ITERATION AT THE BOUNDARY OF THE SPACE OF RATIONAL MAPS

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Abstract. Let Rat_d denote the space of holomorphic self-maps of $\mathbf{P}^1$ of degree $d \geq 2$, and μ_f the measure of maximal entropy for $f \in \mathrm{Rat}_d$. The map of measures $f \mapsto \mu_f$ is known to be continuous on Rat_d, and it is shown here to extend continuously to the boundary of Rat_d in $\mathbf{P}\,\mathrm{H}^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(d,1)) \simeq \mathbf{P}^{2d+1}$, except along a locus $I(d)$ of codimension $d+1$. The set $I(d)$ is also the indeterminacy locus of the iterate map $f \mapsto f^n$ for every $n \geq 2$. The limiting measures are given explicitly, away from $I(d)$. The degenerations of rational maps are also described in terms of metrics of non-negative curvature on the Riemann sphere: the limits are polyhedral.

For fixed $d > 1$, let Rat_d denote the space of holomorphic maps $f : \mathbf{P}^1 \to \mathbf{P}^1$ of degree d with the topology of uniform convergence. Fixing a coordinate system on the projective line, each such map can be expressed as a ratio of homogeneous polynomials $f(z:w) = (P(z,w) : Q(z,w))$, where P and Q have no common factors and are both of degree d. Parameterizing the space Rat_d by the coefficients of P and Q, we have

$$\mathrm{Rat}_d \simeq \mathbf{P}^{2d+1} - V(\mathrm{Res}),$$

where $V(\mathrm{Res})$ is the hypersurface of polynomial pairs (P,Q) for which the resultant vanishes. In particular, Rat_d is smooth and affine.

In this paper, we aim to describe the possible limiting behavior of a sequence of rational maps which diverges in Rat_d, in terms of the measures of maximal entropy and corresponding conformal metrics on the Riemann sphere. This is the first step in describing a natural compactification of this space, or a boundary of the corresponding moduli space $\mathrm{Rat}_d \,/\, \mathrm{PSL}_2\,\mathbf{C}$, which is well-behaved under iteration. A boundary of the moduli space has been studied by Milnor [Mi] and Epstein [Ep] in degree 2 and Silverman [Si] in all degrees, but iteration does not extend continuously to this boundary, as first seen in [Ep].

As $\mathrm{Rat}_d \subset \mathbf{P}^{2d+1}$, we associate to each point in $\mathbf{P}^{2d+1}$ a self-map of the Riemann sphere of degree $\leq d$. Namely, every $f \in \mathbf{P}^{2d+1}$ gives coefficients for a pair of homogeneous polynomials, defining a map on $\mathbf{P}^1$ away from finitely many **holes**, the shared roots of the pair of polynomials. Each hole comes with a multiplicity, the **depth** of the hole. See Section 1. We also

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define a probability measure μ_f for each $f \in \mathbf{P}^{2d+1}$. For $f \in \mathrm{Rat}_d$, we let μ_f be the unique measure of maximal entropy for f so that $\mu_f = \mu_{f^n}$ for all iterates of f. See [Ly],[FLM], and [Ma1]. For each $f \in V(\mathrm{Res}) \subset \mathbf{P}^{2d+1}$, the measure μ_f will be a countable sum of delta masses, with atoms at the holes of f and along their backward orbits.

The **indeterminacy locus** $I(d) \subset \mathbf{P}^{2d+1}$ in the boundary of Rat_d consists of degree 0 maps such that the constant value is also one of the holes. See §1. The codimension of $I(d)$ is $d+1$. For each $f \in V(\mathrm{Res}) - I(d)$, we will see in Section 2 that $\mu_f = \mu_{f^n}$ for all iterates of f. We prove,

Theorem 1. *Suppose $\{f_k\}$ is a sequence in Rat_d converging in $\mathbf{P}^{2d+1}$ to $f \in V(\mathrm{Res})$.*

(a) *If $f \notin I(d)$, then the measures of maximal entropy μ_{f_k} for f_k converge weakly to μ_f.*
(b) *If $f \in I(d)$, then any subsequential limit ν of the measures μ_{f_k} must satisfy $\nu(\{c\}) \geq d_c/(d_c+d)$, where $c \in \mathbf{P}^1$ is the constant value of f and a hole of depth $d_c \geq 1$.*

If $f \notin I(d)$ has a hole at $h \in \mathbf{P}^1$ of depth $d_h \geq 1$, then $\mu_f(\{h\}) \geq d_h/d$. In Section 5, Example 1, we provide examples in Rat_d for every $d \geq 2$ which realize the lower bound of part (b) when $d_c = 1$, so that $\nu(\{c\}) = 1/(d+1)$.

The iterate map. Theorem 1 is, in part, an extension of a result of Mañé which states that the measures of maximal entropy vary continuously (in the weak topology) over Rat_d [Ma2, Thm B]. The proof of Theorem 1 relies on the study of the iterate map $\Phi_n : \mathrm{Rat}_d \to \mathrm{Rat}_{d^n}$ which sends a rational map f to its n-th iterate f^n. The iterate map Φ_n extends to a rational map from $\mathbf{P}^{2d+1}$ to $\mathbf{P}^{2d^n+1}$. We obtain,

Theorem 2. *The following are equivalent:*

(i) *$g \in I(d) \subset \mathbf{P}^{2d+1}$,*
(ii) *the iterate map Φ_n is undefined at g for some $n \geq 2$,*
(iii) *the iterate map Φ_n is undefined at g for all $n \geq 2$, and*
(iv) *the map $f \mapsto \mu_f$ is discontinuous at g.*

In other words, the map of measures $f \mapsto \mu_f$ extends continuously from Rat_d to a point $g \in V(\mathrm{Res})$ if and only if the iterate map Φ_n extends continuously to g for some $n \geq 2$. The understanding of the iterate map also motivated Theorem 1, and from it we obtain the following corollary.

Corollary 3. *The iterate map $\Phi_n : \mathrm{Rat}_d \to \mathrm{Rat}_{d^n}$ given by $f \mapsto f^n$ is proper for all n and $d \geq 2$.*

Proof. The measure of maximal entropy μ_f for $f \in \mathrm{Rat}_d$ is always non-atomic, and the map $f \mapsto \mu_f$ is continuous (with the topology of weak convergence on the space of probability measures on the Riemann sphere) on Rat_d [Ma2, Thm B]. Now, suppose $\{f_k\}$ diverges in Rat_d. There exists a subsequence of the maximal measures μ_{f_k} which converges weakly to a measure ν. By Theorem 1, ν has atoms. Recall that for $f \in \mathrm{Rat}_d$, the measure μ_f is also the measure of maximal entropy for all the iterates of f. If for some n the iterates $\{f_k^n\}$ converge in Rat_{d^n}, the measures μ_{f_k} would have to converge to a non-atomic measure. □

Note that properness does *not* hold in degree 1, where $\mathrm{Rat}_1 \simeq \mathrm{PGL}_2\,\mathbf{C}$, since there are unbounded families of elliptic Möbius maps of finite order. Properness of the iterate map for degrees > 1 should be intuitively obvious, but analyzing its behavior near $I(d)$ is somewhat subtle.

Dynamics at the boundary. In Section 3, we describe the dynamics of a map in the boundary of $\mathrm{Rat}_d \subset \mathbf{P}^{2d+1}$. In particular, when $f \notin I(d)$, we can define the escape rate function G_F of f in $\mathbf{C}^2$ and show that it satisfies $dd^c G_F = \pi^* \mu_f$, just as for non-degenerate rational maps (see [HP, Thm 4.1]). In analogy with results for rational maps, the measure μ_f is also the limit of pull-backs of any probability measure on $\mathbf{P}^1$.

Metric convergence. Every rational map determines a conformal metric on $\mathbf{P}^1$ (unique up to scale) with non-negative distributional curvature equal to the measure of maximal entropy (see Section 6). The sphere with this metric can be realized as the intrinsic metric of a convex surface in $\mathbf{R}^3$ by a theorem of Alexandrov [Al, VII §7]. Each degenerate map $f \in V(\mathrm{Res})$ determines a conformal metric with countably many singularities, one for each atom of the measure μ_f. Theorem 1 together with [Re, Thm 7.3.1] implies,

Corollary 4. *Suppose f_k in Rat_d converges in $\mathbf{P}^{2d+1}$ to $f \notin I(d)$. Then the spheres with associated metrics have a convex polyhedral limit with distributional curvature $4\pi\mu_f$.*

The (countably many) cone points in the limiting metric of $f \notin I(d)$ have cone angles given by $2\pi - 4\pi\mu_g(\{p\})$ for each $p \in \mathbf{P}^1$, where a negative cone angle corresponds to a flared end. The metric convergence is uniform away from the flared ends.

The subspace of polynomials in Rat_d is very interesting by itself as the limiting measures should be always atomic with *finitely many* points in the support. See the examples in Section 7. The metrics associated to such measures are polyhedral with finitely many vertices. Thus, a boundary of the moduli space of polynomials could be described in terms of the geometry of convex polyhedra (as in e.g. [Th]). This perspective will be pursued further in a sequel.

Outline. In Section 1, we fix notation and define the probability measure μ_f for each $f \in \mathbf{P}^{2d+1}$. Section 2 is devoted to a study of the indeterminacy of the iterate map at the boundary of Rat_d. In Section 3, we study the dynamics of a map in the boundary, and we show the existence of the escape rate function. Theorem 1 is proved in Section 4, and Theorem 2 is proved in Section 5. The Alexandrov geometry of rational maps and Corollary 4 are described in Section 6. We conclude with some further examples in Section 7.

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1. Definitions and notation

For concreteness, we identify the space of rational maps, Rat_d, with an open subset of $\mathbf{P}^{2d+1}$, the space of coefficients of pairs of homogeneous polynomials of degree d in two variables, up to scale. Each point $f \in \mathbf{P}^{2d+1}$ gives coefficients for a pair of homogeneous polynomials,

$$f(z:w) = (P(z,w) : Q(z,w))$$

which is well-defined as a map on $\mathbf{P}^1$ away from the common roots of P and Q. If P and Q share a root, we say that f is **degenerate**, the common roots $\{h_i \in \mathbf{P}^1\}$ are the **holes** of f, and the multiplicity of h_i as a root of $\gcd(P,Q)$ is called the **depth** d_i of this hole. Note that away from its holes, the map f agrees with a rational map, denoted by φ, of degree $e < d$. Clearly, we have $d = e + \sum_i d_i$.

To fix notation, if $f \in \mathbf{P}^{2d+1}$ is degenerate, we write

$$f = (P : Q) = (Hp : Hq) = H\varphi,$$

where $H = \gcd(P,Q)$ is a homogeneous polynomial of degree $d-e$, and the holes of f are the zeroes of H on $\mathbf{P}^1$. The **indeterminacy locus** $I(d) \subset \mathbf{P}^{2d+1}$ is the set of degenerate maps $f = H\varphi$ for which φ is constant and this constant value is one of the holes of f; that is, f has the form $f(z:w) = (aP(z,w) : bP(z,w))$ for some $(a:b) \in \mathbf{P}^1$ with $P(a,b) = 0$. Equivalently, $I(d)$ is given by

$$I(d) = \{f = H\varphi \in V(\mathrm{Res}) : \deg\varphi = 0 \text{ and } \varphi^* H \equiv 0\}.$$

A simple dimension count shows that $I(d)$ has codimension $d+1$. In fact, the locus $I(d)$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^{d-1}$ by sending $f \in I(d)$ with $\varphi \equiv c$ to the pair (c, the unordered collection of $d-1$ other holes in $\mathbf{P}^1$).

Example. For $d = 1$, the space of coefficients is $\mathbf{P}^3$, the space of all non-zero two-by-two matrices M up to scale. In this degree, we have

$$I(1) = \{M : \mathrm{tr}\, M = \det M = 0\}.$$

Indeed, if $M \in I(1)$, by a change of coordinates, we can assume that M is the constant map infinity on $\mathbf{P}^1$ with one hole at infinity. In coordinates, $M(z:w) = (w:0)$, or rather, M is a matrix with one non-zero entry off the diagonal and zeroes elsewhere. Up to conjugacy, these are precisely the matrices with vanishing trace and determinant.

Holomorphic correspondences. The space of rational maps of degree d sits naturally as a Zariski open subset of $\mathbf{P}\,\mathrm{H}^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(d,1)) \simeq \mathbf{P}^{2d+1}$, the projectivized space of global sections of the line bundle $\mathcal{O}(d,1)$ on $\mathbf{P}^1 \times \mathbf{P}^1$. In particular, the graph of a non-degenerate rational map is identified with the zero locus of a section of $\mathcal{O}(d,1)$. In fact, every point in $\mathbf{P}\,\mathrm{H}^0(\mathbf{P}^1\times\mathbf{P}^1,\mathcal{O}(d,1))$ defines a holomorphic correspondence in $\mathbf{P}^1\times\mathbf{P}^1$ in the following way. A boundary point of Rat_d in $\mathbf{P}\,\mathrm{H}^0(\mathbf{P}^1\times\mathbf{P}^1,\mathcal{O}(d,1))$ corresponds exactly to a degenerate map where a hole of depth k gives rise to a "vertical component" of multiplicity k in the correspondence. More specifically, if $f=(P,Q)=H\varphi\in V(\mathrm{Res})$, the associated section

$$\mathbf{P}^1\times\mathbf{P}^1 \ni ((z:w),(x:y)) \mapsto yP(z,w)-xQ(z,w)$$

has zero locus

$$\Gamma_f = \{(p,\varphi(p))\in\mathbf{P}^1\times\mathbf{P}^1\}\cup\{h\times\mathbf{P}^1 : H(h)=0\},$$

where the vertical components come equipped with a multiplicity. From the definitions, we see that the indeterminacy locus $I(d)$ consists of correspondences with a flat horizontal component intersecting a vertical component on the diagonal. We will see later that if a degenerate map f has a vertical component at h, then f is to be interpreted as mapping the point h over the whole of $\mathbf{P}^1$.

Probability measures. We define a probability measure μ_f on $\mathbf{P}^1$ for each point $f\in\mathbf{P}^{2d+1}$. First, if f is a non-degenerate rational map, let μ_f be its unique measure of maximal entropy, given by the weak limit,

$$\mu_f = \lim_{n\to\infty}\frac{1}{d^n}\sum_{f^n(z)=a}\delta_z,$$

for any non-exceptional point $a\in\mathbf{P}^1$ [Ly],[FLM],[Ma1]. The measure μ_f has no atoms, and its support equals the Julia set of f; it is also the unique measure of maximal entropy for every iterate f^n.

If $f=H\varphi\in V(\mathrm{Res})$ is degenerate and φ is non-constant, we define an atomic measure,

$$\mu_f := \sum_{n=0}^{\infty}\frac{1}{d^{n+1}}\sum_i\sum_{\varphi^n(z)=h_i}\delta_z,$$

where the holes h_i and all preimages by φ are counted with multiplicity. Note that if the hole h_i has depth d_i, then $\mu_f(\{h_i\})\ge d_i/d$. If φ is constant, then the depths of the holes sum to d, and we set

$$\mu_f = \frac{1}{d}\sum_i\delta_{h_i},$$

where again the holes h_i are counted with multiplicity.

We will see in the following section that for every degenerate $f\notin I(d)$, we will have $\mu_f=\mu_{f^n}$ for all iterates of f.

Example. Suppose that $f(z:w)=(P(z,w):w^d)$ where $P\not\equiv 0$ is a homogeneous polynomial such that $P(1,0)=0$. Then f is degenerate with a hole at $\infty=(1:0)$, and φ is a polynomial in $\mathbf{C}$ of degree $<d$. Since the

backwards orbit of ∞ under any polynomial consists only of ∞ itself, we must have $\mu_f = \delta_\infty$.

Generally, when computing the μ_f-mass of a point for $f \in V(\mathrm{Res})$, one needs to count the number of times the forward iterates of the point land in a hole of f. The following lemma follows directly from the definition of the measure μ_f.

Lemma 5. *Let $f = H\varphi \in \mathbf{P}^{2d+1}$ be degenerate with $\deg\varphi > 0$. For each $a \in \mathbf{P}^1$, we have,*

$$\mu_f(\{a\}) = \frac{1}{d}\sum_{n=0}^{\infty} \frac{m(\varphi^n(a))d(\varphi^n(a))}{d^n},$$

where $d(\varphi^n(a))$ is the depth of $\varphi^n(a)$ as a hole of f and $m(\varphi^n(a))$ is the multiplicity of a as a solution of $\varphi^n(z) = \varphi^n(a)$.

The space $M^1(\mathbf{P}^1)$ of probability measures on $\mathbf{P}^1$ is given the weak topology. Note that $M^1(\mathbf{P}^1)$ is metrizable because it is a compact subset of the dual space to the separable $C(\mathbf{P}^1)$, the continuous functions on $\mathbf{P}^1$.

2. Iterating a degenerate map

The iterate map $\Phi_n : \mathrm{Rat}_d \to \mathrm{Rat}_{d^n}$, which sends f to f^n, is a regular morphism between smooth affine varieties. It extends to a rational map $\mathbf{P}^{2d+1} \dashrightarrow \mathbf{P}^{2d^n+1}$ for all $d \geq 1$. In this section, we will give a formula for the iterates of a degenerate map, where defined, and specify the indeterminacy locus of the iterate map Φ_n.

Let $(a_d, \ldots, a_0, b_d, \ldots, b_0)$ denote the homogeneous coordinates on $\mathbf{P}^{2d+1}$. The $2d^n+2$ coordinate functions which define the iterate map $\Phi_n : \mathbf{P}^{2d+1} \dashrightarrow \mathbf{P}^{2d^n+1}$ generate a homogeneous ideal I_n in the ring $A = \mathbf{Z}[a_d, \ldots, b_0]$. The ideal $I_1 = (a_d, \ldots, b_0)$, generated by all homogeneous monomials of degree 1 in A, is the ideal generated by the identity map Φ_1.

Lemma 6. *In the ring A, the ideals I_n are generated by homogeneous polynomials of degree $(d^n-1)/(d-1)$ and satisfy*

$$I_n \subset I_1 \cdot I_{n-1}^d$$

for all $n \geq 2$. In particular, they form a descending chain.

Proof. The affine space $\mathbf{C}^{2d+2}$ parameterizes, by the coefficients, all pairs $F = (P, Q)$ of degree d homogeneous polynomials in two variables. Such a pair defines a map $F : \mathbf{C}^2 \to \mathbf{C}^2$, and the composition map

$$\mathcal{C}_{d,e} : \mathbf{C}^{2d+2} \times \mathbf{C}^{2e+2} \to \mathbf{C}^{2de+2}$$

sending (F, G) to the coefficients of $F \circ G$ is bihomogeneous of degree $(1, d)$ in the coefficients of F and G. In particular, the second iterate Φ_2 is the (projectivization of the) restriction of $\mathcal{C}_{d,d}$ to the diagonal of $\mathbf{C}^{2d+2} \times \mathbf{C}^{2d+2}$, and so its coordinate functions are homogeneous of degree $1 + d$. Thus, the ideal I_2 in A generated by these coordinate functions of Φ_2 is contained in $I_1^{d+1} = I_1 \cdot I_1^d$.

For the general iterate, of course $F^n = F \circ F^{n-1}$, so Φ_n can be expressed as

$$\Phi_n = \mathcal{C}_{d,d^{n-1}} \circ (\mathrm{Id}, \Phi_{n-1}) : \mathbf{C}^{2d+2} \to \mathbf{C}^{2d^n+2}.$$

Consequently, Φ_n is homogeneous of degree $1 + d(\deg \Phi_{n-1})$. By induction, we have $\deg \Phi_n = 1+d+\cdots+d^{n-1} = (d^n-1)/(d-1)$. The above expression for Φ_n and the bihomogeneity of the composition map implies that the coordinate functions of Φ_n must lie in the ideal $I_1 \cdot I_{n-1}^d$. □

Recall from Section 1 that $I(d) \subset \mathbf{P}^{2d+1}$ is defined as the locus of degenerate constant maps such that the constant value is equal to one of the holes.

Lemma 7. *The indeterminacy locus for the iterate map* $\Phi_n : \mathbf{P}^{2d+1} \dashrightarrow \mathbf{P}^{2d^n+1}$ *is* $I(d)$ *for all* $n \geq 2$ *and all* $d \geq 1$. *If* $f = H\varphi \notin I(d)$ *is degenerate, then*

$$f^n = \left(\prod_{k=0}^{n-1} (\varphi^{k*} H)^{d^{n-k-1}} \right) \varphi^n.$$

Proof. Suppose that $f = (P : Q) = (Hp : Hq) = H\varphi$ is degenerate. The second iterate of f has the form,

$$\begin{aligned} f \circ f &= (P(P,Q) : Q(P,Q)) \\ &= (H^d P(p,q) : H^d Q(p,q)) \\ &= (H^d H(p,q) p(p,q) : H^d H(p,q) q(p,q)) \\ &= H^d \varphi^*(H) \varphi \circ \varphi. \end{aligned}$$

Since the map φ is non-degenerate, we will never have $\varphi \circ \varphi(z : w) = (0 : 0)$. However, we have $H(p,q) \equiv 0$ if and only if $\varphi(z : w) = (\alpha : \beta) \in \mathbf{P}^1$ for all $(z : w) \in \mathbf{P}^1$ and $H(\alpha, \beta) = 0$. This exactly characterizes the set $I(d)$ in $\mathbf{P}^{2d+1}$. Thus, for $f \notin I(d)$, the above gives the formula for the second iterate. An easy inductive argument gives the general form of the iterate f^n for all $f \notin I(d)$.

We have shown explicitly that the indeterminacy locus for the second iterate map Φ_2 is exactly $I(d)$. Since the formula for f^n holds and does not vanish identically for any $f \notin I(d)$, the indeterminacy locus of Φ_n must be *contained* in $I(d)$ for each $n \geq 3$. However, by Lemma 6, the chain of ideals defined by the iterate maps is descending. Thus, if the coordinate functions of Φ_2 vanish simultaneously along $I(d)$, the coordinate functions of Φ_n vanish simultaneously along $I(d)$ for all $n \geq 2$. Therefore, the indeterminacy locus is $I(d)$ for all n. □

Note that indeterminacy of the iterate map along $I(d)$ implies that Φ_n can not be extended continuously to any point $g \in I(d)$. See the two examples in Section 5. Observe also that Lemma 7 is a statement about the indeterminacy locus as a *set*. Scheme-theoretically, the indeterminacy depends on the iterate n.

The formula for the iterate map in 7 shows how holes develop when iterating a degenerate map. For example, if $f = H\varphi$ is degenerate of degree d, then the holes of the third iterate of f are the holes of f at depth d^2, the preimages by φ of these holes at depth d, and finally the preimages by φ^2 of the original holes. Comparing the iterate formula with the definition of the measure μ_f given in Section 1 (and Lemma 5), we obtain the following immediate corollary.

Corollary 8. *Let $f \notin I(d)$ be degenerate. If $d_z(f^n)$ denotes the depth of z as a hole of f^n, then*

$$\mu_f(\{z\}) = \lim_{n\to\infty} \frac{d_z(f^n)}{d^n}.$$

Furthermore, $\mu_f = \mu_{f^n}$ for all $n \geq 1$.

When viewed as a holomorphic correspondence (as in Section 1), the second iterate of a point $f \in \mathbf{P}^{2d+1}$ is given by

$$\Gamma_f \circ \Gamma_f = \{(z_1, z_2) \in \mathbf{P}^1 \times \mathbf{P}^1 : z_2 = f(y) \text{ and } y = f(z_1) \text{ for some } y \in \mathbf{P}^1\}.$$

With this notion of composition, it is easy to see that the indeterminacy locus $I(d)$ satisfies

$$I(d) = \{f \in \mathbf{P}\,\mathrm{H}^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(d,1)) : \Gamma_f \circ \Gamma_f = \mathbf{P}^1 \times \mathbf{P}^1\}.$$

For $f \notin I(d)$, the compostion is a well-defined section in $\mathbf{P}\,\mathrm{H}^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(d^2, 1))$, with vertical components carrying multiplicity given by the depths of the holes of the iterate. When iterating, as $n \to \infty$, Corollary 8 says that the vertical components carry an increasing percentage of the total degree, and their locations are at the atoms of μ_f.

3. The dynamics of a degenerate map

In this section we define the Julia set of degenerate map $f \notin I(d)$ and relate it to the support of the measure μ_f. We explain how μ_f is the weak limit of pull-backs by f of any probability measure on the Riemann sphere. Also, it is possible to define the escape rate function in $\mathbf{C}^2$ for every degenerate $f \notin I(d)$, and we will see that it is a potential for the atomic measure μ_f.

Let $f = H\varphi \notin I(d)$ be degenerate. As for rational maps, we define the Fatou set $\Omega(f)$ as the largest open set on which the iterates of f form a normal family. Care must be taken in this definition since we require, first, that the iterates f^n be well-defined for each n. Thus, the family $\{f^n|U\}_n$ can not be normal if $h_i \in f^n(U)$ for some $n \geq 1$ and some hole h_i of f. With this definition, we let $J(f)$ be the complement of $\Omega(f)$.

Let us assume for the moment that $\deg \varphi > 0$. By the definition of the Julia set $J(f)$, it is clear that

$$J(f) = J(\varphi) \cup \overline{\bigcup_{n=0}^{\infty} \bigcup_{i} \varphi^{-n}(h_i)}.$$

Recall that a point $z \in \mathbf{P}^1$ is said to be exceptional for φ if its backward orbit is finite. If at least one of the holes h_i is non-exceptional for the map φ, then $J(\varphi)$ is contained in the closure of the union of the preimages of h_i. Examining again the definition of the measure μ_f, we see that its support must be this closure of the union of all preimages of the holes of f. When $\deg \varphi = 0$, it makes sense to set $J(\varphi) = \emptyset$. We have proved the following.

Proposition 9. *Let $f = H\varphi \notin I(d)$ be degenerate. If one of the holes of f is non-exceptional for φ, then* $\operatorname{supp} \mu_f = J(f)$*. If each hole of f is exceptional for φ, then* $\operatorname{supp} \mu_f$ *is contained in the exceptional set of φ.*

Note that even if $J(\varphi) \subset \operatorname{supp} \mu_f$, it can happen that $\mu_f(J(\varphi)) = 0$, as all holes of f may lie in the Fatou set of φ.

Pullbacks of measures by degenerate maps. The holes of a degenerate map f are identified with the vertical components of the holomorphic correspondence of f, as described in Section 1. The degenerate f should be interpreted as sending each of its holes over the whole of $\mathbf{P}^1$ with appropriate multiplicity (the depth of the hole). In particular, pull-backs of measures can be appropriately defined, at least when $\deg \varphi \neq 0$, by integration over the fibers, as

$$\langle f^*\mu, \psi \rangle = \langle \varphi^*\mu, \psi \rangle + \sum_i \langle \delta_{h_i}, \psi \rangle = \int \sum_{\varphi(y)=z} \psi(y)\; d\mu(z) + \sum_i \psi(h_i),$$

where $\{h_i\} \subset \mathbf{P}^1$ is the set of holes of f and ψ is any continuous function on $\mathbf{P}^1$. All sums are counted with multiplicity. When $\deg \varphi = 0$ but $f \notin I(d)$, we can set

$$\langle f^*\mu, \varphi \rangle = \sum_i \psi(h_i).$$

Proposition 10. *For any degenerate $f \notin I(d)$ and any probability measure μ on $\mathbf{P}^1$, we have $f^{n*}\mu/d^n \to \mu_f$ weakly as $n \to \infty$.*

Proof. As the degree of φ is strictly less than d, we have

$$\frac{1}{d^n} \left| \langle \varphi^{n*}\mu, \psi \rangle \right| \to 0,$$

for all test functions ψ. From Corollary 8, the depths of the holes of the iterates of f limit on the mass μ_f. $\square$

Escape rate functions. The escape rate function of a rational map $f \in \mathrm{Rat}_d$, $d \geq 2$, is defined by

$$G_F(z, w) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F^n(z, w)\|, \tag{1}$$

where $F : \mathbf{C}^2 \to \mathbf{C}^2$ is a homogeneous polynomial map such that $\pi \circ F = f \circ \pi$. Here, $\pi : \mathbf{C}^2 - 0 \to \mathbf{P}^1$ is the canonical projection and $\|\cdot\|$ is any norm on $\mathbf{C}^2$. If F_1 and F_2 are two lifts of f to $\mathbf{C}^2$ (so that necessarily $F_2 = \alpha F_1$ for some $\alpha \in \mathbf{C}^*$), then $G_{F_1} - G_{F_2}$ is constant. The escape rate function is a potential for the measure μ_f in the sense that $\pi^*\mu_f = dd^c G_F$ [HP, Thm 4.1]. We use the notation $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)/2\pi$.

The proofs of the following proposition and Corollary 13 rely on the isomorphism between the space of probability measures on $\mathbf{P}^1$ and (normalized) plurisubharmonic functions U on $\mathbf{C}^2$ such that $U(\alpha z) = U(z) + \log|\alpha|$ for all $\alpha \in \mathbf{C}^*$ [FS, Thm 5.9]. The isomorphism is given by $\mu = \pi_* dd^c U$, from potential functions to measures.

Proposition 11. *For $d \geq 2$, the escape rate function G_F exists for each degenerate $f \notin I(d)$ and satisfies $\pi^*\mu_f = dd^c G_F$.*

We first need a lemma on Möbius transformations. Let σ denote the spherical metric on $\mathbf{P}^1$.

Lemma 12. *Let $E \subset \mathbf{P}^1$ be a finite set and $E(r) = \{z \in \mathbf{P}^1 : d_\sigma(z, E) \leq r\}$. For each Möbius transformation $M \in \mathrm{Rat}_1$, there exists r_0 such that*

$$\bigcup_{k\geq 0} M^k(E(r^k)) \neq \mathbf{P}^1$$

for all $r < r_0$.

Proof. By choosing coordinates on $\mathbf{P}^1 \simeq \overline{\mathbf{C}}$, we can assume that M has the form $M(z) = z + 1$ or $M(z) = \lambda z$ for $\lambda \in \overline{\mathbf{D}}$. In the new coordinate system, the spherical metric is comparable to the given metric σ.

When $|\lambda| = 1$, the statement is obvious for r_0 sufficiently small. When $|\lambda| < 1$, we need only consider the case when $\infty \in E$. For r small, a spherical disk of radius r around ∞ is comparable in size to the complement of the Euclidean disk of radius $1/r$ centered at 0, so we need to choose $r_0 < |\lambda|$.

Finally, suppose $M(z) = z + 1$. Again, we need only consider the case when $\infty \in E$. It is clear that the point $M^k(0) = k$, for example, remains inside the Euclidean disk of radius r^{-k} for all k if $r < 1$. Thus, $M^k(0)$ is outside the spherical disk of radius r^k about ∞ for all k. We can therefore choose any $r_0 < 1$. ☐

Proof of Proposition 11. Let $f = H\varphi \notin I(d)$ be degenerate. Expressing f in homogeneous coordinates defines a polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$ of (algebraic) degree d such that $\pi \circ F = f \circ \pi$ where defined. In particular, F vanishes identically along the lines $\pi^{-1}(h)$ for each hole h of f. We aim to define G_F by equation (1) as for non-degenerate maps, so we need to show that the limit exists. Write $F = H\Phi$, where Φ is a non-degenerate homogeneous polynomial map of degree $e < d$ (so that $\Phi^{-1}(\{0\}) = \{0\}$), and H is the gcd of the coordinate functions of F.

The iterate formula for f in Lemma 7 holds also for F so that

$$(2) \quad G_n(x) := \frac{1}{d^n} \log \|F^n(x)\| = \sum_{k=0}^{n-1} \frac{1}{d^{k+1}} \log |H(\Phi^k(x))| + \frac{1}{d^n} \log \|\Phi^n(x)\|$$

for all $x \in \mathbf{C}^2$.

Suppose first that $e = \deg \varphi = 0$, so that $F(z, w) = (aH(z, w), bH(z, w))$ and $H(a, b) \neq 0$. The above expression for G_n is simply

$$G_n(z, w) = \frac{1}{d} \log |H(z, w)| + \sum_{1}^{n-1} \frac{1}{d^{k+1}} \log |H(a, b)| + \frac{1}{d^n} \log \|(a, b)\|,$$

which converges to

$$G_F(z, w) = \frac{1}{d} \log |H(z, w)| + \frac{1}{d-1} \log |H(a, b)|$$

locally uniformly on $\mathbf{C}^2 - 0$ as $n \to \infty$. Furthermore, this function G_F is clearly a potential for the atomic measure

$$\mu_f = \frac{1}{d} \sum_{z \in \mathbf{P}^1 : H|\pi^{-1}(z)=0} \delta_z$$

on $\mathbf{P}^1$, where the zeros of H are counted with multiplicity.

Now suppose that $e = \deg \varphi > 0$. Then there exists a constant $K > 1$ so that for all $x \in \mathbf{C}^2$,

$$K^{-1} \|x\|^e \leq \|\Phi(x)\| \leq K \|x\|^e$$

and therefore if $x \neq 0$,

$$|\log \|\Phi(x)\|| \leq e |\log \|x\|| + \log K.$$

Replacing x with the iterate $\Phi^{n-1}(x)$ we obtain by induction on n,

$$|\log \|\Phi^n(x)\|| \leq e^n |\log \|x\|| + (1 + e + \cdots + e^{n-1}) \log K.$$

Dividing by d^n gives

$$(3) \qquad \frac{1}{d^n} \log \|\Phi^n(x)\| \to 0$$

locally uniformly on $\mathbf{C}^2 - 0$ as $n \to \infty$, since $e < d$. Similiarly, the quantity

$$(4) \qquad \sum_{0}^{n-1} \frac{1}{d^{k+1}} \log \|\Phi^k(x)\| - \sum_{0}^{n-1} \frac{e^k}{d^{k+1}} \log \|x\|$$

is uniformly bounded in n on $\mathbf{C}^2 - 0$.

Consider the plurisubharmonic function

$$g_n(x) = \sum_{k=0}^{n-1} \frac{1}{d^{k+1}} \log |H(\Phi^k(x))|,$$

on $\mathbf{C}^2$. Notice that g_n defines a potential function for the atomic measure

$$\mu_n = \sum_i \sum_{k=0}^{n-1} \frac{1}{d^{k+1}} \sum_{z:\varphi^k(z)=h_i} \delta_z,$$

where the h_i are the holes of f, counted with multiplicity; that is, $\pi^*\mu_n = dd^c g_n$ on $\mathbf{C}^2 - 0$. Note also that the measure μ_n has total mass $1-(e/d)^n$, and g_n scales by

$$g_n(\alpha x) = g_n(x) + (1-(e/d)^n)\log|\alpha|$$

for all $\alpha \in \mathbf{C}^*$. The measures μ_n converge weakly to μ_f in $\mathbf{P}^1$. By (2) and (3), for any $\varepsilon > 0$, we have

$$|G_n(x) - g_n(x)| < \varepsilon \tag{5}$$

for all sufficiently large n, locally uniformly in $\mathbf{C}^2 - 0$. We will show that the functions g_n converge in L^1_{loc} to the unique potential function of μ_f.

If the sequence g_n is uniformly bounded above on compact sets and does not converge to $-\infty$ locally uniformly, then some subsequence converges in L^1_{loc} [FS, Thm 5.1]. For an upper bound, note first that

$$\sup\{\log|H(x)| : \|x\| \le 1\} < \infty. \tag{6}$$

If for each $x \neq 0$ in $\mathbf{C}^2$, we set $x^1 := x/\|x\|$, then $H(x) = \|x\|^{d-e}H(x^1)$, and therefore,

$$g_n(x) = \sum_0^{n-1} \frac{d-e}{d^{k+1}} \log\|\Phi^k(x)\| + \sum_0^{n-1} \frac{1}{d^{k+1}} \log|H(\Phi^k(x)^1)|. \tag{7}$$

The bound on (4) together with (6) show that $\{g_n\}$ is uniformly bounded above on compact sets.

To obtain a convergent subsequence of $\{g_n\}$ in L^1_{loc}, it suffices now (by [FS, Thm 5.1]) to show that $g_n \not\to -\infty$ uniformly on compact sets. If g_{n_j} converges to v in L^1_{loc}, then by [FS, Thm 5.9], v is the potential function for the measure μ_f, unique up to an additive constant. To conclude, therefore, that the full sequence g_n converges so that $\pi^*\mu_f = dd^c G_F$, it suffices to show that there exists a single point $x \in \mathbf{C}^2 - 0$ for which $\lim_{n\to\infty} g_n(x)$ exists and is finite.

When $e \ge 2$, choose any point $x \in \mathbf{C}^2 - 0$ which is periodic for Φ and whose orbit does not intersect the complex lines $\pi^{-1}(h_i)$ over the holes of f. Then the orbit $\Phi^k(x)$, $k \ge 0$, remains a bounded distance away from the zeroes of H. Therefore, $\log|H(\Phi^k(x))|$ is bounded above and below, so that the definition of g_n together with (5) imply that

$$G_F(x) = \lim_{n\to\infty} G_n(x) = \lim_{n\to\infty} g_n(x)$$

exists and is finite. Therefore, $G_F \in L^1_{loc}$ is the potential function for μ_f.

When $e = 1$ a further estimate is required. The map φ on $\mathbf{P}^1$ is a Mobius transformation. Let E be the set of zeroes of H projected to $\mathbf{P}^1$. Let σ

denote the spherical metric on $\mathbf{P}^1$, and observe that there exists $C > 1$ and a positive integer m such that

$$C \geq |H(x)| \geq C^{-1} d_\sigma(\pi(x), E)^m \text{ for all } \|x\| = 1. \tag{8}$$

By Lemma 12 applied to $M = \varphi^{-1}$, the set $A = \mathbf{P}^1 - \cup_{k\geq 0}\varphi^{-k}E(r^k)$ is non-empty for some $r > 0$. Fix $z \in A$ and choose $Z \in \pi^{-1}(z)$ with $\|Z\| = 1$. Using (8), our choice of Z implies that

$$\log C \geq \log |H(\Phi^k(Z)^1)| \geq mk \log r - \log C,$$

for all $k \geq 1$, where $\Phi^k(Z)^1 = \Phi^k(Z)/\|\Phi^k(Z)\|$. Note that since Φ is linear, there is a constant $C' > 1$ such that

$$|\log \|\Phi^k(Z)\|| \leq k \log C'.$$

Combining these estimates, we obtain

$$\sum_n^m \left| \frac{d-1}{d^{k+1}} \log \|\Phi^k(Z)\| + \frac{1}{d^{k+1}} \log |H(\Phi^k(Z)^1)| \right| \leq C'' \sum_n^m \frac{k}{d^k},$$

for some constant C'' which implies, by (7), that $g_n(Z)$ has a finite limit as $n \to \infty$. □

As a corollary to Theorem 1(a), we obtain

Corollary 13. *Suppose that a sequence f_k in Rat_d converges in $\mathbf{P}^{2d+1}$ to $f \notin I(d)$. Then for suitably normalized lifts F_k of f to $\mathbf{C}^2$ and F of f, the escape rate functions G_{F_k} converge to G_F in L^1_{loc}.*

Proof. Again by [FS, Thm 5.9], weak convergence of measures $\mu_{f_k} \to \mu_f$ by Theorem 1(a) implies that the potentials converge in L^1_{loc}. A normalization is required to guarantee convergence; it suffices to choose the unique lifts F_k of f_k such that $\sup\{G_{F_k}(x) : \|x\| = 1\} = 0$. □

4. Proof of Theorem 1

Here we provide the statements needed for the proof of Theorem 1. The argument relies on a fundamental fact about holomorphic functions in $\mathbf{C}$: a proper holomorphic function from a domain U to a domain V in $\mathbf{C}$ has a well-defined degree. This together with the invariance property of the maximal measure, $f^*\mu_f/d = \mu_f$ for all $f \in \mathrm{Rat}_d$, and the continuity of the iterate map away from $I(d)$ will give Theorem 1. For simplicity, we will regularly identify the point $(z : w) \in \mathbf{P}^1$ with $z/w \in \overline{\mathbf{C}}$.

Lemma 14. *Suppose a sequence $f_k \in \mathrm{Rat}_d$ converges in $\mathbf{P}^{2d+1}$ to degenerate $f = (P : Q)$ and f has a hole at h of depth d_h. If neither P nor Q are $\equiv 0$, then any neighborhood N of h contains at least d_h zeroes and poles of f_k (counted with multiplicity) for all sufficiently large k.*

Proof. As the coefficients of $f_k = (P_k : Q_k)$ converges to those of f, then so must the roots of the polynomials P_k and Q_k converge to those of P and Q. If the degenerate map f has a hole at $h \in \mathbf{P}^1$ of depth d_h, then at least d_h roots of P and d_h roots of Q must limit on h. □

Lemma 15. *Suppose $f_k \in \mathrm{Rat}_d$ converge to $f = H\varphi$ in $\mathbf{P}^{2d+1}$. Then as maps, $f_k \to \varphi$ locally uniformly on $\mathbf{P}^1 - \{h_i\}_i$ where the h_i are the holes of f.*

Proof. Write $f_k = (P_k : Q_k)$ and $f = (P : Q)$. By changing coordinates, we may assume that neither P nor Q are $\equiv 0$ and also that no holes lie at the point $\infty = (1 : 0)$. Writing $f(z : w) = (H(z, w)p(z, w) : H(z, w)q(z, w))$, where $H = \gcd(P, Q)$ we may assume that H is monic as a polynomial in z of degree $d - e$. Fix an open set $U \subset \mathbf{P}^1$ containing all holes of f. By the previous lemma there are homogeneous factors $A_k(z, w)$ of P_k and $B_k(z, w)$ of Q_k of degree $d - e$ with all roots inside U. As polynomials of z, we may assume that A_k and B_k are monic, and thus A_k and B_k both tend to H as $k \to \infty$. On the compact set $\mathbf{P}^1 - U$, the ratio A_k/B_k must tend uniformly to 1 as $k \to \infty$, so we have $f_k = (P_k : Q_k)$ limiting on $\varphi = (p : q)$ uniformly on $\mathbf{P}^1 - U$. □

For the proof of the theorem, we will need a uniform version of Lemma 14; namely, there should be preimages of almost every point (with respect to the maximal measure of f_k) inside a small neighborhood of the holes. As we shall see, this can be done when the limit map is not in $I(d)$. Uniformity fails in general, and this failure leads to the discontinuity of the measure in Theorem 2.

Proposition 16. *Suppose the sequence f_k in Rat_d converges to $f = H\varphi \in V(\mathrm{Res}) - I(d)$. If f has a hole at h of depth d_h, then any weak limit ν of the maximal measures μ_{f_k} of f_k must have $\nu(\{h\}) \geq d_h/d$.*

Proposition 17. *Suppose the sequence f_k in Rat_d converges to $f = H\varphi \in I(d)$, If f has a hole of depth d_c at c where $\varphi \equiv c$, then any weak limit ν of the maximal measures μ_{f_k} of f_k must have $\nu(\{c\}) \geq d_c/(d_c + d)$.*

The proof of Proposition 16 follows from the following two lemmas.

Lemma 18. *Suppose under the hypotheses of Proposition 16 that φ is non-constant. Then for any neighborhood N of the hole h, there exists an $M > 0$ such that*

$$\#\{f_k^{-1}(a) \cap N\} \geq d_h,$$

for all $a \in \mathbf{P}^1$ and all $k > M$, where the preimages are counted with multiplicity.

Proof. Suppose that in local coordinates at h and $\varphi(h)$, we can write $\varphi(z) = cz^m + O(z^{m+1})$, $m > 0$. Choose a disk D around $\varphi(h)$ small enough that

(i) D does not contain both 0 and ∞,
(ii) the component of $\varphi^{-1}(D)$ containing h is a disk inside N, and

(iii) the component of $\varphi^{-1}(D)$ containing h maps m-to-1 over D.

Let this component of the preimage of D be denoted by E.

By uniform convergence of f_k to φ away from the holes of f (Lemma 15), for all sufficiently large k, f_k maps a curve close to ∂E m-to-1 over ∂D, and by Lemma 14, d_h zeros or poles lie very close to h. (Note that the hypothesis of Lemma 14 is automatically satisfied when φ is non-constant.) Let E_k denote the disk containing h bounded by the component of $f_k^{-1}(\partial D)$ which is very close to ∂E. Consider the preimage $A_k = f_k^{-1}(\mathbf{P}^1 - D) \cap E_k$. As f_k is proper on A_k, it has a well-defined degree. Counting zeros or poles in A_k, we find the degree is d_h. The map f_k then has degree d_h also on the boundary of A_k. Now, f_k is also proper on the complement of A_k in E_k. Counting degree on its boundary, $d_h + m$, we find that f_k has at least d_h preimages of *all* points of the sphere inside N. $\square$

Lemma 19. *Suppose under the hypotheses of Proposition 16 that φ is constant. Then for any neighborhood N of h, there exists an $M > 0$ so that*

$$\#\{f_k^{-1}(a) \cap N\} \geq d_h,$$

for all $k > M$ and all $a \in \operatorname{supp}(\mu_{f_k})$, where the preimages are counted with multiplicity.

Proof. Suppose that $\varphi \equiv c$. By changing coordinates if necessary, we can assume that the point c is neither $0 = (0 : 1)$ nor $\infty = (1 : 0)$ so that the hypothesis of Lemma 14 is satisfied. By assumption, c is not one of the holes of f. Let D be a disk around c so that all holes and one of 0 or ∞ lie outside D. Let B be a ball around h contained in N. For all large k, f_k has d_h zeros and poles inside B and f_k maps the complement of B (minus a neighborhood of other holes) to D.

It is clear that for these large k, D does not interesect the Julia set of f_k, since $f_k(D)$ is contained in D so the iterates must form a normal family on D.

Consider the preimage $A_k = f_k^{-1}(\mathbf{P}^1 - D) \cap B$. The map f_k is proper on A_k and has a well-defined degree. Counting zeros or poles, this degree is d_h. Since $\operatorname{supp} \mu_{f_k}$ lies in $\mathbf{P}^1 - D$, the lemma is proved. $\square$

Proof of Proposition 16. This follows immediately from the invariance property of μ_{f_k}:

$$f_k^* \mu_{f_k} = d\, \mu_{f_k}.$$

Fix a disk D around the hole h. Choose a bump function b which is one on a disk of half the radius and 0 outside D. Then

$$\mu_{f_k}(D) \geq \int b\, d\mu_{f_k} = \frac{1}{d} \int \sum_{f_k(x)=y} b(x)\, d\mu_{f_k}(y).$$

By Lemmas 18 and 19, for all sufficiently large k, the sum in the integrand is $\geq d_h$ for every y in the support of μ_{f_k}. Taking limits and letting D shrink down to h gives the result. $\square$

Proof of Proposition 17. By changing coordinates if necessary, we can assume that the constant c is neither $0 = (0:1)$ nor $\infty = (1:0)$ so that the hypothesis of Lemma 14 is satisfied. Let D be a disk around c which does not contain both 0 and ∞. Let N be a neighborhood of the holes of g such that $\overline{N} \cap \partial D = \emptyset$. Now choose M large enough that $f_k(\mathbf{P}^1 - N) \subset\subset D$ for all $k > M$. Let $A_k = f_k^{-1}(\mathbf{P}^1 - D) \cap D$. The map f_k is proper on A_k and has at least d_c zeroes or poles, so it is at least d_c-to-1 over the complement of D.

Let ν be any subsequential weak limit of μ_{f_k}. Let $\nu_D = \nu(\mathbf{P}^1 - \overline{D})$. If $\nu_D = 0$ and this holds for all D, then $\nu(\{c\}) = 1$ and the proposition is proved. Generally, for any $\varepsilon > 0$ we have $\mu_{f_k}(\mathbf{P}^1 - \overline{D}) \geq \nu_D - \varepsilon$ for all large k. By the invariance property of μ_{f_k} (as in the proof of the previous proposition), $\mu_{f_k}(D) \geq (\nu_D - \varepsilon)d_c/d$. Thus $\nu(D) \geq (\nu_D - \varepsilon)d_c/d$. Since ε is arbitrary and since ν_D increases (to some value $\nu_0 \leq 1$) as D shrinks, we obtain $\nu(\{c\}) \geq \nu_0 d_c/d$.

On the other hand,

$$1 = \nu_0 + \nu(\{c\}) \geq \nu_0 + \nu_0 d_c/d = \nu_0(1 + d_c/d),$$

and therefore $\nu_0 \leq d/(d_c + d)$. Consequently, $\nu(\{c\}) \geq d_c/(d_c + d)$. □

Proof of Theorem 1. By Lemma 7, the iterate map Φ_n is continuous on $\mathbf{P}^{2d+1} - I(d)$ for every n. Thus, if $f_k \to f \notin I(d)$, then $f_k^n \to f^n$ where the iterate of a degenerate map is described explicitly in Lemma 7. Note that $1/d^n$ multiplied by the depth of a hole of f^n can only increase as $n \to \infty$. As stated in Corollary 8, the depths limit on the mass given by μ_f.

Since the maximal measure for a rational map is the same as the measure for any iterate, Proposition 16 implies that any subsequential limit ν of the measures has *at least* the correct mass on all the points in $\operatorname{supp} \mu_f$. On the other hand, these masses sum to 1, and the measure is a probability measure, so in fact, $\nu = \mu_f$. This proves part (a). Part (b) is exactly the statement of Proposition 17. □

5. Examples and proof of Theorem 2

In this section we complete the proof of Theorem 2. We begin with some examples demonstrating the discontinuity of the iterate maps and, consequently, the discontinuity of the map of measures $f \mapsto \mu_f$ at each point in $I(d)$. Example 1 realizes the lower bound of Theorem 1(b) when the depth d_c is 1. For notation, we will regularly identify the point $(z:w) \in \mathbf{P}^1$ with the ratio z/w in $\overline{\mathbf{C}}$.

Example 1. Let $g = (wP(z,w) : 0) \in I(d)$ where P is homogeneous of degree $d-1$, $P(0,1) \neq 0$, $P(1,0) \neq 0$, and P is monic as a polynomial in z. Then g has a hole of depth 1 at ∞ and no holes at 0. For each $a \in \mathbf{C}^*$ and $t \in \mathbf{D}^*$, consider

$$g_t^a(z:w) := (atz^d + wP(z,w) : tz^d) \in \mathrm{Rat}_d\,.$$

Clearly $g_t^a \to g$ in $\mathbf{P}^{2d+1}$ as $t \to 0$, so that as maps, g_t^a converges to the constant ∞, uniformly away from ∞ and the roots of P (by Lemma 15). The second iterate $\Phi_2(g_t^a)$ has the form

$$(aw^d P(z,w)^d t + z^d w^{d-1} P(z,w)^{d-1} t + O(t^2) : w^d P(z,w)^d t + O(t^2)),$$

and taking a limit as $t \to 0$, we obtain

$$\Phi_2(g_t^a) \to f^a := (w^{d-1} P(z,w)^{d-1}(awP(z,w) + z^d) : w^d P(z,w)^d).$$

Thus the second iterates converge (uniformly, away from the holes of f^a) to the map φ^a given (in coordinates on $\mathbf{C}$) by

$$\varphi^a(z) = \frac{aP(z) + z^d}{P(z)}.$$

Recall that we are assuming $P(0) \neq 0$ so that φ^a is a non-degenerate rational map of degree d for all $a \in \mathbf{C}^*$. As P is monic and of degree $d-1$, it is clear that each φ^a has a parabolic fixed point at ∞. We see immediately that the limit as $t \to 0$ of second iterates of g_t^a depends on the direction of approach, parameterized here by $a \in \mathbf{C}^*$. The degenerate limits f^a all have holes at ∞ of depth $d-1$ and holes at the roots of P each of depth $d-1$.

The degenerate maps $f^a \in \mathbf{P}^{2d^2+1}$ do not lie in $I(d^2)$, so we are able to compute the limiting measures of $\mu_{g_t^a}$ as $t \to 0$ from Theorem 1(a). Of course, the maximal measure for g_t^a coincides with that of its second iterate, so $\mu_{g_t^a} \to \mu_{f^a}$ weakly for each $a \in \mathbf{C}^*$ as $t \to 0$.

The measures μ_{f^a} cannot be the same for all $a \in \mathbf{C}^*$: the holes are the same for each a but the preimages of the roots of P by φ^a depend on a, and these are atoms of μ_{f^a}. For example, suppose $\alpha \neq 0$ is a simple root of $P(z)$. Then for $a = \alpha$, the d solutions to $\varphi^\alpha(z) = \alpha$ are all at 0, so that by Lemma 5,

$$\mu_{f^\alpha}(\{0\}) = \frac{1}{d^2} \sum_{n=1}^{\infty} \frac{d(d-1)}{d^{2n}} = \frac{1}{d(d+1)}.$$

On the other hand, for the generic $a \in \mathbf{C}^*$, the φ^a-orbit of the point 0 never intersects the roots of P, so that $\mu_{f^a}(\{0\}) = 0$.

Lastly, since the limiting maps f^a have holes of depth $d-1$ at ∞ and $\varphi^a(\infty) = \infty$ for each $a \in \mathbf{C}^*$, we can compute easily from Lemma 5 that

$$\mu_{f^a}(\{\infty\}) = \frac{1}{d^2} \sum_{0}^{\infty} \frac{d-1}{d^{2n}} = \frac{1}{d+1},$$

for all $a \in \mathbf{C}^*$. As the degeneration of g_t^a to g develops a hole of depth $d_\infty = 1$ at ∞, we see that this family achieves the lower bound of Theorem 1(b).

Example 2. Let $g = (w^k P(z,w) : 0) \in I(d)$ where P is homogeneous of degree $d-k$, $k > 1$, $P(0,1) \neq 0$, $P(1,0) \neq 0$, and P is monic as a polynomial in z (or $\equiv 1$ if $k = d$). Then g has a hole of depth k at ∞ and no holes at 0. Consider first the family, as in Example 1, given by

$$h_t^a = (atz^d + w^k P(z,w) : tz^d) \in \mathrm{Rat}_d,$$

for $a \in \mathbf{C}^*$ and $t \in \mathbf{D}^*$. Computing second iterates and taking a limit as $t \to 0$ gives

$$\Phi_2(h_t^a) \to h^a := (aw^{kd}P(z,w)^d : w^{kd}P(z,w)^d),$$

and the degenerate h^a has associated lower-degree map $\equiv a$. That is, the maps h_t^a converge to the constant ∞ map but their second iterates converge to the constant a. Furthermore, $h^a \in I(d^2)$ if and only if $P(a,1) = 0$. By Theorem 1(a), when $P(a,1) \neq 0$, the maximal measures of h_t^a converge weakly to

$$\mu_{h^a} = \frac{k}{d}\delta_\infty + \frac{1}{d}\sum_{z:P(z)=0} \delta_z = \mu_g.$$

These measures do not depend on a.

Let us now generalize Example 1 in the following way: for each $a \in \mathbf{C}^*$ and $t \in \mathbf{D}^*$, consider

$$g_t^a(z : w) = (at^k z^d + w^k P(z,w) : tz^{d-k+1}w^{k-1}) \in \mathrm{Rat}_d\,.$$

As $t \to 0$, we have $g_t^a \to g$ in $\mathbf{P}^{2d+1}$. The second iterate $\Phi_2(g_t^a)$ has the form

$$\begin{aligned}(at^k w^{kd} P^d + t^k z^{k(d-k+1)} w^{k(k-1)+k(d-k)} P^{d-k} + O(t^{k+1}) :&\\ t^k z^{(d-k+1)(k-1)} w^{(k-1)^2+k(d-k+1)} P^{d-k+1} + O(t^{k+1})),&\end{aligned}$$

so that in the limit as $t \to 0$, $\Phi_2(g_t^a)$ converges to

$$f^a := (w^{k(d-1)}P^{d-k}(aw^kP^k + z^{k(d-k+1)}) : w^{k(d-1)+1}P^{d-k+1}z^{(d-k+1)(k-1)}).$$

Thus, the second iterates converge, away from the holes of f^a, to a map of degree $k(d-k+1)$ given in coordinates on $\mathbf{C}$ by

$$\varphi^a(z) = \frac{z^{k(d-k+1)} + aP(z)^k}{z^{(k-1)(d-k+1)}P(z)}.$$

Since P is monic of degree $d-k$, φ^a has a parabolic fixed point at ∞ for all $a \in \mathbf{C}^*$. The point ∞ is a hole for f^a of depth $k(d-1)$, so with Lemma 5 we compute,

$$\mu_{f^a}(\{\infty\}) = \frac{1}{d^2}\sum_{n=0}^\infty \frac{k(d-1)}{d^{2n}} = \frac{k}{d+1} > \frac{k}{d+k},$$

and we see how the mass at ∞ compares with the lower bound of Theorem 1(b).

Finally, for different values of $a \in \mathbf{C}^*$, the measures μ_{f^a} are distinct. For $k > 1$, the point 0 has the same μ_{f^a}-mass for all $a \in \mathbf{C}^*$ because it is a preimage of the hole at ∞. However, the preimages of 0 by φ^a vary with a, so it's not hard to see that the measures must vary too.

Question. What is the best lower bound in the statement of Theorem 1(b) for $d_c > 1$? In Example 2, the limiting mass at the constant value $c = \infty$ is $d_c/(d+1)$.

Proof of Theorem 2. The equivalence of properties (i), (ii), and (iii) was established by Lemma 7 in Section 2. For the implication (iv) implies (i), note first that within the space Rat_d, the map $f \mapsto \mu_f$ is continuous by [Ma2, Thm B]. Also, if $g \in V(\mathrm{Res}) - I(d)$, Theorem 1(a) implies that $f \mapsto \mu_f$ extends continuously from Rat_d to g. Suppose now that $g_k \to g$ where $g_k \in V(\mathrm{Res}) - I(d)$ for all k and $\mu_{g_k} \not\to \mu_g$. Then there exists an open set U in $M^1(\mathbf{P}^1)$, the space of probability measures on $\mathbf{P}^1$, such that U contains infinitely many of the measures μ_{g_k} but $\mu_g \notin \overline{U}$. For each k with $\mu_{g_k} \in U$, there exists $f_k \in \mathrm{Rat}_d$ with $\mu_{f_k} \in U$ by Theorem 1(a). However, in this way we can construct a sequence of non-degenerate rational maps converging to g in $\mathbf{P}^{2d+1}$ but such that $\mu_{f_k} \not\to \mu_g$. Theorem 1(a) implies that $g \in I(d)$.

Conversely, fix $g = H\varphi \in I(d)$. By the definition of $I(d)$, φ must be constant, so by a change of coordinates we can assume that $g(z:w) = (w^k P(z,w) : 0)$, where P is homogeneous of degree $d-k$, so that φ is the constant infinity map and ∞ is a hole of g of depth k. We can also assume that $P(0,1) \neq 0$ so that 0 is *not* a hole of g. Thus, g is exactly one of Examples 1 or 2 above, depending on the depth k at ∞. In each case, the family g_t^a, $a \in \mathbf{C}^*$, converges to g as $t \to 0$ and demonstrates the discontinuity of both the iterate map Φ_2 and of $f \mapsto \mu_f$ at g. This completes the proof of Theorem 2. $\square$

6. Limiting metrics

In this section, we present details for Corollary 4 which reinterprets Theorem 1(a) in terms of conformal metrics on the Riemann sphere: a degenerating sequence of rational maps has a convex polyhedral limit (with countably many vertices). We apply the work of Reshetnyak on conformal metrics in planar domains [Re] and invoke the realization theorem of Alexandrov [Al, VII §7] for metrics of non-negative curvature on a sphere to put this metric convergence into context.

In Section 3 we defined the escape rate function $G_F : \mathbf{C}^2 \to \mathbf{R} \cup \{-\infty\}$ of a rational map $f \in \mathrm{Rat}_d$. As explained in [De, §12], a hermitian metric on the tautological bundle $\tau \to \mathbf{P}^1$ is defined by

$$\|v\|_F = \exp G_F(v)$$

for all $v \in \mathbf{C}^2$. A metric ρ_f is then induced on the tangent bundle $T\mathbf{P}^1 \simeq \tau^{-2}$, uniquely up to scale. Since G_F is a potential function for the measure of maximal entropy μ_f of f, we find that the curvature form of ρ_f (in the sense of distributions) is $4\pi\mu_f$. In particular, the metric is Euclidean flat on the Fatou components.

Example. Let $p \in \mathrm{Rat}_d$ be a polynomial and $P(z,w) = (p(z/w)w^d, w^d)$ a lift of p to $\mathbf{C}^2$. The escape rate function on $\mathbf{C}$ defined by

$$G_p(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |p^n(z)|$$

satisfies

$$G_P(z,w) = G_p(z/w) + \log|w|,$$

where G_P is the escape rate function in $\mathbf{C}^2$ [HP, Prop 8.1]. The associated conformal metric on $\mathbf{C}$ is given by

$$\rho_p = e^{-2G_p(z)}|dz|,$$

which is isometric to the flat planar metric on the filled Julia set,

$$K(p) = \{z \in \mathbf{C} : G_p(z) = 0\} = \{z \in \mathbf{C} : p^n(z) \not\to \infty \text{ as } n \to \infty\}.$$

An important theorem of Alexandrov says that any intrinsic metric of non-negative curvature on S^2 (or $\mathbf{P}^1$) can be realized as the induced metric on a convex surface in $\mathbf{R}^3$ (or possibly a doubly-covered convex planar domain). In particular, our metrics ρ_f on $\mathbf{P}^1$ can be identified with convex shapes in $\mathbf{R}^3$, unique up to scale and the isometries of $\mathbf{R}^3$. Conversely, too, the induced metric on any convex surface in $\mathbf{R}^3$ can be uniformized to define a conformal metric on $\mathbf{P}^1$. See [Al, VII §7] or [Po, Ch.1].

Corollary 4 addresses the question of how these metrics degenerate in an unbounded family of rational maps. By Theorem 1(a), when a sequence $f_k \in \mathrm{Rat}_d \subset \mathbf{P}^{2d+1}$ converges to $f \notin I(d)$, the curvatures of the associated metrics converge weakly to a purely atomic measure μ_f. Each degenerate map has an associated metric which is **polyhedral**, in the sense that its curvature can be expressed as a countable sum of delta masses.

A detailed description of the relation between conformal metrics in $\mathbf{C}$ and logarithmic potentials and curvature distributions is given in [Re, §7]. For example, if an atomic curvature distribution $4\pi\mu$ has an isolated atom of mass m at point $z_0 \in \mathbf{C}$, then the metric is locally represented by

$$|z - z_0|^{-m/2\pi} e^{h(z)} |dz|,$$

where h is harmonic near z_0. This formulation shows that a neighborhood of z is isometric to a Euclidean cone of angle $2\pi - m$. If the cone angle is negative, then the point z_0 is “at infinity” in this metric; we will say the sphere has a **flared end** at z_0.

Finally, Reshetnyak proved a convergence theorem about these conformal metrics [Re, Thm 7.3.1]. Suppose that ρ_k and ρ are metrics on $\mathbf{P}^1$ with curvature distributions $4\pi\mu_k$ and $4\pi\mu$, where μ_k and μ are probability measures (non-negative distributions). If $\mu_k \to \mu$ weakly, then the metrics ρ_k (as distance functions on $\mathbf{P}^1 \times \mathbf{P}^1$) converge to ρ, locally uniformly on the complement of any points $z \in \mathbf{P}^1$ with $\mu\{z\} \geq 1/2$. That is to say, the convergence is uniform away from any flared ends of the metric ρ.

Let it be remarked that such a convergence theorem of course requires a normalization, a choice of scale for these metrics. If, for example, points

0 and ∞ in $\mathbf{P}^1$ are not flared ends for any of the metrics, we could fix $\rho_k(0,\infty) = \rho(0,\infty) = 1$ for all k.

As a consequence of the general theory of Alexandrov and Reshetnyak, we obtain that the limiting metrics in the boundary of $\mathrm{Rat}_d \subset \mathbf{P}^{2d+1}$ (except possibly in $I(d)$) are convex polyhedral.

Question. Are *all* limiting metrics in the boundary of Rat_d polyhedral? That is, if $f_k \to f \in I(d)$ such that the maximal measures μ_{f_k} converge weakly, is the limiting distribution a countable sum of delta masses?

Polynomial limits. In the next section, we give examples of degenerating sequences of polynomials in $\mathrm{Poly}_d \subset \mathrm{Rat}_d$ with measures converging to

$$\mu = \frac{1}{d} \sum_i \delta_{z_i},$$

where $\{z_i : i = 1, \dots, d\}$ is any collection of d (not necessarily distinct) points in $\mathbf{P}^1$. Metrically, these limits correspond to all convex polyhedra with d vertices of equal cone angle, the objects of study in [Th]. In our case, several or all vertices may coalesce. When the limit measure is δ_∞, for example, the metric is that of the flat plane.

Question. Are all limiting metrics in the boundary of the space of polynomials, Poly_d, polyhedral with *finitely many* vertices? Experimental evidence indicates yes. In the next section, we see the answer is affirmative in degree 2 (without details).

7. Further examples

In this final section, we study the above ideas as they apply to the boundary of the space of polynomials $\mathrm{Poly}_d \subset \mathrm{Rat}_d$ in $\mathbf{P}^{2d+1}$. We also explain how Epstein's sequences of degree 2 rational maps in [Ep] arise and achieve the lower bound of Theorem 1(b).

Polynomials and δ_∞. Suppose that a sequence of polynomials p_k of degree $d \geq 2$ converges to a polynomial of lower degree locally uniformly on $\mathbf{C}$. As a point in $\mathbf{P}^{2d+1}$, the limit map has the form

$$f(z:w) = (w^k p(z,w) : w^d),$$

where p is the limiting polynomial of degree $d-k$ and f has a hole of depth k at ∞. By Theorem 1(a) and the definition of the measure μ_f, we see that the measures of maximal entropy of p_k converge weakly to δ_∞. Of course, the Julia sets of the polynomials are not necessarily going off to infinity (in the Hausdorff topology). Consider for example,

$$p_\varepsilon(z) = \varepsilon z^3 + z^2$$

as $\varepsilon \to 0$. For small ε, these polynomials are polynomial-like of degree 2 in a neighborhood of the unit disk. The Julia set of the polynomial-like restriction of p_ε is part of the Julia set for p_ε.

The indeterminacy locus at the boundary of Poly_d**.** The space of polynomials of degree d is $\mathrm{Poly}_d \simeq \mathbf{C}^* \times \mathbf{C}^d \subset \mathbf{P}^{d+1}$ in $\mathbf{P}^{2d+1}$. The intersection of the indeterminacy locus $I(d)$ with this $\mathbf{P}^{d+1}$ is a $\mathbf{P}^{d-1}$. In coordinates, if $p = (a_d z^d + \cdots + a_0 w^d : b_0 w^d)$, then $p \in I(d)$ if and only if $a_d = b_0 = 0$. These $p \in I(d)$ are "constant ∞" maps with a hole of depth ≥ 1 at ∞.

It is straightforward to construct a sequence of polynomials converging to $I(d)$ for which some iterate has a positive degree limit, simply by forcing a critical point to be periodic. For example, consider the family

$$q_\varepsilon(z) = \varepsilon^3 z^3 + \varepsilon z^2 + c(\varepsilon),$$

where the constant $c(\varepsilon)$ is chosen so that the critical point at 0 has period two and $c(\varepsilon) = O(1/\varepsilon^2)$ for ε small. (This $c(\varepsilon)$ is obtained by choosing the "correct" square root.) Since the constant term is unbounded, q_ε converges to the constant ∞ as $\varepsilon \to 0$, locally uniformly on $\mathbf{C}$. Algebraically, $q_\varepsilon \to (w^3 : 0)$ in $\mathbf{P}^{d+1}$, so that ∞ is a hole of depth 3. Computing the second iterate $\Phi_2(q_\varepsilon)$, we see that as $\varepsilon \to 0$,

$$\Phi_2(q_\varepsilon) \to z^2$$

locally uniformly on $\mathbf{C}$.

Polynomials and polyhedra with d **vertices.** Fix a polynomial p in $\mathbf{C}$ with d distinct roots and consider the sequence

$$p_k(z : w) = (p(z, w) : w^d/k) \in \mathrm{Poly}_d$$

as $k \to \infty$. The limit $f = (p(z, w) : 0)$ is the constant ∞ map with holes at the roots of p, so in particular $f \notin I(d)$. By Theorem 1(a), the measures of maximal entropy for p_k converge weakly to $\mu_f = \sum_{p(z)=0} \delta_z/d$. Metrically, the limit is a polyhedron with d vertices of equal cone angle.

For any measure μ which is a sum of delta masses of mass $1/d$ at a collection of d (not necessarily distinct) points in $\mathbf{P}^1$, there exists a sequence of degree d polynomials for which the measures of maximal entropy converge weakly to μ. This follows easily from the existence of the limits with delta masses at d disjoint points in $\mathbf{P}^1 - \{\infty\}$. Thus, all shapes of polyhedra with d vertices of equal cone angle arise.

General limits at boundary of Poly_d**.** From experimental evidence, it appears that even when a sequence of polynomials converges to $I(d)$, the limiting measure should be supported at finitely many points. In degree two, it is possible to show (using the fact that the moduli of certain annuli separating Green level lines grow unboundedly if the sequence diverges in moduli space) that the limiting measures can be supported in at most three points. Different limiting measures arise from different normalizations. For example, choosing a fixed point and a particular n-th preimage of that fixed point to remain finite, we obtain points in the plane of mass $1/2^n$ with the remaining mass at ∞.

For a general degree, the number of points in the support could be arbitrarily large. For example, various normalizations of the family $\varepsilon z^3 + z^2$

as $\varepsilon \to 0$ can give a limiting measure with 1, 2, or $2+2^n$ points in its support for any desired $n \geq 1$. It would be interesting to know exactly what the possible limits are in every degree, as a choice of normalization is not enough to capture every degeneration.

Epstein's sequences in Rat_2**.** In [Ep, Prop 2], Epstein gave the first example of the discontinuity of the iterate map at the boundary of the space of rational maps. He studied unbounded sequences in the moduli space of degree 2 rational maps, and in particular, examined sequences of rational maps in Rat_2 converging to a degenerate map $f = H\varphi \in V(\mathrm{Res})$ for which φ is an elliptic Mobius transformation of order $q > 1$. If a sequence approaches f from a particular direction in $\mathrm{Rat}_2 \subset \mathbf{P}^5$ (depending on a complex parameter T), there are certain conjugates of this sequence converging to $I(2)$ such that their q-th iterates converge to the degree 2 map, $f_T(z) = z + T + 1/z$ (uniformly away from the holes at 0 and ∞).

When $q = 2$, his examples realize the lower bound in Theorem 1(b). Indeed, he provides a sequence denoted $F_k \in \mathrm{Rat}_2$ such that $F_k \to (zw : 0) \in I(2)$, the constant infinity map with holes at 0 and ∞, each of depth 1. The second iterates of F_k converge to $f_T = (zw(z^2 + Tzw + w^2) : z^2w^2) \notin I(4)$, so that f_T has a hole of depth 1 at ∞. By Theorem 1(a), the measures μ_{F_k} converge weakly to μ_{f_T}, and we can compute with Lemma 5 that

$$\mu_{f_T}(\{\infty\}) = \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{4^n} = \frac{1}{3},$$

which is exactly the lower bound in Theorem 1(a) when the degree is 2 and the depth is 1.

Epstein used these examples in his proof that certain hyperbolic components in the moduli space $\mathrm{Rat}_2 / \mathrm{PSL}_2\,\mathbf{C}$ are bounded. It is my hope that a more systematic understanding of the iterate map and the boundary of the space of rational maps will be applicable to related questions in general degrees.

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