{ and } (1 + \|\xi\|^2)^{\frac{s}{2}} \hat{u}(\xi) \in L^2(\mathbf{R}^n)\} \; . \tag{2.1.1}$$

Lemma 2.1.2 *With the multiplication operation, $H^s_c(\Omega)$ is a (Ω)-module.*

Proof. Take $u \in H^s_c(\Omega)$ and $\varphi \in (\Omega)$. We need to show that $\varphi u \in H^s_c(\Omega)$.

Certainly, $\operatorname{supp} \varphi u \subset \operatorname{supp} u$. Then, we can assume that φ is compactly supported on an open neighborhood of Ω. With this assumption, $\varphi \in \mathcal{S}(\mathbf{R}^n)$ and $u \in \mathcal{S}'(\mathbf{R}^n)$. Consequently, $\widehat{\varphi u} = (2\pi)^{-n}\hat{\varphi} * \hat{u}$. Therefore,

$$\begin{aligned}(1+\|\xi\|^2)^{\frac{s}{2}}\widehat{\varphi u}(\xi) &= (2\pi)^{-n}\int(1+\|\xi\|^2)^{\frac{s}{2}}\hat{\varphi}(\xi-\eta)\hat{u}(\eta)d\eta \\ &= (2\pi)^{-n}\int(1+\|\xi\|^2)^{\frac{s}{2}}(1+\|\eta\|^2)^{\frac{-s}{2}}\hat{\varphi}(\xi-\eta)(1+\|\eta\|^2)^{\frac{s}{2}}\hat{u}(\eta)d\eta\ .\end{aligned}$$

By assumption $(1+\|\eta\|^2)^{\frac{s}{2}}\hat{u}(\eta) \in L^2(\mathbf{R}^n)$. Then, the desired result will follow from the Cauchy- Schwartz inequality, if we prove Peetre's inequality:

$$(1+\|\xi\|^2)^{\frac{s}{2}}(1+\|\eta\|^2)^{\frac{-s}{2}} \le 2^{|s|}(1+\|\xi-\eta\|^2)^{|s|}\ \forall\ s \in \mathbf{R},\ \xi,\eta \in \mathbf{R}^n\ .$$

This comes from the fact that

$$1+\|\xi+\eta\|^2 \le 2(1+\|\xi\|^2)(1+\|\eta\|^2)\ ,$$

after conveniently taking the s-th power. ∎

Theorem 2.1.3 *The family $\{H^{-s}_c(\Omega)\}$ is a (Ω) filtration of $D'_c(\Omega)$, i.e., they are (Ω)-modules and*

1. $H^{-s_1}_c(\Omega) \subset H^{-s_2}_c(\Omega),\ s_1 > s_2$.
2. $D'_c(\Omega) = \cup_s H^{-s}_c(\Omega)$.
3. $_c(\Omega) = \cap_s H^{-s}_c(\Omega)$.

Proof. It clearly follows from (2.1.1) that $H^{-s_1}_c(\Omega) \subset H^{-s_2}_c(\Omega)$ if $s_1 > s_2$. On the other hand, if $u \in D'_c(\Omega)$, using theorem 1.2.10 we find some$N$ such that

$$|\ \hat{u}(\xi)\ | \le C(1+\|\xi\|^2)^{\frac{N}{2}}\ .$$

Therefore,

$$(1+\|\xi\|^2)^{\frac{s}{2}}\ |\ \hat{u}(\xi)\ | \in L^2(\mathbf{R}^n)$$

for all s such that $s+N < -n$. It follows that $D'_c(\Omega) = \cup_s H^{-s}_c(\Omega)$.

The last assertion also follows from theorem 1.2.10, and it is left as an exercise for the reader. ∎

If we drop the condition on the support in (2.1.1), we we obtain the spaces $H^s(\mathbf{R}^n)$. That is to say,

$$H^s(\mathbf{R}^n) = \{u \in \mathcal{S}'(\mathbf{R}^n) : \ (1+ \parallel \xi \parallel^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbf{R}^n)\} \ .$$

For what values of s is the constant function $f(x) = c$ an element of $H^s(\mathbf{R}^n)$.

The geometry of the boundary of Ω creates some troubles when one tries to generalize the spaces $H^s_c(\Omega)$ releasing the constrain over the support. There are several ways of doing this yielding different type of spaces (see [Ad]). We shall restrict our attention to the case where Ω is a bounded domain with smooth boundary but in fact, some of the results to be proven below also hold when Ω has a nice singular boundary.

The idea is to look at the restriction operator

$$D'(\mathbf{R}^n) \longrightarrow D'(\Omega)$$
$$u \longrightarrow u\mid_\Omega$$

over $H^s(\mathbf{R}^n)$, and define $H^s(\Omega)$ as its image. For that, given a closed subset K of $\mathbf{R}^n$, let us denote by $H^s_K(\mathbf{R}^n)$ the space of elements of $H^s(\mathbf{R}^n)$ which are supported on K. This is a closed subspace of $H^s(\mathbf{R}^n)$, and it therefore inherits a Hilbert structure. Thus, if we set $\Omega = \mathbf{R}^n - K$, we define

$$H^s(\Omega) = \frac{H^s(\mathbf{R}^n)}{H^s_K(\mathbf{R}^n)} \ , \tag{2.1.4}$$

and provide it with the quotient Hilbert structure. We obtain a short exact sequence

$$0 \to H^s_K(\mathbf{R}^n) \to H^s(\mathbf{R}^n) \to H^s(\Omega) \to 0 \ .$$

This sequence splits.

Define also

$$\overset{\circ}{H}{}^s(\Omega) = \{u \in H^s(\mathbf{R}^n) : \operatorname{supp} u \subset \overline{\Omega}\} \ . \tag{2.1.5}$$

Using the isomorphism

$$(1+\Delta)^{\frac{s}{2}} : \ H^s(\mathbf{R}^n) \longrightarrow L^2(\mathbf{R}^n) \ ,$$

show that $(H^s(\mathbf{R}^n))^* = H^{-s}(\mathbf{R}^n)$.

Show that $(H^s(\Omega))^* = \overset{\circ}{H}{}^{-s}(\Omega)$.

Finally, consider a diffeomorphism $\phi : \Omega' \to \Omega$, where Ω and Ω' are open subsets in R^n. If we apply the change of variable formula in the integral conditions defining $H^s_c(\Omega)$, we see that $f \circ \phi \in H^s_c(\Omega')$ if, and only if, $f \in H^s_c(\Omega)$. We introduce the local Sobolev spaces

$$H^s_{loc}(\Omega) = \{u \in D'(\Omega) : \ \varphi u \in H^s_c(\Omega)\ \forall\, \varphi \in_c (\Omega)\}\ . \tag{2.1.6}$$

Definition 2.1.7 *Let X be a paracompact manifold. The space $H^s_{loc}(X)$ consists of the set of distributions $u \in D'(X)$ such that $u \circ \varphi \in H^s_{loc}(\varphi(U))$ for any coordinate system $\varphi(q) = (x_1(q), \ldots, x_n(q))$ valid on the open set $U \subset X$.*

The discussion above shows that $H^s_{loc}(X)$ is well-defined.

Let X be a paracompact manifold. Show that $\{H^{-s}_{loc}(X)\}$ is a (X) filtration of $D'(X)$, i.e., $H^{-s}_{loc}(X)$ is a (X)-module for each s and

1. $H^{-s_1}_{loc}(X) \subset H^{-s_2}_{loc}(X),\ s_1 > s_2$.
2. $D'(X) = \cup_s H^{-s}_{loc}(X)$.
3. $(X) = \cap_s H^{-s}_{loc}(X)$.

If E is any vector bundle sitting above X, by taking tensor product over the space of smooth sections, we define $H^s_{loc}(X; E)$, etc.

2.2 Compactness theorems

The way they were defined, it is clear that the spaces $H^s(\mathbf{R}^n)$ become smoother when s increases. More precisely,

Lemma 2.2.1 *(Sobolev embedding theorem). Let k be a non-negative integer and let $s > k + n/2$. If $f \in H^s(\mathbf{R}^n)$, then $f \in C^k_b(\mathbf{R}^n)$ and*

$$\sup_{|\alpha| \le k} \mid D^\alpha f(x) \mid \le\ C \parallel f \parallel_s = C \left(\int (1 + \|\xi\|^2)^s \mid \hat{f}(\xi) \mid^2 d\xi \right)^{1/2}\ .$$

Proof. When $k = 0$ we have

$$\begin{aligned} | f(x) |^2 &= | \int e\hat{f}(\xi)(1+\|\xi\|^2)^{s/2}(1+\|\xi\|^2)^{-s/2}\, d\xi | \\ &\leq \int | \hat{f}(\xi) |^2 (1+\|\xi\|^2)^s\, d\xi (\int (1+\|\xi\|^2)^{-s}\, d\xi) \\ &\leq C \ \| f \|_s^2 , \end{aligned}$$

and the result follows. For general k, we use the fact that $D_x^\alpha \ : \ H^s(\mathbf{R}^n) \to H^{s-\|\alpha\|}(\mathbf{R}^n)$ continuously. ∎

Corollary 2.2.2 *For s and k as above, $H^s(\Omega) \subset C_b^k(\overline{\Omega})$and the inclusion is continuous.*

Proof. Given $u \in H^s(\Omega)$, apply lemma 2.2.1 to an extension $\tilde{u} \in H^s(\mathbf{R}^n)$. The result follows. ∎

On the other hand, let K be a fixed compact subset of $\mathbf{R}^n$.

Lemma 2.2.3 *(Rellich) Suppose $\{f_n\}$ is a uniformly bounded sequence in $H^s(\mathbf{R}^n)$ such that* $\operatorname{supp} f_n \subset K$ *for all n. Then, there exists a subsequence which converges in $H^t(\mathbf{R}^n)$ for all $t < s$.*

Proof. Let φ be a smooth function with compact support such that $\varphi \equiv 1$ on K, and consider $f_n = \varphi f_n$. Since $\hat{f}_n = (2\pi)^{-n}\hat{\varphi} * \hat{f}_n$ and $D_{\xi_j}\hat{f}_n = (2\pi)^{-n} D_{\xi_j}\hat{\varphi} * \hat{f}_n$, wehave

$$\begin{aligned} | D_{\xi_j}\hat{f}_n(\xi) | &= (2\pi)^{-n} | \int D_{\xi_j}\hat{\varphi}(\xi-\eta)\hat{f}_n(\eta) d\eta | \\ &\leq (2\pi)^{-n}\int | D_{\xi_j}\hat{\varphi}(\xi-\eta)\hat{f}_n(\eta) | \, d\eta \\ &\leq (2\pi)^{-n}\int | D_{\xi_j}\hat{\varphi}(\xi-\eta) | \, (1+\|\eta\|^2)^{\frac{-s}{2}}(1+\|\eta\|^2)^{\frac{s}{2}} \, | \hat{f}_n(\eta) | \, d\eta \\ &\leq (2\pi)^{-n}\|f_n\|_{H^s(\mathbf{R}^n)}\int | D_{\xi_j}\hat{\varphi}(\xi-\eta) |^2 \, (1+\|\eta\|^2)^{-s} d\eta , \end{aligned}$$

where the last comes from the Cauchy-Schwartz inequality. From the estimates for $D_{\xi_j}\hat{\varphi}$ that we obtain applying Paley-Wiener theorem, combined with Peetre's inequality, we conclude that for some continuous function g,

$$| D_{\xi_j}\hat{f}_n(\xi) | \leq g(\xi)\|f_n\|_{H^s(\mathbf{R}^n)} .$$

A bound like this also holds for $\hat{f}_n$ itself. By the Arzela-Ascoli theorem, we find a subsequence $\{\hat{f}_{n_j}\}$ which converges uniformly on compact sets. We claim that $\{f_{n_j}\}$ is Cauchy in $H^t(\mathbf{R}^n)$.

Indeed,

$$\begin{aligned}\|f_{n_j} - f_{n_k}\|^2_{H^s} &= \int (1+\|\xi\|^2)^t \mid (\hat{f}_{n_j} - \hat{f}_{n_k})(\xi) \mid^2 d\xi \\ &= \int_{\|\xi\|\le r} (1+\|\xi\|^2)^t \mid (\hat{f}_{n_j} - \hat{f}_{n_k}) \mid^2 d\xi + \int_{\|\xi\|>r} (1+\|\xi\|^2)^t \mid (\hat{f}_{n_j} - \hat{f}_{n_k}) \mid^2 d\xi \,.\end{aligned}$$

For fixed r, the first integral on the right is arbitrarily small for large j,k because of the uniform convergence of $\{\hat{f}_{n_j}\}$. We choose r so that the second integral is also small. In fact,

$$\begin{aligned}\int_{\|\xi\|>r} (1+\|\xi\|^2)^t \mid (\hat{f}_{n_j} - \hat{f}_{n_k}) \mid^2 d\xi &= \int_{\|\xi\|>r} (1+\|\xi\|^2)^{t-s} (1+\|\xi\|^2)^s \mid (\hat{f}_{n_j} - \hat{f}_{n_k}) \mid^2 d\xi \\ &\le (1+r^2)^{t-s} \|\hat{f}_{n_j} - \hat{f}_{n_k}\|_{H^s} \,,\end{aligned}$$

which can be made arbitrarily small for large r since $t < s$. $\blacksquare$

Corollary 2.2.4 *The space $H^s_K(\mathbf{R}^n)$ is compactly included in $H^t_K(\mathbf{R}^n)$ for any $t < s$.*

Corollary 2.2.5 *Assume Ω is a bounded domain with smooth boundary. Then $H^s(\Omega)$ is compactly included in $H^t(\Omega)$ for any $t < s$. If X is a compact manifold and E is a vector bundle over X, $H^s(X;E)$ is compactly included in $H^t(X;E)$.*

Remark 2.2.6 *Although the results have been stated for smooth domains, they can be generalized to domains with nice singularities. One proceeds by "doubling" the domain across the sides and extending the distributions conveniently. In the double manifold so obtained, one applies the results here discussed and reaches the desired conclusions by taking restrictions.*

2.3 Trace theorems

In the setting of Sobolev spaces, we want to consider the problem of taking the restriction of a distribution to a hypersurface, or for that matter, to a submanifold. Later on, we shall reconsider the same question in some other circumstances.

Let us start by looking at the case $\mathbf{R}^{n-1} = \mathbf{R}^{n-1} \times \{0\} \subset \mathbf{R}^n$. The map

$$\begin{aligned} {}_c(\mathbf{R}^{n-1}) &\longrightarrow D'(\mathbf{R}^n) \\ \varphi &\longrightarrow \varphi \otimes D^j_{x_n} \delta \end{aligned} \tag{2.3.1}$$

is continuous, and for any s such that $s+j<-1/2$, we have

$$\begin{aligned}\|\varphi\otimes D_{x_n}\delta\|_{H^s} &= \int(1+\|\xi\|^2)^s \mid \xi_n^j\hat{\varphi}(\xi')\mid^2 d\xi \\ &\le \int(1+\|\xi\|^2)^{s+j+\frac{1}{2}}(1+\|\eta\|^2)^s \mid \eta^2\hat{\varphi}(\xi')\mid^2 d\xi' d\eta \\ &\le C\|\varphi\|_{H^{s+j+\frac{1}{2}}}\int(1+\|\eta\|^2)^s \mid \eta^{2j}\mid \\ &\le C\|\varphi\|_{H^{s+j+\frac{1}{2}}} .\end{aligned}$$

Therefore, (2.3.1) extends by continuity and produces an injective closed map

$$\begin{array}{c} H^{s+j+\frac{1}{2}}(\mathbf{R}^{n-1}) \longrightarrow H^s(\mathbf{R}^n) \\ \varphi \longrightarrow \varphi\otimes D^j_{x_n}\delta \end{array} \quad , \ s+j<-\frac{1}{2} . \tag{2.3.2}$$

It is clear that $\varphi\otimes D_{x_n}\delta\notin H^s(\mathbf{R}^n)$ for any s such that $s+j\ge -1/2$.

Proposition 2.3.3 *For $s<-1/2$, $H^s_{\mathbf{R}^{n-1}}(\mathbf{R}^n)$ equals to*

$$\{u:\ u=\sum_{0\le j<-s-1/2}\varphi_j\otimes D^j_{x_n}\delta,\ \varphi_j\in H^{s+j+\frac{1}{2}}(\mathbf{R}^{n-1})\} .$$

If $s\ge -1/2$, $H^s_{\mathbf{R}^{n-1}}(\mathbf{R}^n)=\{0\}$.

Proof. We have proved that the space above is contained in $H^s(\mathbf{R}^n)$. We must show that any element in $H^s_{\mathbf{R}^{n-1}}(\mathbf{R}^n)$ can be expressed in this form.

Let $u\in H^s_{\mathbf{R}^{n-1}}(\mathbf{R}^n)$. Then for any test function $\varphi\in_c(\mathbf{R}^{n-1})$, $u(\cdot,x_n)(\varphi)$ is supported on $\{x_n=0\}$. Thus,

$$u(\cdot,x_n)(\varphi)=\sum_{j=0}^N u_{\varphi,j}D^j_{x_n}\delta .$$

Observe that $u_{\varphi,j}=i^{-j}u(\varphi\otimes x_n^j)$. It follows then that $\varphi\to u_{\varphi,j}$ is a distribution for all j. Call it u_j. Thus, we have

$$u=\sum_{j=0}^N u_j\otimes D^j_{x_n}\delta .$$

Since $u\in H^s(\mathbf{R}^n)$, we must have

$$\begin{aligned}\|u\|_{H^s} &= \int(1+\|\xi\|^2)^s \mid (\textstyle\sum_{j=0}^N \hat{u}_j(\xi')\xi_n^j)\mid^2 d\xi \\ &= \int(1+\|\xi'\|^2)^s(1+\|\eta\|^2)^s \mid (\textstyle\sum_{j=0}^N \hat{u}_j(\xi')(1+\|\xi'\|^2)^{\frac{j}{2}}\eta^j)\mid^2 (1+\|\xi'\|^2)^{\frac{1}{2}}d\eta \\ &= \int \mid (\textstyle\sum_{j=0}^N \hat{u}_j(\xi')(1+\|\xi'\|^2)^{\frac{s}{2}+\frac{j}{2}+\frac{1}{4}}\eta^j \mid^2 (1+\|\eta\|^2)^s d\eta<\infty .\end{aligned}$$

If $\hat{u}_N(\xi') \neq 0$, then $(1+\|\eta\|^2)^s\eta^{2N}$ must be integrable and, therefore, $2s+2N+1<0$. Thus, $N < -s-\frac{1}{2}$. Also, from the fact that as a function of ξ' the integrand is in L^1 for almost all η, we conclude that $\hat{u}_j(\xi')(1+\|\xi'\|^2)^{s+j+\frac{1}{2}} \in L^2(\mathbf{R}^{n-1})$, which is to say that $u_j \in H^{s+j+\frac{1}{2}}(\mathbf{R}^{n-1})$.

The remark preceding the proposition shows that $H^s_{\mathbf{R}^{n-1}}(\mathbf{R}^n) = \{0\}$ for $s \geq -1/2$. ∎

Theorem 2.3.4 *The restriction map*

$$_c(\mathbf{R}^n) \longrightarrow C_c^\infty(\mathbf{R}^{n-1})$$
$$u \longrightarrow D^j_{x_n} u \mid_{\mathbf{R}^{n-1}}$$

extends to a continuous surjective map

$$H^s(\mathbf{R}^n) \longrightarrow H^{s-j-\frac{1}{2}}(\mathbf{R}^{n-1})$$

for all s such that $s-j > 1/2$.

Proof. Let φ be a test function. Then we can write

$$D_{x_n} u \mid_{\mathbf{R}^{n-1}} (\varphi) = D_{x_n} u(\varphi \otimes \delta) = (-1)^j u(\varphi \otimes D^j_{x_n}\delta) \ ,$$

as long as $\varphi \otimes D^j_{x_n}\delta$ is in the dual to the space where u belongs. Assume $u \in H^s(\mathbf{R}^n)$ for some s. We must find conditions over s such that $\varphi \otimes D^j_{x_n}\delta \in H^{-s}(\mathbf{R}^n)$. From proposition 2.3.3, we must have $-s+j < -1/2$.

The restriction operator is the transpose of (2.3.2). Hence, it is a surjective map. ∎

Corollary 2.3.5 *Assume Ω is a smooth domain with boundary $\partial\Omega$ and let V be a transversal vector field with respect to $\partial\Omega$. Then for any s such that $s-j > 1/2$, the map*

$$H^s(\Omega) \longrightarrow H^{s-j-\frac{1}{2}}(\Omega)$$
$$u \longrightarrow V^j u \mid_{\partial\Omega}$$

is continuous. ∎

We finish this section discussing an extension of corollary 2.3.5.

When $s - j \leq 1/2$ the restriction map discussed above is not continuous, but we can recover from that in the following way. Let P be a differential operator of order $m \geq 0$ which is elliptic on $\overline{\Omega}$, i.e., if

$$P(x, D_x) = \sum_{|\alpha| \leq m} p_\alpha(x) D_x^\alpha \ ,$$

then

$$p(x, \xi) = \sum_{|\alpha|=m} p_\alpha(x) \xi^\alpha \neq 0 \ , \ \ \forall\, \xi \in \mathbf{R}^n - 0, \ x \in \overline{\Omega} \ .$$

If V is a vector field as in corollary 2.3.5, then

$$\begin{array}{c} H^s(\Omega) \longrightarrow \oplus_{j=0}^m H^{s-j+\frac{1}{2}}(\Omega) \\ u \longrightarrow \oplus_{j=0}^{m-1} V^j u \mid_{\partial\Omega} \end{array} ,$$

is continuous if and only if $s - m > 1/2$. If $s - m \leq 1/2$, consider

$$H_P^s(\Omega) = \{u : \ u \in H^s(\Omega), \ Pu \in L^2(\Omega)\} \ ,$$

and provide it with the graph norm.

Proposition 2.3.6 *Assume Ω is a smooth domain with boundary $\partial\Omega$ and let V be a transversal vector field with respect to $\partial\Omega$. Then, if P is an elliptic differential operator of order $m \geq 0$, the map*

$$u \longrightarrow \oplus_{j=0}^m V^j u \mid_{\partial\Omega}$$

as a map on smooth functions, extends to a continuous map

$$H_P^s(\Omega) \longrightarrow \oplus_{j=0}^m H^{s-j+\frac{1}{2}}(\Omega)$$

for any value of s. ∎

Note that for $s-m > 1/2$, corollary 2.3.5 is stronger than proposition 2.3.6. The novelty comes in the range $s-m \leq 1/2$. We postpone its proof until further machinery is developed, but before closing up this section, we discuss a formula which will be useful in its proof.

Let (x, y) be a coordinate system near $\partial\Omega$, with x a defining function for $\partial\Omega$ and $V = (-i)\partial_x$. The operator P can be written as

$$P = \sum_{j=0}^m p_j(x, y, D_y) D_x^j \ ,$$

where $p_j(x, y, D_y)$ is a tangential differential operator of order $m - j$. By ellipticity of P, $\partial\Omega$ is non-characteristic and therefore, $p_m \neq 0$. If $u \in (\overline{\Omega})$, let u_c be the extension by zero outside $\overline{\Omega}$. Then,

$$P(u_c) - (Pu)_c = \sum_{k=0}^{m-1} \left(\sum_{j=0}^{m-k-1} p_{j+k+1}(0, y, D_y) D_x^j u \mid_{x=0} \right) \otimes D_x^k \delta(x) \ . \qquad (2.3.7)$$

Observe that the coefficient of $D_x^{m-1}\delta(x)$ in the right hand-side of (2.3.7) is precisely the term $p_m(x, y, D_y)u \mid_{x=0}$, an elliptic differential operator of order zero.

Chapter 3

Conormal distributions of type ρ, δ

Pseudo-differential operators on an arbitrary manifold can be characterized in terms of their Schwartz kernels. These are distributions on the product manifold which have nice singularities: they remains stable, that is in the same Sobolev space, when acted on by any number of vector fields tangent to the diagonal. A distribution with that property is called conormal with respect to the diagonal and thus, a linear operator is psedo-differential iff its Schwartz kernel is a conormal distribution with respect to the diagonal. In this chapter we prepare the ground for the analysis and discussion of psedo-differential operators by defining the notion of a distribution conormal to a given closed embedded submanifold of the ambient space.

3.1 Submanifolds and vector fields

Let X be a paracompact manifold of dimension n, and S a submanifold of X of codimension $n-m$, i.e., a subset of X with regular inclusion map of rank m. Although in most of the cases we shall consider S to be closed and embedded into X, many of the results below apply to a larger class of submanifolds.

We begin recalling a standard fact. Given $p \in S$ there exists a coordinate system $\varphi = \{x_1, \ldots, x_n\}$ valid on an open neighborhood U of p in X such that $\varphi(p) = (0, \ldots, 0)$ and such that the set

$$U' = \{q \in U : \ x_j(q) = 0, \ m+1 \leq j \leq n\}$$

together with the restrictions of $(x_1, \ldots, x_m)$ to U' form a local chart on S containing p. In other words, locally a submanifold is not any different of the case of $\mathbf{R}^m \times 0 \subset \mathbf{Ru}^n$. The coordinate system above shall be referred to as the preferred coordinate system adapted to S at p. For convenience, sometimes we shall write x' and x'' for $(x_1, \ldots, x_m)$ and $(x_{m+1}, \ldots, x_n)$, respectively.

As usual, we denote by $C^\infty(X; TX)$ the space of smooth sections of the tangent bundle to X. We shall primarily be concerned with the following subsets of $C^\infty(X; TX)$:

$$\begin{aligned} \mathcal{V}(S) &= \{V \in C^\infty(X; TX) : \ V \text{ is tangent to } S \text{ at each } s \in S\} , \\ \mathcal{V}_0(S) &= \{V \in C^\infty(X; TX) : \ V \text{ vanishes at each } s \in S\} . \end{aligned} \tag{3.1.1}$$

Any function $f \in C^\infty(X)$ defines an endomorphism of $\mathcal{V}(S)$ and $\mathcal{V}_0(S)$ by $V \longrightarrow f \cdot V$. Thus, both $\mathcal{V}(S)$ and $\mathcal{V}_0(S)$ are left $C^\infty(X)$-modules.

Definition 3.1.2 *A (X)-module $\mathcal{M}$ is said to be locally finitely generated if each $x \in X$ has an open neighborhood U such that for some $V_1, \ldots, V_k$ in $\mathcal{M}$,*

$$C_c^\infty(U) \cdot \mathcal{M} = \sum_{j=1}^{k} C_c^\infty(U) \cdot V_j \ .$$

Proposition 3.1.3 *Both, and , are locally finitely generated (X)-modules and Lie algebras.*

Proof. It is clear that and are Lie algebras. We just need to prove they are locally finitely generated.

Let $p \in S$ and consider (U, φ), a preferred coordinate system adapted to S at p. Then $\varphi_*\{C_c^\infty(U) \cdot\} = C_c^\infty(\varphi(U)) \cdot (\mathbf{R}^n; T\mathbf{R}^n)$. Consequently, because of the invariance of the conditions defining and , it will be enough to assume that $S = \mathbf{R}^m \times \{0\}$, $X = \mathbf{R}^n$. Then,

$$V = \sum_{j=1}^{n} a_j(x)\partial_{x_j} \text{ is tangent to } S \text{ at each } s \in S \Longleftrightarrow a_j(x', 0) = 0 \ , \ m+1 \leq j \leq n \ .$$

By Taylor's expansion we conclude that $a_j(x) = \sum_{k=m+1}^{n} x_k a_{kj}(x)$. Hence, any vector field in can be written as $\sum_{j=1}^{m} a_j(x)\partial_{x_j} + \sum_{k,j=m+1}^{n} a_{kj}(x)x_k\partial_{x_j}$, showing that

$$= (U) - \text{span of } \{\partial_{x_j}, x_k\partial_{x_l}, 1 \leq j \leq m, m+1 \leq k, l \leq n\} \tag{3.1.4}$$

which certainly is enough to prove the proposition for .

Similarly, we can show that

$$= (U) - \text{span of } \{x_k \partial_{x_l}, 1 \leq l \leq m, m+1 \leq k \leq n\} \ . \tag{3.1.5}$$

which proves the desired result for . ∎

Before going any further, let us define the notion of a differential operator between sections of vector bundles over a manifold X (for details see [GS]). Consider two vector bundles E and F over X and let P be a vector bundle map from E to F, i.e., a section of the bundle $\mathrm{Hom}\,(E,F)$. Therefore, given a smooth section u of E, Pu will be a smooth section of F, inducing in this way a map $P : (E) \longrightarrow (F)$, where (E) (resp. (F)) stands for the space of smooth sections of E (resp. F). Note that for any smooth function f, the multiplication map $u \longrightarrow f \cdot u$ commutes with P, $P(f \cdot u) = f \cdot Pu$. Conversely, any map of sections P which commutes with multiplication by functions defines a section of $\mathrm{Hom}\,(E,F)$. Indeed, if u_1 and u_2 are two sections of E that agree at x_0, then locally $u_1 - u_2 = \sum f_i(x) v_i(x)$ where $\{v_i\}$ trivializes the bundle, and $f_i(x_0) = 0$. Then $P(u_1 - u_2) \mid_{x=x_0} = \sum f_i(x) P v_i \mid_{x=x_0} = 0$. Hence, the value of Pu at x_0 only depends on the value of u at x_0 and therefore, P defines an element of $\mathrm{Hom}\,(E,F)$. A linear operator $P : (E) \longrightarrow (F)$ which commutes with multiplication by functions is, by definition, a differential operator of order zero. The space of all such operators will be denoted by $Diff^0(X;E,F)$. Proceeding by induction, we define a differential operator of order k to be a linear map $P : (E) \longrightarrow (F)$ such that $[P,f] = P \cdot f - f \cdot P \in Diff^{k-1}(X;E,F)$. For trivial bundles of rank one, we shall simply write $Diff^k(X)$ rather than $Diff^k(X;E,F)$. Note that the composition of differential operators $P \in Diff^i(X;E,F)$ and $Q \in Diff^j(X;F,G)$, QP, defines an element of $Diff^{i+j}(X;E,G)$. The union over k, $\cup_k Diff^k(X;E,F)$ is the set of all differential operators and will be denoted by $Diff(X;E,F)$. When $E = F$, it is provided with a ring structure given by composition.

Using the definition show that if $(x_1, \ldots, x_n)$ are local coordinates near p, then $P \in Diff^k(X;E,F)$ can be expressed as

$$P = \sum_{|\alpha| \leq k} A_\alpha(x) D_x^\alpha \ ,$$

where $A_\alpha \in (\mathrm{Hom}\,(E,F))$. Conversely, any linear map P which locally can be written in this form is an element of $Diff^k(X;E,F)$.

In the ring of differential operators on X, $Diff(X)$, consider the subrings and locally generated by and 1 and and 1, respectively, as (X)-modules. That is to say, these subrings consist of finite sums of the type $\varphi(x)V_1 \dots V_k$, where $\varphi(x) \in (X)$ and $V_1, \dots, V_k$ are in and , respectively. Note that is the ring of totally characteristic differential operators at S.

Proposition 3.1.6 *In a preferred coordinate system adapted to S at p, the differential operators $D^{\alpha}_{x'} D^{\beta}_{x''} (x'')^{\gamma}$, $\mid \gamma \mid = \mid \beta \mid$, form a local basis for over (X). Similarly, $D^{\alpha}_{x'} D^{\beta}_{x''} (x'')^{\gamma}$, $\mid \gamma \mid = \mid \alpha \mid + \mid \beta \mid$, form a local basis for over (X).*

Proof. By induction over the order of $P \in$. Observe that

$$= \cup_k \mathcal{M}^k(S) \ ,$$

where $\mathcal{M}^k(S) = \cap Diff^k(X)$. By (3.1.4) the result is true for $k = 1$. Assume it for $\mathcal{M}^k(S)$ and consider $P \in \mathcal{M}^{k+1}(S)$. Then we must show that

$$P = \sum_{|\alpha|+|\beta|\leq k+1, |\alpha|=|\gamma|} a_{\alpha,\beta,\gamma}(x) D^{\alpha}_{x'} D^{\beta}_{x''} (x'')^{\gamma} \ .$$

It will suffice to assume that $P = V_1 \dots V_{k+1}$, with each V_j in . But then, by the induction hypothesis and (3.1.4), we have

$$V_1 \dots V_{k+1} = \Big(\sum_{|\alpha|+|\beta|\leq k, |\alpha|=|\gamma|} a_{\alpha,\beta,\gamma}(x) D^{\alpha}_{x'} D^{\beta}_{x''} (x'')^{\gamma} \Big) \Big(\sum_{j=1}^{m} a_j(x) D_{x_j} + \sum_{k,j=m+1}^{n} a_{kj}(x) x_k D_{x_j} \Big) \ ,$$

for some functions $a_{\alpha,\beta,\gamma}$, a_j, a_{kj}. An easy commutation argument will complete the proof.

Similarly, $= \cup_k \mathcal{M}^k_0(S)$, where $\mathcal{M}^k_0(S) = \cap Diff^k(X)$. The same proof will apply if instead of (3.1.4) we now use (3.1.5). ∎

Neither nor are subbundles of TX. In fact, their fiber dimension is not even locally constant on open sets intersecting S. However, when restricted to S, they are subbundles of $T_S X$, with $|_S$ the zero section of $T_S X$. Away from S, there is no restriction over either or . As a matter of fact, for any vector field $V \in$, given $p \in X - S$, there exists a neighborhood U of p, disjoint of S, such that $V \mid_U$ can be considered as the restriction to U of a vector field in or . If

$$V_p = \{ V \in T_p X : \text{ such that } \exists\, W \in \text{ with } W(p) = V \} \ ,$$
$$V^0_p = \{ V \in T_p X : \text{ such that } \exists\, W \in \text{ with } W(p) = V \} \ ,$$

then

Proposition 3.1.7 *The manifold S can be recovered by*

$$S = \{p \in X : \; V_p \neq T_pX\} = \{p \in X : \; V_p^0 \neq T_pX\} \; .$$

Proof. Using (3.1.4) or (3.1.5) one can easily see that for $p \in S$, V_p and V_p^0 are not equal to T_pX. Thus, S is contained in the sets on the right hand side of the expression above. The remark preceding the proposition shows that for $p \notin S$, V_p and V_p^0 equal T_pX. The result follows. ∎

In general $\subset$ and the inclusion is strict unless the codimension of S equals the dimension of X. For more information on these and related notions, as well as applications, the reader should see [MM], [MR1], [MR2], [Me], or [Si].

(see [MR1]). In any smooth manifold X, by a -variety we mean a collection $\mathcal{L} = \{S_1 \ldots S_N\}$ of embedded submanifolds of X which are pairwise disjoint, have closed union and are such that the space $\mathcal{V}(\mathcal{L})$ of smooth vector fields on X, tangent to each of the S_i, is everywhere locally finitely generated as a (X)-module. Given a -variety $\mathcal{L}$, is $\mathcal{V}(\mathcal{L})$ a lie algebra ?

(see [MR1]). In $\mathbf{R}^3$ with coordinates (t, x, y) consider the submanifolds $S_1 = \{x+t = 0\}$, $S_2 = \{t^2 = x^2 + y^2 \; ; \; t \neq 0\}$. Show that:

1. $\mathcal{V}(S_2)$ is spanned by $t\partial_t + x\partial_x + y\partial_y$, $x\partial_t + t\partial_x$, $y\partial_t + t\partial_y$, $y\partial_x - x\partial_y$.

2. $\mathcal{V}(S_1, S_2)$ is spanned by $t\partial_t + x\partial_t + y\partial_y$, $x\partial_x + t\partial_x$, $y(\partial_t - \partial_x) + (x+t)\partial_y$.

(see [MR2]). Consider two embedded plain curves, γ_1 and γ_2, tangent at the origin but only simply so. Find local coordinates (x, y) near the origin such that

$$\gamma_1 = \{y = 0\}, \; \gamma_2 = \{y = x^2\}.$$

Consider the -variety $\mathcal{K}_1 = \{\gamma_1 - 0, 0\}$ $\mathcal{K}_2 = \{\gamma_2 - 0, 0\}$. Show that $\mathcal{V}(\mathcal{K}_1)$ is generated by $y\partial_y$, $y\partial_x$ $x\partial_x$. Similarly, $\mathcal{V}(\mathcal{K}_2)$ is generated by $(y - x^2)\partial_y, (y - x^2)\partial_x, x(\partial_x + 2x\partial_y)$.

3.2 Conormal distributions of type ρ, δ

In this section, we want to define a set of distributions with singularities nicely placed along S. To motivate the subject, consider the Dirac delta function $\delta(x)|dx|$ in $D'(\mathbf{R}; \Omega\mathbf{R})$. It is a distribution whose only singularity is located at $S = \{x = 0\}$, a closed embedded submanifold of $\mathbf{R}$. But a lot more is true about it. Observe that $\delta(x) \in H^{-1/2-\varepsilon}(\mathbf{R})$ for any $\varepsilon > 0$, and as a distribution, $(xD_x)^k \delta(x) = i^k \delta(x)$ for any k. In other words, the Sobolev regularity of $\delta(x)$ is stable under the action of any element in .

The example above motivates the following generalization. Let ρ and δ be real numbers such that $0 \le \delta \le 1$, $0 < \rho \le 1$, and $1 - \rho \le \delta$. We consider distributions $u \in D^{'}(X)$ for which there exists s such that $u \in H^s_{loc}(X)$ and such that its regularity decreases by δ and $1 - \rho$ when acted on by V in and , respectively.

Definition 3.2.1 *The space $I_{\rho,\delta}(X, S)$ of conormal distributions to S of type ρ, δ, consists of the set of distributions u for which, given an open set U with compact closure, there exists a real number $s = s(u, U)$ such that if $\varphi \in_c (X)$ and* $\operatorname{supp} \varphi \subset U$,

$$V_1 \ldots V_k W_1 \ldots W_l \varphi u \in H^{s-l(1-\rho)-k\delta}(X)\ , V_i \in, W_j \in\ , \tag{3.2.2}$$

for any k, l.

Note that $\subset$. Therefore the restriction $1 - \rho \le \delta$ is natural. Otherwise we conclude that $V_1 \ldots V_k W_1 \ldots W_l \varphi u \in H^{s-(k+l)\delta}(X)$ when we view $W_1, \ldots, W_l$ as elements in . If $1 - \rho > \delta$, thiswould be contained in $H^{s-l(1-\rho)-k\delta}(X)$. If $\rho = 0$ then $\delta = 1$ and we know that for any family of vector fields $V_1, \ldots, V_r$ and for any $u \in D^{'}(X)$, there exists s such that $V_1 \ldots V_r \varphi u \in H^{s-r}(X)$. Thus $\rho > 0$ is also natural.

The definition above may be rephrased stating that

$$W_1 \ldots W_l V_1 \ldots V_k \varphi u \in H^{s-l(1-\rho)-k\delta}(X)\ , V_i \in, W_j \in\ .$$

Just observe that $VW = WV + [V, W]$ and that $[,] \subset$ and $[,] \subset$.

Let $u \in I_{\rho,\delta}(X, S)$ and suppose for simplicity that u is compactly supported with support contained on the coordinate pacth of a preferred coordinate system $(x^{'}, x^{''}) = (x_1, \ldots, x_n)$ adapted to S at p. Note that $\partial_{x_1}, \ldots, \partial_{x_m}$ are not elements of . Then the condition defining $I_{\rho,\delta}(X, S)$ says that for some fixed s,

$$\begin{array}{c} x_k \partial_{x_l} u \in H^{s-(1-\rho)}(X)\ ,\ 1 \le l \le n,\ m+1 \le k \le n \\ \partial_{x_j} u \in H^{s-\delta}(X)\ ,\ 1 \le j \le m \end{array} .$$

Iterating we can see that

$$\begin{matrix} (x'')^{\alpha} D_x^{\beta} u \in H^{s-|\alpha|(1-\rho)}(X) \ , \ \forall\, \alpha, \, \beta \text{ such that } \mid \alpha \mid = \mid \beta \mid \\ D_{x'}^{\gamma} u \in H^{s-|\gamma|\delta}(X) \ , \ \forall\, \gamma \end{matrix} \quad . \tag{3.2.3}$$

This coordinatized version is frequently useful in dealing with distributions in $I_{\rho,\delta}(X, S)$.

Example 3.2.4 Consider the Dirac delta function δ_{x^0} centered at $x^0 \in \mathbf{R}^n$. Then, if $x^0 = (x_1^0, \ldots, x_n^0)$, $((x_j - x_j^0) D_{x_i})^k \delta_{x^0} \in H^{-\frac{n}{2}-\varepsilon}(\mathbf{R}^n)$ for any $\varepsilon > 0$. Thus, $\delta_{x^0} \in I_{1,0}(\mathbf{R}^n, \{x = 0\})$.

Example 3.2.5 Consider any smooth function $\varphi \in (\mathbf{R}^{n-1})$ and let $u = \varphi(x') \otimes \delta(x_n)$. Then, $u \in I_{1,0}(\mathbf{R}^n, \{x_n = 0\})$. Moreover, if $u \in I_{1,0}(\mathbf{R}^n, \{x_n = 0\})$, then the distribution $\bar{u}(.) = \langle u(., x_n), \psi(x_n) \rangle$ is smooth. Consequently, any element of $I_{1,0}(\mathbf{R}^n, \{x_n = 0\})$ supported on $\{x_n = 0\}$ is of the form

$$\sum \varphi_j(x') \otimes D_{x_n}^j \delta(x_n) \ ,$$

where $\varphi \in (\mathbf{R}^{n-1})$ (see proposition 2.3.3).

Show that for every z, x_+^z and x_-^z are distributions in $I_{1,0}(\mathbf{R}, \{x = 0\})$.

Let f be an element of $I_{\rho,\delta}(\mathbf{R}^n, S)$ with compact support. Here S is a closed embedded submanifold of $\mathbf{R}^n$. Consider the heat kernel $g_t(x) = \dfrac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\|x\|^2}{4t}}$. What type of distribution is the convolution $g_t * f$?

One can point out some generalizations of the notions here defined. Given a Lie algebra of vector fields $\mathcal{L} \subset$ which is a locally finitely generated (X)-module and involutive, i.e., $[\mathcal{L}, \mathcal{L}] \subset \mathcal{L}$, we define

$$I(X, \mathcal{L}) = \{u \in D'(X) : \ V_1 \ldots V_k u \in H_{loc}^s(X) \text{ for some } s, \ \forall\, k, \ V_j \in \mathcal{L}\} \ .$$

Note that $I(X,)$ corresponds to $I_{\rho,\delta}(X, S)$ when $\rho = 1$ and $\delta = 0$. This generalization although slightly artificial at this stage, is highly useful in the treatment of certain type of partial differential equations. The space $I(X,)$ has turn out to be useful in the study of mixed boundary value problems [Si] or in the study of Hodge cohomology of negatively curved manifolds [Ma], and the space $I(\mathbf{R}^n, \mathcal{L})$, where $\mathcal{L}$ is the Lie algebra of vector fields tangent to the cone $C = \{x : \ x_n^2 = \sum_{j=1}^{n-1} x_j^2 \ , \ x_n \neq 0\}$, is useful in the study of the wave equation [MR1]. Observe that in this case, C is not a closed submanifold of $\mathbf{R}^n$.

Lemma 3.2.6 *For any open set $U \subset X$, the restriction*

$$u \longrightarrow u\mid_U$$

defines a map from $I_{\rho,\delta}(X,S)$ into $I_{\rho,\delta}(X\cap U, S\cap U)$.

Proof. Let A be an open subset of U with compact closure and consider $\varphi \in C_c^\infty(U)$ such that $\varphi \equiv 1$ on a neighborhood of A. Let $\{V_1,\dots,V_k\}$ and $\{W_1,\dots,W_l\}$ be arbitrary families in $\mathcal{V}(S\cap U)$ and $\mathcal{V}_0(S\cap U)$, respectively. Extended by zero outside the support of φ, the families $\{\varphi V_1,\dots,\varphi V_k\}$ and $\{\varphi W_1,\dots,\varphi W_l\}$ can be considered as families in and . Then

$$V_1\dots V_k W_1\dots W_k u\mid_A = \varphi V_1\dots\varphi V_k \varphi W_1\dots\varphi W_l u\mid_A$$

which belongs to $H^{s-l(1-\rho)-k\delta}(A)$ for some $s=s(u,A)$. The result follows. ∎

Corollary 3.2.7 *If $u \in I_{\rho,\delta}(X,S)$, then sing supp $U \subset S$.*

Proof. Let $U = X - S$. Then U is open in X. By lemma 3.2.8,

$$u\mid_U \in I_{\rho,\delta}(X\cap U, S\cap U) = I_{\rho,\delta}(X-S,\emptyset)\ .$$

By proposition 3.1.8, given an arbitrary number of vector fields $V_1,\dots,V_k$ in $(U;TU)$, $V_1\dots V_k u\mid_A \in H_{loc}^{s-k(1-\rho)}(A)$ for some fixed s. Here, A is an open set as the one used in the proof of the lemma. Therefore, $u \in H_{loc}^{s+k\rho}(A)$ for any k. The result follows from the Sobolev embedding theorem. ∎

Corollary 3.2.9 gives a description of $I_{\rho,\delta}(X,S)$ away from the submanifold S. Near S the situation is not so simple. We can think of elements in $I_{\rho,\delta}(X,S)$ as smooth functions in S with values in the space of distributions on a transversal submanifold. We finish this section proving this assertion.

Working locally we may assume that $X=\mathbf{R}^n$ with $S=\mathbf{R}^m\times\{0\}\subset\mathbf{R}^n$. Let us quickly look at the case $\rho=1$ and $\delta=0$. If u is an element in $I_{1,0}(\mathbf{R}^n,S)$ with compact support, its order is finite and therefore, there exists $s\in\mathbf{R}$ such that

$$D_{x'}^\alpha D_{x''}^\beta (x'')^\gamma u \in H^s(\mathbf{R}^n)\ \forall\, \alpha,\beta,\gamma,\ \mid\beta\mid=\mid\gamma\mid\ \ .$$

If $s \geq 0$, we have

$$D^{\alpha}_{x'} D^{\beta}_{x''} (x'')^{\gamma} u \in L^2(\mathbf{R}^n) \; .$$

On the other hand, if $s < 0$, we can show that for any $k > -s$,

$$D^{\alpha}_{x'} D^{\beta}_{x''} (x'')^{\gamma} u \in (1+\Delta_{x'})^k (1+\Delta_{x''})^k L^2(\mathbf{R}^n) \; .$$

Here, $\Delta_{x'} = \sum_{j=1}^{m} D^2_{x_j}$, $\Delta_{x''} = \sum_{j=m+1}^{n} D^2_{x_j}$. We can therefore conclude that $u(x', .)$ is a distribution with Sobolev regularity s, and since there is no restriction on α, continuous as a function of x'. The same is true of any number of derivatives in the x' directions. Thus, $u \in C^{\infty}(\mathbf{R}^m_{x'}; H^s(\mathbf{R}^{n-m}_{x''}))$. With some effort and modifications, the same can be proven for $u \in I_{\rho,\delta}(\mathbf{R}^n, S)$.

Let u be any element of $D'(\mathbf{R}^{n-1})$ and consider the distribution $v = u(x') \otimes \delta(x_n)$. Show that $v \in I_{\rho,\delta}(\mathbf{R}^n, \{x_n = 0\})$ if, and only if, $u \in (\mathbf{R}^{n-1})$. Conclude then that $v \in I_{1,0}(\mathbf{R}^n, \{x_n = 0\})$.

Proposition 3.2.8 *Let $u \in I_{\rho,\delta}(\mathbf{R}^n, \{x = (x', x'') : \; x'' = 0\}$ be a distribution with compact support. Then $u \in C^{\infty}(\mathbf{R}^m_{x'}; D'_c(\mathbf{R}^{n-m}_{x''}))$.*

Proof. Since u has finite order, there exists $s \in \mathbf{R}$ such that

$$D^{\alpha}_{x'} D^{\beta}_{x} (x'')^{\gamma} u \in H^{s-|\alpha|\delta-|\beta|(1-\rho)}(\mathbf{R}^n) \; \forall \; \alpha, \beta, \gamma, \; |\beta| = |\gamma| \; . \tag{3.2.9}$$

For simplicity, call $u_{\beta,\gamma}$ the distribution $D^{\beta}_{x}(x'')^{\gamma} u$. Given a test function φ in $D(\mathbf{R}^{n-m}_{x''})$ and for fixed β, γ with $|\beta| = |\gamma|$, consider the distribution

$$v(x') = u_{\beta,\gamma}(\varphi) \; .$$

It is easy to see that

$$\hat{v}(\eta') = \frac{1}{(2\pi)^m} \int \hat{u}_{\beta,\gamma}(\eta) \check{\varphi}(\eta'') d\eta'' \; ,$$

where $\check{\varphi}(\eta'') = \hat{\varphi}(-\eta'')$.

Consider (3.2.12) with $|\alpha| = 0$. If $s' = s - |\beta| (1-\rho) \geq 0$, then

$$\begin{aligned}
\|v\|_{s'} &= \int (1 + \|\eta'\|^2)^{s'} \; |\hat{v}(\eta')|^2 \, d\eta' \\
&\leq C \int \left((1 + \|\eta'\|^2)^{s'} \int |\hat{u}_{\beta,\gamma}(\eta)|^2 \, d\eta'' d\eta' \right) \int |\check{\varphi}(\eta'')|^2 \, d\eta'' \\
&\leq C \int (1 + \|\eta\|^2)^{s'} \; |\hat{u}_{\beta,\gamma}(\eta)|^2 \, d\eta \\
&\leq C \|u_{\beta,\gamma}\|_{s'} \; .
\end{aligned}$$

On the other hand, if $s^{'} < 0$, then for some k large enough, $(1 + \|\eta^{''}\|^2)^{-k}\hat{u}_{\beta,\gamma}(\eta)$ in in L^2 as a function of $\eta^{''}$. Consequently,

$$\begin{aligned}
\|v\|_{s'} &= \int (1 + \|\eta^{'}\|^2)^{s^{'}} \mid \hat{v}(\eta^{'}) \mid^2 d\eta^{'} \\
&\leq C\int (1 + \|\eta^{'}\|^2)^{s^{'}} (1 + \|\eta^{''}\|^2)^{-2k} \mid \hat{u}_{\beta,\gamma}(\eta) \mid^2 d\eta \int (1 + \|\eta^{''}\|^2)^{2k} \mid \check{\varphi}(\eta^{''}) \mid^2 d\eta^{''} \\
&\leq C\int (1 + \|\eta^{'}\|^2)^{s^{'}} (1 + \|\eta^{''}\|^2)^{-2k} \mid \hat{u}_{\beta,\gamma}(\eta) \mid^2 d\eta \\
&\leq \sup \frac{(1 + \|\eta^{'}\|^2)^{s^{'}} (1 + \|\eta^{''}\|^2)^{-2k}}{(1 + \|\eta\|^2)^{s^{'}}} \int (1 + \|\eta\|^2)^{s^{'}} \mid \hat{u}_{\beta,\gamma}(\eta) \mid^2 d\eta \\
&\leq C\|u_{\beta,\gamma}\|_{s'} \ .
\end{aligned}$$

Therefore, $v \in H_c^{s^{'}}(\mathbf{R}^m)$. If we work with $D^{\alpha}_{x'}v$ instead of v, we conclude that $D^{\alpha}_{x'}v \in H_c^{s^{'}-|\alpha|\delta}(\mathbf{R}^m)$. By regularity theorems for differential operators, this implies that $v \in H_c^{s^{'}+|\alpha|(1-\delta)}(\mathbf{R}^m)$ for any α. But $1 - \delta \geq \rho > 0$. Hence, $v \in C^\infty(\mathbf{R}^m)$. This finishes the proof. ∎

In contrast with the case of elements in $I_{1,0}(\mathbf{R}^n, \{x;\ x^{''} = 0\})$, we cannot conclude that $u \in C^\infty(\mathbf{R}^m_{x'}; H^s_c(\mathbf{R}^{n-m}_{x''}))$ for fixed s. The reason is that differentiation in the $x^{'}$ directions decreases the regularity in (3.2.12) by δ.

Show that $H_{\mathbf{R}^{n-1}}(\mathbf{R}^n) \subset I(\mathbf{R}^n, \mathcal{V}_0(\mathbf{R}^{n-1}))$ for all s (compare with exercise 3.2.10).

Show that the Schwartz kernel of any differential operator P is a distribution in $I_{1,0}(X \times X, \Delta; \Pi_2^*\Omega X)$, where Δ is the diagonal in $X \times X$ and Π_2 is the projection $X \times X \to X$ onto the second factor (see exercise 1.3.12).

3.3 Sobolev order filtration

Let us recall that the space of distributions with compact support, $D_c^{'}(X)$, is equal to $\cup_s H^s_c(X)$, where $H^s_c(X)$ is the space of compactly supported distributions in X with Sobolev order s. Therefore, if

$$I_{\rho,\delta,c}(X,S) = I_{\rho,\delta}(X,S) \cap D_c^{'}(X) \ , \tag{3.3.1}$$

we define $I_{\rho,\delta,c,s}(X,S)$ by

$$\begin{aligned} I_{\rho,\delta,c,s}(X,S) = \{u \in I_{\rho,\delta,c}(X,S) :\ & QPu \in H^{-s-l(1-\rho)-k\delta}(X), \\ & P \in \mathcal{M}^k(S),\ Q \in \mathcal{M}_0^l(S),\ k,l \text{ arbitrary}\} \ . \end{aligned} \tag{3.3.2}$$

Lemma 3.3.3 *$I_{\rho,\delta,c,s}(X,S)$ is a filtration of $I_{\rho,\delta,c}(X,S)$.*

Proof. Clearly $I_{\rho,\delta,c}(X,S) = \cup_s I_{\rho,\delta,c,s}(X,S)$. We need to prove that $I_{\rho,\delta,c,s}(X,S)$ is a $C^\infty(X)$-submodule of $I_{\rho,\delta,c}(X,S)$, which is fairly clear, and the remark $I_{\rho,\delta,c,s}(X,S) \subset I_{\rho,\delta,c,s'}(X,S)$ if $s \le s'$ will complete the proof. ∎

More important are the following three propositions.

Proposition 3.3.4 *If $T = V_1 \dots V_m$ with $V_i \in -$, then*

$$T \,:\, I_{\rho,\delta,c,s}(X,S) \longrightarrow I_{\rho,\delta,c,s+m\delta}(X,S)\ .$$

If $T = V_1 \dots V_m$ with $V_i \in$, then

$$T \,:\, I_{\rho,\delta,c,s}(X,S) \longrightarrow I_{\rho,\delta,c,s+m(1-\rho)}(X,S)\ .$$

Proof. Suppose $u \in I_{\rho,\delta,c,s}(X,S)$. Since sing supp $u \subset S$, the statement $Tu \in I_{\rho,\delta,c,s+m\delta}(X,S)$ is local near S. Hence, working on a preferred coordinate system adapted to S at p, the result (in fact, both results) will follow from (3.2.3). ∎

Proposition 3.3.5 *Let T be a differential operator of order j. Then*

$$P : I_{\rho,\delta,c,s}(X,S) \longrightarrow I_{\rho,\delta,c,s+j}(X,S)\ .$$

Proof. Let us recall that a differential operator of order j maps a distribution in $H^s_c(X)$ into a distribution in $H^{s-j}_c(X)$. Since we need to show that the condition

$$QPu \in H_c^{-s-l(1-\rho)-k\delta}(X)\ \forall\, P \in \mathcal{M}^k(S),\ Q \in \mathcal{M}^l_0(S)$$

implies that

$$QP(Tu) \in H_c^{-s-j-l(1-\rho)-k\delta}\ \forall\, P \in \mathcal{M}^k(S),\ Q \in \mathcal{M}^l_0(S),$$

the result will follow from repeated applications of the fact that for all $V \in \mathcal{M}^k(S)$ (resp. $V \in \mathcal{M}^l_0(S)$)

$$VT = TV + [V,T],$$

and $[V,T]$ is a differential operator of the same order as the order of T.

Proposition 3.3.6 *If $\varphi \in C^\infty(X)$ vanishes to order j on S, then*

$$\varphi : I_{\rho,\delta,c,s}(X,S) \longrightarrow I_{\rho,\delta,c,s-j\rho}(X,S)\ , \tag{3.3.7}$$

where the map (3.3.7) *is defined by multiplication.*

Proof. Working in a preferred coordinate system $(x^{'}, x^{''})$ adapted to S at p, we conclude that

$$\varphi = \sum_{\|\gamma\|=j} \varphi_\gamma(x)(x^{''})^\gamma$$

for some $\varphi_\gamma \in C^\infty(X)$. It suffices then to show the result for $\varphi = (x^{''})^\gamma$, $\mid \gamma \mid = j$, and indeed, it suffices to do so for $j = 1$. Thus, we need to show that

$$W_1 \dots W_l\ V_1 \dots V_k\ (x^{''})^\gamma u\ \in\ H^{-s+j\rho-l(1-\rho)-k\delta}\ V_i \in V(S)\ W_j \in V_0(S) \forall k, l.$$

We can commute through to show that

$$D^\alpha_{x''} W_1 \dots W_l\ V_1 \dots V_k\ (x^{''})^\gamma u\ \in\ H^{-s-(l+1)(1-\rho)-k\delta}\ \forall\ \mid \alpha \mid \le \mid \gamma \mid = 1$$

and

$$D^\alpha_{x'} W_1 \dots W_l\ V_1 \dots V_k\ (x^{''})^\gamma u\ \in\ H^{-s-l(1-\rho)-(k+1)\delta}\ .$$

Hence, since $1 - \delta \le \rho$, this leads to the desired conclusion. ∎

Chapter 4

Symbol isomorphism

In this chapter we prove the symbol isomorphism theorem for conormal distributions of type ρ, δ. To avoid notational and technical complications which do not sheed extra light into this result, we advise the reader to assume that $\rho = 1$ and $\delta = 0$, reducing the work to the consideration of $I(X,)$. To state a simplified version of the result to be proved, we assume momentarily that X is compact and we let and stand for the normal and conormal bundle of S. If $n = \dim \; = \dim X$ and $d = \operatorname{codim} O_S =$ fiber dimension of $\; = \operatorname{codim} S$, then the composition of a normal fibration and the invariant Fourier transform defines an isomorphism

$$\sigma \; : \; \frac{I^m(X,S)}{I^{m-1}(X,S)} \longrightarrow \frac{S^{m-\frac{d}{2}+\frac{n}{4}}() \otimes \Omega_{\text{fiber}}()}{S^{m-1-\frac{d}{2}+\frac{n}{4}}() \otimes \Omega_{\text{fiber}}()} \; .$$

The filtration $I^m(X,S)$ on the left is equivalent to the Sobolev order filtration introduced before, and the normalization is chosen so that pseudo-differential operators of order m correspond to symbols of the same order, as we shall see later.

4.1 Linearization

Let S and X be as before, with $\dim X = n$ and $\operatorname{codim} S = d$. We do not assume here that X is compact. Given $p \in S$, the tangent space T_pS sits inside T_pX and the normal space to S at p is obtained by taking the quotient

$$= \frac{T_pX}{T_pS} \; .$$

We get a set by setting $= \{(p,v) : v \in\}$. The triple $(, S, \Pi)$ where Π is the projection map $\Pi : \longrightarrow S$, produces a vector bundle of rank d. Indeed, if $(x^{'}, x^{''})$ is a coordinate system adapted to p in U, i.e., $\tilde{U} = S \cap U = \{p \in U : x^{''}(p) = (x_{n-d+1}(p), \ldots, x_n(p)) = 0\}$ and the restriction of $x^{'}$ to $\tilde{U}$ defines coordinates on S, then the vector fields $\partial_{x_{n-d+1}}, \ldots, \partial_{x_n}$ restricted to $p \in \tilde{U}$ give normal vectors on and moreover, they form a basis of . Thus $\cong R^d$ and $|_{\Pi^{-1}\tilde{U}} \cong \tilde{U} \times R^d$. This shows simultaneously how to provide with a differentiable structure as well as the local triviality condition of the bundle. We shall call $(, S, \Pi)$ the normal bundle of S, and we shall often denote it by . Its dual bundle, , will be called the conormal bundle to S. In both cases, S sits inside the bundle as its zero section O_S. Under the duality pairing between T^*X and TX, correspond to the annihilator of $TS \subset T_S X$, justifying the terminology employed.

Theorem 4.1.1 *Let $S \subset X$ be an embedded submanifold with normal bundle . Then there is a neighborhood G of S in X and a smooth diffeomorphism*

$$f : G \longrightarrow$$

onto a neighborhood $G^{'}$ of the zero section $O_S \cong S$ in , which restricts to S as the natural identification with O_S and such that the map $f_* : TG \longrightarrow TG^{'}$ induces the identity isomorphism

$$f_* : \longrightarrow NO_S \, .$$

Any two such maps are homotopic (in a sufficiently small neighborhood of S) through a smooth curve of maps of the same type.

Proof. Give X a riemannian structure and fix f by using geodesic flow, in TX, starting at . The property of f_* is proven once we observe that $NO_S = \dfrac{T_{O_S}}{TO_S} \cong$. Conversely, given f, choose a riemannian structure consistent with f being geodesic flow near S. The homotopy can be constructed from a homotopy of riemann metrics. ∎

Maps as in the theorem above will be referred to as normal fibration or tubular neighborhood maps.

Under the following assumptions, prove theorem 4.1.1 as follows:

1. If S is compact and contained in R^n, show there is an $\varepsilon > 0$ such that if $q \in U_\varepsilon = \{q \in R^n : \text{ dist } (q, S) < \varepsilon\}$, there exist a unique $p \in S$ minimizing the distance from q to S, and that the map

$$\begin{aligned} \Pi \, : \, U_\varepsilon &\longrightarrow S \\ q &\longrightarrow p(q) \end{aligned}$$

is smooth (this will prove the theorem when $X = R^n$).

2. Assume that X is also compact. Using Whitney embedding theorem, deduce the normal fibration theorem from i.

It follows directly from the definition that if $f \, : \, X \longrightarrow X^{'}$is a diffeomorphism of manifolds and $S^{'} = f(S)$, then

$$f^* \, : \, I_{\rho,\delta,c,s}(X^{'}, S^{'}) \longrightarrow I_{\rho,\delta,c,s}(X, S) \; .$$

Via a normal fibration, we lift distributions in $I_{\rho,\delta,s,c}(X, S)$ to distributions in $I_{\rho,\delta,s,c}(, O_S)$. Exploiting the vector space structure of the fibers in , we associate with each u in $I_{\rho,\delta,s,c}(X, S)$ a symbol in . Of course, we have to see up to what extend the symbol so obtained depends on the choice of the fibration.

Hence, consider a fibration $f \; : \; G \longrightarrow f(G) \subset$ as in theorem 4.1.1. Given $\varphi \in (X)$ supported inside G and such that $\varphi \equiv 1$ near S, define the map

$$\begin{aligned} I_{\rho,\delta}(X, S) &\xrightarrow{L_f} I_{\rho,\delta}(, O_S) \\ u &\longrightarrow (f^{-1})^* \varphi u \end{aligned} \quad . \tag{4.1.2}$$

By corollary 3.2.9, the map above is an isomorphism independent of φ up to smooth errors. Indeed, if $\varphi^{'}$ is another function like φ, then $(f^{-1})^* \varphi u - (f^{-1})^* \varphi^{'} u = (f^{-1})^* (\varphi - \varphi^{'}) u$ which is smooth since supp$\,(\varphi - \varphi^{'})$ does not intersect S. Moreover, since $u = f^*((f^{-1})^* u) - (1 - \varphi) u$, the map (4.1.3) is itself an isomorphism up to smooth errors. We need to see how this isomorphism depends on the choice of f.

Theorem 4.1.3 *Let f and g be normal fibrations of X at S. Then, for $u \in I_{\rho,\delta,c,s}(X, S)$,*

$$L_f u - L_g u \in I_{\rho,\delta,c,s-(\rho-\delta)}(, O_S) \; .$$

Before proving theorem 4.1.4 let us make the following remarks. Since both, f and g are normal fibrations of X at S, the map $h = g \circ f^{-1}$ (wherever defined) has the following properties:

$$\begin{aligned} &1.\ h(s) = s \text{ for all } s \in O_S. \\ &2.\ h_*(V) = V + T_S(O_S) \text{ for all } V \in T_S(\Pi^{-1}(s)). \end{aligned} \tag{4.1.4}$$

Note that the identity map has properties (4.1.5), and because any two normal fibrations are homotopic through normal fibration maps, there exists a one parameter family h_t of local diffeomorphism with properties (4.1.5) such that $h_0 = i_d$ and $h_1 = h = g \circ f^{-1}$.

Lemma 4.1.5 *Let h_t be a one parameter family of local diffeomorphism with properties* 1 *and* 2 *above. Then, for any $\varphi \in ()$,*

$$\frac{d}{dt} h_t^* \varphi = h_t^*(V_t \varphi) ,$$

where V_t is a t-dependent vector field in which locally can be expressed as a finite sum $\sum \mu_j V_j$ with μ_j a smooth function vanishing at O_S, $\mu_j \mid_{O_S} = 0$, and $V_j \in \mathcal{V}(O_S)$.

Proof. In a local trivialization of the normal bundle , consider coordinates $(y, w) = (y_1, \ldots, y_{n-d}, w_1, \ldots, w_d)$ where y is a coordinate in S and w is a linear coordinate on the fiber of . We can write

$$h_t(y, w) = (Y_t(y, w), W_t(y, w)),$$

with $Y_t = (Y_t^1, \ldots, Y_t^{n-d})$ and $W_t = (W_t^1, \ldots, W_t^d)$. Therefore

$$\frac{d}{dt} h_t^* \varphi(y, w) = \frac{d}{dt} \varphi(Y_t(y, w), W_t(y, w)) = \left(\left(\frac{dY_t}{dt} \cdot \partial_y + \frac{dW_t}{dt} \cdot \partial_w \right) \varphi \right) (Y_t, W_t) . \tag{4.1.6}$$

But $Y_t(y, 0) = y$. Hence,

$$Y_t(y, w) = y + w \cdot A_t(y, w)$$

for certain smooth matrix valued function A_t locally defined. Thus,

$$\frac{d}{dt} Y_t(y, w) = w \cdot \frac{d}{dt} A_t(y, w) , \tag{4.1.7}$$

showing that $\frac{d}{dt} Y_t$ vanishes at $w = 0$. On the other hand, using the second property we obtain

$$(h_t)_*(\partial_{w_i}) = \frac{\partial Y_t^j}{\partial w_i} \frac{\partial}{\partial y_j} + \frac{\partial W_t^j}{\partial w_i} \frac{\partial}{\partial w_j} ,$$

where $\dfrac{\partial W_t^j}{\partial w_i}(y,0) = \delta_{ij}$. In the expression above, the summation convention has been used. Since $W_t(y,0) = 0$, expansion in Taylor series will show that $W_t(y,0) = w + o_t(\|w\|^2)$, and consequently

$$\frac{\partial W_t^j}{\partial t} = w_j f_{jk}(y,w) w_k \ , \tag{4.1.8}$$

for some functions f_{jk}. Inserting (4.1.8) and (4.1.9) in (4.1.7), we obtain the desired result. ∎

Remark 4.1.9 *From* (4.1.7) *we see that the vector fields V_t of the lemma could be in $\mathcal{V}_0(O_S)$ or in $\mathcal{V}(O_S) - \mathcal{V}_0(O_S)$.*

Proof of theorem 4.1.4. Since modulo smooth errors the pullback map

$$g^* \ : \ I_{\rho,\delta,c,s}(,O_S) \longrightarrow I_{\rho,\delta,c,s}(X,S)$$

is an isomorphism, it will be enough to prove that for $u' \in I_{\rho,\delta,c,s}(,O_S)$, with support sufficiently close to O_S, we have

$$(g\circ f^{-1})^* u' - u' \in I_{\rho,\delta,c,s-(\rho-\delta)}(,O_S) \ . \tag{4.1.10}$$

Indeed, if $(f^{-1})^*\varphi u - (g^{-1})^*\varphi u \in I_{\rho,\delta,c,s-(\rho-\delta)}(,O_S)$, modulo smooth errors we can write φu as $g^* u'$ for some u' out of $I_{\rho,\delta,c,s}(,O_S)$. Then

$$(g\circ f^{-1})^* u' - u' = (f^{-1})^* g^* u' - (g^{-1})^* g^* u' \in I_{\rho,\delta,c,s-(\rho-\delta)}(,O_S) \ .$$

Conversely, if condition (4.1.11) holds for all $u' \in I_{\rho,\delta,c,s}(,O_S)$ setting $u = g^*u'$ we conclude that $(f^{-1})^*u - (g^{-1})^*u \in I_{\rho,\delta,c,s-(\rho-\delta)}(,O_S)$ for all $u \in I_{\rho,\delta,c,s}(X,S)$.

In order to prove (4.1.11), let γ_t be a homotopy of normal fibrations connecting f and g, i.e., $\gamma_0 = f$, $\gamma_1 = g$, and γ_t is a normal fibration for all $t \in [0,1]$. Hence $h_t = \gamma_t \circ f^{-1}$ is a curve with properties (4.1.5) above, connecting the identity map and $g \circ f^{-1}$. Consequently, given an arbitrary distribution u in $I_{\rho,\delta,c,s}(,O_S)$ and using lemma 4.1.6, we have:

$$(g\circ f^{-1})^* u' - u' = \int_0^1 \frac{d}{dt}\left(h_t^* u'\right) dt = \int_0^1 h_t^*\left(V_t u'\right) dt \ ,$$

where of course, we have extended the lemma in the obvious way so that it can be applied to distributions instead of smooth functions. By the local form of V_t (see (4.1.7) and remark

4.1.10) and from propositions 3.3.4 and 3.3.6, we conclude that $V_t u'$ is in $I_{\rho,\delta,c,s-\rho+\delta}(, O_S)$. The result follows. ∎

Corollary 4.1.11 *The map L_f restricted to $I_{\rho,\delta,s,c}(X,S)$ induces an isomorphism*

$$\frac{I_{\rho,\delta,c,s}(X,S)}{{}_c(X)} \longrightarrow \frac{I_{\rho,\delta,c,s}(,O_S)}{{}_c()} \ , \tag{4.1.12}$$

independent of φ, and the induced map of quotient spaces

$$\frac{I_{\rho,\delta,c,s}(X,S)}{I_{\rho,\delta,c,s-(\rho-\delta)}(X,S)} \xrightarrow{L_s} \frac{I_{\rho,\delta,c,s}(,O_S)}{I_{\rho,\delta,c,s-(\rho-\delta)}(,O_S)} \tag{4.1.13}$$

is an isomorphism independent of the normal fibration f. ∎

The identification (4.1.14) is the key tool in the proof of the symbol isomorphism theorem. It indicates the price one pays when reducing the analysis of conormal distributions to S in X of type ρ, δ, to conormal distributions to O_S in of the same type: the distribution $L_f u$ corresponding to $u \in I_{\rho,\delta,c,s}(X,S)$ is independent of f up to distributions of the same type with Sobolev regularity $\rho-\delta$ units better. Note that if $\rho = \delta$ the map L_s is the obvious isomorphism between spaces consisting of just one element.

Let P be a smooth differential operator on X of order m.

i. Consider m functions $f_i \in (X)$ vanishing on S. Show that the function

$$S \longrightarrow \mathrm{C}$$
$$p \longrightarrow P(f_1.f_2 \ldots f_m)(p)$$

only depends on df_i, $i = 1, \ldots, m$. Thus, it defines a symmetric multilinear map Q by $Q(df_1, \ldots, df_m)(p) = P(f_1 \ldots f_m)(p)$.

ii. Let ϕ be a smooth function on and consider $T_m\phi$, the part of the Taylor series at $(p,0) \in O_S$ homogeneous of degree m in the fibers. Define

$$P_S(\phi)(p,0) = Q(T_m\phi)(p) \ .$$

Note that P_S involves only differentiation along the fibers. Extend it to all of by demanding translation invariance on the fibers. The resulting operator, P_S, is a differential operator on .

Consider a preferred coordinate system $(x^{'}, x^{''})$ adapted to S at p, with $x^{'} = (x_1, \ldots, x_{n-d})$ and $x^{''} = (x_{n-d+1}, \ldots, x_n)$ as before. Then if

$$P = \sum_{|\alpha^{'}|+|\alpha^{''}|\leq m} p_{(\alpha^{'},\alpha^{''})}(x^{'}, x^{''}) D_{x^{'}}^{\alpha^{'}} D_{x^{''}}^{\alpha^{''}} ,$$

show that

$$P_S = \sum_{|\alpha^{''}|=m} p_{(0,\alpha^{''})}(x^{'}, 0) D_{z^{''}}^{\alpha^{''}} ,$$

where $(x^{'}, z^{''})$ are the set of coordinates in induced by $(x^{'}, x^{''})$.

By proposition 3.3.5 if $u \in I_{\rho,\delta,c,s}(X, S)$ and P is a differential operator of order m, then $Pu \in I_{\rho,\delta,c,s+m}(X, S)$. Show that

$$L_{s+m}(Pu) = P_S L_s u ,$$

where L_s is the map (4.1.14).

4.2 Symbols of type ρ, δ

Although the notions we introduce in this section shall be only used when working with , they can be defined more generally on vector bundles over X. Here we consider these general versions.

Let $E \xrightarrow{\Pi} X$ be a vector bundle over X of rank r with zero section O_E. The non-zero real numbers act on E by multiplication along the fibers. Thus, for each $t \in R - 0$ we obtain an isomorphism $m_t : E \longrightarrow E$ such that $\Pi \circ m_t = \Pi$. A subset $\Gamma \subset$ is said to be conic if, and only if, given any point $\gamma \in \Gamma$, $m_t \gamma \in \Gamma$ for all $t > 0$.

Definition 4.2.1 *A vector field $L \in (E, TE)$ is said to be homogeneous of degree zero along the fibers if and only if $(m_t)_* L = L$.*

The set of all homogeneous vector fields of degree zero along the fibers will be denoted by $\mathcal{V}_{\mathcal{H}}(E)$.

Consider an open set O of the base manifold X above which the bundle E is trivial. Take coordinates (x, ξ) trivializing the bundle over $\Pi^{-1}O$,with x coordinate on O and ξ a linear coordinate in R^r. If

$$L_{(x,\xi)} = \sum_{j=1}^{n} a_j(x,\xi)\partial_{x_j} + \sum_{j=1}^{r} b_j(x,\xi)\partial_{\xi_j} \ ,$$

then

$$((m_t)_* L)_{(x,\xi)} = \sum_{j=1}^{n} a_j(x,\xi)\partial_{x_j} + \sum_{j=1}^{r} \frac{1}{t} b_j(x,\xi)\partial_{\xi_j} \ .$$

Thus, if $(m_t)_* L = L$ then $a_j(x,\xi) = a_j(x,0)$, $j = 1,\dots,n$, and $b_j(x,t\xi) = tb_j(x,\xi)$, $j = 1,\dots,r$. Observe that the set $\mathcal{V}_{\mathcal{H}}(E)$ has two distinguished subsets, namely $\mathcal{V}^0_{\mathcal{H}}(E) = \{L \in \mathcal{V}_{\mathcal{H}}(E) : \ L\mid_{O_E} = 0\}$ and $\mathcal{V}^1_{\mathcal{H}}(E) = \mathcal{V}_{\mathcal{H}}(E) - \mathcal{V}^0_{\mathcal{H}}(E)$.

For the case of an open set Ω in $\mathbf{R}^n$, a symbol of order m is a function $a \in (T^*\mathbf{R}^n)$ such that on any compact subset K of Ω,

$$\mid D^\alpha_x D^\beta_\xi a(x,\xi) \mid \leq C_{K,\alpha,\beta}(1 + \|\xi\|)^{m-|\beta|} \ ,$$

for some constant $C_{K,\alpha,\beta}$, and for all α, β. This can be rephrased by saying that

$$\mid D^\alpha_x \xi^\gamma D^\beta_\xi a(x,\xi) \mid \leq C_{K,\alpha,\beta}(1 + \|\xi\|)^m \text{ for all } \alpha,\beta,\gamma, \ \mid \beta \mid = \mid \gamma \mid \ .$$

The operator D^α_x can be expressed as composition of elements in $\mathcal{V}^1_{\mathcal{H}}(T^*\mathbf{R}^n)$, while $\xi^\gamma D^\beta_\xi$ is a composition of elements in $\mathcal{V}^0_{\mathcal{H}}(T^*\mathbf{R}^n)$ when $\mid \beta \mid = \mid \gamma \mid$. Precisely speaking, a symbol of order m and type $(1,0)$ is a function in $(T^*\mathbf{R}^n)$ for which the estimates

$$\mid V_1 \dots V_k W_1 \dots W_l a(x,\xi) \mid \leq C(1 + \|\xi\|)^m \ , \ V_i \in \mathcal{V}^1_{\mathcal{H}}(T^*\mathbf{R}^n), \ W_j \in \mathcal{V}^0_{\mathcal{H}}(T^*\mathbf{R}^n)$$

holds on compact sets, for any k, l.

Let $\| \ \|$ be an euclidean metric on the fibers of E.

Definition 4.2.2 *Let m, ρ, δ be real numbers with $0 < \rho \leq 1$, $0 \leq \delta \leq 1$. We denote by $S^m_{\rho,\delta}(E)$ the set of all smooth functions $a \in (E)$, such that for every compact subset K of the base manifold X and for all k, l, the estimate*

$$\sup_{x \in K} \mid V_1 \dots V_k W_1 \dots W_l a \mid_{\Pi^{-1}(K)} \leq C_{K,k,l}(1 + \|\xi\|)^{m + l(1-\rho) + k\delta} \ , \ V_i \in \mathcal{V}^1_{\mathcal{H}}(E), \ W_j \in \mathcal{V}^0_{\mathcal{H}}(E) \ , \tag{4.2.3}$$

for some constant $C_{K,k,l}$. The elements of $S^m_{\rho,\delta}(E)$ shall be called symbols of order m and type ρ, δ. Given an open conic set Γ, $S^m_{\rho,\delta}(\Gamma)$ is the space of smooth functions on Γ which satisfies symbol estimates along directions in Γ.

Proposition 4.2.4 *Consider an open set O on which E is trivial an let (x,ξ) be coordinates trivializing E. Then, if $a \in S^m_{\rho,\delta}(E)$, and K is a compact subset of O, we have*

$$| D^\alpha_x D^\beta_\xi a(x,\xi) | \leq C_{K,\alpha,\beta}(1+\|\xi\|)^{m-|\beta|\rho+|\alpha|\delta} \,, x \in K \,. \tag{4.2.5}$$

Conversely, any function a in (E) satisfying these estimates above every set on which E is trivial, belongs to $S^m_{\rho,\delta}(E)$.

Proof. Let $C^0_{fiber}(E)$ be the set of smooth functions on E which are constant along the fibers. We have seen that locally, in coordinates (x,ξ) trivializing the bundle E,

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(E) &= C^0_{fiber}(E) - \text{span}\{\partial_{x_i}, \xi^j\partial_{\xi_k},\ 1 \leq j \leq n,\ 1 \leq j,k \leq r\} \\ \mathcal{V}^1_{\mathcal{H}}(E) &= C^0_{fiber}(E) - \text{span}\{\partial_{x_i},\ 1 \leq j \leq n\} \\ \mathcal{V}^0_{\mathcal{H}}(E) &= C^0_{fiber}(E) - \text{span}\{\xi^j\partial_{\xi_k},\ 1 \leq j,k \leq r\} \end{aligned} \quad . \tag{4.2.6}$$

The homogeneous differential operator D^α_x is the composition of $| \alpha |$ elements in $\mathcal{V}^1_{\mathcal{H}}(E)$. On the other hand, for any β, γ with $| \beta |=| \gamma |$, $\xi^\gamma D^\beta_\xi$ is a linear combination of the form $\sum_{j=0}^{|\beta|} P_j$, where P_j is the composition of j-elements in $\mathcal{V}^0_{\mathcal{H}}(E)$. Hence, if $a \in S^m_{\rho,\delta}(E)$ and β and γ are as above, we have

$$| D^\alpha_x \xi^\gamma D^\beta_\xi a(x,\xi) | \leq C_{K,\alpha,\beta,\gamma}(1+\|\xi\|)^{m+|\beta|(1-\rho)+|\alpha|\delta} \ \ x \in K \,,$$

where K is any compact subset of open neighborhood of the base manifold above which the bundle is trivial. The result on this type of sets follows dividing by $\|\xi\|^{|\gamma|}$ away from $\xi = 0$. For general compact sets K, we cover it by a finite number of open sets above which the bundle is trivial. The result will follow applying several times the special case already proved.

For the converse, we localize the problem using a partition of unity subordinated to a locally finite covering of X by open sets on which E is trivial. Then using (4.2.6) one easily obtains (4.2.3) over compact subsets sitting inside the open sets of the covering. The result follows. ∎

The significance of proposition 4.2.4 is that it allows us to check for symbol estimates locally in order to conclude that a smooth function is a symbol.

Using (4.2.5) we can topologize $S^m_{\rho,\delta}(E)$ as follows: firstly, let $S^m_{K,\rho,\delta}(E)$ be the space of symbols of type ρ, δ and order m with support contained in $\Pi^{-1}(K)$, K a compact subset of X. Then

$$p_{K,j,m}(a) = \sup_{(x,\xi)\in\Pi^{-1}(K),|\alpha|+|\beta|\leq j} \frac{| D^\alpha_x D^\beta_\xi a(x,\xi) |}{(1+\|\xi\|)^{m-|\beta|\rho+|\alpha|\delta}} \tag{4.2.7}$$

is a family of seminorms on $S^m_{K,\rho,\delta}(E)$, providing it with a Frechét space structure. A set $S \subset S^m_{K,\rho,\delta}(E)$ is declared to be open if, and only if, $S \cap S^m_{K,\rho,\delta}(E)$ is open in $S^m_{K,\rho,\delta}(E)$ for all $K \subset\subset X$. Note that on bounded sets, the topologies of $S^m_{\rho,\delta}(E)$ and (E) coincide. Using this topology, the reader can easily prove the following

Proposition 4.2.8 *For the spaces $S^m_{\rho,\delta}(E)$ we have:*

1. *$S^m_{\rho,\delta}(E) \subset S^{m'}_{\rho,\delta}(E)$ if $m' \geq m$, and the inclusion is continuous.*

2. *Let $V \in \mathcal{V}^1_{\mathcal{H}}(E)$. Then V induces a continuous map*

$$\begin{aligned} S^m_{\rho,\delta}(E) &\longrightarrow S^{m+\delta}_{\rho,\delta}(E) \\ a &\longrightarrow Va \end{aligned} .$$

Similarly, if $V \in \mathcal{V}^0_{\mathcal{H}}(E)$, it induces a map as above but with range in $S^{m-(\rho-1)}_{\rho,\delta}(E)$.

3. *The multiplication map*

$$\begin{aligned} S^m_{\rho,\delta}(E) \otimes S^{m'}_{\rho,\delta}(E) &\longrightarrow S^{m+m'}_{\rho,\delta}(E) \\ (a,b) &\longrightarrow a.b \end{aligned}$$

is continuous. ∎

We set

$$S^\infty_{\rho,\delta}(E) = \cup_m S^m_{\rho,\delta}(E), \quad S^{-\infty}_{\rho,\delta}(E) = \cap_m S^m_{\rho,\delta}(E) . \tag{4.2.9}$$

Remark 4.2.10 *The symbols of order $-\infty$ are independent of ρ, δ, i.e., for any pairs ρ, δ and ρ', δ' as in definition 4.2.2, $S^{-\infty}_{\rho,\delta}(E) = S^{-\infty}_{\rho',\delta'}(E)$. Therefore, we shall simply write $S^{-\infty}(E)$.*

Let $a \in S^1_{1,0}(T^*\mathbf{R}^n)$ such that $a(x,\xi) \geq 0$ for all $(x,\xi) \in T^*\mathbf{R}^n$. Show that $e^{-a(x,\xi)}$ is a symbol in $S^0_{\frac{1}{2},\frac{1}{2}}(T^*\mathbf{R}^n)$.

The following proposition shall be useful later on, when dealing with the construction of certain parametrices for pseudo-differential operators.

Proposition 4.2.11 *(Asymptotic summation) Let $a_j \in S^{m_j}_{\rho,\delta}(E)$, $j = 0, 1, 2, \ldots$, and assume that $m_j \to -\infty$. Set $m'_k = \max_{j\geq k} m_j$. Then there exists $a \in S^{m'_0}_{\rho,\delta}(E)$ such that for every k*

$$a - \sum_{j<k} a_j \in S^{m'_k}_{\rho,\delta}(E)\ . \tag{4.2.12}$$

The symbol a is uniquely determined modulo $S^{-\infty}(E)$.

Proof. Using proposition 4.2.4 we conclude that for existence it will be enough to assume that $E = O \times \mathbf{R}^r$, where O is some open set in $\mathbf{R}^n$.

Take a smooth function φ in $\mathbf{R}^r$ such that $\varphi \equiv 0$ for $\|\xi\| \leq 1/2$ and $\varphi \equiv 1$ for $\|\xi\| \geq 1$. If $\{\varepsilon_j\}$ is a decreasing sequence of positive real numbers with zero limit, then

$$a(x,\xi) = \sum_{j=0}^{\infty} \varphi(\varepsilon_j\xi) a_j(x,\xi)$$

converges as a smooth function because on compact subsets of $O \times \mathbf{R}^r$, there are only a finite number of non-zero terms. We want to choose the ε_j's such that

$$\sum_{j=k}^{\infty} \varphi(\varepsilon_j\xi) a_j(x,\xi) \tag{4.2.13}$$

converges in $S^{m'_k}_{\rho,\delta}(O \times \mathbf{R}^r)$. Certainly, this will prove the desired statement. Indeed, we would have

$$a(x,\xi) - \sum_{j<k}^{\infty} \varphi(\varepsilon_j\xi) a_j(x,\xi) = \sum_{j=k}^{\infty} \varphi(\varepsilon_j\xi) a_j(x,\xi) + \sum_{j<k} (1 - \varphi(\varepsilon_j\xi)) a_j(x,\xi) \in S^{m'_k}_{\rho,\delta}(O \times \mathbf{R}^r)\ ,$$

since the last term in the expression above is a rapidly decreasing symbol.

Firstly, let us prove that for any m, m' with $m' < m$, given an arbitrary compact subset K of O and $\mu \in \mathbf{R}_+$, there exists $\varepsilon > 0$ such that if $c \in S^m_{\rho,\delta}(O \times \mathbf{R}^r)$ and $c_\varepsilon(x,\xi) = \varphi(\varepsilon\xi) c(x,\xi)$, then

$$p_{K,j,m}(c_\varepsilon) \leq \mu p_{K,j,m'}(c)\ . \tag{4.2.14}$$

Consider multi-indices α and β with $\mid \alpha \mid + \mid \beta \mid \leq j$. Then,

$$\frac{\mid D_x^\alpha D_\xi^\beta \varphi(\varepsilon\xi) c(x,\xi) \mid}{(1+\|\xi\|)^{m-|\beta|\rho+|\alpha|\delta}} \leq \sum_{\gamma\leq\beta} \left(\begin{array}{c} \beta \\ \gamma \end{array} \right) \varepsilon^{|\gamma|} \frac{\mid D_\xi^\gamma \varphi(\varepsilon\xi) D_x^\alpha D_\xi^{\beta-\gamma} c(x,\xi) \mid}{(1+\|\xi\|)^{m-|\beta|\rho+|\alpha|\delta}} \tag{4.2.15}$$

Let c be a generic constant. If $\mid \gamma \mid < m - m^{'}$, the terms in the sum of the right hand-side of (4.2.16) are bounded by 0 on $\|\xi\| \leq 1/2\varepsilon$, and by

$$C\varepsilon^{|\gamma|} \frac{\mid D_x^\alpha D_\xi^{\beta-\gamma} c(x,\xi) \mid}{(1+\|\xi\|)^{m^{'}-|\beta|\rho+|\alpha|\delta}(1+\|\xi\|)^{m-m^{'}-|\gamma|}} \ , \tag{4.2.16}$$

in $\|\xi\| > 1/2\varepsilon$. Here C is a bound for the derivatives of φ up to order $\mid \beta \mid$. Hence the nontrivial contributions in this case are bounded by

$$C\varepsilon^{m-m^{'}}(1+2\varepsilon)^{-(m-m^{'}-|\gamma|)} \frac{\mid D_x^\alpha D_\xi^{\beta-\gamma} c(x,\xi) \mid}{(1+\|\xi\|)^{m^{'}-|\beta|\rho+|\alpha|\delta}} \ .$$

If $\mid \gamma \mid \geq m - m^{'}$, we still obtain trivial contributions when $\|\xi\| \leq 1/2\varepsilon$, but now the contributions are also trivial when $\|\xi\| > 1/\varepsilon$. In between, i.e., for $1/2\varepsilon \leq \|\xi\| \leq 1/\varepsilon$, we have estimates (4.2.17). Since $m - m^{'} - \mid \gamma \mid$ is now less or equal than zero, we can bound these terms by

$$C\varepsilon^{|\gamma|}(1+\varepsilon)^{-(m-m^{'}-|\gamma|)} \frac{\mid D_x^\alpha D_\xi^{\beta-\gamma} c(x,\xi) \mid}{(1+\|\xi\|)^{m^{'}-|\beta|\rho+|\alpha|\delta}} \ .$$

In either case, (4.2.15) will follow after taking the supremum over all the multi-indices whose sum has weight bounded by j. Note that the ε obtained depends only on $\mu, j, m - m^{'}$ and K.

Using proposition 4.2.8 and the hypothesis, we can assume that the sequence of orders m_j is strictly decreasing. Then, $m_k^{'} = m_k$. With this assumption, we proceed to find the ε_j's by induction. Let $\{K_j\}$ be an exhaustive sequence of compact subsets of O. Take $\varepsilon_0 = 1$ and once $\varepsilon_1, \ldots, \varepsilon_{q-1}$are found, choose ε_q such that $\varepsilon_q < \varepsilon_{q-1}/2$, and

$$p_{K_j,j,m_{j-1}}(a_{\varepsilon_j}) < \frac{1}{2^j} \ .$$

Such a choice is possible. Indeed, since $a_j \in S_{\rho,\delta}^{m_j}(E)$ and $m_j < m_{j-1}$, we can take μ in (4.2.15) to be $(2^j p_{K_j,j,m_j}(a_j))^{-1}$ if $a_j \neq 0$, or $(2^j)^{-1}$ otherwise. Once this choice is made, take ε such that (4.2.15) holds. Then we take ε_j to be the minimum of $\varepsilon_{j-1}/2$ and ε.

With this choice, (4.2.14) converges in $S^{m_k}_{\rho,\delta}(O \times \mathbf{R}^r)$. For let K be any compact subset of O, and l be a nonnegative integer. Take $d > l$ such that $K \subset K_d$. For integers i, n greater than d, with $i < n$,

$$p_{K,l,m_k}(\sum_{j=i}^{n} a_{\varepsilon_j}) \leq \sum_{j=i}^{n} p_{K_d,d,m_k}(a_{\varepsilon_j}) \ .$$

If $j > d$ we have $p_{K_d,d,m_k}(a_{\varepsilon_j}) \leq p_{K_j,j,m_{j-1}}(a_{\varepsilon_j})$, as long as $m_k \leq m_{j-1}$, which happens for any $k \geq j-1$. Hence, once m_k is fixed, for large i, n we have

$$\begin{aligned} p_{K,l,m_k}(\textstyle\sum_{j=i}^{n} a_{\varepsilon_j}) &\leq \textstyle\sum_{j=i}^{n} p_{K_d,d,m_k}(a_{\varepsilon_j}) \\ &\leq \textstyle\sum_{j=i}^{n} \dfrac{1}{2^j} \longrightarrow 0 \end{aligned}$$

as $i, n \to \infty$. Thus, (4.2.14) is a Cauchy sequence in $S^{m_k}_{\rho,\delta}(O \times \mathbf{R}^r)$ on sets of the form $\Pi^{-1}(K)$. The desired result follows.

The statement about uniqueness is clear. ∎

A symbol a with the properties of the symbol in the proposition above will be called the asymptotic sum of the symbols a_j, and we shall write

$$a \sim \sum a_j \ .$$

For reference purpose, we define separately a subset of the space of symbols. Let $\rho = 1$ and $\delta = 0$, and as usual, drop them from the notation. A function $a \in (E - O_E)$ is said to be homogeneous of degree m along the fibers if, and only if,

$$(m_t)^* a = t^m a \ .$$

Such a function clearly determines a symbol in $S^m(E)$ modulo symbols in $S^{-\infty}(E)$.

Definition 4.2.17 *By $S^m_{cl}(E)$ (or classical symbols) we denote the subspace of elements in $S^m(E)$ which consists of asymptotic sums*

$$\sum a_j \ ,$$

where a_j is homogeneous of degree $m - j$ on the complement of some neighborhood of O_E. When a is a symbol with values in some bundle G, it is classical if it has an expansion as above with a_j a homogeneous symbol of degree $m - j$ valued on G.

Another important subclass of symbols are those which are elliptic.

Definition 4.2.18 *A symbol $a \in S^m(E)$ is elliptic if, and only if, for every relatively compact subset K of X, there are constants C, C' such that $\mid a(x,\xi) \mid \geq C\|\xi\|^m$ for all $x \in K$, $\|\xi\| \geq C'$. When a is a symbol in $S^m(E)$ with values in some bundle of homomorphism, it is said to be elliptic if over $\Pi^{-1}(K)$ the homomorphism $a(x,\xi)$ is invertible for large values of ξ.*

The key feature of elliptic symbols is that they can be inverted modulo $S^{-\infty}(E)$. This will turn out to be relevant in the study of elliptic pseudo-differential operators, and we postpone any extra discussion until then.

4.3 Equivalent filtration and symbol isomorphism

Starting with an element u of $I_{\rho,\delta}(X,S)$, we would like to study the distribution $\mathcal{F} \circ L_f u$. Here $\mathcal{F}$ is the Fourier transform of chapter 1, and L_f is the map defined in (4.1.3). Note that $L_f u$ has compact support along the fibers on and it is therefore, a tempered distribution along the fibers. Hence, $\mathcal{F}(L_f u)$ makes sense if we think of $\mathcal{F}$ as the map dual to that in (1.5.8).

Modulo smooth errors, adding a rapidly decreasing function to a distribution does not modify the distribution. It will be rather convenient to modify $I_{\rho,\delta}(,O_S)$ and consider the space

$$IS_{\rho,\delta}(,O_S) = I_{\rho,\delta}(,O_S) + S() \ ,$$

where $S()$ is the space of rapidly decreasing functions along the fibers. Here we do not require compactness of the base manifold S. The Sobolev order filtration introduced in chapter 3 carries over,

$$IS_{\rho,\delta,c,s}(,O_S) = I_{\rho,\delta,c,s}(,O_S) + S_c() \ , \tag{4.3.1}$$

and proposition 3.3.4, 3.3.5 and 3.3.6 hold at the level of $IS_{\rho,\delta,c,s}(,O_S)$.

Theorem 4.3.2 *(full symbol isomorphism) The Fourier transform induces an isomorphism*

$$\frac{IS_{\rho,\delta}(,O_S;\Omega_{fiber})}{()} \longrightarrow \frac{S^{\infty}_{\rho,\delta}()}{S^{-\infty}()} \ . \tag{4.3.3}$$

Proof. Via corollary 3.2.9, we see that it is always possible to choose a representer $u \in I_{\rho,\delta}(, O_S; \Omega_{fiber})$ of the class $[u]$ with compact support along the fibers. Using a partition of unity, and by linearity of the Fourier transform, we can also assume that the support of u is contained inside $\Pi^{-1}(O)$, where O is an open set of S above which, is trivial.

Choose coordinates $\zeta = (y, z) \in O \times \mathbf{R}^d$ that trivialize $|_O$. The condition defining u as a compactly supported element of $I_{\rho,\delta}(, O_S; \Omega_{fiber})$ says that for some fixed s,we have

$$\begin{array}{c} z^\alpha D_\zeta^\beta u \in H^{s-|\alpha|(1-\rho)}(O \times \mathbf{R}^d; \Omega\mathbf{R}^d),\ \mid \alpha \mid = \mid \beta \mid \\ D_y^\gamma u \in H^{s-|\gamma|\delta}(O \times \mathbf{R}^d; \Omega\mathbf{R}^d),\ \forall\, \gamma \end{array} . \tag{4.3.4}$$

Over $\Pi^{-1}(O)$, u can be expressed as $u(y, z) \mid dz \mid$. Then

$$\hat{u}(y, \xi) = \mathcal{F}u(y, \xi) = u(e^{-i\langle \cdot, \xi \rangle}) . \tag{4.3.5}$$

By proposition 3.2.11, this is a smooth function of y; it is clearly a smooth function in ξ. By the support condition, we can conclude that $\hat{u} \in ()$. It remains to show that estimates of the type (4.2.5) hold.

Choose vector fields $V_1, \ldots, V_l$ in $\mathcal{V}^1_{\mathcal{H}}()$ and $W_1, \ldots, W_k$ in $\mathcal{V}^0_{\mathcal{H}}()$. The operator $V_1 \ldots V_l$ can be written in the form $\sum_{|\alpha| \le l} a_\alpha(y) D_y^\alpha$, while the operator $W_1 \ldots W_k$ can be expressed as $\sum_{|\beta| \le k, |\beta| = |\gamma|} b_{\beta,\gamma}(y) D_\xi^\beta \xi^\gamma$. Hence, it is enough to show the existence of $m \in R$ such that

$$\mid D_\xi^\beta \xi^\gamma D_y^\alpha \hat{u}(y, \xi) \mid \le C(1 + \|\xi\|)^{m-|\beta|(1-\rho)+|\alpha|\delta} \quad \mid \beta \mid = \mid \gamma \mid \ .$$

In the distributional sense, $D_\xi^\beta \xi^\gamma D_y^\alpha \hat{u}$ is the Fourier transform of $z^\beta(-D_z)^\gamma D_y^\alpha u$. Using proposition 3.3.4, we conclude that

$$z^\beta(-D_z)^\gamma D_y^\alpha u \in H^{s-|\beta|(1-\rho)-|\alpha|\delta}(O \times \mathbf{R}^d; \Omega\mathbf{R}^d),\ \mid \beta \mid = \mid \gamma \mid \ . \tag{4.3.6}$$

Consider this expression with α and β set to be the null multi-indices. Using the fact that $(1 + \Delta_z)^k$ has constant coefficients, we can show that $(1 + \|\xi\|^2)^k \hat{u}(y, \xi)$ is in L^1 as a function of ξ for any $k < s/2 - d/4$. Since $\hat{u}$ is smooth, the function $(1 + \|\xi\|^2)^k \hat{u}(y, \xi)$ must be bounded. We conclude that

$$\mid \hat{u} \mid \le C(1 + \|\xi\|)^{-2k} .$$

If we now use (4.3.6) for any multi-indices α, β, γ (with $\mid \beta \mid = \mid \gamma \mid$), replacing k by $k - \frac{|\beta|(1-\rho)-|\alpha|\delta}{2}$ in the argument above, we conclude that

$$\mid D_\xi^\beta \xi^\gamma D_y^\alpha \hat{u}(y,\xi) \mid \leq C(1+\|\xi\|)^{-2k-|\beta|(1-\rho)+|\alpha|\delta} , \tag{4.3.7}$$

leading to the desired conclusion.

Conversely, let $a(y,\xi)$ be a symbol of order m and type ρ, δ, with y-support compact in O. We need to show that $u(y,z) \mid dz \mid$ is a conormal distribution of type ρ, δ, where

$$u(y,z) = \left(\frac{1}{2\pi}\right)^d \int e^{i\langle z,\xi\rangle} a(y,\xi) d\xi . \tag{4.3.8}$$

If we choose k such that $k > \mid m \mid + d$, then $(1+\|\xi\|^2)^{-k} a(y,\xi)$ is such that

$$(1+\|\xi\|^2)^{-k} a(y,\xi) \leq C(1+\|\xi\|)^{m-2k} \leq C(1+\|\xi\|)^{m-2|m|-2d} ,$$

and therefore, it is square integrable in ξ. Since it is smooth in y we have:

$$u(y,z) = (1+\Delta)^k \left(\frac{1}{2\pi}\right)^d \int e^{i\langle z,\xi\rangle} \frac{1}{(1+\|\xi\|^2)^k} a(y,\xi) d\xi \ \in H^{-2k}(O \times R^d) .$$

Moreover, for any j, $\partial_{y_j} a(y,\xi)$ can be bounded by $(1+\|\xi\|)^{m+\delta}$. Then, replacing k by $k+\delta/2$ in the argument above, we conclude that $\partial_{y_j} u \in H^{-2k-\delta}(O \times \mathbf{R}^d)$. Similarly, using the fact that $\mid D_{\xi_k} \xi_j a(y,\xi) \mid \leq C(1+\|\xi\|)^{m+(1-\rho)}$, we conclude that $z_k D_{z_j} u \in H^{-2k-(1-\rho)}(O \times \mathbf{R}^d)$ for any j,k. Iterating, we obtain estimates 3.2.3 for $s = -2k$. The proof is now completed since $\mathcal{F}$ is an isomorphism. ∎

Observe that the order of the symbol $\mathcal{F}u$ over each set of the form $\Pi^{-1}(O)$, with O relatively compact in X, depends only on the Sobolev order of $u \mid_{\Pi^{-1}(O)}$, and the dimension and fiber dimension of .

To proceed any further at this point, there are two choices: the first is to determine exactly the order of $\mathcal{F}u$ given $u \in I_{\rho,\delta,s,c}(X,S)$ in terms of s, n and d. The second, easier in some sense, is to define a new order filtration in $I_{\rho,\delta}(X,S)$ in terms of the order of the symbols, and study how the change in the order of the symbol affects the order so defined. We follow this last choice.

Definition 4.3.9 *By $IS^m_{\rho,\delta}(E, O_E; \Omega^\alpha_{fiber}E) \subset IS_{\rho,\delta}(E, O_E; \Omega^\alpha_{fiber}E)$ we shall denote the subspace of distributions u such that*

$$\mathcal{F}u \in S^{m-\frac{d}{2}+\frac{n}{4}}_{\rho,\delta}(E^*; \Omega^{1-\alpha}_{fiber}E^*) ,$$

where d is the fiber dimension of the vector bundle E, and n is its total dimension.

For reference purposes, we write analogs to propositions 3.3.4, 3.3.5 and 3.3.6 in the specific case of $E =$. We leave to the reader the task of checking their validity in this new setting.

Proposition 4.3.10 *1. If $T = V_1 \dots V_m$ with $V_i \in \mathcal{V}(O_S) - \mathcal{V}_0(O_S)$, then*

$$T : IS^s_{\rho,\delta,c}(, O_S) \longrightarrow IS^{s+m\delta}_{\rho,\delta,c}(, O_S) \ .$$

If $T = V_1 \dots V_m$ with $V_i \in \mathcal{V}_0(O_S)$ then it induces a map as above with range in $IS^{s+m(1-\rho)}_{\rho,\delta,c}(, O_S)$.

2. If $P \in Diff^j()$, then

$$P : IS^s_{\rho,\delta,c}(, O_S) \longrightarrow IS^{s+j}_{\rho,\delta,c}(, O_S) \ .$$

3. If φ is a smooth function that vanishes to order j on O_S, then

$$\varphi : IS^s_{\rho,\delta,c}(, O_S) \longrightarrow IS^{s-j\rho}_{\rho,\delta,c}(, O_S) \ ,$$

where the map above is given by multiplication. ∎

We now prove that the filtrations (4.3.1) and (4.3.9) are equivalent.

Proposition 4.3.11 *There exists $M > 0$ such that*

$$IS_{\rho,\delta,c,s-M}(, O_S) \subset IS^s_{\rho,\delta,c}(, O_S) \subset IS_{\rho,\delta,c,s+M}(, O_S) \ .$$

Proof. The first inclusion follows from the proof of theorem 4.3.2 because the order of the symbol $\mathcal{F}u$ over relatively compact subsets of X depends only on n and d.

For the second, we work in local coordinates $\zeta = (y, z)$ over $O \times \mathbf{R}^d$as before, and assume that u has compact y-support. Then, in the distributional sense,

$$u(y, z) = \left(\frac{1}{2\pi}\right)^d \int e^{i\langle z,\xi\rangle} a(y, \xi) d\xi \ ,$$

for some $a(y, \xi) \mid d\xi \mid \in S^{m-\frac{d}{2}+\frac{n}{4}}_{\rho,\delta}() \otimes \Omega_{fiber}()$, which is compactly supported in y.

From the symbol estimates for a we obtain

$$(1+\|\xi\|)^{-m+\frac{d}{2}-\frac{n}{4}-\frac{d}{2}-\varepsilon}a(y,\xi)\in L^2()\ \forall\varepsilon>0\ ,$$

and therefore,

$$a(y,\xi)\in(1+\|\xi\|)^{\frac{d}{2}+\varepsilon+m-\frac{d}{2}+\frac{n}{4}}L^2_c()\ .$$

Hence,

$$u(y,z)\in H^{-m+\frac{d}{2}-\frac{n}{4}-\frac{d}{2}-\varepsilon}(O\times\mathbf{R}^d)\ .$$

From the identity

$$D_y^\alpha D_z^\beta z^\gamma u=\mathcal{F}^{-1}(D_y^\alpha\xi^\beta(-D_\xi)^\gamma a)\ ,$$

we conclude that for α,β,γ with $\mid\beta\mid=\mid\gamma\mid$, $D_y^\alpha D_z^\beta z^\gamma u$ corresponds with a density-valued symbol in $S_{\rho,\delta}^{m-\frac{d}{2}+\frac{n}{4}+|\beta|(1-\rho)+|\alpha|\delta}()\otimes\Omega_{fiber}$. Using the same argument we obtain that

$$D_y^\alpha D_z^\beta z^\gamma u\in H^{-m-(\frac{n}{4}+\varepsilon)-|\beta|(1-\rho)-|\alpha|\delta}(O\times\mathbf{R}^d)\ ,$$

which leads to the conclusion that

$$V_1\ldots V_l W_1\ldots W_k u\in H^{-m-(\frac{n}{4}+\varepsilon)-k(1-\rho)-l\delta}(O\times\mathbf{R}^d)$$

for all l,k, $V_i\in\mathcal{V}(O_S)$, $W_j\in\mathcal{V}_0(O_S)$. Thus, $u\in I_{\rho,\delta,c,-m-(\frac{n}{2}+\varepsilon)}(,O_S)$. ∎

Definition 4.3.12 *Let S be a closed embedded submanifold of a smooth manifold X, and let $f:G\longrightarrow G'$ be a normal fibration of S. By $I^m_{\rho,\delta}(X,S)$ we denote the subspace of $I_{\rho,\delta}(X,S)$ consisting of those distributions that, modulo (X), can be written as the pull-back to X via f of $IS^m_{\rho,\delta}(,O_S)\cap D'_c(G')$.*

Lemma 4.3.13 *The spaces $I^m_{\rho,\delta}(X,S)$ are well-defined, i.e., they are independent of the normal fibration and produce a filtration of $I_{\rho,\delta}(X,S)$ equivalent to the Sobolev order filtration.*

Proof. Once the independency of f is established, the other statements will follow. The equivalency with the Sobolev filtration is obtained using proposition 4.3.11.

As in the proof of theorem 4.1.4, it will be enough to show that $IS^m_{\rho,\delta}(,O_S)+()$ is invariant under the pull-back by local diffeomorphism h near O_S with properties i and ii.

Let h_t be a homotopy between the identity and h. Using lemma 4.1.6, we conclude that if $u \in IS^m_{\rho,\delta}(,O_S) + ()$ has support sufficiently close to the zero section,

$$h^*u - u = \int_0^1 h_t^*(V_t u)dt \;, \tag{4.3.14}$$

for some vector field that can be written locally as $\sum \mu V$ with $V \in \mathcal{V}(O_S)$ and $\mu \mid_{O_S} = 0$. Iterating this argument, we establish that

$$\begin{aligned} h^*u - u = \sum_{j=0}^{k-1} \int_0^1 \dots \int_0^1 V_{t_j \dots t_0}(\dots (V_{t_0}u)\dots)dt_j \dots dt_0 + \\ \int_0^1 \dots \int_0^1 h^*_{t_k \dots t_0} V_{t_k \dots t_0}(\dots (V_{t_0}u)\dots)dt_k \dots dt_0 \;, \end{aligned} \tag{4.3.15}$$

for vector fields $V_{t_j \dots t_0}$ which locally can be written as the one above. Using proposition 4.3.10, the last term in (4.3.15) is the integral in the parameters of the pull-back under a diffeomorphism of an element in $IS^{m+k(\delta-\rho)}_{\rho,\delta}(,O_S)$. For $\rho > \delta$ and k sufficiently large, $-k(\delta-\rho) > 2M$, with M as in proposition 4.3.11.Therefore, this last term will be the integral with respect to the parameters of the pull-back under a diffeomorphism of an element in $I_{\rho,\delta,s-M,c}(,O_S) +_c ()$, which is once again included in $IS^m_{\rho,\delta}(,O_S)$. The desired invariance follows. ∎

Theorem 4.3.16 *(principal symbol) Suppose $S \subset X$ is closed and embedded, and let L_f be the map defined in* (4.1.3)*. Then the composition $\mathcal{F} \circ L_f$ induces an isomorphism*

$$\frac{I^m_{\rho,\delta}(X,S;\Omega^{\frac{1}{2}}X)}{(X)} \longrightarrow \frac{S^{m-\frac{d}{2}+\frac{n}{4}}_{\rho,\delta}() \otimes \Omega^{\frac{1}{2}}()}{S^{-\infty}() \otimes \Omega^{\frac{1}{2}}()} \;,$$

independent of φ, and it projects down to an isomorphism independent of f:

$$\frac{I^m_{\rho,\delta}(X,S;\Omega^{\frac{1}{2}}X)}{I^{m-(\rho-\delta)}_{\rho,\delta}(X,S;\Omega^{\frac{1}{2}}X)} \longrightarrow \frac{S^{m-\frac{d}{2}+\frac{n}{4}}_{\rho,\delta}() \otimes \Omega^{\frac{1}{2}}()}{S^{m-(\rho-\delta)-\frac{d}{2}+\frac{n}{4}}_{\rho,\delta}() \otimes \Omega^{\frac{1}{2}}()} \;. \tag{4.3.17}$$

Proof. If $u \in IS^m_{\rho,\delta}(,O_S;\Omega^{\frac{1}{2}}X)$, it follows from (4.3.14) that if f and g are normal fibrations, $L_f \circ (L_g)^{-1}u - u$ is an element of $IS^{m-(\rho-\delta)}_{\rho,\delta}(,O_S;\Omega^{\frac{1}{2}}X)$. The rest of the proof is a simple verification of how the densities transform under the different maps involved in the statement of the theorem. ∎

Note that if $u \in I^m_{\rho,\delta}(X,S)$, $\mathcal{F} \circ L_f u$ will be an element in $S^{m-\frac{d}{2}+\frac{n}{4}}_{\rho,\delta}() \otimes \Omega fiber()$, invariantly defined up to density-valued symbols of order $\rho - \delta$ units lower. In either case,

$u \in I^m_{\rho,\delta}(X,S;\Omega^{\frac{1}{2}}X)$ or $u \in I^m_{\rho,\delta}(X,S)$, we shall denote its principal symbol by $\sigma(u)$, and call σ the map in (4.3.17).

4.4 Wave front set of conormal distributions

We have shown that for any normal fibration $f : G \longrightarrow G'$, given $u \in I^m_{\rho,\delta}(X,S)$, the composition $\sigma_f(u) = \mathcal{F}((f^{-1})^*\varphi u)$ defines an element of $S^{m-\frac{d}{2}+\frac{n}{4}}_{\rho,\delta}() \otimes \Omega_{fiber}/S^{-\infty}() \otimes \Omega_{fiber}$, independent of φ. Here φ is a smooth function with support contained in G, identically equals to one on a neighborhood of S. We now want to address the problem of defining intrinsically a subset of on which the principal symbol $\sigma(u)$ is zero, i.e., rapidly decreasing. In principle, one could do this using chosing a normal fibration f and making the corresponding statement for $\mathcal{F}((f^{-1})^*\varphi u)$, but it is not a priori obvious that having a rapidly decreasing symbol along certain directions is a notion independent of f.

Consider two normal fibrations, f and g, and assume that modulo smooth errors and lower order terms, $u = f^*v = g^*v$ for some $v \in IS^m_{\rho,\delta}(,O_S)$, with compact support along the fiber. Hence, the principal symbol of u can be computed either from $\mathcal{F}v$ or from $\mathcal{F}((f^{-1})^*g^*v) = \mathcal{F}((g \circ f^{-1})^*v$. From (4.3.15) of the previous section, we have

$$v - (g \circ f^{-1})^*v = \sum_{j=1}^{k} v_j + v_{(k)} \ ,$$

where $v_j \in IS^{m-j(\rho-\delta)}_{\rho,\delta}(,O_S)$, $v_{(k)} \in IS^{m-k(\rho-\delta)}_{\rho,\delta}(,O_S)$. Each v_j is of the form

$$v_j = \int_0^1 \ldots \int_0^1 V_j \ldots V_1 v dt_1 \ldots dt_j \ ,$$

where the V_i's are vector fields in $\mathcal{V}(O_S)$ depending smoothly on the parameters $t_1, \ldots, t_j$, each of which can be expressed locally as $V_i = \sum_q \mu_{iq} V_{iq}$, $V_{iq} \in \mathcal{V}(O_S)$, $\mu_{iq} \mid_{O_S} = 0$. We can expand the functions μ_{iq} in Taylor series about $z = 0$ to conclude that for every k and modulo $IS^{m-k(\rho-\delta)}_{\rho,\delta}(,O_S)$, v_j is given by a differential operator acting over v, whose coefficients are functions in $C^0_{fiber}()$, constant along the fiber. Hence, for any k, modulo elements in $S^{m-k(\rho-\delta)-\frac{d}{2}+\frac{n}{4}}_{\rho,\delta}() \otimes \Omega_{fiber}$, $\mathcal{F}v_j$ is given by a differential operator acting over $\mathcal{F}v$. Consequently, if along a direction ξ on the fiber the symbol $\sigma_f(u) = \mathcal{F}v$ decreases rapidly, so will $\mathcal{F}((g \circ f^{-1})^*v$ and $\mathcal{F}((g^{-1})^*u)$. Thus, this notion is independent of the fibration.

The fact that $\mathcal{F}v$ decreases rapidly in the direction ξ_0 is a condition over the behavior of $\mathcal{F}v(y,\xi)$ when $\xi = t\xi_0 \to \infty$, i.e., when $t \to \infty$. Hence, given a open conic set Γ and $u \in I^m_{\rho,\delta}(X,S)$, it makes sense invariantly to say that the symbol $\sigma(u)$ decreases rapidly along directions in Γ. One simply takes a normal fibration f and consider $\mathcal{F}((f^{-1})^*\varphi u)$ for some smooth function φ as before. Then we demand from this function that it decreases rapidly along directions in Γ, and this notion is independent of both, f and φ.

Definition 4.4.1 *Given $u \in I^m_{\rho,\delta}(X,S)$, its wave front set is the set of all directions along which $\sigma(u)$ is not rapidly decreasing:*

$$WF(u) = \cap_\Gamma\{-(\Gamma \cup O_S) : \ \sigma(u) \in S^{-\infty}(\Gamma), \ \Gamma \subset \text{ open conic}\} \ . \tag{4.4.2}$$

We finish by proving the following

Proposition 4.4.3 *Assume that $u \in I^m_{\rho,\delta}(X,S)$ satisfies the condition*

$$WF(u) \cap = \emptyset \ . \tag{4.4.4}$$

Then u is a smooth function.

Proof. Modulo smooth errors, u can be lifted to a distribution $v \in S^m_{\rho,\delta}(, O_S)$ with fiber support compact, and $\sigma(u) = \mathcal{F}v$. Using (4.4.4) we conclude that $\mathcal{F}v \in S^{-\infty}(, O_S) \otimes \Omega_{fiber}$. Hence, v is rapidly decreasing along the fibers of and consequently smooth. Thus, u is smooth. ∎

Chapter 5

Push-forward and pull-back of conormal distributions

In this chapter we define the notion of push-forward and pull-back of distributions under a very general setting, and then we study the behavior of these operations over conormal distributions. The idea behind all this is to express the action of pseudo-differential and Fourier integral operators over distributions in terms of their kernels, as the restriction of $K(x, y) \otimes u(z)$ to $y = z$, followed by integration along the fiber $x = constant$. Hence, we must understand when these operations are valid as well as the type of distributions one gets when applying them. In that sense, this chapter contains results of technical nature, which are needed for other purposes.

5.1 Push-forward

Consider a fiber map $\Pi : X \to Y$, i.e., a smooth map such that for every $y \in Y$ there exists a neighborhood U of y in Y and a diffeomorphism $\tau : U \times \Pi^{-1}(y) \to \Pi^{-1}(U)$ making commutative the following diagram:

$$\begin{array}{ccc} U \times \Pi^{-1}(y) & \xrightarrow{\tau} & \Pi^{-1}(U) \\ \Pi \downarrow & & \downarrow \Pi \\ U & \xrightarrow{i_d} & U \end{array} . \tag{5.1.1}$$

Each fiber over U is diffeomorphic to $\Pi^{-1}(y)$. It follows that

$$\Omega X \cong \Omega_{fiber} X \otimes \Pi^* \Omega Y \ . \tag{5.1.2}$$

Moreover, if $\Pi : X \to Y$, $\Pi' : X' \to Y'$ are fiber maps and we have diffeomorphism $f : X \to X'$ and $g : Y \to Y'$ with $\Pi' \circ f = g \circ \Pi$, then

$$\begin{aligned} f^*(\Omega X') &\cong f^* \Omega_{fiber} X' \otimes f^*((\Pi')^* \Omega Y') \\ &\cong \Omega_{fiber} X \otimes \Pi^* \Omega Y \qquad \cong \Omega X \ . \end{aligned}$$

Thus (5.1.2) is natural with respect to diffeomorphism of fiber maps.

By integration along the fibers we obtain a map

$$\begin{gathered} \Pi_* : {}_c(X, \Omega_{fiber} X) \longrightarrow_c (Y) \\ \Pi_* u = \int_{fiber} u \end{gathered} \quad , \tag{5.1.3}$$

where in interpreting the right hand-side as a function of Y, we make use of (5.1.1).

Proposition 5.1.4 *The map* (5.1.3) *extends by continuity to a linear map*

$$\Pi_* : D_c'(X; \Omega_{fiber} X) \longrightarrow D_c'(Y) \ . \tag{5.1.5}$$

Proof. Take $u \in D_c'(X; \Omega_{fiber} X)$ and consider a sequence $\{u_n\} \subset_c (X, \Omega_{fiber} X)$ such that $u_n \to u$. By the principle of uniform boundedness, it will be enough to show that $\Pi_* u_n(\varphi)$ is bounded by a constant times the m-th norm of φ over any compact set. But for some m, on compact sets K contained in U, a set as in (5.1.1), we have

$$\begin{aligned} | \, \Pi_* u_n(\varphi) \, | = & \ \ | \int_{fiber} u_n(\Pi^* \varphi) \, | \\ \leq & \ \ \sup_{\Pi^{-1}(K) \cap \operatorname{supp} u_n, |\alpha| \leq m} | \, D^\alpha_{(y,z)} \Pi^* \varphi \, | \leq \sup_{y \in K, |\alpha| \leq m} | \, D^\alpha_y \varphi \, | \ , \end{aligned} \tag{5.1.6}$$

where we have used the fact that $\Pi^* \varphi$ is constant along the fiber, as well as the fact that $u_n \to u$. The result for general compact sets follows using a partition of unity subordinated to a covering of Y by sets as in (5.1.1). ∎

Remark 5.1.7 *If u is not compactly supported, the push-forward $\Pi_*(u)$ can still be defined if the fibers of $\Pi : X \to Y$ are compact or if u is compactly supported along the fibers. We obtain a map as in* (5.1.5) *among distributions which are not necessarily compactly supported.*

5.2 Pull-back

Given a submersion $f : X \to Y$, the pull-back over smooth densities $f^*\nu = \nu \circ f$ extends by continuity and produce a map $f^* : D^{'}(Y, \Omega^\alpha Y) \to D^{'}(X, \Omega^\alpha X)$ (see exercise 1.4.8). Indeed, the push-forward $f_*\varphi$ of a smooth density $\varphi \in_c (X, \Omega^{1-\alpha}X)$ is an element of $_c(Y, \Omega^{1-\alpha}Y)$ and we can define $f^*\nu$ by

$$f^*\nu(\varphi) = \nu(f_*\varphi) \ .$$

If ν is smooth, $f^*\nu$ can be defined by simple composition, with no requirement over f. These cases are extremes of a more general situation to be discussed after the notion of wave front set is introduced in the following chapter. Here we restrict ourselves, and discuss the simplified version of the pull-back of an element $\nu \in I_{\rho,\delta}(Y, S^{'})$ under an immersion f which intersects $S^{'}$ transversally.

Before going any further, the reader should be convinced that the pull-back notion is not always defined is one insists that it be a continuous extension to distributions of the usual operation over smooth densities. In fact, even for the simple case of the diagonal map $\mathbf{R} \ni x \to (x, x) \in \mathbf{R}^2$ and $\nu(x, y) = \delta(x) \otimes \delta(y)$, it is impossible to define $f^*\nu$ since there is no way to make sense of the expression $\delta^2(x)$. In this case, for $f^*\nu$ to be defined, one should be able to approach the submanifold $y = x$ along transversal directions on which ν is smooth.

Consider an immersion $f : X \to Y$ and let $S^{'}$ be a closed embedded submanifold of Y. The map f is said to be transversal to $S^{'}$ if, and only if, at any point $s \in f^{-1}(S^{'})$,

$$f_*T_sX + T_{f(s)}S^{'} = T_{f(s)}Y \ . \qquad (5.2.1)$$

Note that in the case where $S^{'}$ is just a single point, the transversality condition coincides with the notion of a submersion.

Lemma 5.2.2 *Suppose $f : X \to Y$ is a smooth map transversal to $S^{'}$. Then $S = f^{-1}(S^{'})$ is a submanifold of X.*

Proof. Let $\psi_1, \ldots, \psi_d$ be a set of defining functions of $S^{'}$, i.e.,

$$S^{'} = \{y \in Y : \ \psi_1(y) = \ldots = \psi_d(y) = 0\}$$

and $d\psi_1, \ldots, d\psi_d$ are linearly independent over $S^{'}$. Then

$$S = \{x \in X : \ \psi_1 \circ f(x) = \ldots = \psi_d \circ f(x) = 0\}$$

and to show that S is a submanifold, it will be enough to prove that $d(\psi_1 \circ f), \ldots, d(\psi_d \circ f)$ is a linearly independent set over S. Take a point s in S. If a linear combination $\sum \alpha_i d(\psi_i \circ f)_s$ equals zero, then by the chain rule

$$\sum \alpha_i d(\psi_i \circ f)_s(V) = 0 = \sum \alpha_i (d\psi_i)_{f(s)} f_* V \quad \forall \ V \in T_s X \ .$$

Thus, $\sum \alpha_i (d\psi_i)_{f(s)}$ is identically zero over Image f_s^*. Since the ψ_i's are defining functions for $S^{'}$, $\sum \alpha_i (d\psi_i)_{f(s)}$ is identically zero over $T_{f(s)} S^{'}$. By transversality we have that $\sum \alpha_i (d\psi_i)_{f(s)}$ is identically zero on $T_{f(s)} Y$ which implies that $\alpha_1 = \ldots = \alpha_l = 0$. ∎

Proposition 5.2.3 *Suppose $f : X \to Y$ is a smooth immersion transversal to $S^{'}$. Then*

$$f^* :_c (Y) \longrightarrow_c (X)$$

extends by continuity to a map

$$f^* : I_{\rho,\delta}(Y, S^{'}) \longrightarrow D^{'}(X) \ .$$

Proof. By a partition of unity, we can assume that u is supported on a small set near $S^{'}$ and that f is an embedding of X into Y. Then, we can find coordinates $(y_1, \ldots, y_n)$ valid on U, such that

$$S^{'} = \{y^{'} = (y_1, \ldots, y_d) = 0\} \ , \quad f(X) = \{y^{'''} = \{y_{n-k+1}, \ldots, y_n) = 0\} \ ,$$

with $d + k \leq n$ and supp $u \subset U$. We expect that the restriction $u \mid_{f(X)}$ defines an element of $D^{'}(f(X)) \cong D^{'}(X)$.

In the coordinates chosen above,

$$u(y) = (2\pi)^{-d} \int e^{iy^{'}\eta^{'}} a(y^{''}, y^{'''}, \eta^{'}) d\eta^{'} \ ,$$

where $y^{''} = (y_{d+1}, \ldots, y_{n-k})$ and $a(y^{''}, y^{'''}, \eta^{'}) \mid d\eta^{'} \mid \in S(N^* S^{'}) \otimes \Omega_{fiber} N^* S^{'}$ is the full symbol of u. Then, for $\varphi(y^{'}, y^{''}) \mid dy^{'} dy^{''} \mid \in_c (f(X); \Omega f(X))$, we have

$$u \mid_{f(X)} (\varphi) = (2\pi)^{-d} \int e^{iy^{'}\eta^{'}} a(y^{''}, 0, \eta^{'}) d\eta^{'} \varphi(y^{'}, y^{''}) dy^{'} dy^{''} \ ,$$

from which it clearly follows that

$$| u |_{f(X)} (\varphi) | \leq C \|\varphi\|_m ,$$

where m is the order of u as a distribution with compact support. ∎

Remark 5.2.4 *For an immersion f as in the proposition above,*

$$N_f^* = \{(f(x), \eta) : (f_*)_x^t \eta = 0\}$$

does not intersect N^*S'. Hence, $WF(u) \cap N_f^* = \emptyset$. Later on we shall see that it is this relation between wave front set and N_f^* what makes possible the definition of f^*u as an extension of the usual definition on smooth functions. Under this hypothesis, the conormality assumption simply says that f^*u is also conormal. We shall postpone that discussion until the following chapter.

5.3 Push-forward and Pull-back of conormal distributions

We now proceed to study the operations defined in the previous two sections on the level of conormal distributions. We start by considering the push-forward.

Given a fiber map $\Pi : X \longrightarrow Y$ and a closed embedded submanifold S, the set $\Pi(S)$ can be a rather complicated one. We therefore restrict our attention to the case where $\Pi(S)$ is an embedded submanifold of Y and

$$\Pi |_S : S \longrightarrow Y \text{ is a fiber bundle over } \Pi(S) . \tag{5.3.1}$$

The bundle of densities along the fibers for the map above shall be denoted as $\Omega_{fiber} S$.

Observe that $\Pi^* : T^*_{\Pi(x)} Y \longrightarrow T^*_x X$ is injective. On the other hand, if $V \in$ then $\Pi_* V \in \mathcal{V}(\Pi(S))$. Thus, if $\alpha \in N^*(\Pi(S))$ then $\Pi^* \alpha(V) |_S = \alpha(\Pi_* V) |_{\Pi(S)} = 0$. Thus, $\Pi^*(N^*(\Pi(S))) \subset$ and Π^* induces a bundle map

$$\Pi^* : N^*(\Pi(S)) \longrightarrow \tag{5.3.2}$$

which identifies the fiber of $N^*\Pi(S)$ at $\Pi(x)$ with a subspace of the fiber of at x. On the other hand, let $x \in S$ and denote by F and F_S the fibers at x of Π and $\Pi\mid_S$, respectively. Then,

$$N_xS \;=\; \frac{T_xX}{T_xS} \;\;\cong \frac{T_{\Pi(x)}Y + T_xF}{T_{\Pi(x)}\Pi(S) + T_xF_S} \;\;\cong \frac{T_{\Pi(x)}Y}{T_{\Pi(x)}\Pi(S)} + \frac{T_xF}{T_xF_S}$$
$$\cong N_{\Pi(x)}\Pi(S) + N_xF_S\ ,$$

where N_xF_S is the normal to the fiber F_S as a submanifold of F. It then follows that

$$\Omega^{-1}N_xS \sim \Omega^{-1}N_{\Pi(x)}\Pi(S) \otimes \Omega^{-1}N_xF_S\ .$$

But recall that $\Omega^\alpha V \sim \Omega^{-\alpha}V^*$. Also, since $T_xF \sim T_xF_S + N_xF_S$ we have

$$\Omega_{fiber}X \cong \Omega_{fiber}S \otimes \Omega_{fiber}NF_S\ .$$

Thus, by proposition 1.4.4 we derive

$$\Omega_{fiber} \otimes \Omega_{fiber}X \cong \Omega_{fiber}N^*\Pi(S) \otimes \Omega_{fiber}S\ . \tag{5.3.3}$$

Note that the right hand-side of (5.3.3) reduces to $\Omega N^*\Pi(S)$ if $\Pi\mid_S$ is an embedding of S into Y. Putting together (5.3.2) and (5.3.3) we obtain a map

$$\begin{aligned}\Pi_\sharp : S^k_{\rho,\delta}() \otimes \Omega_{fiber} \otimes \Omega_{fiber}X \longrightarrow S^k_{\rho,\delta}(N^*\Pi(S)) \otimes \Omega_{fiber}N^*\Pi(S)\\ \otimes\Omega_{fiber}S\ ,\end{aligned} \tag{5.3.4}$$

by restricting the symbols to $N^*\Pi(S)$. This map will be needed to define $\sigma(\Pi_*u)$ in terms of $\sigma(u)$ when $u \in I^m_{\rho,\delta,c}(X,S;\Omega_{fiber}X)$. But to prove that $\Pi_*I_{\rho,\delta,c}(X,S;\Omega_{fiber}X) \subset I_{\rho,\delta,c}(Y,\Pi(S))$ we shall need the following

Lemma 5.3.5 *Assume* $\Pi : X \longrightarrow Y$ *is a fiber map for which* (5.3.1) *holds. Then, for each* $V \in \mathcal{V}(\Pi(S))$ *(resp.* $\mathcal{V}_0(\Pi(S))$*) there exists* $W \in$ *(resp.) such that*

$$\Pi_*W = V\ .$$

Proof. The result follows readily using the local splitting of T_xX into $T_{\Pi(x)}Y + T_xF$. We then can take $W = V + 0$. Details are left to the reader. ∎

Theorem 5.3.6 *Assume $\Pi : X \longrightarrow Y$ is a fiber map for which* (5.3.1) *holds. Then the restriction of* (5.1.5) *induces a map*

$$\Pi_* : I^m_{\rho,\delta,c}(X,S;\Omega_{fiber}X) \longrightarrow I^{m+\frac{N_S}{2}-\frac{N}{4}}_{\rho,\delta,c}(Y,\Pi(S)) \ .$$

Here N and N_S are the fiber dimension of Π and $\Pi\mid_S$, respectively. The symbol $\sigma(\Pi_* u)$ is the projection of $\Pi_\sharp\sigma(u)$ to $S^{m'-\frac{d'}{2}+\frac{n'}{4}}_{\rho,\delta}(N^*\Pi(S))\otimes\Omega_{fiber}N^*\Pi(S)\otimes\Omega_{fiber}S/S^{m'-\frac{d'}{2}+\frac{n'}{4}-(\rho-\delta)}_{\rho,\delta}($ $N^*\Pi(S))\otimes\Omega_{fiber}N^*\Pi(S)\otimes\Omega_{fiber}S$, where $m'=m+\frac{N_S}{2}-\frac{N}{4}$, $d'=\mathrm{codim}\,\Pi(S)$ and $n'=\mathrm{dimY}$.

Proof. Let u be a distribution with compact support along the fibers. Then, for some s, we have $\varphi u \in H^s_c(X,\Omega_{fiber}X)$ where $\varphi_c(X)$. By continuity of Π_*, $\Pi_* u \in H^{s'}_c(Y)$ for some s', which obviously can be taken to be s. If V is a vector field,

$$Vu(\psi) = u(V^t\psi) \ \ u \in D'(X,\Omega X), \ \psi \in D(X) \ ,$$

defines a vector field V^t. Moreover, if $V \in$ (resp.), so does V^t. Take V and W in $\mathcal{V}(\Pi(S))$ and (resp. $\mathcal{V}_0(\Pi(S))$ and) related as in lemma 5.3.5. Then:

$$\begin{aligned} V\Pi_* u(\psi) \ &= \Pi_* u(V^t\psi) \ = u(\Pi^* V^t\psi) \ = u(W^t\Pi^*\psi) \\ &= \Pi_* W u(\psi) \ , \end{aligned}$$

from which it easily follows that $\Pi_* u \in I_{\rho,\delta,c}(Y,\Pi(S))$ if $u \in I_{\rho,\delta,c}(X,S;\Omega_{fiber}X)$.

Since the symbol is invariantly defined, to check that the projection of $\Pi_\sharp\sigma(u)$ produces the symbol of $\Pi_* u$, we just need to do it in coordinates. In so doing, we shall see the relation between the standard order of u and $\Pi_* u$.

Take coordinates $(y,x)=(y',y'',x',x'')$ such that $S=\{y=0\}$, $\Pi(S)=\{y''=0\}$ and the fibers are given by $y''\circ\Pi = constant$, $x'\circ\Pi = constant$. Then, $u \in I^m_{\rho,\delta,c}(X,S;\Omega_{fiber}X)$ is represented as

$$u(y,x) = \left(\frac{1}{2\pi}\right)^d \int e^{iy\eta}a(x,\eta)d\eta \mid dy'dx'' \mid ,$$

for some section $a(x,\eta)\mid dy'dx''\mid$ of $S^{m-\frac{d}{2}+\frac{n}{4}}_{\rho,\delta}()\otimes\Omega_{fiber}X$. Hence,

$$\begin{aligned} \Pi_* u(y'',x') &= (2\pi)^{-d}\int\int e^{iy\eta}a(x,\eta)d\eta dy'dx'' \\ &= (2\pi)^{-d'}\int e^{iy''\eta''}b(x,0,\eta'')d\eta''dx'' \ , \end{aligned}$$

where

$$b(x,0,\eta^{''}) = (2\pi)^{d^{'}-d} \int e^{iy\eta} a(x,\eta) d\eta^{'} dy^{'} = a(x,0,\eta^{''}) \ .$$

By introducing a conormal error of order ρ units lower (see proposition 3.3.6), we can set $x^{''} = 0$ in b. Hence,

$$\sigma(\Pi_* u) = b(x^{'},0,0,\eta^{''}) \mid dx^{''} \mid$$

which, of course, is only defined modulo symbols of order $\rho - \delta$ units lower. This symbol has the same order as a. Thus, $m - \frac{d}{2} + \frac{n}{4} = m^{'} - \frac{d^{'}}{2} + \frac{n^{'}}{4}$. But $n^{'} = n - N$ and $d^{'} = d - N_S$. Hence,

$$m^{'} = m + \frac{N_S}{2} - \frac{N}{4}$$

as stated. ∎

Let us now consider the pull-back. If $f : X \longrightarrow Y$ is a smooth immersion intersecting $S^{'} \subset Y$ transversally, then by lemma 5.2.2, $S = f^{-1}(S^{'})$ is a submanifold. As it was pointed out in remark 5.2.4, the image of a conormal distribution under f^* is itself conormal. Here we shall prove this observation and work out the symbol behavior.

For any $s \in S$, in view of (5.2.1), $f_* T_s X$ contains a complements of $T_{f(s)} S^{'}$ in $T_{f(s)} Y$. Therefore,

$$f_* : N_s S = \frac{T_s X}{T_s S} \longrightarrow \frac{T_{f(s)} Y}{T_{f(s) S^{'}}} = N_{f(s)} S^{'}$$

is surjective, and therefore an isomorphism. Thus, the dual map induces an isomorphism

$$f^* : N^*_{f(X)} S^{'} \longrightarrow \subset T^* X \ .$$

By naturality of the pull-back, we have

$$\Omega_{fiber} \cong f^* \Omega_{fiber} N^* S^{'} \ .$$

We thus obtain a map

$$f^\sharp : S^m_{\rho,\delta}(N^* S^{'}) \otimes \Omega_{fiber} N^* S^{'} \longrightarrow S^m_{\rho,\delta}() \otimes \Omega_{fiber} \qquad (5.3.7)$$

by just restriction of the symbols to $N^*_{f(X)} S^{'} \cong$.

Theorem 5.3.8 *Assume $f : X \to Y$ is a smooth immersion transversal to $S^{'} \subset Y$. Then the pull-back operation on smooth functions extends by continuity to a map*

$$f^* : I^m_{\rho,\delta}(Y, S^{'}) \longrightarrow I^{m+\frac{k}{4}}_{\rho,\delta}(X, f^{-1}(S^{'})) ,$$

where $k = \dim Y - \dim X$. Moreover,

$$\sigma(f^*u) = f^\sharp \sigma(u)$$

where $f^\sharp$ is the map (5.3.7).

Proof. As before, let $S = f^{-1}(S^{'})$. If $V \in$ (resp.), it is rather clear that $f_*V \in \mathcal{V}(f(X) \cap S^{'})$ (resp. $\mathcal{V}_0(f(X) \cap S^{'})$) as a vector field in $f(X)$. We can extend f_*V to a vector field in Y which belongs to $\mathcal{V}(S^{'})$ (resp. $\mathcal{V}_0(S^{'})$). Hence, from the identity $V f^*u = f^*(Wu)$, it follows easily that $f^*u \in I_{\rho,\delta}(X, S)$ if $u \in I_{\rho,\delta}(Y, S^{'})$.

To prove that $\sigma(f^*u) = f^\sharp \sigma(u)$ we can work in local coordinates. Let $(y_1, \ldots, y_n)$ be a set of coordinates as in the proof of proposition 5.2.3, valid on an open set U. If u has support in U, we can represent it as

$$u(y) = (2\pi)^{-d} \int e^{iy^{'}\eta^{'}} a(y^{''}, y^{'''}, \eta^{'}) d\eta^{'}$$

for a symbol $a(y^{''}, y^{'''}, \eta^{'}) \mid d\eta^{'} \mid$ in $S^{m-\frac{d}{2}+\frac{n}{4}}_{\rho,\delta}(N^*S^{'}) \otimes \Omega_{fiber} N^*S^{'}$. Therefore,

$$u \mid_{f(X)} = (2\pi)^{-d} \int e^{iy^{'}\eta^{'}} a(y^{''}, 0, \eta^{'}) d\eta^{'} ,$$

showing that $\sigma(f^*u) = f^\sharp \sigma(u)$ as stated. Moreover, since the order of the symbols of u and f^*u coincide, we must have

$$m - d/2 + n/4 = m^{'} - d^{'}/2 + n^{'}/4 ,$$

where $m^{'}$ is the standard order of f^*u, $d^{'}$ is the codimension of S in X and $n^{'}$ is the dimension of X. Since $d^{'} = d$ and $n^{'} = n - k$, it follows that

$$m^{'} = n + k/4 = m + (\dim Y - \dim X)/4 .$$

This finishes the proof. ∎

Chapter 6

Pseudo-differential operators

In this chapter we use the space of conormal distributions and the symbol isomorphism theorem to define and study pseudo-differential operators. As pointed out by Hörmander (see [Ho1]), these operators are introduced to handle variable coefficients differential operators as one would handle the constant coefficients via the Fourier transform. It turns out that the inverse of elliptic differential operators can be found in the algebra of pseudo-differential operators, making this particularly useful in the study of elliptic equations. Some non-elliptic equations demand a more general type of operators but a similar technique can be applied. These, the Fourier integral operators, will not be studied here and the reader is advised to consult [Ho1], [Ho3] or [Me].

6.1 Pseudo-differential operators

Let X be a smooth paracompact manifold and consider the product $X \times X$. Sitting inside $X \times X$ is the diagonal Δ consisting of all the points of the form (x, x). It is a closed embedded submanifold of $X \times X$. Let Π_i be the projection of $X \times X$ onto the i-th factor, $i = 1, 2$.

Using the Schwartz kernel theorem (see theorem 1.3.4 and exercise 1.4.9), we see that any continuous linear map

$$T : D(X) \longrightarrow D^{'}(X) , \tag{6.1.1}$$

has associated with it a unique distribution $K_T \in D'(X \times X; \Pi_2^*\Omega X)$ such that

$$T(\varphi)(\psi) = K_T(\varphi \otimes \psi)$$

for every $\varphi \in D(X)$, $\psi \in D(X; \Omega X)$.

Definition 6.1.2 *For any smooth manifold X, the space of pseudo-differential operators of type ρ, δ, $\Psi_{\rho,\delta}(X)$, consists of those linear maps T as in* (6.1.1) *such that $K_T \in I_{\rho,\delta}(X \times X, \Delta; \Pi_2^*\Omega X)$. The space of pseudo-differential operators of type ρ, δ and order m, $\Psi^m_{\rho,\delta}(X)$, consists of those elements $T \in \Psi_{\rho,\delta}(X)$ such that $K_T \in I^m_{\rho,\delta}(X \times X, \Delta; \Pi_2^*\Omega X)$.*

As usual, we shall drop ρ, δ from the notation when $\rho = 1$ and $\delta = 0$.

If $T \in \Psi^m_{\rho,\delta}(X)$ then $T = T_1 + R$ with $K_{T_1} \in I^m_{\rho,\delta}(X \times X, \Delta; \Pi_2^*\Omega X)$ and $K_R \in (X \times X; \Pi_2^*\Omega X)$, respectively. The principal symbol of K_T, $\sigma(K_T)$, will be an element of $S^{m-\frac{d}{2}+\frac{n}{4}}_{\rho,\delta}(N^*\Delta) \otimes \Omega_{fiber} N^*\Delta \otimes \Pi_2^*\Omega X$ modulo a section of the same bundle of order $\rho - \delta$ units lower. Here, d and n are the codimension of Δ in $X \times X$ and dimension of $X \times X$, respectively. It then follows that $\sigma(K_T)$ is an element of $S^m_{\rho,\delta}(N^*\Delta) \otimes \Omega_{fiber} N^*\Delta \otimes \Pi_2^*\Omega X / S^{m-(\rho-\delta)}_{\rho,\delta}(N^*\Delta) \otimes \Omega_{fiber} N^*\Delta \otimes \Pi_2^*\Omega X$. Some identifications between T^*X and $N^*\Delta$ will allow us to simplify the symbol isomorphism when dealing with pseudo-differential operators, erasing conveniently the density factors. However, the reason for the normalization introduced in chapter 4 which makes elements in $\Psi^m_{\rho,\delta}(X)$ correspond to elements in $S^m_{\rho,\delta}(N^*\Delta) \otimes \Omega_{fiber} N^*\Delta \otimes \Pi_2^*\Omega X$ deserves to be explained. It is a choice dictated by the relation between differential operators and symbols of the same order obtained using the Fourier transform. Indeed, if $x = (x_1, \dots, x_n)$ are local coordinates on a fixed coordinate patch U, then we can represent $u \in_c (U)$ by

$$u(x) = \left(\frac{1}{2\pi}\right)^n \int e^{ix\xi} \hat{u}(\xi) d\xi \ .$$

Thus, if $P = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha$, we have

$$Pu(x) = \left(\frac{1}{2\pi}\right)^n \int e^{ix\xi} \Big(\sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha \Big) \hat{u}(\xi) d\xi \ ,$$

which says that in the distributional sense, the kernel of P is given by

$$K_P = \frac{1}{2\pi^n} \int e^{i(x-y)\xi} p(x, \xi) d\xi |\Pi_2^* dy| \ ,$$

where $p(x,\xi)=\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$. Note that $z=x-y$ defines a fiber of $N^*\Delta$ and $p(x,\xi)|d\xi|$ can be thought of as a section of $S^m(N^*\Delta)\otimes\Omega_{fiber}N^*\Delta$. Thus, K_P defines an element of $I^m(U\times U,\Delta;\Pi_2^*\Omega U)$. By using another coordinate system, the reader can check that $\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is invariantly defined.

Proposition 6.1.3 *Assume $T\in\Psi^m_{\rho,\delta}(X)$. Then*

$$T\ :_c(X)\longrightarrow (X)\ .$$

Proof. By definition we know that $T=T_1+R$ where T_1 and R have kernels in $I_{\rho,\delta}(X\times X,\Delta;\Pi_2^*\Omega X)$ and $(X\times X;\Pi_2^*\Omega X)$, respectively. Hence, if $u\in_c(X)$, we have $Ru\in(X)$. On the other hand, $K_{T_1}\otimes u\in I_{\rho,\delta}(X\times X\times X,\Delta\times X;\Pi_2^*\Omega X)$. Indeed, if V is a vector field in $X\times X\times X$ tangent to (resp. vanishing at) $\Delta\times X$, then $V=V_1+V_2$ where V_1 is the lift under the projection map of an element in $\mathcal{V}(\Delta)$ (resp. $\mathcal{V}_0(\Delta)$) and V_2 is the lift of a vector field in the third factor. The condition on the local Sobolev order of $K_{T_1}\otimes u$ follows easily from this.

Since the map

$$i\ :X\times X\longrightarrow X\times X\times X$$
$$(x,y)\longrightarrow(x,y,y)$$

is transversal to $\Delta\times X$ and $i^{-1}\Delta\times X=\Delta$, by theorem 5.3.8 the restriction $i^*(K_{T_1}\otimes u)$ is a well-defined element of $I_{\rho,\delta}(X\times X,\Delta;\Pi_2^*\Omega X)$ with compact support in y. By theorem 5.3.6, the push-forward of this distribution under Π_2 will then produce an element of $I_{\rho,\delta}(X,\Pi_2(\Delta))=I_{\rho,\delta}(X,X)=(X)$. Thus, $T_1(u)=(\Pi_2)_*i^*(K_{T_1}\otimes u)\in(X)$. The result follows. ∎

Let E and F be smooth complex vector bundles over the manifolds X and Y, respectively. Generalize the Schwartz kernel theorem ot he case of continuous linear operators

$$T\ :D(X;E)\longrightarrow D^{'}(Y;F)\ . \tag{6.1.4}$$

Using this, define the space of pseudo-differential operators $\Psi^m_{\rho,\delta}(X;E,F)$ as those maps as above (with $Y=X$) for which $K_T\in I^m_{\rho,\delta}(X\times X,\Delta;\mathrm{Hom}\,(E,F)\otimes\Pi_2^*\Omega X)$.

Show that any differential operator is a pseudo-differential operator.

i) Show that the operator

$$T : D(\mathbf{R}) \longrightarrow D'(\mathbf{R})$$
$$u \longrightarrow \int_{-\infty}^{x} u(\tau)d\tau$$

is an element of $\Psi^{-1}(\mathbf{R})$.

ii) Show that the Volterra operator

$$T : D(\mathbf{R}) \longrightarrow D'(\mathbf{R})$$
$$u \longrightarrow \int_{0}^{x} u(\tau)d\tau$$

is not a pseudo-differential operator.

Given an operator T as in (6.1.5), one can define its transpose by

$$T^t(\varphi)(\psi) = \varphi(T\psi)$$

for all $\psi \in_c (X; E)$, $\varphi \in_c (Y; F^* \otimes \Omega Y)$. We thus obtain a linear map

$$T^t : D(Y; F^* \otimes \Omega Y) \longrightarrow D'(X; E^* \otimes \Omega X)$$
$$\varphi \longrightarrow T^t(\varphi) .$$

It is continuous since $T\psi \in (Y; F)$ and given a sequence $D(Y; F^* \otimes \Omega Y) \ni \varphi_j \to 0$, then for some fixed compact set K, supp $\varphi_j \subset K$ for all j and $\varphi_j \to 0$ on K uniformly, together with all its derivatives. Then $T^t(\varphi_j)(\psi) \to 0$ for all ψ. It is straightforward to check that if $K_T(x, y)$ is the kernel of T, then $K_{T^t}(x, y) = K_T^t(y, x)$, i.e., the kernel of T^t is obtained by applying the isomorphism $X \times Y \cong Y \times X$ and by taking the transpose in the fibers $\mathrm{Hom}\,(E, F) \xrightarrow{t} \mathrm{Hom}\,(F^*, E^*)$.

Proposition 6.1.5 *Let $T \in \Psi^m_{\rho,\delta}(X; E, F)$. Then there exists a continuous extension*

$$T : D_c'(X; E) \longrightarrow D'(X; F) .$$

Proof. In view of the relation between the kernels of T and T^t, it is easy to conclude that $T^t \in \Psi^m_{\rho,\delta}(X; F^* \otimes \Omega X, E^* \otimes \Omega X)$. Therefore, for $u \in D_c'(X; E)$ we can define

$$Tu(\varphi) = u(T^t\varphi) \quad \varphi \in D(X; F^* \otimes \Omega X) .$$

Indeed, the use of sectional basis and proposition 6.1.3 shows that $T^t\varphi \in (X; E^* \otimes \Omega X) \cong (D_c'(X; E))^*$. ∎

Note that $\Psi^{-\infty}_{\rho,\delta}(X;E,F) = \Psi^{-\infty}(X;E,F)$ consists of those operators whose kernels are in $(X \times X; \mathrm{Hom}\,(E,F) \otimes \Pi_2^*\Omega X)$. We know these operators regularize distributions (see exercise 1.4.11), i.e., for $T \in \Psi^{-\infty}(X;E,F)$, $Tu \in (X;F)$ for all $u \in D'_c(X;E)$. Many properties of pseudo-differential operator T depends only on its class in

$$\frac{\Psi^m_{\rho,\delta}(X;E,F)}{\Psi^{-\infty}_{\rho,\delta}(X;E,F)}\,. \tag{6.1.6}$$

This is the case of the result in proposition 6.1.3 as the reader can easily check. Often, we shall identify T_1 and T_2 in $\Psi^m_{\rho,\delta}(X;E,F)$ if $T_1 - T_2 \in \Psi^{-\infty}_{\rho,\delta}(X;E,F)$.

Given a class $[T]$ in (6.1.9), it is convenient to have a representer of that class with a good behavior so that the composition with other operators can be defined. We end this section describing and studying this representer.

Definition 6.1.7 *An operator as in* (6.1.5) *is said to have proper support if for each compact set $K \subset X$, there exists a compact set $K' \subset Y$ such that* $\mathrm{supp}\, T\varphi \subset K'$ *if* $\mathrm{supp}\,\varphi \subset K$. *The set of properly supported elements of $\Psi_{\rho,\delta}(X;E,F)$ (resp. $\Psi^m_{\rho,\delta}(X;E,F)$) shall be denoted by $\Psi_{p,\rho,\delta}(X;E,F)$ (resp. $\Psi^m_{p,\rho,\delta}(X;E,F)$).*

Properly supported operators induce continuous linear maps $D(X;E) \longrightarrow D'_c(Y;F)$. In particular, if $T \in \Psi^m_{p,\rho,\delta}(X;E,F)$, by proposition 6.1.8 we obtain a continuous extension $T:\, D'_c(X;E) \longrightarrow D'_c(Y;F)$.

Proposition 6.1.8 *Assume $T \in \Psi^m_{\rho,\delta}(X;E,F)$. Then,*

$$T = T_1 + R\,,$$

where $T_1 \in \Psi^m_{p,\rho,\delta}(X;E,F)$ and $R \in \Psi^{-\infty}(X;E,F)$.

Proof. For a properly supported operator T, its kernel $K_T(x,y)$ has support in $Y \times X$ such that the projections Π_Y and Π_X, restricted to $\mathrm{supp}\, K_T$, are proper maps. In fact, this characterizes T as a properly supported operator.

Let $T \in \Psi^m_{\rho,\delta}(X;E,F)$. Choose $\rho \in (X \times X)$ such that $\rho \equiv 1$ in a neighborhood of the diagonal $\Delta \subset X \times X$ and such that the restriction of Π_i $(i = 1,2)$ to its support is a proper map. Then $K_T = \rho K_T + (1-\rho)K_T$. Let T_1 and R be the operators whose kernels are

ρK_T and $(1-\rho)K_T$, respectively. Since K_T is conormal, $(1-\rho)K_T$ is smooth and therefore, $R \in \Psi^{-\infty}(X;E,F)$. By the remark of the previous paragraph, $T_1 \in \Psi^m_{p,\rho,\delta}(X;E,F)$. ∎

We have seen that a pseudo-differential operator can be extended by continuity to act on $D'_c(X;E)$. For properly supported operators more is true.

Proposition 6.1.9 *Assume that $T \in \Psi_{p,\rho,\delta}(X;E,F)$. Then, there exists a continuous extension*

$$T : D'(X;E) \longrightarrow D'(X;F) .$$

Proof. Since $K_{T^t}(x,y) = K^t_T(y,x)$, T^t is also properly supported and consequently, it maps $D(X;F^* \otimes \Omega X)$ continuously into $D(X;E^* \otimes \Omega X)$. Given $u \in D'(X;E)$ we can define

$$Tu(\varphi) = u(T^t\varphi) \ , \quad \varphi \in D(X;F^* \otimes \Omega X) .$$

This clearly define an extension of T. ∎

6.2 Operations with pseudo-differential operators

We have observed already that for a linear operator T as is (6.1.5), there is a transpose operator defined by duality as $T^t\varphi(\psi) = \varphi(T\psi)$. Since $K_{T^t}(x,y) = K_T(y,x)$, it follows that the space of pseudo-differential operators of order m is closed under this map. We collect this result in the following

Proposition 6.2.1 *The transpose induces an order preserving isomorphism*

$$\begin{array}{c} {}^t : \Psi^m_{\rho,\delta}(X;E,F) \longrightarrow \Psi^m_{\rho,\delta}(X;F^* \otimes \Omega X, E^* \otimes \Omega X) \\ T \longrightarrow T^t \end{array} ,$$

where $K_{T^t}(x,y) = K_T(y,x)$. If T is properly supported, so is T^t.

On the other hand, let $\langle \ , \ \rangle_E$ be a sesquilinear pairing between $D(X;E)$ and $D'(X;E^* \otimes \Omega X)$. Given a linear operator T as above, its adjoint T^* is defined by duality as $\langle T^*\varphi, \psi\rangle_E = \langle \varphi, T\psi\rangle_F$, and produces a linear map

$$T^* : D(Y;F^* \otimes \Omega X) \longrightarrow D'(X;E^* \otimes \Omega X) ,$$

where E^*, F^*, stand now for the sesquilinear dual bundles. If $K_T(x,y)$ is the kernel associated with T, then $K_{T^*}(x,y) = K_T^*(y,x)$. It is then clear that the following proposition holds.

Proposition 6.2.2 *The adjoint induces an order preserving isomorphism*

$$\begin{array}{c} ^* : \Psi^m_{\rho,\delta}(X;E,F) \longrightarrow \Psi^m_{\rho,\delta}(X;F^*\otimes\Omega X, E^*\otimes\Omega X) \\ T \longrightarrow T^* \end{array},$$

where $K_{T^*}(x,y) = K_T^*(y,x)$. If T is properly supported, so is T^*.

Finally, we shall consider the composition of pseudo-differential operators.

Proposition 6.2.3 *Assume P and Q are pseudo-differential operators. Then, if either P or Q is properly supported, the composition PQ is well-defined and it is a pseudo-differential operator.*

Proof. Assume for simplicity that Q is properly supported. Then, combining propositions 6.1.3 and 6.1.12 and the remark following definition 6.1.10, we see that Q maps $D(X;E)$ into $D(X;F)$. Consequently, for any $u \in D(X;E)$, $P(Qu)$ is a well-defined distribution, obtaining in this way a map

$$\begin{array}{c} D(X;E) \longrightarrow D'(X;G) \\ u \longrightarrow P(Qu) \end{array},$$

which is obviously linear. It remains to show that this map is pseudo-differential.

Proceeding formally, one sees that the kernel of the composition is given by $K_{PQ} = \Pi_*\rho^*(K_P\otimes K_Q)$, where ρ is the inclusion map $(x,y,z)\longrightarrow(x,y,y,z)$ and Π is the fiber map $(x,y,z)\longrightarrow(x,z)$. The problem is that the push-forward and pull-back are rather delicate operations, making hard to see what type of distribution K_{PQ} actually is. We shall prove that $\rho^*(K_P\otimes K_Q)$ is conormal to $\bar\Delta = \{(x,x,x)\}$ in $X\times X\times X$. Since $\Pi\mid_{\bar\Delta}$ is an embedding onto Δ, it will follow from proposition 5.3.4 that $K_{PQ}\in I^m_{\rho,\delta}(X\times X,\Delta;\mathrm{Hom}\,(E,G)\otimes\Pi_2^*\Omega X)$.

Recall that by proposition 4.4.3, both K_P and K_Q are smooth along directions which do not belong to $N^*\Delta$. Therefore, the tensor product $K_P\otimes K_Q$ fails to be smooth only along directions in $N^*\Delta\times N^*\Delta\cup N^*\Delta\times O_{T^*X\times X}\cup O_{T^*X\times X}\times N^*\Delta$. The conormal bundle of the image of $X \times X \times X$ under ρ does not intersect this set. Thus, $K_P \otimes K_Q$ is smooth along directions in

$$N_\rho^* = \{(x, y, y, z, 0, \xi, -\xi, 0)\} \ ,$$

and one can approach the manifold $M = X \times \Delta \times X$ transversally to define the restriction $K_P(x, y) \otimes K_Q(y', z) \mid_M = K_P(x, y) \otimes K_Q(y, z)$. That this distribution belongs to $I_{\rho,\delta}(X \times X \times X; \bar{\Delta}; \mathrm{Hom}\,(E, G) \otimes \Pi_2^* \Omega X)$ is now a consequence of the conormality of the distributions K_P and K_Q, respectively. ∎

Remark 6.2.4 *Note that we have avoided talking about $WF(K_P \otimes K_Q)$ since $K_P \otimes K_Q$ is not necessarily conormal and therefore, definition* 4.4.1 *does not apply. In section* 6.6*, we shall extend the notion of wave front set to any distribution. With that in mind we note that the result above amounts just to exploit the property that $WF(K_P \otimes K_Q) \subset WF(K_P) \times WF(K_Q) \cup WF(K_P) \times O_{T^*(X \times X)} \cup O_{T^*(X \times X)} \times WF(K_Q)$.*

6.3 Symbol isomorphism for pseudo-differential operators

If T is a pseudo-differential operator as in (6.1.5), of order m, its kernel K_T belongs to $I^m_{\rho,\delta}(X \times X, \Delta; \mathrm{Hom}\,(E, F) \otimes \Pi_2^* \Omega X)$. Hence, $\sigma(K_T) \in S^m_{\rho,\delta}(N^*\Delta) \otimes \Omega_{fiber} N^*\Delta \otimes \mathrm{Hom}\,(E, F) \otimes \Pi_2^* \Omega X$. The isomorphism given by the diagonal map $X \ni x \to (x, x) \in X \times X$, induces a bundle isomorphism

$$N^*\Delta \longrightarrow T^*X \tag{6.3.1}$$

which identifies the fibers of both bundles. By the local trivialization condition of a vector bundle, we have

$$\Omega_{fiber} N^*\Delta \otimes \Pi_2^* \Omega X \cong \Omega_{fiber} T^*X \otimes \Pi^* \Omega X \cong \Omega T^*X \ . \tag{6.3.2}$$

We now use the symplectic structure of the cotangent bundle: T^*X is provided with a canonical one-form α whose exterior derivative $\omega = d\alpha$ is non-degenerate. If (x, ξ) are coordinates in T^*X, then $\alpha = \sum \xi_i dx_i$ and $\omega = \sum d\xi_i \wedge dx_i$. Consequently, $\mid \omega \wedge \ldots \wedge \omega \mid$ defines a canonical trivializing section of ΩT^*X.

Theorem 6.3.3 *There exists an isomorphism*

$$\sigma_m : \frac{\Psi^m_{\rho,\delta}(X;E,F)}{\Psi^{m-(\rho-\delta)}_{\rho,\delta}(X;E,F)} \longrightarrow \frac{S^m_{\rho,\delta}(T^*X;\Pi^*\mathrm{Hom}\,(E,F))}{S^{m-(\rho-\delta)}_{\rho,\delta}(T^*X;\Pi^*\mathrm{Hom}\,(E,F))} .$$

Moreover, if the composition of $P \in \Psi^m_{\rho,\delta}(X;E,F)$ *and* $Q \in \Psi^n_{\rho,\delta}(X;F,G)$ *is defined,* $PQ \in \Psi^{m+n}_{\rho,\delta}(X;E,G)$ *and* $\sigma_{m+n}(PQ) = \sigma_m(P)\sigma_n(Q)$.

Proof. We already know that

$$\sigma_m : \frac{\Psi^m_{\rho,\delta}(X;E,F)}{\Psi^{m-(\rho-\delta)}_{\rho,\delta}(X;E,F)} \longrightarrow \frac{S^m_{\rho,\delta}(N^*\Delta)\otimes \mathrm{Hom}\,(E,F))\otimes \Omega_{fiber}N^*\Delta\otimes\Pi_2^*\Omega X}{S^{m-(\rho-\delta)}_{\rho,\delta}(N^*\Delta)\otimes \mathrm{Hom}\,(E,F))\otimes \Omega_{fiber}N^*\Delta\otimes\Pi_2^*\Omega X}$$

is an isomorphism. By (6.3.1), any element of $S^m_{\rho,\delta}(N^*\Delta)$ is identify uniquely with an element of $S^m_{\rho,\delta}(T^*X)$. By (6.3.2), $\Omega_{fiber}N^*\Delta\otimes\Pi_2^*\Omega X$ is identify with ΩT^*X. Under the projection $\Pi : T^*X \longrightarrow X$, the bundle $\mathrm{Hom}\,(E,F)$ pulls back to the bundle $\Pi^*\mathrm{Hom}\,(E,F)$ over T^*X. Since T^*X is symplectic with form ω, the bundle ΩT^*X is canonically trivialized. Thus, we can cancel the density factors and obtain

$$\frac{S^m_{\rho,\delta}(N^*\Delta)\otimes \mathrm{Hom}\,(E,F))\otimes \Omega_{fiber}N^*\Delta\otimes\Pi_2^*\Omega X}{S^{m-(\rho-\delta)}_{\rho,\delta}(N^*\Delta)\otimes \mathrm{Hom}\,(E,F))\otimes \Omega_{fiber}N^*\Delta\otimes\Pi_2^*\Omega X} \cong \frac{S^m_{\rho,\delta}(T^*X)\otimes\Pi^*\mathrm{Hom}\,(E,F)}{S^{m-(\rho-\delta)}_{\rho,\delta}(T^*X)\otimes\Pi^*\mathrm{Hom}\,(E,F)} .$$

It remains to show that whenever the composition of pseudo-differential operators P and Q of order m and n is defined, then $\sigma_{m+n}(PQ) = \sigma_m(P)\sigma_n(Q)$. Since the symbol of PQ is invariantly defined and does not depend upon smooth perturbations of either P or Q, we shall compute it in local coordinates assuming that both P and Q are properly supported. Thus, we assume that the kernels K_P and K_Q of P and Q, respectively, are compactly supported distributions with support contained in $\mathcal{O}\times\mathcal{O}$, $\mathcal{O}$ a coordinate neighborhood. Let (x,y) be coordinates in $\mathcal{O}\times\mathcal{O}$. The maps $f_L(x,y) = (x,x-y)$ and $f_R(x,y) = (y,x-y)$ are normal fibrations of $\Delta\subset\mathcal{O}\times\mathcal{O}$, with inverses $f_L^{-1}(x,v) = (x,x-v)$ and $f_R^{-1}(x,v) = (x+v,x)$ (see section 6.4 below). With respect to the fibration f_L, the full symbol of P is given by taking Fourier transform along the fibers of the conormal bundle:

$$\sigma_L(P)\mid d\xi dy\mid = \langle K_P(x,x-\cdot), e^{-i(\cdot,\xi)}\rangle \mid d\xi\mid \ .$$

Observe that this section of $S^m_{\rho,\delta}(N^*\Delta)\otimes\Omega_{fiber}N^*\Delta\otimes\Pi^*\Omega\mathcal{O}$ can be identified with a section of $S^m_{\rho,\delta}(T^*\mathcal{O})\otimes\Omega T^*\mathcal{O}$. Similarly,

$$\sigma_R(Q)\mid d\xi dy\mid = \langle K_Q(y+\cdot,y), e^{-i(\cdot,\xi)}\rangle \mid d\xi\mid \ .$$

For $u \in_c (\mathcal{O}, E)$, we can write

$$Pu(x) = \langle K_P(x,\cdot), u(\cdot)\rangle = (2\pi)^{-n}\int \sigma_L(P)(x,\xi)e^{i(x-y,\xi)}u(y)dyd\xi$$
$$= (2\pi)^{-n}\int \sigma_L(P)(x,\xi)e^{i(x,\xi)}\hat{u}(\xi)d\xi \ ,$$

$$Qu(x) = \langle K_Q(x,\cdot), u(\cdot)\rangle = (2\pi)^{-n}\int \sigma_R(Q)(y,\xi)e^{i(x-y,\xi)}u(y)dyd\xi \ .$$

Hence,

$$PQu(x) = (2\pi)^{-n}\int e^{i(x,\xi)}\sigma_L(P)(x,\xi)\sigma_R(Q)(y,\xi)u(y)dyd\xi \ ,$$

which is nothing but the action of a pseudo-differential operator on $\mathcal{O}$ whose kernel is the inverse Fourier transform along the fibers of $N^*\Delta$ of the symbol $a(x,y,\xi) = \sigma_L(P)(x,\xi)\sigma_R(Q)(y,\xi) \in S^{m+n}_{\rho,\delta}(N^*\Delta)\otimes\Omega_{fiber}N^*\Delta\otimes\Pi^*\Omega\mathcal{O}$. Note that $\sigma_m(P)$ and $\sigma_n(Q)$ are represented by $\sigma_L(P)$ and $\sigma_R(Q)$, respectively. We just need to identify the invariant part of $a(x,y,\xi)$. The expansion

$$a(x,y,\xi) = \sigma_L(P)(x,\xi)\sigma_R(Q)(x,\xi) + (x-y)\cdot\tilde{a}(x,y,\xi) \ ,$$

combined with proposition 3.3.6 assures that

$$\sigma_{m+n}(PQ)(x,\xi) = a(x,\xi) \ (\mathrm{mod}\, S^{m+n-(\rho-\delta)}_{\rho,\delta}) = \sigma_L(P)(x,\xi)\sigma_R(Q)(x,\xi) \ (\mathrm{mod}\, S^{m+n-(\rho-\delta)}_{\rho,\delta})$$
$$= \sigma_m(P)(x,\xi)\sigma_n(Q)(x,\xi) \ .$$

The result follows. ∎

Corollary 6.3.4 *If i stands for the inclusion map, the short sequence*

$$0 \to \Psi^{m-(\rho-\delta)}_{\rho,\delta}(X;E,F) \xrightarrow{i} \Psi^m_{\rho,\delta}(X;E,F) \xrightarrow{\sigma_m} \frac{S^m_{\rho,\delta}(T^*X;\Pi^*\mathrm{Hom}\,(E,F))}{S^{m-(\rho-\delta)}_{\rho,\delta}(T^*X;\Pi^*\mathrm{Hom}\,(E,F))} \to 0$$

is exact. ∎

We stop momentarily to discuss three important examples of differential operators widely used in differential geometry.

Example 6.3.5 Let X be a smooth manifold and E be the complexification of the exterior algebra $\Lambda^*(X) = \oplus_p\Lambda^p(X)$. Here, $\Lambda^p(X)$ is the set of all p-forms on X. If $(x_1,\ldots,x_n)$ is a coordinate system valid on an open set U, any section of E over U can be expressed as

$$\omega(x) = \sum_{p=0}^{n}\sum_{i=(i_1,\ldots,i_p)} f_{i_1,\ldots,i_p}(x)dx_{i_1}\wedge\ldots\wedge dx_{i_p} \ , \qquad (6.3.6)$$

for some complex-valued function $f_{i_1,\ldots,i_p}$. This indicates how to trivialize $\Lambda(X)$ over U: the collection $\{dx_{i_1}\wedge\ldots\wedge dx_{i_p} \,:\, 1\le i_1<\ldots<i_p\le n\,,\ 0\le p\le n\}$ form a basis for the fiber of $\Lambda^*(X)$ over any $x\in U$. Thus, $\Lambda^*(X)$ is a vector bundle of rank 2^n. The exterior derivative d is the unique $\mathbf{R}$-linear map

$$d : (X;\Lambda^*(T^*X)) \longrightarrow (X;\Lambda^*(T^*X))$$

such that $d((X;\Lambda^p(T^*X))\subset(X;\Lambda^{p+1}(T^*X)$ for all $p\ge 0$, $df(X)=Xf$ if $f\in(X)$ and X is a vector field, $d\circ d=0$ and $d(\omega_1\wedge\omega_2)=d\omega_1\wedge\omega_2+(-1)^r\omega_1\wedge d\omega_2$ where $\omega_1\in(X;\Lambda^r(T^*X))$. We still call d the operator we obtain on sections of the complexification E of $\Lambda^*(T^*X)$. Let P be any zeroth order differential operator acting on sections of $(X;E)$. Then $P\omega=p(x)\omega$ for all $\omega\in(X;E)$ and for some $p\in\mathrm{Hom}(E,E)$ (see chapter 3 for definitions). Thus

$$\begin{aligned} dP\omega - Pd\omega &= d(p\omega)-pd\omega \\ &= dp\wedge\omega + pd\omega - pd\omega \;=\; dp\wedge\omega \end{aligned} .$$

The operator $\omega\longrightarrow dp\wedge\omega$ is differential and has order 0. Hence $[d,P]\in Diff^0(X;E)$ for any $P\in Diff^0(X;E)$, proving that d is a differential operator of order 1. Thus $d\in\Psi^1(X;E,E)=\Psi^1(X;E)$. Its symbol is an element of $\dfrac{S^1(T^*X;\Pi^*\mathrm{Hom}\,(E,E))}{S^0(T^*X;\Pi^*\mathrm{Hom}\,(E,E))}$. Since we know a priori it is invariantly defined, we can proceed to compute it in coordinates. Consider a compactly supported section ω as in (6.3.6). From the properties of d we see that

$$\begin{aligned} d\omega &= \textstyle\sum_{p,i}\dfrac{\partial f_{i_1\ldots i_p}}{\partial x_j}(x)dx_j\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_p} \\ &= \dfrac{1}{(2\pi)^n)}\displaystyle\int e^{i(x-y)\xi}i\xi\wedge\textstyle\sum_{p,i} f_i(y)dy_{i_1}\wedge\ldots\wedge dy_{i_p} \\ &= \dfrac{1}{(2\pi)^n)}\displaystyle\int e^{i(x-y)\xi}\sigma(d)(x,\xi)\omega(y)dyd\xi\ , \end{aligned}$$

where $\sigma(d)(x,\xi)$ is the full symbol of d in the (x,ξ) coordinates of T^*X. It follows that the principal symbol of d is the element represented by the homomorphism $\sigma(d)(x,\xi)(\alpha)=i\xi\wedge\alpha$.

Example 6.3.7 In addition to the hypothesis on X in the previous example, suppose it is a compact riemannian manifold with riemannian metric $(\ ,\ )$. This metric induces a metric on the fibers of the exterior algebra which integrated defines a pairing between $D(X;\Lambda^*(T^*X))$ and $D'(X;(\Lambda^*(T^*X))^*\otimes\Omega X)$. The density factor can be ignored because the riemannian structure trivializes the density bundle in a canonical way. Also, the dual bundle $(\Lambda^*(T^*X))^*$ is isomorphic to $\Lambda^*(T^*X)$. Complexifying all this construction, we obtain a sesquilinear pairing $\langle\ ,\ \rangle$ between $D(X;E)$ and $D^{'}(X;E)$.

Using proposition 6.2.2, we see that the adjoint δ of d is pseudo-differential operator of order 1. Since it clearly preserves support, it is a differential operator. Its symbol can be computed using the relation $\langle d\omega,\alpha\rangle = \langle\omega,\delta\alpha\rangle$. It is the element of $\dfrac{S^1(T^*X;\Pi^*\mathrm{Hom}\,(E,E))}{S^0(T^*X;\Pi^*\mathrm{Hom}\,(E,E))}$ given by $\sigma_1(\delta)(x,\xi)\alpha = -i\xi\rfloor\alpha$, the contraction of α by $-i\xi$ in the first slot.

The Hodge Laplacian Δ is defined as $(d+\delta)^2$. Since $d^2=0$ it follows that $\delta^2=0$. Hence, $\Delta = d\delta+\delta d$. Therefore,

$$\sigma_2(\Delta)(x,\xi)\alpha = \sigma_1(d)\sigma_1(\delta)\alpha + \sigma_1(\delta)\sigma_1(d)\alpha = \|\xi\|^2\alpha\ .$$

This can be checked using some local trivialization of the bundle E. Notice that the symbol above is, at every point of T^*X-0, an invertible element of $\Pi^*\mathrm{Hom}\,(E,E)$.

Example 6.3.8 Let V be a vector space with inner product $\langle\ ,\ \rangle$. In the free tensor algebra $\sum_r \otimes^r V$, consider the ideal $\mathcal{I}$ generated by all elements of the form $v\otimes v + \langle v,v\rangle\cdot 1$, $v\in V$. The Clifford algebra is defined as

$$Cl(V,\langle\ ,\ \rangle) = \frac{\sum_r \otimes^r V}{\mathcal{I}}\ .$$

Since $V\subset \sum_r\otimes^r V$, it projects down to $Cl(V,\langle\ ,\ \rangle)$. This projection is 1-1. Thus, we think of V as included in $Cl(V,\langle\ ,\ \rangle)$. An endomorphism of V preserving the inner product, extends canonically to an inner automorphism of $Cl(V,\langle\ ,\ \rangle)$. As a vector space, $Cl(V,\langle\ ,\ \rangle)$ is isomorphic to the exterior algebra Λ^*V, and the isomorphism preserves the obvious filtrations. $Cl(V,\langle\ ,\ \rangle)$ decomposes as a direct sum of subspaces, $Cl_0(V,\langle\ ,\ \rangle)$ and $Cl_1(V,\langle\ ,\ \rangle)$, generated by products $v_{i_1}\cdot v_{i_2}\ldots\cdot v_{i_k}$ with k even and odd, respectively, defining a Z_2-graded algebra structure. Given an element $x = av_{i_1}\cdot v_{i_2}\ldots\cdot v_{i_k}$, define $\bar{x} = (-1)^r\bar{a}v_{i_k}\cdot\ldots\cdot v_{i_1}$. Then $x\cdot\bar{x}$ is a algebra norm on $Cl(V,\langle\ ,\ \rangle)$. The spin group is defined as

$$\mathrm{Spin}(V) = \{x\in Cl_0(V,\langle\ ,\ \rangle):\ x\cdot\bar{x}=1,\ xvx^{-1}\in V \text{ if } v\in V\}\ .$$

When $V=R^n$ we shall simply write $\mathrm{Spin}(n)$. This group is a double covering of $\mathrm{SO}(n)$.

There is a natural representations of Spin(n), $\rho : \mathrm{Spin}(n) \longrightarrow \mathrm{Aut}(V)$, given by $\rho(g)v = gvg^{-1}$.

With the tuple $(V, \langle\ ,\ \rangle)$ we can associate the space of spinors $S(V, \langle\ ,\ \rangle)$. Assume then that $n = 2k$ and that V has a complex structure J which is an isometry relative to the inner product. The complex space (V, J) leads to the complex exterior algebra

$$\Lambda^*_{\mathbf{C}} V = \oplus_{j=0}^{k} \Lambda^j_{\mathbf{C}} V \ .$$

For $v \in (V, J)$, the map $\varphi \to v \wedge \varphi - v \rfloor \varphi$ is $\mathbf{R}$-linear in V and its square equals $-\langle v, v\rangle I$. Thus, it extends to a homomorphism of $\mathbf{C}$-algebras

$$\mu : Cl(V, \langle\ ,\ \rangle) \subset Cl(V, \langle\ ,\ \rangle) \otimes_{\mathbf{R}} \mathbf{C} \longrightarrow \mathrm{End}_{\mathbf{C}}(\Lambda^*_{\mathbf{C}} V) \ .$$

Set $S(V, \langle\ ,\ \rangle, J) = \Lambda^*_{\mathbf{C}} V$. The restriction of μ to Spin(n) is reducible. This μ-action of Spin(n) preserves both, $S_+(V, \langle\ ,\ \rangle, J) = \oplus_{j=2l} \Lambda^j_{\mathbf{C}} V$ and $S_-(V, \langle\ ,\ \rangle, J) = \oplus_{j=2l+1} \Lambda^j_{\mathbf{C}} V$. One obtains irreducible representations

$$D^{\pm}_{\frac{1}{2}} : \mathrm{Spin}(n) :\longrightarrow \mathrm{Aut}(S_{\pm}(V, \langle\ ,\ \rangle, J) \ .$$

Given an oriented riemannian manifold X of dimension $n = 2k$, let $F(X)$ be the bundle of frames. It is an SO(n)-bundle. We assume that the transition functions of $F(X)$ can be lifted preserving the cocycle condition, to functions in Spin(n). In this form, we obtain a Spin(n)-bundle $\tilde{F}(X)$. The manifold is said to be a spin-manifold if this lifting is possible. The obstruction to it is measured by the second Stiefel-Whitney class $w_2(x) \in H^2(X; Z_2)$.

If for a G-bundle $P \to X$ one has a representation $\rho : G \longrightarrow \mathrm{Aut}(W)$, then the bundle $P \times_\rho W$ is defined as $P \times W / \{(pg, w) \sim (p, \rho(g)w)\}$. Sections of this bundle are in 1-to-1 correspondence with maps $f : P \longrightarrow W$ satisfying the property that $f(pg) = \rho(g^{-1})f(p)$ for all $g \in G$. Given a connection form on P and a tangent vector $Y \in T_x X$, the notion of the horizontal vector field $\tilde{Y}$ along $\Pi^{-1}(x)$ is well defined. If f represents a section of $P \times_\rho W$, by checking the compatibility condition, we can see that $\tilde{Y} f$ also represents a section of the same bundle. Thus, the connection on P induces a linear connection

$$\nabla : (X; P \times_\rho W) \longrightarrow (X; T^* X \otimes P \times_\rho W) \ .$$

Let μ be the representation of Spin$(2k)$ on $S(\mathbf{R}^{2k}, \langle\ ,\ \rangle, J) = S(\mathbf{R}^{2k})$, where $\langle\ ,\ \rangle$ and J are the usual structures. Consider also the representation ρ of Spin$(2k)$ defined before, extended to the whole algebra of $\mathbf{R}^{2k}$. We obtain bundles

$$\begin{gathered} S(\tilde{F}) = \tilde{F} \times_\mu S(\mathbf{R}^{2k}) \\ Cl(TX) = \tilde{F} \times_\rho Cl(\mathbf{R}^{2k}, \langle\ ,\ \rangle) \end{gathered} .$$

The spinor bundle $S(\tilde{F})$ is a natural $Cl(TX)$-module. Indeed, if $u \in (X; Cl(TX))$ and $\varphi \in (X; S(\tilde{F}))$, we can regard u and φ as functions on $\tilde{F}$ with values in $Cl(\mathbf{R}^{2k}, \langle\ ,\ \rangle)$ and $S(\mathbf{R}^{2k})$, respectively. Then, $u \cdot \varphi$ is a function on $\tilde{F}$ with values on $S(\mathbf{R}^{2k})$ and

$$(u \cdot \varphi)(pg) = \mu(g^{-1})u(p)\rho(g^{-1})\varphi(p) = u(p)g \cdot g^{-1}\varphi(p)g = \mu(g^{-1})(u \cdot \varphi)(p) .$$

The Dirac operator is defined on $(X; S(\tilde{F}))$ as follows: the connection on $\tilde{F}$ induces a connection on $S(\tilde{F})$. The metric on X identifies TX and T^*X. We write ∇ for the composite map

$$(X; S(\tilde{F})) \longrightarrow (X; T^*X \otimes S(\tilde{F})) \longrightarrow (X; TX \otimes S(\tilde{F})) .$$

Since $TX \subset Cl(TX)$, Clifford multiplication induces a map

$$m : TX \otimes S(\tilde{F}) \longrightarrow S(\tilde{F}) .$$

The Dirac operator is the composition of m and ∇:

$$\begin{gathered} D : (X; S(\tilde{F})) \longrightarrow (X; S(\tilde{F})) \\ \varphi \longrightarrow m\nabla\varphi \end{gathered} .$$

If $\{e_1, \ldots, e_n\}$ is an orthonormal frame on TX, locally it follows that

$$D\varphi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\varphi .$$

Consequently, the symbol $\sigma(D)(x, \xi)$ is the homomorphism given by Clifford multiplication $\varphi \to \frac{1}{i}\xi \cdot \varphi$.

6.4 Representation in local coordinates

We now proceed to describe the elements of $\Psi^m_{\rho,\delta}(X;E,F)$ locally. Therefore we assume that $X = R^n$ with its usual coordinate system. Since every vector bundle over a contractible manifold is trivial, in order to achieve the desired description it will be enough to assume that E and F are trivial vector bundles of rank one, i.e., we assume that we are in the scalar case, because otherwise, after chosing some sectional basis, we shall simply deal with matrices of operators whose entries are scalar pseudo-differential operators.

Hence, let $T \in \Psi^m_{\rho,\delta}(\mathbf{R}^n)$. We want to write down a coordinate formula for the action of T over $u \in_c (\mathbf{R}^n)$. Recall that what characterizes T is the fact that its kernel K_T belongs to $I^m_{\rho,\delta}(\mathbf{R}^n \times \mathbf{R}^n, \Delta; \Pi_2^*\Omega\mathbf{R}^n)$. By proposition 6.1.1, T can be written as $T_1 + R$ where T_1 is properly supported and R is regularizing. Let $\rho \in \mathcal{S}(\mathbf{R}^n)$ with $\rho > 0$. If we set

$$\sigma(R)(x,y,\xi) = \frac{K_R(x,y)}{\rho(x-y)}\hat{\rho}(\xi) \ ,$$

we obtain a rapidly decreasing symbol such that

$$Ru(x) = (2\pi)^{-n}\int e^{i(x-y)\xi}\sigma(R)(x,y,\xi)u(y)dyd\xi \ . \tag{6.4.1}$$

Under the assumption that T is properly supported, we shall show that it admits a similar representation with a symbol $\sigma(T)$ in $S^m_{\rho,\delta}(T^*\mathbf{R}^n)$. In fact, there are different representations of Tu which depend upon the normal fibration of $\Delta \subset \mathbf{R}^n \times \mathbf{R}^n$ employed to lift K_T to an element of $I^m_{\rho,\delta}(N\Delta, \mathcal{O}_\Delta; \Pi_2^*\Omega R^n)$. We disgress momentarily to study three particular examples of fibrations.

Let (x,y) be coordinates in $R^n \times R^n$ and consider the maps

$$\begin{aligned} f_L(x,y) &= (x, x-y) \\ f_R(x,y) &= (y, x-y) \\ f_W(x,y) &= (x+y/2, x-y) \end{aligned} \quad . \tag{6.4.2}$$

These maps are diffeomorphism of $\mathbf{R}^n \times \mathbf{R}^n$ onto itself. Moreover, $f_L(x,x) = f_R(x,x) = f_W(x,x) = (x,0)$ which we consider a typical element of the zero section of $N\Delta$. It is rather clear that under any of these maps, $T\Delta$ gets mapped onto $T\mathcal{O}_\Delta$ and the quotient

$T_\Delta \mathbf{R}^n \times \mathbf{R}^n / T\Delta$ is identified with $T_{\mathcal{O}_\Delta} N\Delta / T\mathcal{O}_\Delta$. Thus, f_L, f_R and f_W are normal fibrations of Δ, the left, right, and Weyl fibrations, respectively.

If T is properly supported, the pull-back of $K_T(x,y) \mid dy \mid$ under the inverse of any of the fibrations (6.4.2) is a tempered distribution along the fibers of $N\Delta$. The Fourier transform produces left, right and Weyl symbols:

$$\begin{aligned}\sigma_L(T)(x,\xi) &= \int K_T(x, x-v) e^{-iv\xi} dv \mid dyd\xi \mid \\ \sigma_R(T)(y,\xi) &= \int K_T(y+v, y) e^{-iv\xi} dv \mid dyd\xi \mid \\ \sigma_W(T)(x+y/2,\xi) &= \int K_T(x+y/2+v/2, x+y/2-v/2) e^{-iv\xi} dv \mid dyd\xi \mid\end{aligned} .$$

These are the full symbols of the operator T in the coordinates chosen, and are elements of $S^m_{\rho,\delta}(N^*\Delta) \otimes \Omega_{fiber} N^*\Delta \otimes \Pi_2^* \Omega \mathbf{R}^n)$, which can be thought of as elements of $S^m_{\rho,\delta}(T^*\mathbf{R}^n) \otimes \Omega T^*\mathbf{R}^n$. Note that in the coordinates above, $\mid dyd\xi \mid$ is the density ω^n, where ω is the canonical form $dy \wedge d\xi$ of $T^*\mathbf{R}^n$.

Proposition 6.4.3 *Any $T \in \Psi^m_{\rho,\delta}(\mathbf{R}^n)$ has a representation*

$$Tu(x) = (2\pi)^{-n} \int e^{i(x-y)\xi} \sigma(T)(x,y,\xi) u(y) dyd\xi \ ,$$

for some ρ,δ symbol $\sigma(T)$ of order m. Conversely, any such operator defines an element of $\Psi^m_{\rho,\delta}(\mathbf{R}^n)$.

Proof. In view of (6.4.1) and the linearity of the map $T \to \sigma(T)$, it is enough to assume that T is properly supported. In that case, T can be represented as above, where the symbol $\sigma(T)$ is determined once a choice of normal fibration is made. In particular, for the left, right or Weyl fibration, T is represented by $\sigma_L(T)$, $\sigma_R(T)$ or $\sigma_W(T)$, respectively.

Conversely, any operator as in the proposition has kernel given by the "oscillatory integral"

$$K_T(x,y) \mid dy \mid = \int e^{i(x-y)\xi} \sigma(T)(x,y,\xi) \mid dy \mid ,$$

which is a conormal distribution. ∎

Notice that the maps $g_L(x,v) = (x, x-v)$ and $g_R(x,v) = (x+v, x)$ are inverses of f_L and f_R, respectively. The one parameter family of maps

$$h_t(x,v) = (x+tv, x-(1-t)v)$$

defines a homotopy between g_L and g_R. This homotopy is the inverse of a homotopy of normal fibrations connecting f_L and f_R, which we knew exists in view of theorem 4.1.1. In terms of the coordinates (x, y) on $\mathbf{R}^n \times \mathbf{R}^n$, $v = x - y$. Consider the homotopy $g_t(x, y) = h_t \circ f_L(x, y)$ between the identity and $g_R \circ f_L$. If u is a distribution on $\mathbf{R}^n \times \mathbf{R}^n$, we have

$$(g_R \circ f_L)^* u - u = \int_0^1 \frac{d}{dt} g_t^* u dt = \int_0^1 g_t^*(v \cdot (\partial_x + \partial_y)) u dt \ .$$

Iterating this N-times we obtain

$$(g_R \circ f_L)^* u - u = \sum_{j=1}^{N} \frac{(v \cdot (\partial_x + \partial_y))^j}{j!} u + \int_0^1 \int_0^{s_1} \ldots \int_0^{s_N} g_t^*(v \cdot (\partial_x + \partial_y))^{N+1} u ds_{N+1} ds_N \ldots ds_1 dt \ .$$

If u is an element of $I^m_{\rho,\delta}(\mathbf{R}^n \times \mathbf{R}^n, \Delta)$, since the vector field $v \cdot (\partial_x + \partial_y)$ is tangent to and vanishes on Δ, the last term in the right hand-side of the expression above belongs to $I^{m-(N+1)(\rho-\delta)}_{\rho,\delta}(\mathbf{R}^n \times \mathbf{R}^n, \Delta)$. It then follows (see proposition 4.2.12) that

$$f_L^* g_R^* u \sim \sum_{j \geq 0} \frac{(v \cdot (\partial_x + \partial_y))^j}{j!} u \ .$$

Notice that if (x, v) are trivializing coordinates for $N\Delta$, $(f_L)_*((x - y) \cdot (\partial_x + \partial_y)) = v \cdot \partial_x$. Consequently,

$$g_R^* u \sim \sum_{j \geq 0} \frac{(v \cdot \partial_x)^j}{j!} g_L^* u \ .$$

If, additionally, we assume that the projections onto the first and second factor restricted to the support of u are proper maps, after taking invariant Fourier transform along the fibers in the expression above, we conclude that

$$\sigma_R(u) \sim \sum_{j \geq 0} \frac{\mathrm{i}^j}{j!} (\partial_\xi \cdot \partial_x)^j \sigma_L(u) \ , \tag{6.4.4}$$

where $\sigma_L(u)$ and $\sigma_R(u)$ are the full symbols of u computed using the left and right fibrations, respectively. Clearly, we also have

$$\sigma_L(u) \sim \sum_{j \geq 0} \frac{(-\mathrm{i})^j}{j!} (\partial_\xi \cdot \partial_x)^j \sigma_R(u) \ , \tag{6.4.5}$$

Assume that T is represented in its more general form

$$Tu(x) = (2\pi)^{-n} \int e^{i(x-y)\xi} a(x, y, \xi) u(y) dy d\xi$$

for some symbol $a(x,y,\xi)$. Show that its left reduced symbol can be obtained as

$$\sigma_L(T)(x,\xi) \sim \sum_{j\geq 0} \frac{(-i)^j}{j!} (\partial_\xi \cdot \partial_y)^j a(x,y,\xi) \mid_{y=x} \ .$$

Using the fact that if the Taylor series of f is written in terms of the differentials $f^{(k)}$ as

$$f(x+y) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x;y_1,\ldots,y_j)}{j!} \ ,$$

then $f^{(k)}(x;y_1,\ldots,y_j) = \sum_{|\alpha|=k} k! \dfrac{\partial_x^\alpha f(x) y^\alpha}{\alpha!}$, conclude that

$$\sigma_L(T)(x,\xi) \sim \sum_{|\alpha|\geq 0} \frac{(-i)^j}{\alpha!} \partial_\xi^\alpha \partial_y^\alpha a(x,y,\xi) \mid_{y=x} \ .$$

Hint: In the left fibration the symbol of $g_L^* u$ on the fiber over x_0 depends only on u on the submanifold $x = x_0$. Freeze the coefficient of $K_T(x,y)$ at $x = x_0$ and use (6.4.5).

Notice that if the role of the amplitude a in the second expression above is played by the right symbol $\sigma_R(P)$ of some pseudo-differential operator P, we obtain an equivalence version of (6.4.5) relating the right and left symbol of P.

Theorem 6.4.6 *Let $P \in \Psi^m_{p,\rho,\delta}(\mathbf{R}^n)$ and $Q \in \Psi^{m'}_{p,\rho,\delta}(\mathbf{R}^n)$, respectively. Then $PQ \in \Psi^{m+m'}_{p,\rho,\delta}(\mathbf{R}^n)$ and*

$$\sigma_L(PQ) \sim \sum_\alpha \frac{(-\mathrm{i})^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_L(P)(x,\xi) \partial_x^\alpha \sigma_L(Q)(x,\xi) \ .$$

Proof. By proposition 6.2.3 and theorem 6.3.3, we know that the composition is an element of $\Psi^{m+n}_{p,\rho,\delta}(\mathbf{R}^n)$ whose principal symbol is the product of the principal symbols of P and Q, respectively. It only remains to show that asymptotic expansion for $\sigma_L(PQ)$.

We have left and right representations

$$Pu(x) = (2\pi)^{-n} \int e^{i(x-y)\xi} \sigma_L(P)(x,\xi) u(y) dy d\xi = (2\pi)^{-n} \int e^{ix\xi} \sigma_L(P)(x,\xi) \hat{u}(\xi) d\xi \ ,$$

and

$$Qu(x) = (2\pi)^{-n} \int e^{i(x-y)\xi} \sigma_R(Q)(y,\xi) u(y) dy d\xi = (2\pi)^{-n} \int e^{ix\xi} \left(\int e^{-iy\xi} \sigma_R(Q)(y,\xi) u(y) dy \right) d\xi \ ,$$

for P and Q, respectively. Then, computing the Fourier transform of Qu and inserting it in the expression for Pu, we obtain

$$PQu(x) = (2\pi)^{-n} \int\int e^{i(x-y)\xi} \sigma_L(P)(x,\xi) \sigma_R(Q)(y,\xi) u(y) dy d\xi \ .$$

Let us set $c(x,y,\xi) = \sigma_L(P)(x,\xi)\sigma_R(Q)(y,\xi)$. This is a symbol of order $m+m'$. By Taylor series expansion about $x = y$, we see that

$$c(x,y,\xi) = \sum_{0\le|\gamma|\le N} \frac{\mathrm{i}^{|\gamma|}((y-x)\cdot D_y)^\gamma c}{\gamma!}(x,x,\xi) + \sum_{|\gamma|=N+1}(y-x)^\gamma c_\gamma(x,y,\xi) \; .$$

Since the symbol c_γ has order $m+m'-N(\rho-\delta)$, it follows that the kernel of PQ is asymptotically equals to

$$\begin{aligned} K_{PQ} &\sim (2\pi)^{-n}\sum_{|\gamma|\ge 0}\int \frac{\mathrm{i}^{|\gamma|}}{\gamma!}e^{i(x-y)\xi}((y-x)\cdot D_y)^\gamma c(x,x,\xi)d\xi \\ &\sim (2\pi)^{-n}\sum_{|\gamma|\ge 0}\frac{\mathrm{i}^{|\gamma|}}{\gamma!}\int (i\nabla_\xi)^\gamma e^{i(x-y)\xi}\cdot(D_y^\gamma c)(x,x,\xi)d\xi \\ &\sim (2\pi)^{-n}\sum_{|\gamma|\ge 0}\int e^{i(x-y)\xi}\frac{(\partial_y\cdot D_\xi)^\gamma c}{\gamma!}(x,x,\xi)d\xi \end{aligned} \; .$$

Hence, taking Fourier transform along the fiber $v = x-y$ to compute the left reduced symbol of the composition, and writing the differentials in terms of partial derivatives, we obtain

$$\sigma_L(PQ) \sim \sum_{|\gamma|\ge 0}\sum_{\beta\le\gamma}\frac{(-\mathrm{i})^{|\gamma|}}{\gamma!}\begin{pmatrix}\gamma\\ \beta\end{pmatrix}\partial_\xi^{\gamma-\beta}\sigma_L(P)\partial_x^\gamma\partial_\xi^\beta\sigma_R(Q) \; .$$

Changing the summation index $\gamma-\beta$ to α we conclude that

$$\begin{aligned} \sigma_L(PQ) &\sim \sum_\alpha \frac{(-\mathrm{i})^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\sigma_L(P)\partial_x^\alpha\left(\textstyle\sum_\beta \frac{(-\mathrm{i})^{|\beta|}}{\beta!}\partial_x^\beta\partial_\xi^\beta\sigma_R(Q)\right) \\ &\sim \sum_\alpha \frac{(-\mathrm{i})^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\sigma_L(P)\partial_x^\alpha\sigma_L(Q) \; , \end{aligned}$$

where, to obtain the last equivalence, we have used (6.4.5) (see also exercise 6.4.6 and remark thereafter). The desired result is proven. ∎

Corollary 6.4.7 *Let X be a smooth manifold and consider $P \in \Psi^m_{\rho,\delta}(X)$ and $Q \in \Psi^{m'}_{\rho,\delta}(X)$ such that its composition is defined. Then, $[P,Q] = PQ - QP \in \Psi^{m+m'-(\rho-\delta)}_{\rho,\delta}(X)$ and its symbol is*

$$\sigma([P,Q]) = \frac{1}{\mathrm{i}}\sum_j(\partial_{\xi_j}\sigma(P)\partial_{x_j}\sigma(Q) - \partial_{\xi_j}\sigma(Q)\partial_{x_i}\sigma(P)) \; .$$

Proof. In view of theorem 6.3.3 and corollary 6.3.4 the only unresolved question is the computation of the symbol of $[P,Q]$ in terms of the symbols of P and Q, respectively. But the symbol is invariantly defined and in coordinates, according to the previous theorem, is equals to the expression given in the statement, since $\sigma_L(P)$ (resp. $\sigma_L(Q)$) represents $\sigma(P)$ (resp. $\sigma(Q)$). The result follows. ∎

Let $P \in \Psi^m_{\rho,\delta}(X; E, F)$. Show that $\sigma_m(P)(x, s\xi)(w) = e^{-isf} P(e^{sf} v)(x)$ where $v \in_c (X; E)$ has value $w \in E_x$ at x and $df(x) = \xi \neq 0$. Conclude that if P is classical, i.e., $\sigma_m(P)$ is a classical symbol (see definition 4.2.18), then $\sigma_m(P) = \lim_{s\to\infty} s^{-m} e^{-isf} P(e^{sf} v)(x)$.

How different from the one above, is the formula for $\sigma([P, Q])$ when P and Q are pseudo-differential operators acting on sections of a bundle E.

6.5 Continuity in Sobolev spaces

In this section we discuss continuity properties of pseudo-differential operators when acting on sections of some Sobolev order. We shall prove that if $P \in \Psi^m_{\rho,\delta}(X; E, F)$ with $\rho > \delta$ then

$$P\,:\, H^s_{loc}(X, E) \longrightarrow H^{s-m}_{loc}(X, F) \tag{6.5.1}$$

continuously. That is, for $u \in H^s_c(X, E)$, $\operatorname{supp} u \subset K$ a fixed compact subset, $\|Pu\|_{s-m} \leq C\|u\|_s$ for some constant C independent of u.

Let U be a coordinate neighborhood in X and (x, ξ) coordinates in T^*U. Using (2.1.1) it is clear that the operator

$$\begin{gathered} {}_c(U, E) \longrightarrow (U, E) \\ \tilde{Q}_s u(x) = (2\pi)^{-n} \int e^{i} (1 + \|\xi\|^2)^{\frac{s}{2}} \hat{u}(\xi) d\xi \end{gathered}$$

maps $H^s_{loc}(X, E) \cap \{u :\ \operatorname{supp} u \subset U\}$ continuously onto $L^2(X, E) \cap \{u :\ \operatorname{supp} u \subset U\}$, and this map with restricted domain and range is an isomorphism. Patching together operators of this type, we obtain globally defined pseudo-differential operators Q_s inducing isomorphisms

$$Q^E_s :\ H^s_{loc}(X, E) \longrightarrow L^2_{loc}(X, E)\ .$$

Therefore, to achieve the desired continuity result for the operator P, it will be enough to show it for the operator

$$\tilde{P} :\ L^2_{loc}(X, E) \longrightarrow L^2_{loc}(X, F)$$

given by $Q^F_{s-m}PQ^E_{-s}$, where in defining the composition one introduces compactly supported functions wherever necessary such that in the end, it is nothing but composition of properly supported operators. Thus the proof of continuity of (6.5.1) is reduced to the case $m = s = 0$.

The need for a trivialization of ΩX demands to fix a positive density once and for all. In the sequel, ν will be one such density and all the L^2-inner products will be computed relative to ν.

Lemma 6.5.2 *Suppose $P \in \Psi^0_{\rho,\delta}(X)$, $\rho > \delta$, is self-adjoint and $\sigma_0(P) \geq c > 0$. Then there exists $A \in \Psi^0_{p,\rho,\delta}(X)$ self-adjoint, such that $P = A^*A + S$, where $S \in \Psi^{-\infty}(X)$.*

Proof. Modulo a smoothing operator we can assume that P is properly supported. If there is one such operator A, the symbol algebra isomorphism will imply that

$$\sigma_0(P) = \sigma_0(A)^2 \geq c > 0\ .$$

This produces a candidate for symbol of A, namely $a = \sqrt{\sigma_0(P)}$. We must check that $a \in S^0_{\rho,\delta}(T^*X)$. Using proposition 4.2.4, it will be enough to check that for $K \subset X$, compact and contained in an open set O on which T^*X is trivial, we have

$$\mid D^\alpha_x D^\beta_\xi a \mid \leq C(1 + \|\xi\|)^{-\rho|\beta|+\delta|\alpha|} \quad , \tag{6.5.3}$$

over $\Pi^{-1}(K)$. Here (x, ξ) are trivializing coordinates over O.

Observe that

$$D^\alpha_x D^\beta_\xi a(x,\xi) = \sum_{\substack{\sum\alpha_i = \mid\alpha\mid,\ \sum\beta_i = \mid\beta\mid \\ N \leq \mid\alpha\mid + \mid\beta\mid}} c^{\alpha,\beta}_{\alpha_1,\beta_1,\ldots,\alpha_N,\beta_N} \frac{D^{\alpha_1}_x D^{\beta_1}_\xi \ldots D^{\alpha_N}_x D^{\beta_N}_\xi \sigma_0(P)(x,\xi)}{\sigma_0(P)^{N-\frac{1}{2}}(x,\xi)}\ .$$

Indeed, a formula as the one above is obviously true when there is no differentiation at all, or when $\mid\alpha\mid + \mid\beta\mid = 1$. The general expression is obtained by induction. Estimates (6.5.3) follows from proposition (4.2.4).

Choose $A_0^{'} \in \Psi^0_{p,\rho,\delta}(X)$ such that $\sigma_0(A_0^{'}) = \sqrt{\sigma_0(P)}$. Then $\sigma_0((A_0^{'})^*) = \sigma_0(A_0^{'})$ and the operator

$$A_0 = \frac{1}{2}(A_0^{'} + (A_0^{'})^*) \in \Psi^0_{p,\rho,\delta}(X)$$

is self-adjoint with $\sigma_0(A_0)^2 = \sigma_0(P)$. Using the short exact sequence of corollary 6.3.4, we have

$$R_1 = P - A_0^2 \in \Psi^{-(\rho-\delta)}_{p,\rho,\delta} \ .$$

Proceeding by induction, assume we have found a sequence of operators $A_l \in \Psi^{-l(\rho-\delta)}_{p,\rho,\delta}(X)$, $l = 0, 1, \ldots, k-1$, such that $A_l^* = A_l$ and

$$R_k = P - \left(\sum_{l=0}^{k-1} A_l\right)^2 \in \Psi^{-k(\rho-\delta)}_{p,\rho,\delta} \ .$$

We want to choose $A_k \in \Psi^{-k(\rho-\delta)}_{p,\rho,\delta}$ such that

$$P - \left(\sum_{l=0}^{k} A_l\right)^2 \in \Psi^{-(k+1)(\rho-\delta)}_{p,\rho,\delta}(X) \ .$$

The expression above implies that

$$R_k - \left(\sum_{l=0}^{k-1} A_l\right) A_k - A_k \left(\sum_{l=0}^{k-1} A_l\right) + A_k^2 = R_k - 2A_0 A_k \mod \Psi^{-(k+1)(\rho-\delta)}_{p,\rho,\delta}(X) \ .$$

Hence, choose $A_k^{'}$ such that

$$\sigma_{-k(\rho-\delta)}(A_k^{'}) = \frac{\sigma_{-k(\rho-\delta)}(R_k)}{2\sigma_0(A_0)} = \frac{\sigma_{-k(\rho-\delta)}(R_k)}{2\sqrt{\sigma_0(P)}} \in S^{-k(\rho-\delta)}_{\rho,\delta}(T^*X) \ ,$$

and set

$$A_k = \frac{1}{2}(A_k^{'} + (A_k^{'})^*) \ .$$

By the asymptotic summation formula, there exists $A^{'} \in \Psi^0_{p,\rho,\delta}(X)$ such that

$$A^{'} - \sum_{l=0}^{N} A_l \in \Psi^{-(N+1)((\rho-\delta)}_{p,\rho,\delta}(X) \ .$$

The operator

$$A = \frac{1}{2}(A^{'} + (A^{'})^*)$$

has the same properties and satisfies the requirements of the lemma. ∎

Theorem 6.5.4 *Suppose* $P \in \Psi^m_{\rho,\delta}(X; E, F)$. *Then*

$$P : \ H^s_c(X; E) \longrightarrow H^{s-m}_{loc}(X; F)$$

continuously.

Proof. As it was mentioned before, we can assume that $s = m = 0$. Using sectional basis we see that E and F can be assumed to be trivial bundles or rank one. Therefore, by density of $_c(X)$, it will be enough to show that there exists a constant C such that

$$\langle Pu, Pu\rangle = \int \mid Pu \mid^2 \nu \leq C \int \mid u \mid^2 \nu = C\langle u, u\rangle \ .$$

This estimate is equivalent to one of the form

$$\langle (C^{''} - P^*P)u, u\rangle \geq -C^{'} \langle u, u\rangle \ ,$$

where $C = C^{'} + C^{''}$ for some constants $C^{'}, C^{''}$. If

$$B = C^{''} - P^*P \ ,$$

then $B \in \Psi^0_{\rho,\delta}(X)$ and $\sigma_0(B) = C^{''} - \mid \sigma_0(P) \mid^2$. Therefore, by choosing $C^{''}$ such that $C^{''} > \sup \mid \sigma_0(P) \mid^2$, we obtain an operator B which is elliptic and self-adjoint. Applying lemma 6.5.2, we find an square root A of B modulo smoothing operators, $B = A^*A + S$, $A \in \Psi^0_{p,\rho,\delta}(X)$, $S \in \Psi^{-\infty}(X)$. Hence:

$$\begin{aligned} \langle Bu, u\rangle &= \langle A^*Au, u\rangle + \langle Su, u\rangle \\ &= \langle Au, Au\rangle + \langle Su, u\rangle \ \geq \langle Su, u\rangle \ \geq -\|Su\|_{L^2}\|u\|_{L^2} \\ & \qquad \geq -C^{'} \langle u, u\rangle \ , \end{aligned}$$

because S is smoothing and u has compact support. The desired result follows. ∎

Observe that in the case $s = m = 0$, we have really proven that

$$\langle Pu, Pu\rangle \leq M^2\langle u, u\rangle + \langle Su, u\rangle \ ,$$

for some smoothing operator S and M any upper bound for $\lim_{\xi\to\infty} \sup \mid \sigma_0(P) \mid$ on the support of u. Hence, we have the following

Corollary 6.5.5 *Let X be a compact manifold and $P \in \Psi^0_{\rho,\delta}(X)$ ($\rho > \delta$) with $\lim_{\xi\to\infty} \sigma_0(P) = 0$ for some metric on the fibers of T^*X. Then*

$$P: \ L^2(X) \longrightarrow L^2(X)$$

is a compact operator. ∎

We can now study how the space of conormal distributions behaves when acted on by pseudo-differential operators.

Lemma 6.5.6 *Assume $P \in \Psi_{p,\rho,\delta}(X)$ has proper support. Then,*

$$P : I_{\rho,\delta}(X,S) \longrightarrow I_{\rho,\delta}(X,S) \ .$$

Proof. Assume that P has order m. Since for any $V \in (X,TX)$ the operator $Q = [V,P]$ is an element of $\Psi^m_{\rho,\delta}(X)$, one can show that

$$V_k \ldots V_1 P = P V_1 \ldots V_k + Q_1 V_2 \ldots V_k + \ldots + Q_{k-1} V_k + Q_k \ ,$$

where the Q_j's are elements of $\Psi^m_{\rho,\delta}(X)$. In particular, this will hold for vector fields in and . Therefore, if $u \in I_{\rho,\delta,c,s}(X,S)$, for vector fields $V_1,\ldots,V_k$, $W_1,\ldots,W_l$ in and , respectively, we have

$$V_k \ldots V_1 W_l \ldots W_1 P u = V_k \ldots V_1 (P W_1 \ldots W_l + \textstyle\sum_{r=1}^{l-1} Q_r W_{r+1} \ldots W_l + Q_l) u \ ,$$

for some operators Q_r of the same order and type as P itself. Using continuity of pseudo-differential operators in Sobolev spaces we conclude that the most singular term in the right hand-side of the expression above is $V_k \ldots V_1 P W_1 \ldots W_l u$. Commuting through the vector fields $V_1, \ldots V_k$ with the operator P and using the continuity result one more time, we see that $V_k \ldots V_1 W_l \ldots W_1 P u$ has local Sobolev order $s - m - l(1-\rho) - k\delta$. This proves the desired result. ∎

This lemma in fact shows that for $P \in \Psi^m_{p,\rho,\delta}(X)$,

$$P : \ I_{\rho,\delta,c,s}(X,S) \longrightarrow I_{\rho,\delta,c,s-m}(X,S) \ .$$

This leads to the question of how is the symbol of $u \in I_{\rho,\delta,c,s}(X,S)$ related to the symbol of Pu and the symbol of P itself.

Proposition 6.5.7 *If $u \in I^s_{\rho,\delta,c}(X,S)$ and $P \in \Psi^m_{\rho,\delta}(X)$, then*

$$P : \ I^s_{\rho,\delta,c}(X,S) \longrightarrow I^{s+m}_{\rho,\delta}(X,S) \ ,$$

and

$$\sigma_{s+m}(Pu) = \sigma(P) \mid \sigma_s(u) \ . \tag{6.5.8}$$

Proof. We have seen already that $Pu \in I_{\rho,\delta}(X,S)$. Since its symbol is invariantly defined, we can compute it in coordinates to establish the validity of (6.5.8). For that, let $(x^{'},x^{''})$ be a preferred coordinate system on U adapted to S at p. Then $S = \{q \in U : x^{''}(q) = (x_{n-d}(q),\ldots,x_n(q)) = 0\}$ and

$$u(x^{'},x^{''}) = \left(\frac{1}{2\pi}\right)^{d} \int e^{ix^{''}\xi^{''}} a(x^{'},\xi^{''}) d\xi^{''} ,$$

where $a(x^{'},\xi^{''}) \mid d\xi^{''} \mid \in S^{s-\frac{d}{2}+\frac{n}{4}}_{\rho,\delta}(N^*S) \otimes \Omega_{fiber} N^*S$.

Let $\chi \in_c (U)$ with $\chi \equiv 1$ on a neighborhood of an open set V with compact closure in U. To compute $\sigma_{s+m}(Pu) \mid_{\Pi^{-1}V}$ it is enough to compute $\sigma_{s+m}(P\chi u)$ since for any other choice $\chi^{'}$, $P\chi u \mid_V = P\chi^{'} u \mid_V$ modulo smooth errors. If $P(x,D_x)$ is the given pseudo-differential operator, we can assume without loss of generality that it is properly supported in a neighborhood of $V \times V$. Then it has a left representation, and

$$P\chi u(x) = \left(\frac{1}{2\pi}\right)^{n+d} \int e^{i(x-y)\gamma + iy^{''}\eta^{''}} \sigma_L(P)(x,\gamma)\chi(y)a(y^{'},\eta^{''}) d\eta^{''} dy d\gamma \ .$$

We need to show that the partial Fourier transform of $P\chi u$ in the $x^{''}$-variables, is a symbol in the conormal bundle to S whose top part is given by the right hand-side of (6.5.8). Call this transform $b(x^{'},\xi^{''}) \mid d\xi^{''} \mid$. Then,

$$b(x^{'},\xi^{''}) = \left(\frac{1}{2\pi}\right)^{n+d} \int e^{-ix^{''}\xi^{''} + i(x-y)\gamma + iy^{''}\eta^{''}} \sigma_L(P)(x,\gamma)\chi(y)a(y^{'},\eta^{''}) d\eta^{''}) dy d\gamma \ .$$

After changing the variables $x^{''},\gamma,\eta^{''}$ to $rx^{''}, r\gamma, r\eta^{''}$, respectively, we see that

$$b(x^{'},\xi^{''}) = \left(\frac{1}{2\pi}\right)^{n+d} r^{n+2d} \int e^{ir(x^{''}\xi^{''} + (x-y)\gamma + y^{''}\eta^{''})} \sigma_L(P)(x^{'},rx^{''},r\gamma)\chi(y)a(y^{'},r\eta^{''}) d\eta^{''}) dy d\gamma \ .$$

Consider the phase function $\varphi((x^{''},y,\gamma,\eta^{''}) = x^{''}\xi^{''} + (x-y)\gamma + y^{''}\eta^{''}$. It has non-degenerate critical points given by the equations $x^{'} = y^{'}$, $x^{''} = 0 = y^{''}$, $\gamma^{'} = 0$ and $\gamma^{''} = \eta^{''} = \xi^{''}$. On these critical points, φ vanishes. Applying the stationary phase formula [GS], we see that for the rescaled variable $r^{-1}\xi^{''}$,

$$b(x^{'},\xi^{''}) \mid d\xi^{''} \mid = \sigma_L(P)(x^{'},0,0,\xi^{''})\chi(x^{'},0)a(x^{'},\xi^{''}) \mid d\xi^{''} \mid \ .$$

Over V, $\chi \equiv 1$. Hence,

$$b(x^{'},\xi^{''}) \mid d\xi^{''} \mid \mid_{\Pi^{-1}V} = \sigma_L(P)(x^{'},0,0,\xi^{''})a(x^{'},\xi^{''}) \mid d\xi^{''} \mid = \sigma(P) \mid_{N^*S} \sigma(u) \mid_{\Pi^{-1}V} \ .$$

This completes the proof. ∎

Without using stationary phase formula, prove formula (6.5.8) for the case where P is a differential operator.

6.6 Elliptic operators and elliptic complexes

As an illustration of the power of the ideas developed so far, we discuss a particular class of operators, the elliptic pseudo-differential operators.

Let $P \in \Psi^m_{\rho,\delta}(X; E, F)$. Its symbol is an element of $S^m_{\rho,\delta}(T^*X; \Pi^*\mathrm{Hom}(E,F))$. P is said to be elliptic if for every relatively compact subset K of X, $\sigma(P)(x,\xi)$ is an invertible element of $\Pi^*\mathrm{Hom}(E,F)$ for all $x \in K$ when ξ is large (see definition 4.2.19). If the ellipticity condition only holds for directions (x,ξ) on an open cone Γ, we shall say that P is elliptic on Γ. The open set of elliptic points of P will be denoted by $Ell(P)$.

Define the operator wave front set by

$$WF'(P) = \{(x,y,\xi,\eta) \in T^*X \times X \;:\; (x,y,\xi,-\eta) \in WF(K_P)\}\;. \tag{6.6.1}$$

We are abusing notation in the expression above by ignoring the role played by the bundle $\mathrm{Hom}(E,F)$. This is done just for notational convenience. Note that the bundle isomorphism (6.3.1) permits to identify $WF'(P)$ with a subset of T^*X. We shall frequently do this, without making any reference to it.

Proposition 6.6.2 *Let $P \in \Psi^m_{\rho,\delta}(X; E, F)$ be an elliptic operator on an open cone Γ. Then, for any closed cone $\Gamma' \subset \Gamma$, there exists $Q \in \Psi^{-m}_{\rho,\delta}(X; F, G)$ such that*

$$WF'(QP - I) \cap \Gamma' = WF'(PQ - I) \cap \Gamma' = \emptyset\;.$$

Proof. Choose a closed cone K such that $\Gamma' \subset \overset{\circ}{K} \subset K \subset \Gamma$. If $p = \sigma_m(P)$, we can find a symbol $q \in S^{-m}_{\rho,\delta}(T^*X; \Pi^*\mathrm{Hom}(E,F))$ such that

$$pq = 1 + a + b\;,$$

where $a \in S^{-\infty}(T^*X; \Pi^*\mathrm{Hom}(E,F))$ and $b \in S^0_{\rho,\delta}(T^*X; \Pi^*\mathrm{Hom}(E,F))$ with $\operatorname{supp} b \subset T^*X - K$. We can choose $Q_0 \in \Psi^{-m}_{p,\rho,\delta}(X; F, E)$ such that $\sigma_{-m}(Q_0) = q$. Then

$$\sigma(PQ - I) = b \mod S^{-(\rho-\delta)}_{\rho,\delta}(T^*X; \Pi^*\mathrm{Hom}(E,F))\;.$$

We now choose $B_0 \in \Psi^0_{\rho,\delta}(X;F,E)$ with $\sigma(B_0) = b$. It follows that $WF'(B_0) \cap K = \emptyset$ and

$$PQ_0 - (I + B_0) = E_1 \in \Psi^{-(\rho-\delta)}_{\rho,\delta}(X;F,E) \ .$$

We proceed by induction to get rid of the error term. Assume we have found Q_j, B_j, $j = 0,\ldots,k-1$, such that $Q_j \in \Psi^{-m-j(\rho-\delta)}_{\rho,\delta}(X;F,E)$ and $B_j \in \Psi^{-j(\rho-\delta)}_{\rho,\delta}(X;F,E)$, respectively, with $WF'(B_j) \cap K = \emptyset$ and

$$P(\sum_{j=0}^{k-1} Q_j) - (I + \sum_{j=0}^{k-1} B_j) = E_k \in \Psi^{-k(\rho-\delta)}_{\rho,\delta}(X;F,E) \ .$$

One construct Q_k, B_k, such that

$$P(\sum_{j=0}^{k} Q_j) - (I + \sum_{j=0}^{k} B_j) = E_k \in \Psi^{-(k+1)(\rho-\delta)}_{\rho,\delta}(X;F,E) \ ,$$

and $WF'(B_k) \cap K = \emptyset$. The two relations above imply that

$$\sigma(P)\sigma(Q_k) - \sigma(B_k) = \sigma(E_k) \ .$$

By ellipticity of $\sigma(P)$ on Γ, we can find $Q_k \in \Psi^{-m-k(\rho-\delta)}_{\rho,\delta}(X;F,E)$ such that

$$\sigma p(Q_k) = \sigma(E_k) + b_k + a_k \ ,$$

with $a_k \in S^{-\infty}(T^*X, \Pi^*\mathrm{Hom}(E,F))$, $b_k \in S^{-k(\rho-\delta)}_{\rho,\delta}(T^*X; \Pi^*\mathrm{Hom}(E,F))$ and $\mathrm{supp}\, b_k \cap K = \emptyset$. Choose $B_k \in \Psi^{-k(\rho-\delta)}_{\rho,\delta}(X;F,E)$ such that $\sigma(B_k) = b_k$. Using the asymptotic summation formula, we set

$$\begin{array}{c} Q \sim \sum Q_j \\ B \sim \sum B_j \end{array} \ ,$$

which gives

$$PQ - (I + B) \in \Psi^{-\infty}(X;F,E) \ ,$$

with $WF'(B) \cap K = \emptyset$.

We leave to the reader to show that Q can be chosen to be also a left parametrix on Γ'.

∎

Corollary 6.6.3 *Let* $P \in \Psi^m_{\rho,\delta}(X; E, F)$ *be an elliptic operator. Then there exists a parametrix* $Q \in \Psi^{-m}_{rode}(X; F, E)$ *such that*

$$\begin{aligned} S &= QP - I \in \Psi^{-\infty}(X; E, F) \\ S' &= PQ - I \in \Psi^{-\infty}(X; F, E) \end{aligned} .$$

If $u \in D'(X, E)$ *is such that* $Pu \in_c (X, F)$*, then* $u \in (X, E)$*.*

Proof. The only unsettled point is the last part of the statement. But if $u \in D'(X, E)$ is such that $Pu = f \in_c (X, F)$, then

$$Qf - Su = u \ ,$$

for S a regularizing operator. It follows that $Qf - Su$ is smooth. Then, so is u. ∎

The results above are enough to obtain a great deal of information about elliptic equations on compact manifolds. Let $P \in \Psi^m_{\rho,\delta}(X; E, F)$ be elliptic and consider the one-parameter family of continuous linear operators

$$\begin{aligned} H^{s+m}(X, E) &\xrightarrow{P_s} H^s(X, F) \\ u &\longrightarrow Pu \end{aligned} . \tag{6.6.4}$$

The continuity is, of course, a consequence of theorem 6.5.4.

Theorem 6.6.5 *The map* (6.6.4) *is Fredholm.*

Proof. Let u be a distribution in the kernel of P_s. Using the parametrix Q of corollary 6.6.3, we see that

$$u = Su \ .$$

Since S is a regularizing operator, it follows that $u \in (X; E)$. Thus, $\ker P_s \subset (X; E)$ is independent of s. By Rellich's theorem (see corollary 2.2.5), $\ker P_s$ is a finite dimensional space.

We show next that the range of P_s is closed. Let f_n be a sequence in the range converging to f. For each n, there exists a unique u_n perpendicular to $\ker P_s$ such that $Pu_n = f_n$. We then obtain

$$Qf_n = QPu_n = u_n + Su_n \ .$$

Since S is regularizing, Rellich's theorem implies it is a compact operator. Assume that the sequence of norms $\|u_n\|$ goes to ∞. This will imply that some subsequence of the sequence $v_n = u_n/\|u_n\|$ converges to an element in $\ker P_s$, contradicting the fact that every element of $\{v_n\}$ has norm one and is perpendicular to $\ker P_s$. Thus, $\{\|u_n\|\}$ is a bounded sequence. Hence, some subsequence of $\{u_n\}$ converges, say to u, and $Pu = \lim Pu_n = \lim f_n = f$.

Finally, using a riemannian structure to identify the corresponding dual bundles with the bundles themselves, we see that the adjoint of P_s is again defined by an elliptic pseudo-differential operator. This is a simple consequence of theorem 6.2.2. Thus, $\operatorname{coker} P_s = \ker P^*$ is also independent of s, contained in $(X;F)$, and therefore, a finite dimensional space. This finishes the proof. ∎

Consider now a graded bundle $E = \oplus_{j=0}^{k} E_j$ over X and a graded classical pseudo-differential operator $P = \oplus_{j=0}^{k-1} P_j$ of order m, i.e., $P_j \in \Psi^m(X; E_j, E_{j+1})$. The sequence

$$(X;E_0) \stackrel{P_0}{\to} (X;E_1) \stackrel{P_1}{\to} (X;E_2) \stackrel{P_3}{\to} \dots \stackrel{P_{k-1}}{\to} (X;E_k)$$

is called a complex if $P_j P_{j-1} = 0$. Notice that in this case, $\sigma(P_j)\sigma(P_{j-1}) = 0$ follows from theorem 6.3.3. We shall refer to the complex (E, P). A particular example is the De Rham complex essentially defined in 6.3.5.

A complex (E, P) is said to be elliptic if for every $(x,\xi) \in T^*X - 0$, $\ker \sigma(P_j)(x,\xi) = \operatorname{range} \sigma(P_{j-1})(x,\xi)$. It follows that the operator $\Delta_j = P_j^* P_j + P_{j-1} P_{j-1}^*$ is an elliptic pseudo-differential operator of order $2m$. Indeed, if $v \in \ker \sigma(\Delta_j)(x,\xi)$, taking the inner product of $\sigma(\Delta_j)(v)$ with v itself, we conclude that $0 = \|\sigma(P_j)(v)\|^2 + \|\sigma(P_{j-1}^*)(v)\|^2$. Thus, v is the kernel of $\sigma(P_j)$ and therefore, $v = \sigma(P_{j-1})(w)$ for some w. But then, $\langle \sigma(\Delta_j)(v), w\rangle = \langle \sigma(P_{j-1}^*)\sigma(P_{j-1})(w), w\rangle = \|\sigma(P_{j-1})(w)\|^2 = \|v\|^2 = 0$. Thus, $\sigma(\Delta_j)$ is invertible. Conversely, if $\sigma(\Delta_j)$ is invertible then the complex (E, P) is elliptic. This only amounts to show that any v in $\ker \sigma(P_j)$ belongs to $\operatorname{range} \sigma(P_{j-1})$ since the other inclusion follows from the complex condition. From invertibility of $\sigma(\Delta_j)$ we find w such that $v = \sigma(\Delta_j)(w)$. This implies that $\sigma(P_j)\sigma(P_j^*)\sigma(P_j)(w) = 0$. From this it follows that $\sigma(P_j^*)\sigma(P_j)(w) = 0$ and therefore, $v = \sigma(P_{j-1})\sigma(P_{j-1}^*)(w)$, an element of the range of $\sigma(P_{j-1})$. An example of an elliptic complex is the De Rham complex. This was proven in 6.3.7.

The j-th cohomology group of the complex (E,P) is defined by

$$H^j(X;E,P) = \frac{\ker P_j}{\operatorname{range} P_{j-1}} \,.$$

Theorem 6.6.6 *(Hodge) Let (E,P) be an elliptic complex. Then,*

$$L^2(X;E_j) = \ker \Delta_j \oplus \operatorname{range} P_{j-1} \oplus \operatorname{range} P_j^*$$

as an orthonormal direct sum, $\ker \Delta_j$ is a finite dimensional vector space contained in $(X;E_j)$, and $H^j(X;E,P) \equiv \ker \Delta_j$.

Proof. By theorem 6.6.5 we know that Δ_j has a finite dimensional kernel entirely contained in $(X;E_j)$. Regarding it as a map from $H^{2m}(X;E_j)$ to $L^2(X;E_j)$, it is Fredholm and we obtain $L^2(X;E_j) = \ker \Delta_j \oplus \operatorname{range} \Delta_j$. Clearly, the range of Δ_j is contained in the span of the ranges of P_{j-1} and P_j^*. Since $P_j P_{j-1} = 0$, these two spaces are orthogonal to each other. It is clear that $\ker \Delta_j = \ker P_j \cap \ker P_{j-1}^*$. It follows that $\operatorname{range} \Delta_j$ contains the span of the ranges of P_{j-1} and P_j^*. Since $\operatorname{range} \Delta_j$ is a closed subspace of $L^2(X;E_j)$, the desired decomposition is proven.

There is an inclusion of $\ker \Delta_j$ in $H^j(X;E,P)$. This inclusion is injective because the range of P_{j-1} is orthogonal to $\ker \Delta_j$. We leave the proof that this map is surjective to the reader. ∎

Assume $P \in \Psi^m(X;E,E)$ is elliptic and self-adjoint, with $m > 0$. Let Π be the orthogonal projection onto the kernel of P, and consider the operator $A = (1-\Pi)P^{-1}(1-\Pi)$. Comparing this operator with the parametrix Q of P given by corollary 6.6.3, show that A is a self-adjoint operator in $\Psi^{-m}(X;E,E)$. Using compactness of A as an operator in $L^2(X;E)$, establish the spectral decomposition for the unbounded operator in $L^2(X;E)$ defined by P.

We finish this section by proving proposition 2.3.6, where continuity of elliptic operators in Sobolev spaces is going to be used one more time:

Proof of proposition 2.3.6. Let (x,y) be a coordinate system near $\partial\Omega$, with x a defining function for $\partial\Omega$ and $V = (-i)\partial_x$, and write the operator P as $P = \sum_{j=0}^m p_j(x,y,D_y)D_x^j$, where $p_j(x,y,D_y)$ is a tangential differential operator of order $m-j$. The result has been proven when $s-m>1/2$. Otherwise, (i.e., when $s-m\le 1/2$) distributions in $H^{s-m}(\Omega)$ can be approximated by smooth functions with with compact support in Ω.

The jump formula (2.3.7) applied to $u\in(\overline{\Omega})$ gives

$$\|\sum_{k=0}^{m-1}\left(\sum_{j=0}^{m-k-1}p_{j+k+1}(0,y,D_y)D_x^j u\mid_{x=0}\right)\otimes D_x^k\delta(x)\|_{s-m}=\|P(u_c)-(Pu)_c\|_{s-m}\ .$$

If $u\in H_P^s(\Omega)$, then $(Pu)_c\in L^2(\mathbf{R}^n)$, and since $s-m\le 1/2$, the term $\|P(u_c)\|_{s-m}$ can be bounded by the s Sobolev norm of u since P is elliptic and continuous on Sobolev spaces, and u can be approximated with functions supported in the interior. Hence, applying the triangle inequality, we obtain

$$\|\sum_{k=0}^{m-1}\left(\sum_{j=0}^{m-k-1}p_{j+k+1}(0,y,D_y)D_x^j u\mid_{x=0}\right)\otimes D_x^k\delta(x)\|_{s-m}\le C(\|u\|_s+\|Pu\|_{L^2})\ .$$

It follows that that the distribution whose norm is computed in the left hand-side belongs to $H^{s-m}(\Omega)$ and it is supported by $\partial\Omega$. Pairing it with a test function which near the border is equal to x^{m-1} we conclude that $p_m(x,y,D_y)u\mid_{x=0}\otimes D_x^{m-1}\delta(x)\in H^{s-m}(\Omega)$. Since $\partial\Omega$ is compact, proposition 2.3.3 implies that $p_m(x,y,D_y)u\mid_{x=0}\in H^{s+\frac{1}{2}}(\partial\Omega)$, and since $p_m(0,y,D_y)$ is elliptic of order 0, this implies that the first trace $u\mid_{x=0}$ is in $H^{s+\frac{1}{2}}(\partial\Omega)$. Knowing this, one can go back to the jump formula once again, and by induction conclude that the remaining traces $V^j u\mid_{\partial\Omega}$ are in $H^{s-j+\frac{1}{2}}(\partial\Omega)$ for $1\le j\le m-1$. ∎

6.7 Wave front set of a distribution. Pull-back revisited

The kernel K_P of a pseudo-differential operator P is an element of $I(X\times X,\Delta;\Pi_2\Omega X)$. Therefore, we have $WF(K_P)\subset N^*\Delta$. From the definitions and the identification $\Pi: N^*\Delta\longrightarrow T^*X$ we see that $T^*X-0-WF^{'}(P)$ is the largest open cone in T^*X-0 where $\sigma(P)$ is rapidly decreasing.

Define

$$\gamma(P)=\{\rho\in T^*X-0\ :\ \sigma(P)\text{ is not elliptic in a cone around }\rho\}\ . \tag{6.7.1}$$

We proceed to prove a useful microlocal partition of unity result.

Proposition 6.7.2 *If $\{\varphi_j\}_{j\in I} \subset S^0(T^*X)$ form a partition of unity of T^*X, then there exist $P_j \in \Psi^0(X)$ such that $WF^{'}(P_j) \subset \operatorname{supp}\varphi_j$, and*

$$\sum P_j = I + S\ , \quad S \in \Psi^{-\infty}(X)\ .$$

Proof. We can assume that the partition of unity is locally finite. Take $P_j^0 \in \Psi^0(X)$ such that $\sigma_0(P_j^0) = \varphi_j$. Without loss of generality we can also assume that the collection of supports of the kernels of the P_j^0's is locally finite in $X \times X$. Then $WF^{'}(P_j^0) \subset \operatorname{supp}\varphi_j$ and

$$I + E_1 = P^0 = \sum P_j^0 \in \Psi^0(X)$$

for some $E_1 \in \Psi^{-1}(X)$.

Proceeding by induction, assume that we have found P^k with $P^k = \sum P_j^k$, $WF^{'}(P_j^k) \subset \operatorname{supp}\varphi_j$, and $P^k - I = E_{k+1} \in \Psi^{-(k+1)}(X)$. We want to find $Q_j^k \in \Psi^{-(k+1)}(X)$ such that

$$\sum (P_j^k + Q_j^k) - I \in \Psi^{-(k+2)}(X)$$

and $WF^{'}(P_j^k + Q_j^k) \subset \operatorname{supp}\varphi_j$.Since $P^k - I \in \Psi^{-(k+1)}(X)$, we must have

$$\sum \sigma_{-(k+1)}(Q_j^k) = -\sigma_{-(k+1)}(\sum P_j^k - I) = -\sum_r \varphi_r \sigma_{-(k+1)}(E_{k+1})\ .$$

Choose Q_j^k such that $\sigma_{-(k+1)}(Q_j^k) = -\varphi_j \sigma_{-(k+1)}(E_{k+1})$. The result will follow after using the asymptotic summation formula. ∎

Refining the notion of singular support of a distribution, we now introduce the following concept:

Definition 6.7.3 *If $u \in D^{'}(X)$, its wave front set is the subset of $T^*X - 0$ defined by*

$$WF(u) = \cap\{\gamma(P) :\ P \in \Psi_p^0(X),\ Pu \in (X)\}\ .$$

It clearly follows that pseudo-differential operators are microlocal, i.e., $WF(Pu) \subset WF(u)$. Our most immediate goal is to show that the definition above coincides with 1.2.12 when expressed in local coordinates. Note that $WF(u)$ is invariant under m_t, the operator multiplication along the fibers by $t \in R_+$. That is to say, $WF(u)$ is a conic subset of $T^*X - 0$.

Lemma 6.7.4 *Suppose $u \in D^{'}(X)$ and let $P \in \Psi_p^0(X)$ be such that $WF^{'}(P) \cap WF(u) = \emptyset$. Then $Pu \in (X)$.*

Proof. Let Γ_1 and Γ_2 be conic open sets such that $WF(u) \subset \Gamma_1$, $WF^{'}(P) \cap \Gamma_1 = \emptyset$, $\Gamma_1 \cup \Gamma_2 = T^*X - 0$ and $WF(u) \subset T^*X - 0 - \Gamma_2$. Using definition 6.7.3, we conclude that there is a family of operators $\{P_i\}_{i \in I}$, $P_i \in \Psi_p^0(X)$, such that $P_i u \in (X)$ and at least one of the P_i's is elliptic on Γ_2. We can assume that the collection $\{\operatorname{supp} K_{P_i}\}_{i \in I}$ is locally finite and that

$$\cap \gamma(P_i) = \emptyset \ .$$

Choose an operator P_0 with symbol identically one on a conic neighborhood of $T^*X - 0 - \Gamma_2$, and rapidly decreasing on the complement of Γ_1. Consider the operator

$$Q = P_0^* P_0 + \sum_{i \in I} P_i^* P_i \ .$$

It is elliptic and therefore, it has a parametrix T. Thus,

$$u - TQu \in (X) \ .$$

Since for i in I we have $P_i u \in (X)$, this implies that $u - TP_0^* P_0 u \in (X)$. Hence, $Pu - PTP_0^* P_0 u \in (X)$. Using the formula for the symbol of the composition, we see that the operator $PTP_0^* P_0$ is regularizing. Thus, $PTP_0^* P_0 u \in (X)$, and therefore, $Pu \in (X)$, as desired. ∎

Proposition 6.7.5 *Let* $\Pi : T^*X \to X$ *be the natural projection. Then* $\Pi(WF(u)) = \operatorname{sing\,supp} u$*.*

Proof. Clearly $\Pi(WF(u)) \subset \operatorname{sing\,supp} u$ since for $x \notin \operatorname{sing\,supp} u$, we can find a function $\varphi \in_c (X)$ such that $\varphi(x) \neq 0$ and u is smooth on its support. The operator multiplication by φ belongs to $\Psi^0(X)$ and $\varphi u \in_c (X)$. Hence, the fiber above x is not in $WF(u)$ showing that $x \notin \operatorname{sing\,supp} u$.

For the converse, let $x \notin \Pi(WF(u))$. Since $T_x^*X - 0/R_+ = S_x^*X$ is compact, there are operators $P_j \in \Psi_p^0(X)$, $i \leq j \leq k$, such that

$$\cap_j \{\gamma P_j\} \cap T_x^* = \emptyset$$

and $P_j u \in (X)$ for all j. This condition says that at each $\rho \in T_x^*X$, at least one of the P_j's is elliptic. Consider a microlocal partition of unity $A_1, \ldots, A_k, A_{k+1}$ subordinated to the sets $\{Ell(P_1), \ldots, Ell(P_k), T^*X - T_x^*X\}$. Then, $\sum A_i = I + S$, where S is some smoothing operator. Since $WF'(A_i) \subset Ell(P_i)$ for $1 \leq i \leq k$, we can find operators B_i and S_i, with the first of order zero and the second a smoothing operator, such that $A_i = B_iP_i + S_i$. Choose $\varphi \in_c (X)$ such that it is equal to one on a neighborhood of x, and such that the projection of $WF'(A_{k+1})$ does not intersect its support. Then we have

$$\varphi u = (\sum A_i - S)\varphi u = \sum (B_iP_i + S_i)\varphi u + A_{k+1}\varphi u - S\varphi u \; .$$

Since $P_iu \in (X)$ and pseudo-differential operators are microlocal, the terms $B_iP_i\varphi u$ are smooth. By lemma 6.7.4, the term $A_{k+1}\varphi u$ is also smooth. And the remaining terms are smooth because the operators are regularizing. Then, φu is smooth, implying that x is not in its singular support. ∎

Proposition 6.7.6 *Let (x, ξ) be a local coordinate system on T^*X valid on $\Pi^{-1}U$ with $x_0 \in U$. Let $\gamma_0 = (x_0, \xi_0)$ be a point in T^*X. Then $\gamma_0 \notin WF(u)$ if and only if, there exists a function $\varphi \in_c (U)$ such that $\varphi(x_0) \neq 0$ and the localized Fourier transform, $\widehat{\varphi u}$, is rapidly decreasing ina cone about ξ_0.*

Proof. Suppose $\widehat{\varphi u}$ is rapidly decreasing in the cone $\Gamma_\varepsilon = \{\xi : \; \|\xi/\|\xi\| - \xi_0/\|\xi_0\|\| < \varepsilon\}$. Choose a symbol ψ of order zero with support contained in Γ_ε and such that ψ is elliptic about ξ_0. Then the tempered distribution $v(\xi) = \psi(\xi)\widehat{\varphi u}(\xi)$ is smooth and rapidly decreasing. Thus, its inverse Fourier transform is smooth:

$$w(x) = (2\pi)^{-n} \int e^{ix\xi} v(\xi) d\xi = 2\pi)^{-n} \int e^{i(x-y)\xi} \varphi(y)\psi(\xi)u(y)dyd\xi \; .$$

That is to say, the smooth function w is nothing but the action of a pseudo-differential operator P over u whose right symbol is $\varphi(y)\psi(\xi) \mid d\xi \mid$. Since P is elliptic about γ_0, it follows that γ_0 is not in the wave front set of u.

Conversely, suppose $\gamma_0 \notin WF(u)$. Since $WF(u)$ is a closed conic set, there exists an $\varepsilon > 0$ such that

$$WF(u) \cap \{x : \; \|x - x_0\| < \varepsilon\} \times \Gamma_\varepsilon = \emptyset \; .$$

Consider an operator P with right symbol $\varphi(y)\psi(\xi)$ as above. Then, by lemma 5.7.4, Pu is smooth. It follows that $\widehat{Pu} = \psi(\xi)\widehat{\varphi u}(\xi)$ is rapidly decreasing, which implies that $\widehat{\varphi u}$ is rapidly decreasing along ξ_0. ∎

We have establish the desired equivalence between the coordinatized definition 1.2.12 and the invariant one given in 6.7.3. We now enlarge the discussion concerning the pull-back operation using the availability of the wave front set concept as one that has invariant meaning.

Let X and Y be smooth manifolds and consider a smooth map $f : X \to Y$. Define the set of conormals of the map f by

$$N_f^* = \{(f(x), \eta) :\ (f_*)_x^t \eta = 0\} .$$

This set was already found in the discussion of pull-back of a conormal distribution (see remark 5.2.4).

If Γ is a closed conic subset of T^*X, define $D'_\Gamma(X) = \{u \in D'(X) :\ WF(u) \subset \Gamma\}$. This is topologized by declaring $u_j \in D'_\Gamma(X)$ to converge to $u \in D'_\Gamma(X)$ if the sequence converges weakly to u and $Pu_j \to Pu$ for all $P \in \Psi_p(X)$ with $WF'(P) \cap \Gamma = \emptyset$.

Theorem 6.7.7 *If $u \in D'_\Gamma(Y)$ and $WF(u) \cap N_f^* = \emptyset$, the pull-back f^*u can be defined in one and only one way so that it is equal to the composition $u \circ f$ when u is continuous and it is sequentially continuous from $D'_\Gamma(Y)$ to $D'(X)$ for any closed cone $\Gamma \subset T^*Y - 0$ such that $\Gamma \cap N_f^* = \emptyset$. The following relation holds:*

$$WF(f^*u) \subset f^*WF(u) = \{(x, (f_*)_x^t \eta) :\ (f(x), \eta) \in WF(u)\} .$$

Proof. The result is local and therefore we assume that $X = R^n$, $Y = R^m$. If $u \in (R^m)$ and φ is a test function with support near x_0,

$$(f^*u(\varphi) = (2\pi)^{-m} \int \hat{u}(\xi)\varphi(x)e^{i(f(x),\eta)}dxd\xi = \hat{u}((2\pi)^{-m} \int \varphi(x)e^{i(f(x),\cdot)}dx) = \hat{u}(\tilde{\varphi}) .$$

If $(f_*)_x^t \eta \neq 0$ when $x \in \operatorname{supp} \varphi$, then we can find a vector field L homogeneous of degree -1 along the fibers such that $e^{i(f(x),\eta)}$ is an eigenfunction for L of eigenvalue 1. Thus, $\tilde{\varphi}$ is rapidly decreasing. If Γ is a conic neighborhood of $\{\eta :\ (f_*)_{x_0}^t \eta = 0\}$, then $\tilde{\varphi}$ is rapidly decreasing outside Γ if the support of φ is sufficiently closed to x_0. On the other hand, if Γ is sufficiently small, $\hat{u}$ is going to be rapidly decreasing on Γ, even if u is replaced by a sequence converging to an element with wave front set outside Γ. Breaking the integral above in pieces associated with Γ and its complement, we conclude that the desired pull-back operation can be defined by continuity.

The second part of the theorem can be proven using the local characterization of the wave front set discussed in proposition 6.7.6. Breaking the integral of the localized Fourier transform into integrals on an open conic neighborhood Γ_1 of $\{(\xi,\eta):\ ((f_*)_{x_0})\eta=\xi\}$ and its complement Γ_2, we see that this Fourier transform is rapidly decreasing on Γ_2 and rapidly decreasing on Γ_1 if $(f(x_0),\eta)\ni WF(u)$. ∎

Notice that if $P\in\Psi_{\rho,\delta}(X)$ and $u\in D^{'}(X)$, then the product $K_P\otimes u\in D^{'}(X\times X\times X;\Pi_2^*\Omega X)$ has wave front set contained in $WF(K_P)\times WF(u)\cup WF(K_P)\times 0_{T^*X}\cup 0_{T^*(X\times X)}\times WF(u)$. By conormality of K_P, this last set is contained in $N^*\Delta\times WF(u)\cup N^*\Delta\times 0_{T^*X}\cup 0_{T^*(X\times X)}\times WF(u)$. This set does not intersect the conormal set of the map $\rho(x,y)=(x,y,y)$. Hence the pull-back $\rho^*K_P\otimes u=K_P(x,y)u(y)$ is well-defined, and has wave front set contained in the set of points $(x,x,\xi,-\xi)$ such that $((x,-\xi)\in WF(u)$. If u is compactly supported then the distribution above is compactly supported along the fibers of the map $\Pi(x,y)=x$. Hence, pushing forward we obtain that Pu can be expressed as

$$Pu\Pi_*\rho^*K_P\otimes u\ .$$

Given a fiber map $\Pi:X\to Y$, let $\Pi^{-1}(y)$ denote the fiber through x, where $\Pi(x)=y$. Show that for $u\in D_c^{'}(X;\Omega_{fiber}X)$, $WF(\Pi_*u)\subset\{\gamma\in T_y^*Y:\ \text{there exists}\ \gamma^{'}\in N_x^*\Pi^{-1}(y),\ \gamma^{'}\in WF(u),\ \Pi_y^*\gamma=\gamma^{'}\}$, where $N_x^*\Pi^{-1}(y)$ is the conormal to the fiber.

Using purely wave front set considerations, show that convolution with a smooth function is a regularizing operator.

When is it possible to multiply two distributions u_1 and u_2? When is it possible to restrict a distribution u to a submanifold S?

We finish this section proving a result concerning linear operators with general Schwartz kernels.

Theorem 6.7.8 *Assume the operator $T:D(X)\to D^{'}(Y)$ is linear and continuous with Schwartz kernel $K_T\in D^{'}(Y\times X;\Pi_X\Omega X)$. If $WF(K_T)\cap T^*Y\times 0_{T^*X}=\emptyset$, then $T\varphi\in D(Y)$ for all $\varphi\in D(X)$. In this case, T^t has an extension from $D_c^{'}(Y)$ to $D^{'}(X)$. Furthermore, if*

$WF(K_T) \cap 0_{T^*Y} \times T^*X = \emptyset$, T can be extended by continuity from $D_c'(X)$ to $D'(Y)$. When both conditions hold, the wave front set of Tu is contained in

$$\{(y,\eta):\ (y,x,\eta,-\xi) \in WF(K_P) \text{ for some } (x,\xi) \in WF(u)\}\ .$$

Proof. Consider the distribution $w = K_T(y,x) \otimes \varphi(z)$ for $\varphi \in D_c'(X)$. If φ is smooth it follows that the wave front set of w is contained in the set $\{(y,x,z,\eta,\xi,0):\ (y,x,\eta,\xi) \in WF(K_T)\}$. Let $\rho(y,x) = (y,x,x)$. The conormal set $N_\rho^* = \{(y,x,x,0,\xi,-\xi)\}$ does not intersect the wave front set of w. Hence $\rho^* w(y,x) = K_T(y,x) \otimes \varphi(x)$ is well-defined and has, according to theorem 6.7.7, wave front set contained in the wave front set of K_T. Let $\Pi : Y \times X \to Y$ be the projection onto the first factor. To show that $T\varphi$ is smooth, we use the result of exercise 6.7.8. The conormal at (y,x) of the fiber $\Pi^{-1}(y)$ consists of points of the form $(y,x,\eta,0)$. The hypothesis of $WF(K_T)$ implies that the intersection of these two sets is empty. Hence, $T\varphi = \Pi_* w$ is smooth.

That under this condition T^t has a continuous extension from $D_c'(Y)$ to $D'(X)$ follows easily from this result and the closed graph theorem.

Using an entirely symmetric argument, it follows that if $WF(K_T) \cap 0_{T^*Y} \times T^*X = \emptyset$ then T can be extended by continuity from $D_c'(X)$ to $D'(Y)$. If both conditions are satisfied, the local version of the wave front set, with an argument similar to the one employed in the proof of theorem 6.7.8, implies the statement made about the wave front set of Tu. ∎

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