Quasisymmetric homeomorphisms

Definition. A homeomorphism $Z \xrightarrow{\phi} Z'$ between metric spaces is **quasisymmetric** if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for all triples of distinct points $p, x, y \in Z$, we have

$$\frac{d(\phi(p), \phi(x))}{d(\phi(p), \phi(y))} \leq \eta\left(\frac{d(p, x)}{d(p, y)}\right)$$

Poincare inequalities on metric measure spaces (after Heinonen-Koskela). Say that a Riemannian manifold M satisfies a $(1, p)$-Poincare inequality if there are constants $C \geq 1$, $K < \infty$ with the following property. For every $p \in M$, $0 \leq r \leq \text{diam}(M)$, if $B := B(p, r)$, $CB := B(p, Cr)$, and $u : CB \to \mathbb{R}$ is a C^1 function, then

$$\fint_B |u - u_B| d\,\text{vol}_M \leq Kr \left(\fint_{CB} |\nabla u|^p d\,\text{vol}_M \right)^{\frac{1}{p}},$$

where

$$u_B := \fint_B u \, d\,\text{vol}_M \,.$$

If M is complete and $\text{Ricci}_M \geq 0$, then it satisfies a $(1, n)$-Poincare inequality where $C = C(n)$, $K = K(n)$.

We would like to make sense of such an inequality when we replace M with a metric space Z, and the Riemannian measure $d\,\text{vol}_M$ with a Borel measure μ.

Let $u : Z \to \mathbb{R}$ be a continuous function. A Borel measurable function $\rho : Z \to \mathbb{R}$ is an **upper gradient for** u if for every rectifiable curve $\gamma : [0,1] \to Z$, we have

$$|u(\gamma(1)) - u(\gamma(0))| \leq \int_\gamma \rho ds.$$

Using this definition, we can make sense of Poincare inequalities in metric measure spaces.

Definition. A metric measure space (Z, μ) satisifes a $(1,p)$-Poincare inequality if there are constants $C > 1$, $K < \infty$, such that for all balls $B = B(p,r) \subset Z$, $0 \leq r \leq \text{diam}(Z)$, every continuous function $u : CB \to \mathbb{R}$, and every upper gradient $\rho : CB \to \mathbb{R}$ for u, we have

$$\fint_B |u - u_B| d\mu \leq Kr \left(\fint_{CB} \rho^p d\mu \right)^{\frac{1}{p}}.$$

Examples. 1. $\mathbb{R}^n$, equipped with Lebesgue measure.

2. S^3 equipped with the usual Carnot metric, and the associated 4-dimensional Hausdorff measure (which coincides with Lebesgue measure up to normalization).

3. Limits of sequences of Riemannian n-manifolds with Ricci curvature ≥ 0, equipped with weak limits of the Riemannian measures.

4. (Semmes) Any linearly locally contractible Ahlfors n-regular n-sphere.

5. (Bourdon-Pajot) The boundaries of certain 2-dimensional hyperbolic buildings. These are homeomorphic to the Menger curve (= Menger sponge).

Nonexample. The standard square Sierpinski carpet.

Henceforth the phrase Z **satisfies a Poincare inequality** will be shorthand for: Z is compact, Ahlfors Q-regular, and satisfies a $(1, Q)$-Poincare inequality with respect to Q-dimensional Hausdorff measure for some $Q > 1$.

Heinonen-Koskela

There is a good theory of quasiconformal/quasisymmetric homeomorphisms between spaces satisfying Poincare inequalities.

-QC$\leftrightarrow$QS.

-Absolute continuity.

-ACL.

Bourdon-Pajot

Using the Heinonen-Koskela theory, one can prove rigidity for quasi-isometries of certain hyperbolic buildings.

Cheeger

-Rademacher's theorem.

Cheeger's version:

$\{U_i\}$ countable disjoint collection of Borel sets in Z whose union has full measure.

$f_i : U_i \to \mathbb{R}^{d_i}$ Lipschitz mappings ("coordinates"), where d_i is uniformly bounded.

Every Lipschitz function $u : Z \to \mathbb{R}$ is differentiable a.e. with respect to the relevant f_i, and the derivative is L^∞.

For all i, the component functions of f_i have linearly independent derivatives at almost every $x \in U_i$.

(Heinonen-Koskela-Shanmugalingam-Tyson, Keith) Cheeger's theory applies to functions in $W^{1,Q}$, and $QC(Z)$ acts on the Cheeger cotangent bundle.

Suppose Z, Z' satisfy Poincare inequalities.

Theorem. (Infinitesimal dilatation controls dilatation) If $Z \xrightarrow{f} Z'$ is quasiconformal, and its derivative is almost everywhere K-quasiconformal, then the dilatation of f satisfies $H(f,x) \leq K' = K'(K)$.

Corollary. If $\dim(T^*_z Z) = 1$ almost everywhere, then $QC(Z, Z')$ is automatically uniformly quasiconformal.

Suppose Z, Z' satisfy a Poincare inequality for $Q > 1$, h is a measurable Riemannian metric on T^*Z and h' is a measurable Riemannian metric on Z'.

Definition. A homemorphism $(Z, h) \to (Z', h')$ is **conformal** if it is quasiconformal, and its derivative is conformal almost everywhere. The **conformal group of** h, Conf(h) is defined similarly.

Corollary. If Z satisfies a Poincare inequality, and h is **bounded** Riemannian metric on T^*Z, then Conf(Z, h) is a group of uniformly quasiconformal homeomorphisms. In particular, it is a convergence group, i.e. it acts properly on the space of triples.

(Idiotic) Problem. Is Conf(Z, h) a closed subgroup of Homeo(Z)?

Conformal actions

Definition. A **conformal tuple** is a quadruple (G, Z, h, ρ), where G is a group, Z satisfies a Poincare inequality, h is a bounded measurable Riemannian metric on Z, and ρ is an action of G on Z by h-conformal homeomorphisms, which is cocompact on triples.

Main example. Suppose G is hyperbolic group, and $\partial G \overset{qs}{\sim} Z$.

(Uniformly qc implies conformal) By a construction of Sullivan, it follows that if $G \curvearrowright Z$ is a uniformly qc action of a countable group on a space satisfying a Poincare inequality, then there is a bounded measurable Riemannian metric h on T^*Z with respect to which G acts conformally.

Theorem. (Mostow rigidity) If (G, Z, h, ρ), (G', Z', h', ρ') are conformal tuples, then any isomorphism $G \to G'$ induces an equivariant conformal map $(Z, h) \to (Z', h')$.

Remark. If Conf(h) were a closed subgroup, this could be restated as an assertion about cocompact lattices in the locally compact group Conf(h).

Remark. This shows that association $G \mapsto$ Conf(h) is canonical.

Remark. One can reasonably expect a version of Gromov-Tukia here.

Q: Which hyperbolic groups have boundaries qs to a space satisfying a Poincare inequality?

Necessary conditions:

Bourdon-Pajot:

Definition (Pansu). The **conformal dimension** of a metric space X is the infimal Hausdorff dimension of the spaces quasisymmetric to it. The **Ahlfors regular conformal dimension** of X is the infimal Hausdorff dimension of the Ahlfors regular metric spaces quasisymmetric to X.

Theorem (Tyson). If Z satisfies a Poincare inequality, then $\text{HausDim}(Z) = \text{ConfDim}_{AR}(Z)$.

Theorem (Bonk-K.). If Z is qs to the boundary of a hyperbolic group, Z is Q-regular, and $\text{HausDim}(Z) = \text{ConfDim}_{AR}(Z)$, then Z satisfies a Poincare inequality.

Extremal problem. When is it possible to realize $\text{ConfDim}(X)$ (or $\text{ConfDim}_{AR}(X)$)?