QUASIPERIODIC $\mathrm{SL}(2,\mathbb{R})$ COCYCLES WHICH ARE NOT HOMOTOPIC TO THE IDENTITY

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Abstract. We show that for every $\alpha \in \mathbb{R}$, and for "almost every" $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ non-homotopic to the identity, the cocycle (α, A) is non-uniformly hyperbolic. In the course of the proof we develop a non-perturbative "local theory" for cocycles non-homotopic to the identity, which is based on complexification ideas, and does not use Diophantine assumptions. The complexification technique is centered around the idea of monotonicity (it covers cocycles satisfying a twist condition). It allow us to obtain a quite complete description of the dynamics in the local setting, including the rigidity and minimality of the dynamics, and a surprising result (in this not uniformly hyperbolic setting): analiticity of the Lyapunov exponent. Those results extend to the smooth setting "à la Lyubich" (through the use of asymptotically holomorphic extensions).

Contents

1. Introduction

A *one-dimensional quasiperiodic C^r-cocycle in* $\mathrm{SL}(2,\mathbb{R})$ (briefly, a C^r-cocycle) is a pair (α, A) where $\alpha \in \mathbb{R}$ and $A \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. A cocycle should be viewed as a *skew-product*:

$$(\alpha, A) : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \quad (1.1)$$
$$(x, w) \mapsto (x + \alpha, A(x) \cdot w).$$

We call α the *frequency* of the cocycle.

There is a fairly developed theory of cocycles homotopic to the identity. This is partially motivated by the theory of Schrödinger operators, which lead to the study of a certain family of cocycles homotopic to the identity.

The aim of this paper is to develop a theory in the case of cocycles not homotopic to the identity. It turns out that such a theory is quite rich in its own right. In particular, we will develop a "local theory" which is completely different, much more robust, and sometimes much more complete than that of cocycles homotopic to the identity. Indeed, the results we obtain are quite surprising at first sight: for instance, while in the local theory of cocycles homotopic to the identity KAM schemes play a determinant role, arithmetic properties of the frequency turn out to be irrelevant for all applications considered here.

While this local theory is the main novelty of this work, our original motivation was to obtain global results from local results via renormalization. We will start by discussing those.

1.1. **Global results: typical non-uniform hyperbolicity.** The Lyapunov exponent of (α, A) is defined as

$$L(\alpha, A) = \lim \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n(x)\| dx \geq 0, \tag{1.2}$$

where $A_n(x) = \prod_{j=n-1}^{0} A(x + j\alpha) = A(x + (n-1)\alpha) \cdots A(x)$ (we will keep the dependence on α implicit).

A key property of a cocycle (and other dynamical systems) is whether it has a positive Lyapunov exponent: this is a very good starting point to a description of the dynamics.

In the case of cocycles homotopic to the identity, there is a severe obstruction for a smooth cocycle (α, A) to have a positive Lyapunov exponent, namely, to be conjugate to a constant elliptic matrix $A_0 \in \mathrm{SL}(2, \mathbb{R})$:

$$A(x) = B(x + \alpha) A_0 B(x)^{-1}, \tag{1.3}$$

where $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2, \mathbb{R})$ is smooth. We say that this obstruction is severe because it is reflected on a positive measure set in parametrized families (this is a consequence of KAM theory, see [E] for the most sophisticated results).

In [AK1] the following result was proved. Let

$$R_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}. \tag{1.4}$$

Theorem 1.1 ([AK1])**.** *Let $\alpha \in RDC$ (a subset of $\mathbb{R}$ of full Lebesgue measure). For every $A \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ which is homotopic to the identity, and for almost every $\theta \in \mathbb{R}/\mathbb{Z}$, eiher $L(\alpha, R_\theta A) > 0$ or $(\alpha, R_\theta A)$ is C^r-conjugate to a constant elliptic matrix.*

In other words, *a typical quasiperiodic cocycle which is homotopic to the identity is either non-uniformly hyperbolic or conjugate to a constant.* In this result, typical corresponded both to a full measure set of frequencies and a "full measure" set of cocycles.

We believe that an exclusion of frequencies is unavoidable. This should hold at least for cocycles associated to the Almost Mathieu Operator.

Work more on this comment.

In this paper we consider quasiperiodic cocycles non-homotopic to the identity. In this case, the obstruction discussed above vanishes: if (α, A) is not homotopic to the identity then it can not be conjugate to a cocycle homotopic to the identity, it can not be reducible (conjugate to a constant matrix).

The main global result of this paper is the following: *a typical quasiperiodic cocycle which is not homotopic to the identity has a positive Lyapunov exponent.* However, we are able to show this result without exclusion of frequencies.

Theorem 1.2. *Let $\alpha \in \mathbb{R}$. Let $Z_\alpha^r \subset C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ be the set of all A non-homotopic to the identity such that $L(\alpha, A) = 0$. Then Z_α^r can be written as a union $MZ_\alpha^r \cup NMZ_\alpha^r$, where*

1. *MZ_α^r has positive codimension*[1],
2. *For every $A \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ which is not homotopic to the identity, the set of $\theta \in \mathbb{R}$ such that $R_\theta A \notin NMZ_\alpha^r$ has zero Lebesgue measure.*

Notice that item (1) in the above description can not be removed: if $A \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SO}(2, \mathbb{R}))$ then $L(\alpha, R_\theta A) = 0$ for every $\theta \in \mathbb{R}/\mathbb{Z}$. This event is "essentially" (modulo conjugation) what item (1) covers, more on this later.

1.2. **Local theory: cocycles with a twist.** In order to prove the theorem above we develop a "local" theory of cocycles non-homotopic to the identity. In the case of cocycles homotopic to the identity, "local" stands for "cocycles close to a constant". In our case, there are no constant cocycles, so instead we discuss cocycles which are close to the simplest examples of cocycles not homotopic to the identity, which are of the form (α, A) with $A(x) = R_{\theta + \deg x}$, $\theta \in \mathbb{R}/\mathbb{Z}$, $\deg \neq 0$ (notice that $A : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2, \mathbb{R})$ has degree deg). The key reason to study such cocycles (and for us to call this setting "local") comes from renormalization and will be clear later.

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[1] By this we mean that, locally, MZ_α^r is contained in the zero set of a C^r real-valued function for which 0 is a regular value.

Let us note that, previously, a local theory had been developed using a KAM scheme [K], thus it was perturbative (it depends on arithmetic properties of α). Here we will follow a different path, based on complexification ideas. One of the key advantage of this approach is that it is non-perturbative. Indeed the local setting is quite robust: it is C^1-open (notice that a satisfactory theory is impossible in the C^0 topology by [Bo]).

Our local setting consists of the class of "monotonic cocycles" (cocycles satisfying a twist condition). More precisely, a C^1 cocyle (α, A) is said to be monotonic if its projectivized skew-product action $\mathbb{R}/\mathbb{Z} \times \mathbb{P}^1 \to \mathbb{R}/\mathbb{Z} \times \mathbb{P}^1$ takes the horizontal foliation $\{\mathbb{R}/\mathbb{Z} \times \{z\}\}_{z\in\mathbb{P}^1}$ to a foliation transverse to the horizontal foliation.

Notice that whether (α, A) is monotonic only depends on A, so we may define the set of monotonic functions $M^r \subset C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$. Clearly monotonic cocycles can not be homotopic to the identity.

In the context of cocycles homotopic to the identity, one has to work a lot to obtain regularity properties of the Lyapunov exponent, even just continuity[2]. Perhaps the most surprising result of our analysis is the following:

Theorem 1.3. *Let $r = \omega, \infty$. The Lyapunov exponent of (α, A) is a C^r function of $A \in M^r$.*

This result is quite easy to prove (at least in the analytic case), once we have chosen the right framework. Notice that in the case of cocycles homotopic to the identity, one only expects smoothness of the Lyapunov exponent in the uniformly hyperbolic regime, while such cocycles simply do not exist in our context.[3]

It is not much more difficult to vary the frequency.

Theorem 1.4. *The Lyapunov exponent of (α, A) is a C^∞ function of $(\alpha, A) \in \mathbb{R} \times M^\infty$.*

We have the following result which characterizes monotonic cocycles with a zero Lyapunov exponent:

Theorem 1.5. *Let $(\alpha, A) \in \mathbb{R} \times M^r$, $r = \omega, \infty$. If $L(\alpha, A) = 0$ then $L(\alpha, A)$ is C^r conjugate to a cocycle of rotations.*

This theorem can be seen as a rigidity result: a priori, a zero Lyapunov exponent is related to measurable conjugation to some standard models, such as cocycles of rotations [T].

We believe many other results on monotonic cocycles are accessible by the techniques we use here. As an example of a result going in a quite different direction, we get:

Theorem 1.6. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $A \in P^\infty_\alpha$. Then the projective skew-product action of (α, A) is minimal.*

1.2.1. *Premonotonic cocycles.* The class of monotonic cocycles is not dynamically natural: it is not invariant by real-analytic conjugacy. Thus it is natural to define the class of premonotonic cocycles $P^r \subset \mathbb{R} \times C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ as the set of cocycles (α, A) which are real-analytically conjugate to a monotonic cocycle. We let $P^r = \cup_{\alpha\in\mathbb{R}} P^r_\alpha$ (whether (α, A) is premonotonic depends both on α and on A).

Premonotonic cocycles form a C^1 open set, and all properties of monotonic cocycles that we discussed transfer automatically to this larger setting. It is not difficult to see that a cocycle $(\alpha, A) \in (\mathbb{R} \setminus \mathbb{Q}) \times \mathrm{SO}(2, \mathbb{R}))$ is always premonotonic.

In the statement of Theorem 1.2 we make reference to a partition $Z^r_\alpha = ZM^r_\alpha \cup ZNM^r_\alpha$ of cocycles with zero Lyapunov exponent. We may give a precise definition of those sets now: $ZM^r_\alpha = Z^r_\alpha \cap P^r_\alpha$ is the set of premonotonic A such that $L(\alpha, A) = 0$.

It is easy to see that the first derivative of $A \mapsto L(\alpha, A)$ vanishes in ZM^r_α, and it is possible to show (using the rigidity theorem) that its second derivative does not vanish. It immediately follows:

Corollary 1.7. *The set ZM^r_α has positive codimension in P^r_α.*

This gives item (1) of Theorem 1.2.

It could be hoped that all cocycles with irrational frequencies (non-homotopic to the identity) are premonotonic. We will show however that this is not the case: there are many (positive measure set in an open set of parametrized families) cocycles which are not premonotonic. The examples we will construct build on results of Young [Y].

[2]See [GS] (which also addresses Hölder regularity, and [BJ1] for continuity in the global analytic setting. There are also perturbative local results in the analytic and smooth setting, see [E] and [AK2].

[3]The Lyapunov exponent may not be better than 1/2-Hölder, even for the Almost Mathieu Operator with Diophantine frequencies (which falls in the local setting of Eliasson [E]). Indeed, in this setting, the Lyapunov exponent vanishes in the spectrum but near the endpoints of the gaps one has square root behavior.

At this point, it is not clear if premonotonic cocycles form the largest class of cocycles that behave dynamically as monotonic cocycles. For instance, it is not clear if the class of premonotonic cocycles is fully invariant under iteration or renormalization. However, this class is more than enough for several purposes. Some (possibly larger) classes of cocycles which are invariant under several reasonable operations (including renormalization) and share the properties of monotonic cocycles will be discussed in this work. None of them includes the examples we mentioned above.

1.3. **Reduction from global to local.** In order to prove Theorem 1.2, we will proceed by reduction to the local theory via a renormalization scheme. Such a scheme was shown in [AK1] to cover typical cocycles with zero Lyapunov exponent. The argument is slightly more complicated here because we want to take care of all frequencies. Thus, instead of considering limits of renormalization we are led to use cancelation arguments to show convergence (of an appropriate sequence of renormalizations) to "standard models" (of the form (α, A) with $A(x) = R_{\theta+\deg x}$). (Cancellation schemes were developed in [K], though we will present an alternative argument closer in spirit to [AK1].) In particular, the renormalization scheme leads us to monotonic cocycles. The following consequence implies item (2) of Theorem 1.2.

Theorem 1.8. *Let $(\alpha, A) \in (\mathbb{R} \setminus \mathbb{Q}) \times C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $r = \omega, \infty$, be non-homotopic to the identity. For almost every $\theta \in \mathbb{R}/\mathbb{Z}$, either $L(\alpha, R_\theta A) > 0$ or (α, A) is C^r-conjugate to a cocycle of rotations (and automatically premonotonic).*

All results discussed here have analogues in finite differentiability, which involve loss of derivatives. Although we did not want to consider those distractions in the introduction, we will state and proof the results also in the case of finite differentiability, and we will give an estimate on the loss of derivatives. Many of the results are much stronger than we state in the introduction, for instance, in Theorem 1.2 one is not forced to consider families of the type $\theta \mapsto R_\theta A$, for instance, any perturbation of such a family will do.

1.4. **Structure of the paper.** The key feature of monotonic cocycles is that they are amenable to complexification techniques. The typical complexification procedure, which is quite developed for Schrödinger cocycles, is to perturb the cocycle by a complex parameter (the energy in the Schrödinger case). As far as we know, such complexification techniques were restricted so far to very special kinds of perturbation, where the dependence on the complex parameter is a holomorphic function with very specific global properties. In this work, we show that the construction is much more robust, and the sole feature that enables it is monotonicity of the perturbation parameter. Our first step is to generalize such parameter techniques to the case of general analytic dependence. Then, using a technique introduced for dynamical systems by Lyubich [Ly1] (in the context of unimodal maps), we extend those results to the smooth setting.

After obtaining several parameter style results, we obtain the properties of monotonic cocycles by observing that the phase variable can be considered as a parameter.

2. Monotonicity in parameter space

Let $I \subset \mathbb{R}$ be an interval. We say that a continuous function $f : I \to \mathbb{R}$ is ϵ-monotonic if for every $x \neq x'$ we have

$$\frac{|f(x') - f(x)|}{|x' - x|} \geq \epsilon. \tag{2.1}$$

This definition naturally extends to functions defined on (or taking values on) $\mathbb{R}/\mathbb{Z}$ (by considering lifts) and on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2 \equiv \mathbb{C}$ (by considering the identification with $\mathbb{R}/\mathbb{Z}$ given by $x \mapsto e^{2\pi i x}$).

We say that a continuous one-parameter family of matrices $A_\theta \in \mathrm{SL}(2,\mathbb{R})$ is ϵ-monotonic if, for every $w \in \mathbb{R}^2 \equiv \mathbb{C}$, the function $\theta \mapsto \frac{A_\theta \cdot w}{\|A_\theta \cdot w\|}$ is ϵ-monotonic.

A continuous one-parameter family $A_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ is said to be monotonic if for every $x \in \mathbb{R}/\mathbb{Z}$, the family $x \mapsto A_\theta(x)$ is monotonic.

We will now discuss one-parameter families of cocycles displaying monotonicity with respect to the parameter variable. Many arguments here come from well known results first obtained for Schrödinger cocycles. Through this section, in order to be definite and keep the notation simple, we shall describe the arguments when the parameter space is $\mathbb{R}/\mathbb{Z}$.

2.1. **Complexification.** Much of the information we will get from matrices in $\mathrm{SL}(2,\mathbb{C})$ will come from their action on the Riemann Sphere $\overline{\mathbb{C}}$ through Möebius transformations. The usual action is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times z = \frac{az+b}{cz+d}, \tag{2.2}$$

but we prefer to work with a conjugate action, which we write

$$A \cdot z = QAQ^{-1} \times z, \tag{2.3}$$

where

$$Q = \frac{-1}{1+i} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \tag{2.4}$$

If $A \in \mathrm{SL}(2,\mathbb{C})$ we will write

$$A \cdot \begin{pmatrix} z \\ w \end{pmatrix} = QAQ^{-1} \times \begin{pmatrix} z \\ w \end{pmatrix}, \tag{2.5}$$

where the product in the right hand side is the usual one. Thus $A \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}$ is a complex multiple of $\begin{pmatrix} A \cdot z \\ 1 \end{pmatrix}$.

In this form, $\mathrm{SL}(2,\mathbb{R})$ matrices are the ones preserving the unit disk (as opposed the the upper half plane). Let us identify the real one-dimensional projective space $\mathbb{P}^1$ with the unit circle $\partial\mathbb{D}$ by associating to the line through $(0,0) \neq (x,y) \in \mathbb{R}^2$ the complex number $\frac{x+iy}{x-iy}$. Then the action of $A \in \mathrm{SL}(2,\mathbb{R})$ on $\mathbb{P}^1$ is given precisely by $z \mapsto A \cdot z$.

Define Υ as the space of $\mathrm{SL}(2,\mathbb{C})$ matrics A such that $A \cdot \mathbb{D} \subset \mathbb{D}$.

Let $A \in C^0(\mathbb{R}/\mathbb{Z}, \Upsilon)$. Define $\tau \equiv \tau_A : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{C} \setminus \{0\}$ by

$$A(x) \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} = \tau(x,z) \cdot \begin{pmatrix} A(x) \cdot z \\ 1 \end{pmatrix}. \tag{2.6}$$

Let $\hat{\tau} \equiv \hat{\tau}_A \in C^0(\mathbb{R} \times \mathbb{D}, \mathbb{C})$ satisfy $\hat{\tau}(x,z) = e^{2\pi i \hat{\tau}(x,z)}$. Two choices of $\hat{\tau}$ differ by a constant integer. In particular, $\hat{\tau}(x+1,z) - \hat{\tau}(x,z)$ is an integer deg, which is readily seen to coincide with the degree of A. Thus we can define $\xi \equiv \xi_A \in C^0(\mathbb{R}/\mathbb{Z} \times \mathbb{D}, \mathbb{C})/\mathbb{Z}$ by setting $\xi(x,z) = \hat{\tau}(x,z) - x \deg$.

Given $\alpha \in \mathbb{R}$, define $\xi_n \equiv \xi_{n,\alpha,A} \in C^0(\mathbb{R}/\mathbb{Z} \times \mathbb{D}, \mathbb{C})/\mathbb{Z}$ by

$$\xi_n(x,z) = \frac{1}{n} \sum_{k=0}^{n-1} \xi(x, (\prod_{k=0}^{n-1} A(x+k\alpha)) \cdot z). \tag{2.7}$$

A simple computation shows that $|\Re(\xi_n(x,z) - \xi_n(x,z'))| < \frac{1}{n}$. Thus we can define a function $\hat{\rho} \equiv \hat{\rho}_{\alpha,A} \in \mathbb{C}^0(\mathbb{R}/\mathbb{Z}, \mathbb{R})/\mathbb{Z}$ by

$$\hat{\rho}(x) = \lim \Re \xi_n(x,z). \tag{2.8}$$

Clearly, $\hat{\rho}(x+\alpha) = \hat{\rho}(x)$, so $\hat{\rho}$ is constant if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In general, let

$$\rho \equiv \rho_{\alpha,A} = \lim \int_{\mathbb{R}/\mathbb{Z}} \hat{\rho}(x) dx \in \mathbb{R}/\mathbb{Z}. \tag{2.9}$$

We shall call ρ the *fibered rotation number* of (α, A). It is a continuous function $C^0(\mathbb{R}/\mathbb{Z}, \Upsilon) \to \mathbb{R}/\mathbb{Z}$.

We define $\zeta \equiv \zeta_{\alpha,A} \in \overline{\mathbb{H}}/\mathbb{Z}$ by taking

$$\zeta = -(\rho + 2\pi i L). \tag{2.10}$$

Notice that

$$\frac{-1}{2\pi} \ln \left\| \prod_{k=n-1}^{0} A(x+k\alpha) \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} \right\| = \Im(\chi_n(x,z)) \ln \left\| \begin{pmatrix} \prod_{k=n-1}^{0} A(x+k\alpha) \cdot z \\ 1 \end{pmatrix} \right\|. \tag{2.11}$$

2.1.1. *Invariant section.* Assume that for every $x \in \mathbb{R}/\mathbb{Z}$, we have $A(x) \cdot \overline{\mathbb{D}} \subset \mathbb{D}$. In this case, by the Schwarz Lemma, there exists $m \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{D})$ satisfying

$$m(x+\alpha) = A(x) \cdot m(x), \tag{2.12}$$

and we have for every $(x, z) \in \mathbb{R}/\mathbb{Z} \times \mathbb{D}$,

$$\lim \prod_{k=-1}^{-n} A(x+k\alpha) \cdot z = m(x). \tag{2.13}$$

It immediately follows that

$$\zeta = -\int_{\mathbb{R}/\mathbb{Z}} \xi(x, m(x)) dx. \tag{2.14}$$

Notice that there is another formula for the Lyapunov exponent in terms of m. Let $B_x : \mathbb{D} \to \mathbb{C}$ be the derivative of $A(x) : \mathbb{D} \to \mathbb{D}$ and set $q(x) = \|B_x(m(x))\|_{m(x)}$, where $\|\cdot\|$ is some conformal Riemannian metric on $\mathbb{D}$. Then

$$L = \frac{1}{2} \int_{\mathbb{R}/\mathbb{Z}} -\ln q(x) dx. \tag{2.15}$$

To get good estimates, it is convenient to consider the Poincaré metric, in order to apply the Schwarz Lemma. For instance, if $A(x) \cdot \mathbb{D} \subset \mathbb{D}_{e^{-\epsilon}}$ for every $x \in \mathbb{R}/\mathbb{Z}$ then the Schwarz Lemma gives

$$q(x)^{-2} \geq \frac{1-|m(x+\alpha)|^2}{e^{-2\epsilon}-|m(x+\alpha)|^2} = e^{2\epsilon}(1+\frac{(e^{2\epsilon}-1)|m(x+\alpha)|^2}{1-e^{-2\epsilon}|m(x+\alpha)|^2} \geq e^{2\epsilon}, \tag{2.16}$$

so that $L \geq \frac{\epsilon}{2}$.

2.2. **Simple examples of applications.** We now turn to one-parameter continuous families $\theta \mapsto A_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. To keep definite and to avoid superfluous notation, we will consider in the proofs below only the case when the parameter space is $\mathbb{R}/\mathbb{Z}$.

The key assumption on the family A_θ will be monotonicity in θ. To fix ideas, we will always assume in the proofs below that A_θ is monotonic increasing. One obvious consequence of monotonicity (following directly from the definitions) is the following.

Lemma 2.1. *Let $A_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, be a one-parameter family monotonic in θ. Then $\theta \mapsto \rho_{A_\theta}$ is either non-increasing or non-decreasing.*

Before dwelving into more complicated matters, let us illustrate the relation of monotonicity and complexification with two simple applications.

Let us say that a family $\theta \mapsto A_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ is C^r in θ if there exists a compact subset $K \subset C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ such that for every $x \in \mathbb{R}/\mathbb{Z}$, $\theta \mapsto A_\theta(x)$ belongs to K.

Let $\Omega_\delta = \{z \in \mathbb{C}/\mathbb{Z}, \Im(z) < \delta\}$, $\Omega_\delta^\pm = \{z \in \mathbb{C}/\mathbb{Z}, 0 < \pm\Im(z) < \delta\}$.

Theorem 2.2. *Let $A_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $\theta \in \mathbb{R}/\mathbb{Z}$, be analytic and monotonic in θ. For every $\theta \in \mathbb{R}/\mathbb{Z}$, if $L(A_\theta) = 0$ then*

$$\frac{d}{d\theta}\rho_{A_\theta} \geq \frac{\epsilon}{2\pi} > 0, \tag{2.17}$$

where ϵ is the monotonicity constant of $\theta \mapsto A_\theta$.

Proof. Since A_θ is monotonic, the Cauchy Riemann equations imply that there exists $\delta > 0$ such that the analytic extension of A_θ to $\theta \in \Omega_\delta^+$ satisfies $A_\theta \cdot \mathbb{D} \subset \mathbb{D}_{e^{-2(\epsilon-\delta(\Im(\theta)))\Im(\theta)}}$, where $0 < \delta(t) < \epsilon$ and $\lim \delta(t) = 0$. Using the analiticity in θ of A_θ, we conclude that $\theta \mapsto \zeta_{A_\theta}$ is a holomorphic function $\Omega_\delta^+ \to \mathbb{H}/\mathbb{Z}$, whose imaginary part is continuous up to $\mathbb{R}/\mathbb{Z}$. In particular, for almost every $\sigma \in \mathbb{R}/\mathbb{Z}$,

$$\Im(\zeta_{A_{\sigma+it}}) = \lim(\Im(\zeta_{A_\sigma}) + t\frac{d}{d\sigma}\rho_{A_\sigma} + o(t). \tag{2.18}$$

Since the Lyapunov exponent is upper semicontinuous, if we know additionally that $L(\sigma) = 0$, we have

$$\lim \frac{1}{2\pi t} L(\sigma + it) = \frac{d}{dt}\rho_{A_\sigma}, \tag{2.19}$$

almost surely, and the result follows (since $L(\sigma + it) \geq (\epsilon - \delta(t))t$). □

Theorem 2.3. *Let $A_{\theta,s} \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$, $\theta \in \mathbb{R}/\mathbb{Z}$, s a one-dimensional parameter, be monotonic in θ and analytic in (θ, s). Let $\alpha \in \mathbb{R}$. Then*

$$s \mapsto \int_{\mathbb{R}/\mathbb{Z}} L(\alpha, A_{\theta,s}) d\theta \tag{2.20}$$

is an analytic function of s.

Proof. Let

$$U(t, s) = \int_{\mathbb{R}/\mathbb{Z}} L(\alpha, A_{\sigma+it,s}) d\sigma = \frac{1}{2} \int_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}} -\ln |\tau_{\alpha, A_{\sigma+it,s}(x)}| d\sigma dx. \tag{2.21}$$

Notice that, from the formula above, for $0 < t < \delta$ then $s \mapsto U(t, s)$ is analytic. Moreover, $t \mapsto U(t, s)$ is an affine function of $0 < t < \delta$ (since $\sigma + it \mapsto \tau_{\alpha, A_{\sigma+it,s}(x)}$ is holomorphic). Notice that $U(t, s) = U(-t, s)$, so by subharmonicity, $U(t, s)$ is an affine function of $|t|$ for $0 \leq |t| < \delta$. Thus, for $0 < t < \delta$ we have

$$\int_{\mathbb{R}/\mathbb{Z}} L(\alpha, A_{\theta,s}) d\theta = 2U(\frac{t}{2}, s) - U(t, s), \tag{2.22}$$

is analytic on s. □

Remark 2.1. With a little bit more work, one can get the formula

$$\int_{\mathbb{R}/\mathbb{Z}} L(\alpha, A_{\theta,s}) d\theta = U(t, s) - \frac{|t \deg|}{2\pi}, \tag{2.23}$$

for $0 < t < \delta$, where deg is the degree of $\theta \mapsto A_{\theta,s}(x)$. Indeed, for fixed s, let $q(\sigma + it, s) : \mathbb{R} \times (0, \delta) \to \mathbb{R}$ be a continuous determination of $\rho_{\alpha, A_{\sigma+it,s}}$, so that $|q(\sigma + it + 1, s) - q(\sigma + it, s)|$ is $|\deg|$. Then the function

$$\int_{\sigma}^{\sigma+1} 2\pi i L(\alpha, A_{y+it,s}) + q(y + it, s) dy \tag{2.24}$$

is holomorphic in $\sigma + it \in \mathbb{R} \times (0, \delta)$ and its real part is an affine function of σ of slope $|\deg|$. Thus the function $U(t, s)$ defined above is an affine function of $0 < t < \delta$ with slope $\frac{|\deg|}{2\pi}$, and (ref form) follows.

This theorem implies for instance that $A \mapsto \int_{\mathbb{R}/\mathbb{Z}} L(\alpha, R_\theta A) d\theta$ is an analytic function of $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$. Indeed, it can be shown (see [AB]) that

$$\int_{\mathbb{R}/\mathbb{Z}} L(\alpha, R_\theta A) d\theta = \int_{\mathbb{R}/\mathbb{Z}} \ln \frac{\|A(x)\| + \|A(x)\|^{-1}}{2} dx. \tag{2.25}$$

This generalization beyond families with specific form such as $R_\theta A$ will be crucial when we start to mix phase and parameter in the next section.

It may appear that analiticity is crucial in order to exploit the complexification approach. This is not the case: in the non-analytic case, we can still complexify the problem using asymptotically holomorphic extensions (this idea is inspired from the work of Lyubich on smooth unimodal maps [Ly1]).

2.3. **General framework.** After the motivation above, we are ready to introduce a more general framework for the complexification argument.

Let $\Delta^\pm_\delta$ be the space of all continuous families $A_{\sigma+it} \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $\sigma+it \in \Omega_\delta$, which are C^1 and real-symmetric in $\sigma+it$, satisfy $A_{\sigma+it} \in \operatorname{int} \Upsilon$, $\sigma+it \in \Omega^\pm_\delta$,

$$\overline{\partial}_z A_z = 0, \quad \Im(z) = 0, \tag{2.26}$$

and such that $\sigma \mapsto A_\sigma$ is monotonic in σ.

Let us fix $\alpha \in \mathbb{R}$, $A \in \Delta^\pm_\delta$. Then we have functions $m^+(\sigma+it,x) \in \mathbb{D}$, $\tau^+(\sigma+it,x) \in \mathbb{C}\setminus\{0\}$, $\sigma+it \in \Omega^\pm_\delta$, $x \in \mathbb{R}/\mathbb{Z}$, characterized by

$$A_{\sigma+it}(x)\cdot\begin{pmatrix} m^+(\sigma+it,x) \\ 1\end{pmatrix} = \tau^+(\sigma+it,x)\begin{pmatrix} m^+(\sigma+it,x+\alpha) \\ 1\end{pmatrix}. \tag{2.27}$$

In the notation above, we have $m^+(\sigma+it,x) = m_{\alpha,A_{\sigma+it}(x)}$, $\tau^+(\sigma+it,x) = \tau_{\alpha,A_{\sigma+it}(x)}$.

Notice that $A_{\sigma+it}(x)^{-1} \in \operatorname{int}\Upsilon$ for $\sigma+it \in \Omega^\mp_\delta$. Thus we have also functions $m^-(\sigma+it,x) \in \mathbb{D}$, $\tau^-(\sigma+it,x)$, $\sigma+it \in \Omega^\mp_\delta$, $x \in \mathbb{R}/\mathbb{Z}$, characterized by

$$A_{\sigma+it}(x)\cdot\begin{pmatrix} m^-(\sigma+it,x) \\ 1\end{pmatrix} = \tau^-(\sigma+it,x)\begin{pmatrix} m^-(\sigma+it,x+\alpha) \\ 1\end{pmatrix}. \tag{2.28}$$

In the previous notation, we have $m^-(\sigma+it,x) = m_{-\alpha,A_{\sigma+it}(x)^{-1}}$, $\tau^-(\sigma+it,x) = \tau^{-1}_{-\alpha,A_{\sigma+it}(x)^{-1}}$.

Since $A_{\sigma+it}$ is real-symmetric in $\sigma+it$, letting

$$m^+(\sigma+it,x) = \frac{1}{\overline{m^+(\sigma-it,x)}}, \quad \tau^+(\sigma+it,x) = \frac{1}{\overline{\tau^+(\sigma-it,x)}}, \quad \sigma+it \in \Omega^m p_\delta \tag{2.29}$$

and

$$m^-(\sigma+it,x) = \frac{1}{\overline{m^-(\sigma-it,x)}}, \quad \tau^-(\sigma+it,x) = \frac{1}{\overline{\tau^-(\sigma+it,x)}}, \quad \sigma+it \in \Omega^\pm_\delta, \tag{2.30}$$

we have that (2.26) and (2.28) are valid for $\sigma+it \in \Omega_\delta \setminus \mathbb{R}/\mathbb{Z}$.

The description of Δ^-_δ being analogous to that of Δ^+_δ, we will state our results for the latter one to simplify the notation.

Our first step is a key computation, which generalizes estimates of Kotani and Deift-Simon.

Lemma 2.4. *Let $\alpha \in \mathbb{R}$ and $A \in \Delta^+_\delta$. Let $\sigma_0 \in \mathbb{R}/\mathbb{Z}$.*

(1) *If*

$$\liminf_{t\to 0} \frac{L(\sigma_0+it)}{t} < \infty \tag{2.31}$$

then

$$\liminf_{t\to 0+} \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{1-|m^+(\sigma_0+it,x)|^2}dx + \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{1-|m^-(\sigma_0-it,x)|^2}dx < \infty. \tag{2.32}$$

(2) *If*

$$\limsup_{t\to 0} \frac{L(\sigma_0+it)}{t} < \infty \tag{2.33}$$

then

$$\limsup_{t\to 0+} \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{1-|m^+(\sigma_0+it,x)|^2}dx + \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{1-|m^-(\sigma_0-it,x)|^2}dx < \infty \tag{2.34}$$

and

$$\liminf_{t\to 0+} \int_{\mathbb{R}/\mathbb{Z}} |m^+(\sigma_0+it,x) - m^-(\sigma_0-it,x)|^2 dx = 0. \tag{2.35}$$

Proof. Let us assume that

$$A(\sigma_0+it,x)^{-1}\partial_t A(\sigma_0+it,x)=u(x)\begin{pmatrix}1&0\\0&-1\end{pmatrix}+C(\sigma_0+it,x), \tag{2.36}$$

where $u(x)>0$ and $\sup_{x\in\mathbb{R}/\mathbb{Z}}\|C(\sigma_0+it,x)\|=\epsilon(t)\to 0$ as $t\to 0$. The general case can be reduced to this one by a conjugacy $B(x+\alpha)A(\sigma_0+it,x)B(x)^{-1}$, where $B:\mathbb{R}/\mathbb{Z}\to \mathrm{SL}(2,\mathbb{R})$ is continuous such that

$$B(x)A(\sigma_0,x)^{-1}\partial_t A(\sigma_0,x)B(x)^{-1} \tag{2.37}$$

is a non-zero diagonal matrix (such a matrix B can be found if and only if the determinant of $A(\sigma_0,x)^{-1}\partial_t A(\sigma_0,x)$ is negative, which follows from monotonicity).

Let us denote for simplicity m^+ for $m^+(\sigma_0+it,x)$, m^- for $m^-(\sigma_0+it,x)$, $\tilde m^+$ for $m^+(\sigma_0+it,x+\alpha)$, $\tilde m^-$ for $m^-(\sigma_0+it,x+\alpha)$, τ^+ for $\tau^+(\sigma_0+it,x)$, τ^- for $\tau^-(\sigma_0+it,x)$, A for $A_{\sigma_0+it}(x)$, L for $L(\sigma_0+it)$ and u for $u(x)$.

Notice that

$$A\cdot\begin{pmatrix}m^+\\1\end{pmatrix}=u\frac{m^+-m^-}{m^++m^-}\begin{pmatrix}m^+\\1\end{pmatrix}-\frac{2m^+}{m^+-m^-}\begin{pmatrix}m^-\\1\end{pmatrix}+c^+\begin{pmatrix}m^+\\1\end{pmatrix}+c^-\begin{pmatrix}m^-\\1\end{pmatrix}, \tag{2.38}$$

where $c^+\equiv c^+(\sigma_0+it,x)$, $c^-\equiv c^-(\sigma+it,x)$. We have the estimate

$$|c^+(\sigma_0+it,x)|+|c^-(\sigma_0+it,x)|\le K\epsilon(t)\left(\frac{1}{1-|m^+(\sigma_0+it)|^2}+\frac{1}{1-|m^-(\sigma_0-it)|^2}\right)dx \tag{2.39}$$

for some constant $K>0$.

Notice that $A(\sigma_0+it,x)\cdot\mathbb{D}$ is contained in $\mathbb{D}_{-t(u(x)-\delta(t))}$, where $\delta(t)\to 0$ as $t\to 0$. By the Schwarz Lemma,

$$L(\sigma_0+it)\ge\int_{\mathbb{R}/\mathbb{Z}}\ln e^{t(u(x)-\delta(t))}\frac{1-|m^+(\sigma_0+it,x)|^2}{1-e^{2t(u(x)-\delta(t))}|m^+(\sigma_0+it,x)|^2}dx. \tag{2.40}$$

This gives

$$L(\sigma_0+it,x)\ge\int_{\mathbb{R}/\mathbb{Z}}-t(u(x)-\delta(t))+\ln\frac{e^{2t(u(x)-\delta(t))}(1-|m^+(\sigma_0+it)|^2)}{1-e^{2t(u(x)-\delta)}|m^+(\sigma_0+it,x)|^2}dx. \tag{2.41}$$

Notice that $e^{2t(u(x)-\delta(t)}|m(\sigma_0+it,x)|^2<1$. Using that for $r>0$ and $0\le s<e^{-r}$ we have

$$\ln\left(\frac{e^r(1-s)}{1-e^rs}\right)\ge\frac{r}{1-s}, \tag{2.42}$$

we get

$$L(\sigma_0+it)\ge\int_{\mathbb{R}/\mathbb{Z}}-t(u(x)-\delta(t))+\frac{2t(u(x)-\delta(t))}{1-|m^+(\sigma_0+it,x)|^2}=t\int_{\mathbb{R}/\mathbb{Z}}(u(x)-\delta(t))\frac{1+|m^+(\sigma_0+it,x)|^2}{1-|m^+(\sigma_0+it,x)|^2}, \tag{2.43}$$

so that

$$\lim_{k\to\infty}\frac{L(\sigma_0+it_k)}{t_k}\ge\limsup_{k\to\infty}\int_{\mathbb{R}/\mathbb{Z}}u(x)\frac{1+|m^+(\sigma_0+it_k,x)|^2}{1-|m^+(\sigma_0+it_k,x)|^2}, \tag{2.44}$$

and an analogous argument gives

$$\lim_{k\to\infty}\frac{L(\sigma_0+it_k)}{t_k}\ge\limsup_{k\to\infty}\int_{\mathbb{R}/\mathbb{Z}}u(x)\frac{1+|m^-(\sigma_0-it_k,x)|^2}{1-|m^-(\sigma_0-it_k,x|^2}, \tag{2.45}$$

whenever $t_k\to 0+$ is such that $\lim\frac{L(\sigma_0+it_k)}{t_k}$ exists. This gives item (1) and the first part of item (2). Moreover, under the hypothesis of item (2) (which we assume from now on), we get

$$\liminf\frac{L(\sigma_0+it)}{t}-\int_{\mathbb{R}/\mathbb{Z}}u(x)\frac{1+|m^+(\sigma_0+it,x)|^2}{1-|m^+(\sigma_0+it,x|^2}\ge 0, \tag{2.46}$$

and an analogous argument also gives

$$\liminf \frac{L(\sigma_0+it)}{t} - \int_{\mathbb{R}/\mathbb{Z}} u(x)\frac{1+|m^-(\sigma_0-it,x)|^2}{1-|m^-(\sigma_0-it,x|^2} \geq 0. \tag{2.47}$$

Differentiating

$$A\cdot\begin{pmatrix} m^+\\ 1\end{pmatrix} = \tau^+\begin{pmatrix}\tilde m^+\\ 1\end{pmatrix} \tag{2.48}$$

with respect to t, and applying A^{-1} to both sides, we get

$$(A^{-1}\partial_t)\begin{pmatrix} m^+\\ 1\end{pmatrix} + \partial_t m^+\begin{pmatrix} 1\\ 0\end{pmatrix} = \partial_t\tau A^{-1}\cdot\begin{pmatrix}\tilde m^+\\ 1\end{pmatrix} + \tau\partial_t\tilde m^+ A^{-1}\begin{pmatrix} 1\\ 0\end{pmatrix}. \tag{2.49}$$

Using that

$$\begin{pmatrix} 1\\ 0\end{pmatrix} = \frac{1}{m^+-m^-}\left(\begin{pmatrix} m^+\\ 1\end{pmatrix}-\begin{pmatrix} m^-\\ 1\end{pmatrix}\right) = \frac{1}{\tilde m^+-\tilde m^-}\left(\begin{pmatrix}\tilde m^+\\ 1\end{pmatrix}-\begin{pmatrix}\tilde m^-\\ 1\end{pmatrix}\right), \tag{2.50}$$

we get

$$(A^{-1}\partial_t A)\begin{pmatrix} m^+\\ 1\end{pmatrix} + \frac{\partial_t m^+}{m^+-m^-}\left(\begin{pmatrix} m^+\\ 1\end{pmatrix}-\begin{pmatrix} m^-\\ 1\end{pmatrix}\right) = \frac{\partial_t\tau^+}{\tau^+}\begin{pmatrix} m^+\\ 1\end{pmatrix} + \frac{\partial_t\tilde m^+}{\tilde m^+-\tilde m^-}\left(\begin{pmatrix} m^+\\ 1\end{pmatrix}-\frac{\tau^+}{\tau^-}\begin{pmatrix} m^-\\ 1\end{pmatrix}\right). \tag{2.51}$$

Using (2.36), taking the coefficient of $\begin{pmatrix} m^+\\ 1\end{pmatrix}$ and integrating with respect to x we get

$$\int_{\mathbb{R}/\mathbb{Z}} u\frac{m^++m^-}{m^+-m^-}dx + \int_{\mathbb{R}/\mathbb{Z}} c^+dx = \int_{\mathbb{R}/\mathbb{Z}}\frac{\partial_t\tau^+}{\tau^+}dx. \tag{2.52}$$

We can now consider the real part, which gives

$$\int_{\mathbb{R}/\mathbb{Z}} u\frac{|m^+|^2|m^-|^{-2}-1}{\left|\frac{m^+}{m^-}-1\right|^2}dx + \int_{\mathbb{R}/\mathbb{Z}}\Re c_1 dx = -\partial_t L. \tag{2.53}$$

Using (2.39) we conclude

$$\lim\int_{\mathbb{R}/\mathbb{Z}} u\frac{|m^+|^2|m^-|^{-2}-1}{\left|\frac{m^+}{m^-}-1\right|^2}dx + \partial_t L = 0. \tag{2.54}$$

Using (2.46), (2.47) and (2.54) we get

$$\liminf\frac{L}{t} - \partial_t L - \frac{1}{2}\int_{\mathbb{R}/\mathbb{Z}} u\frac{1+|m^+|^2}{1-|m^+|^2}dx - \frac{1}{2}\int_{\mathbb{R}/\mathbb{Z}} u\frac{1+|m^-|^{-2}}{1-|m^-|^{-2}}dx - \int_{\mathbb{R}/\mathbb{Z}} u\frac{|m^+|^2|m^-|^{-2}-1}{\left|\frac{m^+}{m^-}-1\right|^2}dx \geq 0. \tag{2.55}$$

Notice that

$$I = \frac{1}{2}\frac{1+|m^+|^2}{1-|m^+|^2} + \frac{1}{2}\frac{1+|m^-|^{-2}}{1-|m^-|^{-2}} + \frac{|m^+|^2|m^-|^{-2}-1}{\left|\frac{m^+}{m^-}-1\right|^2} \geq \left|m^+ - \frac{1}{\overline{m^-}}\right|^2 \geq 0, \tag{2.56}$$

so we can write

$$\liminf\frac{L}{t} - \partial_t L \geq \liminf\int_{\mathbb{R}/\mathbb{Z}} uI dx \geq 0. \tag{2.57}$$

Since $\liminf\frac{L}{t} < \infty$, we must have

$$\liminf\frac{L}{t} - \partial_t L = -t\partial_t\frac{L}{t} \leq 0, \tag{2.58}$$

so $\liminf \int_{\mathbb{R}/\mathbb{Z}} uIdx = 0$, and since u is positive and bounded away from 0 we have $\liminf \int_{\mathbb{R}/\mathbb{Z}} Idx = 0$, which gives the second part of item (2) by (2.56). □

The following estimates will allow us to work in the asymptotically holomorphic setting:

Lemma 2.5. *Let $A^s \in \Delta_\delta$ be a one-parameter family. Fix $\alpha \in \mathbb{R}$. Assume that $s \mapsto A^s_z(x)$ is C^r, $1 \le r < \infty$ and*

$$\|\partial_s^k A_z^s(x)\| = O(1), \quad 0 \le k \le r. \tag{2.59}$$

Then

$$|\partial_s^k m_s^+(z,x)| = O(|\Im(z)|^k), \quad 1 \le k \le r. \tag{2.60}$$

Moreover, if additionally $s \mapsto \overline{\partial}_z A^s_z(x)$ is C^{r-1} and we have the estimate

$$\|\partial_s^k \overline{\partial}_z A_z^s(x)\| = o(|\Im(z)|^{\eta-k-1}), \quad 0 \le k \le r-1, \tag{2.61}$$

for some $\eta \in \mathbb{R}$ then

$$|\partial_s^k \overline{\partial}_z m_s^+(z,x)| = o(|\Im(z)|^{\eta-2k-1}), \quad 0 \le k \le r-1. \tag{2.62}$$

Proof. Let $F(s,z,x,w) = A^s_z(x)\cdot w$, $m(s,z,x) = m^+_s(z,x)$. Our estimates will come from the study of the hyperbolicity of F with respect to the variable w, as measured in the Poincaré metric. The way we exploit this hyperbolicity is contained in the following.

Proposition 2.6. *There exists $K > 0$ such that if $u(s,z,x)$ is continuous then*

$$|u(s,z,x)| \le K|\Im(z)|^{-1} \max_{x\in\mathbb{R}/\mathbb{Z}} |u(s,z,x+\alpha) - (\partial_w F)(s,z,x,m(s,z,x))u(s,z,x)|. \tag{2.63}$$

Proof. For s and z fixed, let x satisfy

$$\frac{|u(s,z,x+\alpha)|}{1-|m(s,z,x+\alpha)|^2} = M = \max_{y\in\mathbb{R}/\mathbb{Z}} \frac{|u(s,z,y)|}{1-|m(s,z,y)|^2}. \tag{2.64}$$

Then for every y,

$$|u(s,z,y)| \le \frac{|u(s,z,y)|}{1-|m(s,z,y)|^2} \le M, \tag{2.65}$$

so it is enough to estimate

$$M \le K|\Im(z)|^{-1}|u(s,z,x+\alpha) - (\partial_w F)(s,z,x,m(s,z,x))u(s,z,x)|. \tag{2.66}$$

We have

$$\begin{aligned} |u(s,z,x+\alpha) - (\partial_w F)(s,z,x,m(s,z,x))u(s,z,x)| &= (1-|m(s,z,x+\alpha)|^2) \\ \cdot \left(\frac{u(s,z,x+\alpha)}{1-|m(s,z,x+\alpha)|^2} - (\partial_w F)(s,z,x,m(s,z,x)) \frac{1-|m(s,z,x)|^2}{1-|m(s,z,x+\alpha)|^2} \frac{u(s,z,x)}{1-|m(s,z,x)|^2} \right). \end{aligned} \tag{2.67}$$

Noticing that

$$\left| (\partial_w F)(s,z,x,m(s,z,x)) \frac{1-|m(s,z,x)|^2}{1-|m(s,z,x+\alpha)|^2} \right| < 1, \tag{2.68}$$

we get

$$\begin{aligned} |u(s,z,x+\alpha) - (\partial_w F)(s,z,x,m(s,z,x))u(s,z,x)| &\ge M(1-|m(s,z,x+\alpha)|^2) \\ \cdot \left(1 - |(\partial_w F)(s,z,x,m(s,z,x))| \frac{1-|m(s,z,x)|^2}{1-|m(s,z,x+\alpha)|^2}\right). \end{aligned} \tag{2.69}$$

Now we have the bound

$$|\partial_w F(s,z,x,m(s,z,x)| \frac{1-|m(s,z,x)|^2}{1-|m(s,z,x+\alpha)|^2} \le e^{-\epsilon\Im(z)} \frac{1-e^{2\epsilon\Im(z)}|m(s,z,x)|^2}{1-|m(s,z,x)|^2}, \tag{2.70}$$

for some constant $\epsilon > 0$, which gives

$$(1-|m(s,z,x+\alpha)|^2)\left(1-|(\partial_w F)(s,z,x,m(s,z,x))|\frac{1-|m(s,z,x)|^2}{1-|m(s,z,x+\alpha)|^2}\right) \geq 1-e^{-\epsilon t}, \tag{2.71}$$

which implies the result. □

Differentiating (taking ∂_s^k)

$$m(s,z,x+\alpha) = F(s,z,x,m(s,z,x)), \tag{2.72}$$

we get

$$\begin{aligned}(\partial_s^k m)(s,z,x) =& (\partial_w F)(s,z,x,m(s,z,x)\cdot(\partial_s^k m)(s,z,x) \\ &+ \sum_{\substack{l\geq 0, 1\leq i_1\leq \ldots\leq i_l<k,\\ i_1+\ldots+i_l=j\leq k}} C\cdot(\partial_s^{k-j}\partial_w^l F)(s,z,x,m(s,z,x))\cdot\prod_{n=1}^{l}(\partial_s^{i_n} m)(s,z,x),\end{aligned} \tag{2.73}$$

where $C \equiv C(k,i_1,...,i_l) > 0$ is a constant. Thus, if

$$|\partial_s^j m(s,z,x)| = O(|\Im(z)|^j), \quad 1\leq j\leq k-1, \tag{2.74}$$

we get

$$|(\partial_s^k m)(s,z,x) - (\partial_w F)(s,z,x,m(s,z,x)\cdot(\partial_s^k m)(s,z,x)| = O(|\Im(z)|^{k-1}), \tag{2.75}$$

which implies by the previous proposition

$$|(\partial_s^k m)(s,z,x)| = O(|\Im(z)|^k). \tag{2.76}$$

The first estimate then follows by induction.

Differentiating (taking $\overline{\partial}_z$) (2.73), we get

$$\begin{aligned}(\partial_s^k\overline{\partial}_z m)(s,z,x) =& (\partial_w F)(s,z,x,m(s,z,x)\cdot(\partial_s^k\overline{\partial}_z m)(s,z,x) \\ &+ \sum_{\substack{l\geq 0, 1\leq i_1\leq \ldots\leq i_l<k,\\ i_1+\ldots+i_l=j\leq k}} C\cdot(\partial_s^{k-j}\overline{\partial}_z\partial_w^l F)(s,z,x,m(s,z,x))\cdot\prod_{n=1}^{l}(\partial_s^{i_n} m)(s,z,x) \\ &+ \sum_{\substack{l\geq 0, 1\leq i_1\leq \ldots\leq i_l<k,\\ i_0\geq 0, i_0+i_1+\ldots+i_l=j\leq k}} D\cdot(\partial_s^{k-j}\partial_w^l F)(s,z,x,m(s,z,x))\cdot(\partial_s^{i_0}\overline{\partial}_z m)(s,z,x)\cdot\prod_{n=1}^{l}(\partial_s^{i_n} m)(s,z,x),\end{aligned} \tag{2.77}$$

where $D \equiv D(k,i_0,...,i_l) > 0$ is a constant. Thus, if

$$|\partial_s^j\overline{\partial}_z m(s,z,x)| = o(|\Im(z)|^{\eta-2j-1}), \quad 0\leq j\leq k-1, \tag{2.78}$$

we get

$$|(\partial_s^k\overline{\partial}_z m)(s,z,x) - (\partial_w F)(s,z,x,m(s,z,x)\cdot(\partial_s^k\overline{\partial} m)(s,z,x)| = o(|\Im(z)|^{\eta-2k}), \tag{2.79}$$

which implies as before

$$|(\partial_s^k\overline{\partial}_z m)(s,z,x)| = o(|\Im(z)|^{\eta-2k-1}). \tag{2.80}$$

The second estimate then follows by induction. □

Remark 2.2. The estimates above are still valid if the parameter space is allowed to be multidimensional, or, more generally, a Banach manifold, but the notation is more cumbersome.

Remark 2.3. As a particular case of the previous estimates (zero-dimensional parameter space), if $A \in \Delta_\delta^+$ satisfies

$$\|\overline{\partial}_z A_z(x)\| = o(|\Im(t)|^{\eta-1}) \tag{2.81}$$

then

$$|\overline{\partial}_z m^+(z,x)| = o(|\Im(t)|^{\eta-2}). \tag{2.82}$$

In order to illustrate the asymptotically holomorphic technique, we generalize Theorems 2.2 and 2.3 to the smooth setting.

Given a function $u : \Omega_\delta^+ \to \mathbb{C}/\mathbb{Z}$ which is continuous with locally integrable derivatives and satisfies

$$|\overline{\partial} u(z)| = O(|\Im(z)|^{\epsilon-1}), \tag{2.83}$$

for some $\epsilon > 0$, let us write a canonical decomposition $u = u^h + u^s$ where $u^h : \Omega_\delta^+ \to \mathbb{C}/\mathbb{Z}$ is holomorphic and $u^s : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ is a real-symmetric continuous function given by the Cauchy transform

$$u^s(z) = \lim_{t\to\infty} \frac{-1}{\pi} \int_{[-t,t]\times[-\delta,\delta]} \frac{\phi(w)}{z-w} dw \wedge d\overline{w}, \tag{2.84}$$

where $\phi(z) = \overline{\partial} u(z)$ if $0 < \Im(z) < \delta$ and $\phi(z) = \overline{\partial} u(\overline{z})$ for $0 < -\Im(z) < \delta$.

Notice that if

$$|\overline{\partial} u(z)| = O(|\Im(z)|^{k+\epsilon}), \tag{2.85}$$

then $u^s(z)$ is complex differentiable at each $z \in \mathbb{R}/\mathbb{Z}$, and $u^s(z) : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is C^{k+1}.

Theorem 2.7. *Let $A_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $\theta \in \mathbb{R}/\mathbb{Z}$, be $C^{2+\eta}$ and monotonic in θ. For almost every $\theta \in \mathbb{R}/\mathbb{Z}$, if $L(\alpha, A_\theta) = 0$ then*

$$\frac{d}{d\theta}\rho_{\alpha,A_\theta} \geq \frac{\epsilon}{2\pi} > 0, \tag{2.86}$$

where ϵ is the monotonicity constant of $\theta \mapsto A_\theta$.

Proof. For $\delta > 0$ small, let us denote by $A \in \Delta_\delta^+$, $z \in \Omega_\delta^+$, some fixed asymptotically holomorphic extension of A_θ satisfying

$$|\overline{\partial}_z A_z(x)| = O(|\Im(z)|^{1+\eta}). \tag{2.87}$$

It is enough to show that our hypothesis imply that for almost every $\sigma \in \mathbb{R}$,

$$\partial_\sigma \rho(\sigma) = \lim_{t\to 0} \frac{L(\sigma+it) - \lim_{t\to 0} L(\sigma+it)}{t}, \tag{2.88}$$

since the result then follows as in Theorem.

We have

$$|\overline{\partial}_z m^+(z,x)| = O(|\Im(z)|^{\eta}), \tag{2.89}$$

which implies

$$|\overline{\partial}_z \zeta(z)| = O(|\Im(z)|^{\eta}) \tag{2.90}$$

as well. Thus $\zeta^s(z)$ is complex differentiable at $z \in \mathbb{R}/\mathbb{Z}$ and $\zeta^s : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is C^1. Since $\Im\zeta > 0$ and $\sigma \mapsto \rho(\sigma) = \lim_{t\to 0+} \zeta(\sigma+it)$ is monotonic, this is enough to conclude that (2.88) holds for almost every σ. □

For further use, let us remark that an argument analogous to the proof of Theorem 2.7 also gives:

Proposition 2.8. *Let $A \in \Delta_\delta$ satisfy*

$$\|\overline{\partial}_z A_z\| = O(|\Im(z)|^{1+\epsilon}). \tag{2.91}$$

Then, for every $\sigma_0 \in \mathbb{R}/\mathbb{Z}$, if

$$\limsup_{\sigma\to\sigma_0} \frac{|\rho_{\alpha,A_\sigma} - \rho_{\alpha,A_{\sigma_0}}|}{|\sigma-\sigma_0|} < \infty \tag{2.92}$$

then

$$\limsup_{t\to 0} \frac{|L(\alpha, A_{\sigma_0+it}) - L(\alpha, A_{\sigma_0+it})|}{|t|} < \infty. \tag{2.93}$$

Theorem 2.9. *Let $A_{\theta,s} \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $\theta \in \mathbb{R}/\mathbb{Z}$, s a one-dimensional parameter, be monotonic in θ and $C^{2r+1+\epsilon}$ in (θ, s). Then*

$$s \mapsto \int_{\mathbb{R}/\mathbb{Z}} L(\alpha, A_{\theta,s})d\theta \tag{2.94}$$

is C^r.

Proof. Let $A^s \in \Delta_\delta^+$ be an asymptotically holomorphic extension of $A_{\theta,s}$ satisfying

$$\|\overline{\partial}_s^k A_z^s(x)\| = O(1), \quad 0 \le k \le 2r, \tag{2.95}$$

$$\|\overline{\partial}_s^k \overline{\partial}_z A_z^s(x)\| = O(|\Im(z)|^{2r-2k+\epsilon}), \quad 0 \le k \le 2r. \tag{2.96}$$

Then we have the estimate

$$|\overline{\partial}_s^k \overline{\partial}_z \zeta_s(z)| = O(|\Im(z)|^{r-2k-1}, \quad 0 \le k \le r-1. \tag{2.97}$$

Thus

$$\partial_s^k \int_{\mathbb{R}/\mathbb{Z}} \Im\zeta_s(\theta)d\theta = \int_{\mathbb{R}/\mathbb{Z}} \partial_s^k \Im\zeta_s(\sigma+it)d\sigma + 2\int_{\mathbb{R}/\mathbb{Z}\times(0,\delta)} \partial_s^k \Re\overline{\partial}_z\zeta_s(z)dz\wedge d\overline{z} + \partial_s^k(2\pi t \deg) \tag{2.98}$$

is a continuous function of s for $0 \le k \le r$. □

2.4. L^2-estimates.

Lemma 2.10. *Let $A : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ be measurable and let $\alpha \in \mathbb{R}$. The following are equivalent:*

1. *There exists a measurable $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ such that $\int_{\mathbb{R}/\mathbb{Z}} \|B(x)\|^2 dx < \infty$ and $B(x+\alpha)A(x)B(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$ for almost every x,*
2. *There exists a measurable $m : \mathbb{R}/\mathbb{Z} \to \mathbb{D}$ such that $\int_{\mathbb{R}/\mathbb{Z}} \frac{1}{1-|m(x)|^2}dx < \infty$ and $A(x)\cdot m(x) = m(x+\alpha)$ for almost every x.*

Proof. Let $QB(x)Q^{-1} = \frac{1}{(1-|m(x)|^2)^{1/2}} \begin{pmatrix} 1 & m(x) \\ \overline{m(x)} & 1 \end{pmatrix}$. □

If $(\alpha, A) \in \mathbb{R} \times C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ satisfies the equivalent conditions of the previous lemma, we will say that (α, A) is L^2-conjugate to a cocycle of rotations.

Our aim in this section is to prove the following.

Theorem 2.11. *Let $A_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $\theta \in \mathbb{R}/\mathbb{Z}$, be $C^{2+\epsilon}$ and monotonic in θ. For every $\theta \in \mathbb{R}/\mathbb{Z}$, if*

$$\liminf_{\theta'\to\theta} \frac{|\rho_{\alpha,A_{\theta'}} - \rho_{\alpha,A_\theta}|}{|\theta'-\theta|} < \infty \tag{2.99}$$

(in particular if $\theta \mapsto \rho_{\alpha,A_\theta}$ is Lipschitz) and $L(\alpha, A_\theta) = 0$ then (α, A_θ) is L^2-conjugate to a cocycle of rotations.

We will need a simple compactness result:

Proposition 2.12. *Let $(\alpha_k, A_k) \in \mathbb{R} \times C^0(\mathbb{R}/\mathbb{Z}, \Upsilon)$ be a sequence converging to (α, A). Assume there exists measurable functions $m_k : \mathbb{R}/\mathbb{Z} \to \mathbb{D}$ satisfying $A_k(x)\cdot m_k(x) = m_k(x+\alpha)$, such that $\liminf \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{1-|m_k(x)|^2}dx < \infty$. Then there exists a measurable $m : \mathbb{R}/\mathbb{Z} \to \mathbb{D}$ such that $A(x)\cdot m(x) = m(x+\alpha)$ and $\int_{\mathbb{R}/\mathbb{Z}} \frac{1}{1-|m(x)|^2}dx < \infty$.*

The proof uses the notion of conformal barycenter [DE], and we leave it for the Appendix B.

Corollary 2.13. *Let $A \in \Delta_\delta$. If $\sigma_0 \in \mathbb{R}/\mathbb{Z}$ satisfies*

$$\liminf_{t\to 0} \frac{L(\sigma_0 + it)}{t} < \infty \tag{2.100}$$

then A_{σ_0} is L^2-conjugate a cocycle of rotations.

Proof. Follows from the previous proposition and Lemma 2.7. □

Proof of Theorem 2.11. It is enough to apply Proposition 2.8 and the corolary above. □

Remark 2.4. If one is only concerned with a result valid for almost every θ, one can bypass the use of the conformal barycenter argument. Indeed, the most usual argument in such situations is to apply the Lemma of Fatou to guarantee convergence of $m^+(\sigma + it, x)$ as $t \to 0+$ for almost every x, and then apply Fubini's Lemma to obtain a set of σ of full Lebesgue measure for which $\lim_{t\to 0+} m^+(\sigma + it, x)$ exists for almost every x.

2.5. **Smooth dependence of conjugacies.**

Theorem 2.14. *Let $\alpha \in \mathbb{R}$ and let $A_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be $C^{r+1+\epsilon}$, $0 \le r < \infty$ and monotonic in θ. If $L(\alpha, A_\theta) = 0$ for every θ in some open interval J then there exists $B_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $\theta \in J$ depending C^r on θ and conjugating (α, A_θ) to a cocycle of rotations.*

Proof. We shall assume that $J = \mathbb{R}/\mathbb{Z}$ for simplicity. Consider an asymptotically holomorphic extension of A_θ satisfying

$$\|\overline{\partial}_z A_z\| = O(|\Im(z)|^{r+\epsilon}). \tag{2.101}$$

Then we have

$$\|\overline{\partial}_z m^+(z,x)\| = O(|\Im(z)|^{r-1+\epsilon}), \quad \Im(z) > 0 \tag{2.102}$$

and analogously

$$\|\overline{\partial}_z m^-(z,x)\| = O(|\Im(z)|^{r-1+\epsilon}), \quad \Im(z) < 0. \tag{2.103}$$

Let

$$\phi(z,x) = \overline{\partial}_z m^\pm(z,x), \quad \pm\Im(z) > 0, \tag{2.104}$$

and let $u : \mathbb{C}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be given by

$$u(z,x) = \lim_{t\to\infty} \frac{-1}{\pi} \int_{[-t,t]\times[-\delta,\delta]} \frac{\phi(w,x)}{z-w} dw \wedge d\overline{w}. \tag{2.105}$$

A compactness argument shows that $u(z,x)$ is continuous on both variables. Moreover, $\mathbb{R}/\mathbb{Z} \ni y \mapsto u(y,x)$ is C^r (uniformly in x). Let

$$m(z,x) = m^\pm(z,x), z \in \Omega_\delta^\pm. \tag{2.106}$$

Then

$$\lim_{t\to 0} m(\sigma + it, x) \tag{2.107}$$

exists for almost every σ and almost every x by Lemma 2.4. Thus for almost every $x \in \mathbb{R}/\mathbb{Z}$, $z \mapsto m(z,x) - u(z,x)$ extends to a holomorphic function defined on Ω_δ. A compactness argument shows that this holds indeed for all $x \in \mathbb{R}/\mathbb{Z}$, and that the function $\Omega_\delta \times \mathbb{R}/\mathbb{Z} \ni (z,x) \mapsto m(z,x) - u(z,x)$ is continuous. It also follows that $\mathbb{R}/\mathbb{Z} \ni y \mapsto m(y,x)$ is C^r (uniformly on x). To conclude, it is enough to show that $m(y,x)$ takes values on $\mathbb{D}$.

If y is such that $m(y,x_0) \in \partial\mathbb{D}$ for some $x_0 \in \mathbb{R}/\mathbb{Z}$, then for every $x \in \mathbb{R}/\mathbb{Z}$ we also have $m(y,x) \in \partial\mathbb{D}$ (by invariance). However, since $L(\alpha, A_y) = 0$ for every y, $\rho_{\alpha,A_y})$ is C^1 (by Schwarz Reflection), so for every σ we have

$$\limsup_{t\to 0+} \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{1 - |m^+(\sigma + it, x)|^2} dx < \infty, \tag{2.108}$$

so by continuity $m(\sigma, x) \in \mathbb{D}$ for almost every x. □

Similar arguments yield the analytic and infinitely differentiable cases, so we will not get into their proof:

Theorem 2.15. *Let $\alpha \in \mathbb{R}$ and let $A_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be C^r, $r = \omega, \infty$ and monotonic in θ. If $L(\alpha, A_\theta) = 0$ for every θ in some open interval J then there exists $B_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $\theta \in J$ depending C^r on θ and conjugating (α, A_θ) to a cocycle of rotations.*

2.6. **Dependence on the frequency of the Lyapunov exponent.** Several of our estimates are still valid when varying the frequency.

Lemma 2.16. *Let $(\alpha(s), A^s) \in \mathbb{R} \times \Delta_\delta^+$ be a one-parameter family. Assume that $s \mapsto \alpha(s)$ and $(s,x) \mapsto A_z^s(x)$ are C^r, $1 \le r < \infty$ and*

$$\|\partial_s^i \partial_x^j A_z^s(x)\| = O(1), \quad 0 \le i + j \le r. \tag{2.109}$$

Then

$$|\partial_s^i \partial_x^j m_s^+(z,x)| = O(|\Im(z)|^k), \quad 1 \le i + j \le r. \tag{2.110}$$

Moreover, if additionally $s \mapsto \overline{\partial} A_z^s(x)$ is C^{r-1} and we have the estimate

$$\|\partial_s^i \partial_x^j \overline{\partial}_z A_z^s(x)\| = o(|\Im(z)|^{\eta-k-1}), \quad 0 \le i + j \le r - 1, \tag{2.111}$$

for some $\eta \in \mathbb{R}$ then

$$|\partial_s^i \partial_x^j \overline{\partial}_z m_s^+(z,x)| = o(|\Im(z)|^{\eta-2k-1}), \quad 0 \le i + j \le r - 1. \tag{2.112}$$

Proof. The proof is essentially the same as before, except that the relevant equations are longer. ☐

Using the previous lemma we get the following result. The argument is the same as before and we won't repeat it.

Theorem 2.17. *Let us consider a family $(\alpha(s), A_{\theta,s}) \in \mathbb{R} \times C^{2r+1+\epsilon}(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $1 \le r < \infty$, $\theta \in \mathbb{R}/\mathbb{Z}$, s a one-dimensional parameter, such that $\theta \mapsto \alpha(\theta)$ and $(\theta, s, x) \mapsto A_{\theta,s}(x)$ is $C^{2r+1+\epsilon}$. Then*

$$s \mapsto \int_{\mathbb{R}/\mathbb{Z}} L(\alpha(s), A_{\theta,s}) d\theta \tag{2.113}$$

is C^r.

Remark 2.5. Even if everything is analytic, we do not, in general, get analytic dependence when varying the frequency, and we believe this is unlikely to happen. A special case that has analytic dependence is $A_{\theta,s} = R_\theta A_s$.

3. Monotonic cocycles

We now turn to the study of cocycles presenting monotonicity in phase space. We shall say that $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ a monotonic cocycle if $x \mapsto A(x)$ is monotonic.

Given a monotonic $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, one can consider a family $A_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ given by $A_\theta(x) = A(x+\theta)$. Notice that A_θ is monotonic in θ. This simple observation has the following remarkable consequences:

Theorem 3.1. *Let $A \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $r = \omega, \infty$ be a monotonic cocycle. If $L(\alpha, A) = 0$ then (α, A) is C^r-conjugate to a cocycle of rotations.*

Theorem 3.2. *Let $A \in C^{r+1+\epsilon}(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $0 \le r < \infty$ be a monotonic cocycle. If $L(\alpha, A) = 0$ then (α, A) is C^r-conjugate to a cocycle of rotations.*

Theorem 3.3. *Let us consider a one-parameter family $A^s \in \mathbb{R} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ of monotonic cocycles. If $(s,x) \mapsto A^s(x)$ is C^ω then*

$$s \mapsto L(\alpha(s), A^s) \tag{3.1}$$

is C^ω.

Theorem 3.4. *Let us consider a one-parameter family $(\alpha(s), A^s \in \mathbb{R} \times C^{2r+1+\epsilon}(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $1 \leq r < \infty$ of monotonic cocycles. If $s \mapsto \alpha(s)$ is $C^{2r+1+\epsilon}$ and $(s,x) \mapsto A^s(x)$ is $C^{2r+1+\epsilon}$ then*

$$s \mapsto L(\alpha(s), A^s) \tag{3.2}$$

is C^r.

For every $\alpha \in \mathbb{R}$, $\rho_{\alpha,A_\theta} = \rho_{\alpha,A} + \deg x$ is Lipschitz. More generally, we have the following result:

Lemma 3.5. *Let us consider a one-parameter family $A_\theta \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ which is monotonic in θ. If for some θ_0, A_{θ_0} is a monotonic cocycle, and*

$$K = \limsup_{\theta \to \theta_0} \frac{1}{|\theta - \theta_0|} \|A_\theta(x) - A_{\theta_0}(x)\| < \infty \tag{3.3}$$

then

$$\limsup_{\theta \to \theta_0} \frac{1}{|\theta - \theta_0|} |\rho_{\alpha,A_\theta} - \rho_{\alpha,A_{\theta_0}}| \leq K', \tag{3.4}$$

where K' depends on K, the monotonicity constant of A_{θ_0}, $\|A_{\theta_0}\|_{C^0}$ and the degree of A_{θ_0}.

Proof. The hypothesis imply that for h close to 0 and $z \in \partial\mathbb{D}$, $A_{\theta_0+h}(x) \cdot z$ lies in the shortest segment of $\partial\mathbb{D}$ determined by $A_{\theta_0}(x - Ch) \cdot z$ and $A_{\theta_0}(x + Ch)$, for some $C > 0$. This implies that $\rho_{\alpha,A_{\theta_0+h}}$ lies between $\rho_{\alpha,A_{\theta_0}(\cdot - Ch)}$ and $\rho_{\alpha,A_{\theta_0}(\cdot - Ch)}$, that is, in the segment $[\rho_{\alpha,A_{\theta_0}} - C|h \deg|, \rho_{\alpha,A_{\theta_0}} + C|h \deg|]$, and the result follows. □

Thus, if A is monotonic then $\theta \mapsto R_\theta A$ is a monotonic family with Lipschitz rotation number. In low regularity, it may be preferrable to work with this family, because it is always analytic in θ. As an application, we have the following result (if we were to use only the family $\theta \mapsto A(\cdot + \theta)$, we would need $C^{2+\epsilon}$).

Theorem 3.6. *Let $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be a monotonic cocycle. If $L(\alpha, A) = 0$ then (α, A) is L^2-conjugate to a cocycle of rotations.*

It also allows us to get continuity results in Lipschitz open sets of cocucles.

Theorem 3.7. *Let $\epsilon > 0$ be fixed. The Lyapunov exponent is a continuous function of ϵ-monotonic cocycles.*

Proof. Let $(\alpha_n, A^{(n)}) \to (\alpha, A)$ be a sequence of ϵ-monotonic cocycles converging in $C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. It follows that $\theta \mapsto L_n(\theta) = L(\alpha_n, R_\theta A^{(n)})$ are Hilbert transforms of (uniformly bounded) Lipschitz functions, so they belong to a compact set of continuous functions (their derivatives being uniformly in BMO). Thus we may assume $L_n \to L_\infty$ in $C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. By upper semicontinuity of the Lyapunov exponent, $L(\alpha, R_\theta A) - L_\infty(\theta)$ is a non-negative continuous function, which we must show to be identically zero. This follows from

$$\begin{aligned} \int_{\mathbb{R}/\mathbb{Z}} L(\alpha, R_\theta A) - L_\infty(\theta) d\theta &= \lim_{n\to\infty} \int_{\mathbb{R}/\mathbb{Z}} L(\alpha, R_\theta A) - L(\alpha_n, R_\theta A^{(n)}) d\theta \\ &= \lim_{n\to\infty} \int_{\mathbb{R}/\mathbb{Z}} \ln \frac{\|A(x)\| + \|A(x)\|^{-1}}{2} - \ln \frac{\|A^{(n)}(x)\| + \|A^{(n)}(x)\|^{-1}}{2} dx \\ &= 0 \end{aligned} \tag{3.5}$$

by [AB]. □

3.1. **Minimality.**

It is an interesting problem to consider the dynamics of a cocycle (α, A) from the topological point of view. For this, one considers (α, A) as a function $\mathbb{R}/\mathbb{Z} \times \partial\mathbb{D} \to \mathbb{R}/\mathbb{Z} \times \partial\mathbb{D}$ (a two-dimensional torus) given by $(x, w) \mapsto (x + \alpha, A(x) \cdot w)$.

It can be shown (see [KKHO]) that if $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ is not homotopic to the identity then for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, (α, A) is transitive. The following question seems much harder however:

Problem 3.1. Let $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be non-homotopic to the identity, and let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Is (α, A) minimal?

Of course the same problem still makes sense under additional smoothness assumptions. In this section we will give a partial result in this direction.

Let us first discuss some known results on the minimal sets of non-uniformly hyperbolic cocycles. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. If $L(\alpha, A) > 0$ then it follows from Oseledets Theorem that there exists two measurable function $u, s : \mathbb{R}/\mathbb{Z} \to \partial\mathbb{D}$ (the unstable and stable directions) such that $A(\theta) \cdot u(\theta) = u(\theta + \alpha)$ and $A(\theta) \cdot s(\theta) = s(\theta + \alpha)$ and for almost every θ, for every $w \in \partial\mathbb{D}$, if $w \neq s(\theta)$ then $|A_n(\theta) \cdot w - u(\theta + n\alpha)| \to 0$ exponentially fast, and if $w \neq u(\theta)$ then $|A_n(\theta - n\alpha)^{-1} \cdot w - s(\theta - n\alpha)| \to 0$ exponentially fast. It follows (from unique ergodicity of $\theta \mapsto \theta + \alpha$) that there are exactly two ergodic invariant measures on $\mathbb{R}/\mathbb{Z} \times \partial\mathbb{D}$, the push-forwards of Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ by $\theta \mapsto (\theta, u(\theta))$ and $\theta \mapsto (\theta, s(\theta))$, which we denote by μ_u and μ_s. Let us denote their (compact) support by K_u and K_s. It follows that any minimal set for (α, A) coincides with either K_u or K_s (and in particular, at least one of K_u, K_s is a minimal set).[4]

We now show that the complexification methods allow one to address the local case, at least if one assumes enough smoothness.

Theorem 3.8. *Let $A \in C^{2+\epsilon}(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be monotonic. Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then (α, A) is minimal.*

Proof. If $L(\alpha, A) = 0$ then (α, A) is C^1 conjugate to a cocycle of rotations. For a cocycle of rotations, transitivity obviously implies minimality, so the result follows from [KKHO].

Let now $L(\alpha, A) > 0$. We consider the analytic case, the smooth case being analogous. Let $m : \Omega_\delta \to \mathbb{D}$ satisfy $A(\sigma + it) \cdot m(\sigma + it) = m(\sigma + it + \alpha)$. Since $L(\alpha, A) > 0$, for almost every $\sigma \in \mathbb{R}/\mathbb{Z}$, $m(\sigma) = \lim m(\sigma + it) \in \partial\mathbb{D}$, and $m|\mathbb{R}/\mathbb{Z}$ coincides with the unstable direction $u : \mathbb{R}/\mathbb{Z} \to \partial\mathbb{D}$ defined above. Since A is not homotopic to the identity, $m|\mathbb{R}/\mathbb{Z}$ is not continuous.

We claim that for every interval $J \subset \mathbb{R}/\mathbb{Z}$ and any interval $J' \subset \partial\mathbb{D}$, there exists a positive measure set of $\sigma \in J$ such that $m(\sigma) \in J'$. This follows from the Schwarz Reflection Principle: otherwise $m|J$ would be analytic so by invariance we would have $m|\mathbb{R}/\mathbb{Z}$ analytic, hence continuous. Thus $K_u = \mathbb{R}/\mathbb{Z} \times \partial\mathbb{D}$. Analogously, we have $K_s = \mathbb{R}/\mathbb{Z} \times \partial\mathbb{D}$, which concludes the proof of minimality. □

3.2. **Premonotonic cocycles.** As remarked in the introduction, the concept of monotonicity is not dynamically natural. The easiest way to extend the concept of monotonicity is the following. We say that (α, A) is premonotonic if it is C^1 conjugate to a monotonic cocycle: there exists $B \in C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ such that $B(x+\alpha)A(x)B(x)^{-1}$. This happens if and only if (α, A) is real-analytic conjugate to a monotonic cocycle (any C^1-perturbation of B which is real analytic will do). This definition is such that all results proved for monotonic cocycles extend in a trivial way to this larger setting. It is also enough for us to prove our global results.

Although premonotonicity is (expressily) invariant under smooth conjugacies, we have not succeeded in proving it is invariant under more general classes of transformations. For instance, it is not clear if a cocycle which admits a monotonic iterate is premonotonic (though it is easy to show that there exists real-analytic premonotonic cocycles with any given frequency which do not admit a monotonic iterate). Worse, this definition does not behave well under renormalization. Thus we were led to study some more general classes of cocycles which can be shown to admit a description similar to monotonic cocycles. Our results are, at the moment, not completely satisfactory: we have identified a natural class which is invariant under renormalization, but we did not succeed in showing that this class is actually bigger than the class of premonotonic cocycles. In order not to distract from the normal flow of our arguments, we have left this discussion for Appendix F. In the same appendix we shall also prove the existence of many cocycles which are not premonotonic.

4. Non-uniform hyperbolicity for typical cocycles

4.1. **Monotonic (and premonotonic) cocycles with a zero Lyapunov exponent.**

4.1.1. *An estimate for the second derivative of the Lyapunov exponent for a cocycle of rotations.* Since the Lyapunov exponent L takes non-negative values, we must have $DL = 0$ whenever $L = 0$. Here we are going to show that if $L = 0$ then $D^2L \neq 0$. This implies that $\{L = 0\}$ is a subvariety of positive codimension.

[4]Although we will not need this fact, if (α, A) is non-uniformly hyperbolic then $K_u \cap K_s = \emptyset$ by a result of Herman [H].

Lemma 4.1. *Let $B \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. Let $A_\theta(x) = R_{nx}B(x-\theta)$. Then*

$$\int_0^1 L(\alpha, A_\theta(x))d\theta = \int_0^1 \ln\left(\frac{\|B(x)\| + \|B(x)\|^{-1}}{2}\right)dx. \tag{4.1}$$

Proof. Let $C_\theta(x) = R_{n\theta}R_{nx}B(x)$. Notice that $A_\theta(x+\theta) = C_\theta(x)$. In particular, $L(\alpha, A_\theta) = L(\alpha, C_\theta)$. The result follows by [AB]. □

Let $s \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{sl}(2,\mathbb{R}))$, that is,

$$s(x) = \begin{pmatrix} a(x) & b(x)+c(x) \\ b(x)-c(x) & -a(x) \end{pmatrix}, \tag{4.2}$$

where $a, b, c : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ are continuous functions. Let $A_{\theta,t} = R_{nx}e^{ts(x-\theta)}$. Then the previous lemma implies that

$$\lim_{t\to 0} \frac{1}{2t^2}\int_0^1 L(\alpha, A_{\theta,t})d\theta = \int_0^1 a^2(x) + b^2(x)dx. \tag{4.3}$$

In particular, the limit above is zero if and only if a and b vanish identically, that is, if and only if s takes values in $\mathrm{so}(2,\mathbb{R})$.

4.2. **Reduction to the case of premonotonic cocycles.**

Theorem 4.2. *Let $(\alpha, A) \in (\mathbb{R}\setminus\mathbb{Q}) \times C^2(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be L^2-conjugate to rotations. Then (α, A) is premonotonic.*

Corollary 4.3. *Let $A_\theta \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $r = \omega, \infty$, be $C^{2+\epsilon}$ and monotonic in θ. For almost every θ, either $L(\alpha, A) > 0$ or (α, A) is C^r-conjugate to a cocycle of rotations.*

Corollary 4.4. *Let $A_\theta \in C^{r+1+\epsilon}(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $1 \le r < \infty$, be $C^{2+\epsilon}$ and monotonic in θ. For almost every θ, either $L(\alpha, A) > 0$ or (α, A) is C^r-conjugate to a cocycle of rotations.*

Appendix A. Asymptotically holomorphic extensions

For $r \in [1, \infty)$, let $AH^r(\mathbb{C}, \mathbb{C})$ be the space of C^r functions $f : \mathbb{C} \to \mathbb{C}$ satisfying

$$\frac{d^k}{dt^k}\overline{\partial}F(\sigma) = 0, \quad \sigma \in \mathbb{R},\ k = 0, ..., [r-1], \tag{A.1}$$

and in particular

$$\overline{\partial}F(\sigma + it) = o(|t|^{r-1}), \tag{A.2}$$

for integer r or

$$\overline{\partial}F(\sigma + it) = O(|t|^{r-1}), \tag{A.3}$$

for non-integer r.

It is easy to see that one can define (linear) sections Φ_r of the restriction operator $AH^r \to C^r(\mathbb{R}, \mathbb{C})$, moreover, Φ_r can be chosen to commute with translations. For instance, one can let

$$\Phi_r(f)(\sigma + it) = \int K(x)f(\sigma + tx)dx, \tag{A.4}$$

where $K : \mathbb{R} \to \mathbb{C}$ is a C^∞ function with compact support satisfying

$$\int x^k K(x)dx = i^k, \quad k = 0, ..., [r+1]. \tag{A.5}$$

In order to obtain asymptotically holomorphic extensions of a matrix valued function $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, it is enough to consider

$$\Phi_r(A) = (\Phi_r(a)\Phi_r(d) - \Phi_r(b)\Phi_r(c))^{-1/2}\begin{pmatrix} \Phi(a) & \Phi(b) \\ \Phi(c) & \Phi(d) \end{pmatrix}, \tag{A.6}$$

which is a well defined function $\Omega_\delta \to \mathrm{SL}(2,\mathbb{R})$, where δ only depends on the C^1-norm of A.

Appendix B. Conformal barycenter

Let $\mathcal{M}$ be the set of probability measures on $\mathbb{D}$, and for $\mu \in \mathcal{M}$, let $\Phi(\mu) = \int_{\mathbb{D}} \frac{1}{1-|z|^2} d\mu(z)$. For $w \in \mathbb{D}$, let $\Phi_w(\mu) = \Phi(\mu')$ where μ' is the pushforward of μ by some Moebius transformation of $\mathbb{D}$ taking w to 0. Notice that if $\Phi(\mu) < \infty$ then $\Phi_w(\mu) < \infty$ for every w. For every $1 \leq K < \infty$, let $\mathcal{M}_K = \{\mu \in \mathcal{M}, \Phi(\mu) \leq K\}$, and let $\mathcal{M}_\infty = \cup \mathcal{M}_K$. Notice that $\mathcal{M}_K$ is compact in the weak-* topology for every $K < \infty$.

The next proposition can be proved using the conformal barycenter of Douady-Earle [DE]. The construction is sufficiently simple for us to give the details here.

Proposition B.1. *There exists a Borelian function $\mathcal{B} : \mathcal{M}_0 \to \mathbb{D}$, equivariant with respect to Möebius transformations of $\mathbb{D}$ and such that $\Phi(\delta_{\mathcal{B}(\mu)}) \leq \Phi(\mu)$.*

Proof. Following an idea of Yoccoz, let us define a pairing $\mathbb{D} \times \mathbb{D} \to \mathbb{D}$ by setting $z * w$ as the midpoint of the hyperbolic geodesic passing through z and w if $z \neq w$, and $z * z = z$. This pairing is continuous and equivariant, and we have

$$u_s(z,w) \equiv \Phi_s\left(\frac{\delta_z + \delta_w}{2}\right) - \Phi_s(\delta_{z*w}) = (2\Phi_s(\delta_{z*w}) - 1)\Phi_s(\delta_{z*w}) \geq 0, \tag{B.1}$$

with equality if and only if $z = w$. Notice that

$$u_s(z,s) = \Phi_s(\delta_z) - \Phi_s(\delta_z)^{1/2}. \tag{B.2}$$

Extend the pairing $*$ to $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ linearly. Thus

$$\mu * \nu = \int_{\mathbb{D}\times\mathbb{D}} \delta_{z*w} d\mu(z) d\nu(w). \tag{B.3}$$

If $\mu, \nu \in \mathcal{M}_\infty$ then

$$u_s(\mu,\nu) \equiv \Phi_s(\frac{1}{2}(\mu+\nu)) - \Phi_s(\mu * \nu) = \int_{\mathbb{D}\times\mathbb{D}} u_s(z,w) d\mu(z) d\nu(w) \geq 0, \tag{B.4}$$

with equality if and only if $\mu = \nu$ is a Dirac mass. Notice that $u_s : \mathcal{M} \times \mathcal{M} \to [0,\infty]$ is lower semicontinuous, so if $\mu_k \to \mu$ and $u_s(\mu_k,\mu_k) \to 0$ then μ is a Dirac mass. If $\mu_k \to \delta_s$ we have

$$\limsup_{k\to\infty} u_s(\mu_k,\mu_k) \geq \limsup_{k\to\infty} u_s(\mu_k,\delta_s) = \limsup_{k\to\infty} \int_{\mathbb{D}} \Phi_s(\delta_z) - \Phi_s(\delta_z)^{1/2} d\mu_k(z), \tag{B.5}$$

and in particular if additionally $\lim u_s(\mu_k,\mu_k) = 0$ then $\lim \Phi_s(\mu_k) = 1$.

Given $\mu \in \mathcal{M}$, define $\mu^{(k)}$ inductively by $\mu^{(0)} = \mu$ and $\mu^k = \mu^{(k-1)} * \mu^{(k-1)}$. If $\mu \in \mathcal{M}_\infty$ then $\mu^{(k)} \in \mathcal{M}_\infty$ and we have $\Phi(\mu^{(k+1)}) = \Phi(\mu^{(k)}) - u(\mu^{(k)},\mu^{(k)})$. Thus $u_s(\mu^{(k)},\mu^{(k)}) \to 0$, and any limit of $\mu^{(k)}$ (which exists by compactness) must be a Dirac mass. Moreover, if $\mu^{(n_k)} \to \delta_s$ then $\Phi_s(\mu^{(n_k)}) \to 1$, so $\Phi_s(\mu^{(n)}) \to 1$ as well and δ_s must be the unique limit of $\mu^{(n)}$. Now we can set $\mathcal{B}(\mu) = s$, which is clearly Borelian.[5] □

The estimates above allow us to obtain compactness result for invariant sections of cocycles. For instance, we have the following.

Proposition B.2. *Let $(\alpha_k, A_k) \in \mathbb{R} \times C^0(\mathbb{R}/\mathbb{Z}, \Upsilon)$ be a sequence converging to (α, A). Assume there exists measurable $m_k : \mathbb{R}/\mathbb{Z} \to \mathbb{D}$ satisfying $A_k(x) \cdot m_k(x) = m_k(x+\alpha)$, such that*

$$H \equiv \liminf_{K,k\to\infty} \int_{\mathbb{R}/\mathbb{Z}} \min\{K, \frac{1}{1-|m_k(x)|^2} dx < \infty. \tag{B.6}$$

Then there exists a measurable $m : \mathbb{R}/\mathbb{Z} \to \mathbb{D}$ such that $A(x) \cdot m(x) = m(x+\alpha)$ and $\int_{\mathbb{R}/\mathbb{Z}} \frac{1}{1-|m(x)|^2} dx \leq H$.

[5] Although we do not need this fact, it is easy to see that $\mathcal{B}$ is continuous in each $\mathcal{M}_K$, $1 \leq K < \infty$.

Proof. Let $X_{K,k} = \{x \in \mathbb{R}/\mathbb{Z}, \frac{1}{1-|m_k(x)|^2} < K\}$, and let $\mu_{K,k} = \int_{X_{K,k}} \delta_{m(x)} dx$. Let μ be any limit of $\mu_{K,k}$ along a sequence $K_i \to \infty$, $k_i \to \infty$ attaining the lim inf in (B.6). Then μ is a probability measure which projects onto Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ satisying $\int_{\mathbb{R}/\mathbb{Z}\times\mathbb{D}} \frac{1}{1-|z|^2} d\mu(x,z) \leq H$. Let μ_x, $x \in \mathbb{R}/\mathbb{Z}$ be a desintegration of μ: $\int_{\mathbb{R}/\mathbb{Z}\times\mathbb{D}} \phi(x,z) d\mu(x,z) = \int_{\mathbb{R}/\mathbb{Z}} (\int_{\mathbb{D}} \phi(x,z) d\mu_x(z)) dx$. Then $\mu_{x+\alpha}$ is the pushforward of μ_x by $A(x)$, and $m(x) \in \mathcal{M}_\infty$ for almost every x. Let $m(x) = \mathcal{B}(\mu_x)$. Then $m(x+\alpha) = A(x) \cdot m(x)$ and we have $\int \frac{1}{1-|m(x)|^2} dx \leq \int \int (\frac{1}{1-|z|^2} d\mu_x(z) dx \leq H$. □

Appendix C. Codimension of monotonic cocycles with zero Lyapunov exponent

Let us denote by $M^r_{\deg} \subset C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ the open set of monotonic cocycles of degree deg. Here and in what follows, we shall assume that $r \geq 6$, so that $A \mapsto L(\alpha, A)$ is a C^2 function of $A \in M^r$.

In this section we are going to show that for each $\alpha \in \mathbb{R}$, $\{L(\alpha, A) = 0\}$ is contained on a smooth submanifold of M^r of codimension $4 \deg$. Since $L \geq 0$, the derivative of $A \mapsto L(\alpha, A)$ must vanish at $\{L(\alpha, A) = 0\}$. Thus, in order to estimate the codimension of $\{L(\alpha, A) = 0\}$, it is enough to estimate the rank of the second derivative of $A \mapsto L(\alpha, A)$ at any $A \in M^r$ satisfying $L(\alpha, A) = 0$.

We first deal with the case where $\alpha = 0$. It is easy to see that if $A \in M^r$ then $L(0, A) = 0$ if and only if each of the equations $A(x) = \mathrm{id}$ and $A(x) = -\mathrm{id}$ has deg solutions. An obvious estimate shows that the second derivative of $A \mapsto L(\alpha, A)$ has rank $4 \deg$ at such a cocycle. Even without using this fact, this characterization clearly defines $\{L(0, A) = 0\}$ as a C^{r-2} smooth submanifold of M^r of codimension $4 \deg$.

The case of rational frequencies can be studied similar to the case $\alpha = 0$, so we shall concentrate on $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ from now on.

Theorem C.1. *Let $(\alpha, A) \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be conjugated to a cocycle of rotations. Then D^L has rank $4n$.*

Proof. The result is obvious if $\alpha \in \mathbb{Q}$.

It is enough to consider the case where $A(x) = R_{nx}$. If α is close to 0 then $D^2 L$ has rank $4n$. The result follows by renormalization. □

Appendix D. Partially hyperbolic simpletic cocycles

Theorem D.1. *Let $\alpha \in RDC$. Typical cocycles in $PH^r_{d,2}$ are either reducible or non-uniformly hyperbolic.*

Appendix E. Convergence of renormalization

E.1. Convergence of renormalization.

Theorem E.1. *Let $(\alpha, A) \in (\mathbb{R} \setminus \mathbb{Q}) \times C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be L^2-conjugated to rotations. For almost every $x_* \in \mathbb{R}/\mathbb{Z}$, the renormalizations of (α, A) around x_* converge in the C^1-topology to the set of standard models of the same degree as A.*

Corollary E.2. *Let $(\alpha, A) \in (\mathbb{R} \setminus \mathbb{Q}) \times C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be L^2-conjugated to rotations. If A is not homotopic to the identity then (α, A) admits a monotonic renormalization.*

E.1.1. *Strategy.* The stategy of reduction from global to local of [AK1] can be summarized as follows. One considers $(\alpha, A) \in (\mathbb{R} \setminus \mathbb{Q}) \times C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ which is L^2-conjugate to a cocycle of rotations, and analyze the iterates of (α, A) under a renormalization operator. Those renormalizations are essentially cocycles (α_n, A^n) where α_n are the iterates of α by the Gauss map. Using the L^2-conjugacy, we are able to show that the sequence of renormalizations is C^1 precompact and that limits of renormalization are of the type $(\alpha_\infty, A^\infty)$ where A^∞ is essentially (that is, modulo a constant matrix) a cocycle of rotations. If α_∞ is irrational then $(\alpha_\infty, A_\infty)$ can be C^1-conjugated to a cocycle arbitrarily close to a standard model.

This strategy does not cover the case where α_∞ is rational. This excludes even some Diophantine frequencies (for instance, whenever the continued fraction coefficients converge to ∞ we must have $\alpha_\infty = 0$).

Notice that the strategy of [AK1] does not prove convergence of renormalization to a standard model. Rather, it shows convergence to cocycles of rotations, and it implicitly uses that renormalization converges for cocycles of rotations with irrational frequency.

Why renormalization converges for a cocycle of rotations with irrational frequency? One way to see convergence is to observe an averaging effect when iterating ("cancellations"). In this setting, the mechanism of cancellation is obvious: unique ergodicity of irrational rotations together with commutativity of the group of rotations.

In [K], it was shown under slightly different hypothesis (L^∞-conjugacy instead of L^2-conjugacy, C^2 differentiability) that renormalization converges to a standard model. This method is based on finding cancellations directly in $\mathrm{SL}(2,\mathbb{R})$ and is quite different to the approach described above.

In this work, we will show convergence to a standard model by making a simple adaptation of the argument of [AK1]. We first renormalize several times in order to get close to a cocycle of rotations, but we do not take the limit. Here "close" to rotations means more than just C^1-close: it is important to also guarantee that the L^2-conjugacy is L^2-close to rotations. Then we show that the cancellation mechanism for cocycles of rotations still works if we are only "close" to rotations.

E.1.2. *Statements.* Let us fix some notation. If A is L^2-conjugate to a cocycle of rotations, we denote by B the L^2 conjugacy, so that $B(x+\alpha)A(x)B(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$. We let $\phi(x) = \|B(x)\|^2$, and let

$$S(x) = \sup_{k\geq 0} \frac{1}{2k+1} \sum_{j=-k}^{k} \phi(x). \tag{E.1}$$

In [AK1], precompactness of renormalization is based on the following estimate:

Lemma E.3 (see Lemma 3.3 of [AK1])**.** *Let $(\alpha, A) \in (\mathbb{R}\setminus\mathbb{Q}) \times C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be L^2-conjugated to a cocycle of rotations. For almost every $x_* \in \mathbb{R}/\mathbb{Z}$, there exists $K \equiv K(x_*) > 0$, such that for every $d > 0$, for every n sufficiently large such that $\|n\alpha\| \leq \frac{d}{n}$, for every $x \in \mathbb{R}/\mathbb{Z}$ such that $|x - x_*| \leq \frac{d}{n}$, we have*

$$\|A_n(x)\|, \frac{1}{n}\|\partial A_n(x)\| \leq K. \tag{E.2}$$

(In [AK1] higher derivatives are also considered.)

This result is enough to obtain precompactness of iterates of renormalization in topologies weaker than C^1, but not in the C^1 topology, which will be necessary in our argument. We will prove the following estimate, which does stablish C^1-precompactness.

Lemma E.4. *Let $(\alpha, A) \in (\mathbb{R}\setminus\mathbb{Q}) \times C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be L^2-conjugated to a cocycle of rotations. For almost every $x_* \in \mathbb{R}/\mathbb{Z}$, there exists $K \equiv K(x_*) > 0$, such that for every $d > 0$, for every n sufficiently large such that $\|n\alpha\| \leq \frac{d}{n}$, for every $x, x' \in \mathbb{R}/\mathbb{Z}$ such that $|x - x_*|, |x' - x_*| \leq \frac{d}{n}$, we have*

$$\frac{1}{n}\|A_n(x)^{-1}\partial A_n(x) - A_n(x')^{-1}\partial A_n(x')\| \leq Kn|x - x'| + \delta(n), \tag{E.3}$$

where $\lim_{n\to\infty} \delta(n) = 0$.

(This result also implies that limits of iterates of renormalization are $C^{1+\mathrm{Lip}}$, but we will not use this.)

For $x \in \mathbb{R}/\mathbb{Z}$ such that $B(x)$ is defined, let $A^x(y) = B(x)A(y)B(x)^{-1}$. In [AK1], information about the limits of renormalization was based on the following:

Lemma E.5 (see Lemma 3.4 of [AK1])**.** *Let $(\alpha, A) \in (\mathbb{R}\setminus\mathbb{Q}) \times C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be L^2-conjugated to a cocycle of rotations. For almost every $x_* \in \mathbb{R}/\mathbb{Z}$, for every $\delta > 0$, $d > 0$, for every n sufficiently large such that $\|n\alpha\| \leq \frac{d}{n}$, for every $x \in \mathbb{R}/\mathbb{Z}$ such that $|x - x_*| \leq \frac{d}{n}$, we have*

$$\|A_n^{x_*}(x)\| \leq 1 + \delta(n), \tag{E.4}$$

where $\lim_{n\to\infty} \delta(n) = 0$.

This result is enough to stablish C^0-convergence of renormalization to cocycles of rotations (modulo a constant matrix). Because of C^1-precompactness, this is enough to get C^1-convergence.

Our key estimate, which is based on cancellation, is the following:

Theorem E.6. *Let $(\alpha, A) \in (\mathbb{R} \setminus \mathbb{Q}) \times C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be L^2-conjugated to a cocycle of rotations. For almost every $x_* \in \mathbb{R}/\mathbb{Z}$, for every $d > 0$, for every n sufficiently large such that $\|n\alpha\| \leq \frac{d}{n}$, for every $x \in \mathbb{R}/\mathbb{Z}$ such that $|x - x_*| \leq \frac{d}{n}$, we have*

$$\left\| \frac{1}{n} A_n^{x_*}(x)^{-1} \partial A_n^{x_*}(x) - \omega \right\| \leq \delta(n), \tag{E.5}$$

where $\omega \equiv \omega(x_*) \in \mathrm{so}(2,\mathbb{R})$ and $\lim_{n\to\infty} \delta(n) = 0$.

(Actually ω only depends on the degree of A.)

This result implies convergence of renormalization to the standard model (modulo a constant matrix).

The proofs of Lemma E.4 and Theorem E.6 involve quite a bit of the proof of Lemmas E.3 and E.5, so we will give the proof of all of them for completeness.

E.1.3. *Proof of Lemmas E.3 and E.4.* Let $X_1 \subset \mathbb{R}/\mathbb{Z}$ be the set of $x \in \mathbb{R}/\mathbb{Z}$ for which $S(x) < \infty$. It has full Lebesgue measure by the Maximal Ergodic Theorem. Let $X_2 \subset X_1$ be the set of measurable continuity points of S and of B (and in particular of ϕ). It has full Lebesgue measure by the Lebesgue Density Point Theorem.

In what follows, C_i will denote various constants that only depend on A and $C_i(x)$ will denote constants depending on $x \in X_1$ and which are continuous increasing functions of $\phi(x)S(x)$.

Let $y, y' \in \mathbb{R}/\mathbb{Z}$. We have

$$\begin{aligned} \|\,\mathrm{id} - A_j(y)^{-1} A_j(y')\| &\leq \sum_{r=1}^{j} \sum_{0 \leq i_r < \ldots < i_1 \leq j-1} \prod_{l=1}^{r} \|A_{i_l}(y)\|^2 \|\,\mathrm{id} - A(y + i_l\alpha)^{-1} A(y' + i_l\alpha)\| \\ &\leq \sum_{r=1}^{j} \sum_{0 \leq i_r < \ldots < i_1 \leq j-1} \prod_{l=1}^{r} C_1 |y - y'| \|A_{i_l}(y)\|^2 \\ &\leq -1 + \exp\left(C_1 |y - y'| \sum_{l=0}^{j-1} \|A_l(y)\|^2 \right) \end{aligned} \tag{E.6}$$

where C_1 depends only on the Lipschitz norm of A.

Notice that

$$\sum_{l=0}^{j-1} \|A_l(y)\|^2 \leq j\phi(y)S(y), \tag{E.7}$$

so we get

$$\|\,\mathrm{id} - A_j(y)^{-1} A_j(y')\| \leq -1 + \exp(C_1 j |y - y'| \phi(y) S(y)). \tag{E.8}$$

If $|y - x| \leq \frac{1}{j}$, and $x \in X_1$ then this implies

$$\|\,\mathrm{id} - A_j(y)^{-1} A_j(x)\| \leq C_2(x). \tag{E.9}$$

In particular this gives

$$\|A_j(y)\| \leq C_3(x) \|A_j(x)\|. \tag{E.10}$$

If $|y - x| \leq \frac{1}{n}$ and $x \in X_1$ this gives

$$\sum_{j=0}^{l-1} \|A_j(y)\|^2 \leq C_4(x). \tag{E.11}$$

Thus, if $x \in X_1$ and $|y-x|, |y'-x| \leq \frac{1}{j}$ we get

$$\|\operatorname{id} - A_j(y)^{-1}A_j(y')\| \leq -1 + \exp\left(C_1|y-y'|\sum_{l=0}^{j-1}\|A_l(y)\|^2\right) \leq -1 + \exp(C_1|y-y'|C_4(x)) \leq C_5(x)j|y-y'|. \tag{E.12}$$

Let us now estimate

$$A_n(y)^{-1}\partial A_n(y) = \sum_{j=0}^{n-1} A_j(y)^{-1}A(y+j\alpha)^{-1}\partial A(y+j\alpha)A_j(y). \tag{E.13}$$

We have

$$\|A_n(y)^{-1}\partial A_n(y)\| \leq \sum_{j=0}^{n-1}\|A_j(y)\|^2\|A(y+j\alpha)^{-1}\partial A(y+j\alpha)\| \leq C_6\sum_{j=0}^{n-1}\|A_j(y)\|^2, \tag{E.14}$$

where C_6 depends on the C^1 norm of A. Thus if $|y-x| \leq \frac{1}{n}$ where $x \in X_1$ we can estimate

$$\|A_n(y)^{-1}\partial A_n(y)\| \leq C_7(x)n. \tag{E.15}$$

Let us now consider $x \in X_1$ and points y, y' such that $|y-x|, |y'-x| \leq \frac{1}{n}$. We want to estimate

$$\begin{aligned} A_n(y)^{-1}\partial A_n(y) - A_n(y')^{-1}\partial A_n(y') &= \sum_{j=0}^{n-1} A_j(y)^{-1}(\operatorname{id} - A_j(y)A_j(y')^{-1})A(y+j\alpha)^{-1}\partial A(y+j\alpha)A_j(y) \\ &+ \sum_{j=0}^{n-1} A_j(y)^{-1}(A(y+j\alpha)^{-1}\partial A(y+j\alpha) - A(y'+j\alpha)^{-1}\partial A(y'+j\alpha))A_j(y') \\ &+ \sum_{j=0}^{n-1} A_j(y)^{-1}A(y+j\alpha)^{-1}\partial A(y+j\alpha)(\operatorname{id} - A_j(y)^{-1}A_j(y'))A_j(y). \end{aligned} \tag{E.16}$$

Notice that

$$\left\|\sum_{j=0}^{n-1} A_j(y)^{-1}(A(y+j\alpha)^{-1}\partial A(y+j\alpha) - A(y'+j\alpha)^{-1}\partial A(y'+j\alpha))A_j(y')\right\| \leq \epsilon(n)\sum_{j=0}^{n-1}\|A_j(y)\|^2 \leq \epsilon(n)C_4(x)n, \tag{E.17}$$

where

$$\epsilon(n) = \sup_{|y-y'|\leq\frac{2}{n}}\|A(y)^{-1}\partial A(y) - A(y')^{-1}\partial A(y')\| \tag{E.18}$$

satisfies $\lim_{n\to\infty}\epsilon(n) = 0$.

On the other hand,

$$\begin{aligned} \left\|\sum_{j=0}^{n-1} A_j(y)^{-1}(\operatorname{id} - A_j(y)A_j(y')^{-1})A(y+j\alpha)^{-1}\partial A(y+j\alpha)A_j(y)\right\| &\leq \sum_{j=0}^{n-1} C_6\|A_j(y)\|^2\|\operatorname{id} - A_j(y)A_j(y')^{-1}\| \\ &\leq C_4(x)C_5(x)C_6 n|y-y'|. \end{aligned} \tag{E.19}$$

Similarly

$$\left\|\sum_{j=0}^{n-1} A_j(y)^{-1}A(y+j\alpha)^{-1}\partial A(y+j\alpha)(\operatorname{id} - A_j(y)^{-1}A_j(y'))A_j(y)\right\| \leq C_4(x)C_5(x)C_6 n|y-y'|. \tag{E.20}$$

Thus

$$\|A_n(y)^{-1}\partial A_n(y) - A_n(y')^{-1}\partial A_n(y')\| \leq C_4(x)\epsilon(n)n + C_4(x)C_5(x)C_6 n|y-y'|. \tag{E.21}$$

Let now $x_* \in X_2$. Then for $d > 0$ fixed and for n big, the set of $x \in X_1$ such that $\phi(x) \le \phi(x_*) + 1$, $S(x) \le S(x_*) + 1$ and $|x - x_*| \le \frac{2d}{n}$ has Lebesgue measure bigger than $\frac{4d}{n} - \frac{1}{100n}$. If $\|n\alpha\|_{\mathbb{Z}} \le \frac{d}{n}$, this implies that every y such that $|y - x_*| \le \frac{d}{n}$ is $\frac{1}{10n}$ close to some $x \in X_1$ satisfying such that $x + n\alpha \in X_1$. For such an y we have

$$\|A_n(y)\| \le C_3(x)\|A_n(x)\| \le C_3(x)(\phi(x_*) + 1)^2 \le C_8(x_*), \tag{E.22}$$

$$\|A_n(y)^{-1}\partial A_n(y)\| \le C_7(x)n \le C_9(x_*)n. \tag{E.23}$$

This already gives Lemma E.3. Moreover, if $y' \in \mathbb{R}/\mathbb{Z}$ is such that $|y - y'| \le \frac{1}{2n}$, we also have

$$\|A_n(y)^{-1}\partial A_n(y) - A_n(y')^{-1}\partial A_n(y')\| \le C_4(x)\epsilon(n)n + C_4(x)C_5(x)C_6 n|y - y'| \le C_{10}(x_*)n\epsilon(n) + C_{10}(x_*)n|y - y'|. \tag{E.24}$$

Thus, if $|y - x_*|, |y' - x_*| \le \frac{d}{n}$ we have

$$\|A_n(y)^{-1}\partial A_n(y) - A_n(y')^{-1}\partial A_n(y')\| \le 10dC_10(x_*)n\epsilon(n) + (10d + 1)C_{10}(x_*)n|y - y'|, \tag{E.25}$$

which implies Lemma E.4.

E.1.4. *Proof of Lemma E.5 and Theorem E.6.* Let $x_* \in X_2$. For every $\epsilon > 0$, for n sufficiently large such that $\|n\alpha\|_{\mathbb{Z}} \le \frac{d}{n}$, any $y \in \mathbb{R}/\mathbb{Z}$ such that $|y - x_*| \le \frac{d}{n}$ is at distance at most ϵ from some $x \in X_1$ such that $|x - x_*| \le \frac{d}{n}$ and $\|B(x_*)^{-1}B(x)\|, \|B(x_*)^{-1}B(x + n\alpha)\| \le 1 + \epsilon$. Thus

$$\begin{aligned}\|B(x_*)A_n(y)B(x_*)^{-1}\| &\le \|B(x_*)A_n(x)B_(x_*)^{-1}\| + \|B(x_*)(A_n(y) - A_n(x))B(x_*)^{-1}\| \\ &\le \|B(x_*)B_(x)^{-1}\|\|B(x_*)^{-1}B(x + n\alpha)\| + \|B(x_*)\|^2\|A_n(y) - A_n(x)\| \\ &= (1 + \epsilon)^2 + K(x_*)\|B(x_*)\|^2\epsilon,\end{aligned} \tag{E.26}$$

where $K(x_*)$ is as in Lemma E.3. This already implies Lemma E.5.

Let $\Pi : \mathrm{sl}(2, \mathbb{R}) \to \mathrm{so}(2, \mathbb{R})$ be the orthogonal projection.

Let $n_i \to \infty$ be the increasing sequence of all n_i such that $\|n_i\alpha\| \le \frac{d}{n_i}$. Let $V_i : [-d, d] \to \mathrm{SL}(2, \mathbb{R})$ be given by $V_i(y) = A^{x_*}_{n_i}(x_* + \frac{y}{n_i})$. By Lemma E.4, V_i is C^1 precompact, and by Lemma E.5, any limit V of V_i takes values on $\mathrm{SO}(2, \mathbb{R})$. It follows that

$$\|V_i(x)^{-1}\partial V_i(x) - \Pi(V_i(x)^{-1}\partial V_i(x))\| \to 0. \tag{E.27}$$

In other words, we have proved the following:

Proposition E.7. *For every $x_* \in X_2$,*

$$\lim_{\substack{n\to\infty,\\ \|n\alpha\|_{\mathbb{Z}} \le \frac{d}{n}}} \sup_{|y - x_*| \le \frac{d}{n}} \frac{1}{n}\|A^{x_*}_n(y)\partial A^{x_*}_n(y) - \Pi(A^{x_*}_n(y)\partial A^{x_*}_n(y))\| \to 0. \tag{E.28}$$

For $x \in \mathbb{R}/\mathbb{Z}$ such that $B(x)$ is defined, let $B^x_y = B(x)B(y)B(x)^{-1}$ and let $\phi_x(y) = \|B^x(y)\|^2$.

We may assume $0 < \alpha < 1$. Let $\frac{p_k}{q_k}$, $k \ge 0$ be the sequence of continued fraction approximants of α, see for instance [AK1] §4.2. We will need only a few properties of the approximants. We have $1 = q_0 < q_1 < q_2 < \ldots$ and $\frac{1}{q_k + q_{k+1}} < (-1)^k(q_k\alpha - p_k) \le \frac{1}{q_{k+1}}$.

Let $J_{x,k} = (x + q_{k-1}\alpha - p_{k-1}, x + q_k\alpha - p_k]$ and define $T_{x,k} : J_{x,k} \to J_{x,k}$ the first return map. Thus $T_{x,k}(y) = y + r_{x,k}(y)\alpha$ where $r_{x,k}(y)$ is the first return time of y to $J_{x,k}$. We have $r_{x,k}(y) = q_k$ for $y \in (x + q_{k-1}\alpha, x]$ and $r_{x,k}(y) = q_{k-1}$ for $y \in (x + q_k\alpha]$. Thus x is the only discontinuity of $T_{x,k}$. If we glue the endpoints of $T_{x,k}$ to obtain a circle, the map $T_{x,k}$ becomes an irrational rotation of frequency $G^k(\alpha)$, where G is the Gauss map.

Let $A^{x,k} : J_{x,k} \to \mathrm{SL}(2, \mathbb{R})$ be given by $A^{x,k}(y) = A^x_{r_{x,k}(y)}(y)$.

The return time to $J_{x,k}$ is bounded by q_k which is less that $\frac{2}{|J_{x,k}|}$. This estimate will be crucial for us to use Lemmas E.3, E.4 and E.5 in order to obtain estimates on $A^{x,k}$ (after a convenient choice of x and for k large).

For $x \in X_1$, let $Y_{x,k,\eta}$ be the set of all y such that for all $l > 0$,

$$\sum_{\substack{0 \le j \le l-1, \\ \phi_x(y)\phi_x(T^j_{x,k}(y)) > 1+\eta}} \phi_x(y)\phi_x(T^j_{x,k}(y)) \le \eta. \tag{E.29}$$

Let $X_3 \subset X_2$ be the set of all x such that there exists a sequence $\eta_l \to 0$, $k_l \to \infty$ and $x_l \in X_2$ such that $K(x_l) \le 2K(x)$, $\|B(x) - B(x_l)\| \le \eta$, $x \in \operatorname{int} J_{x_l,k_l}$, and x is a density point of Y_{x_l,k_l,η_l}.

Proposition E.8. *The set X_3 has full Lebesgue measure.*

Proof. For $\eta > 0$, let $Z_{y,l,\eta}$ be the set of points $x \in X_2$ $x \in \operatorname{int} J_{y,l} \cap Y_{y,l,10\eta}$, $\|B(x) - B(y)\| < \eta$ and $2K(x) > K(y)$. It is enough for us to show that

$$\lim_{k\to\infty} |\cup_{l>k} \cup_{y\in X_2} Z_{y,l,\eta}| \ge 1 - 10\eta. \tag{E.30}$$

By the Lebesgue Differentiation Theorem, for almost every $x \in X_2$,

$$\lim_{l\to\infty} \frac{1}{|J_{x,l}|} \int_{J_{x,l}} \phi(y)dy = \phi(x). \tag{E.31}$$

This implies that

$$\lim_{l\to\infty} \frac{1}{|J_{x,l}|} \int_{J_{x,l}} \phi_x(y)dy = 1 \tag{E.32}$$

(one must use that x is a measurable continuity point of B).

By Vitali's Covering Lemma, Lebsgue Density Point Theorem, and the above, for every $k > 0$, there exists a sequence $y_i \in X_2$, $l_i > k$ such that J_{x_i,l_i} form a disjoint cover of almost all of $\mathbb{R}/\mathbb{Z}$, such that

$$\frac{1}{|J_{y_i,l_i}|} \int_{J_{y_i,l_i}} \phi_{y_i}(y) - 1dy \le \eta, \tag{E.33}$$

and the set of $y \in J_{y_i,l_i}$ with $2K(y) > K(y_i)$ and $\|B(y) - B(y_i)\| < \eta$ has measure at least $(1-\eta)|J_{y_i,l_i}|$. By the Maximal Ergodic Theorem, $\frac{|Z_{y_i,l_i,\eta}|}{|J_{y_i,l_i}|} > 1 - 10\eta$. This implies

$$|\cup_{l>k} \cup_{y\in X_2} Z_{y,l,\eta}| \ge \sum |Z_{y_i,l_i,\eta}| \ge 1 - 10\eta \tag{E.34}$$

as required. □

We shall now prove that for any $x_* \in X_3$, the conclusions of Theorem E.6 hold. From now on, we will fix $x_* \in X_3$ and let x_l, k_l, η_l be as in the definition of X_3.

Let $J_l = J_{x_l,k_l}$, $T_l = T_{x_l,k_l}$,

$$s_l(x) = r_{x_l,k_l}(x), \tag{E.35}$$

$$U^l(x) = A^{x_l,k_l}(x) = A^{x_l}_{s^l(x)}(x), \tag{E.36}$$

$$s^j_l(x) = \sum_{i=0}^{j-1} s_l(T^i_l(x)), \tag{E.37}$$

$$U^l_j(x) = \prod_{i=j-1}^{0} U^l(T^i_l(x)) = A_{s^l_j(x)}(x). \tag{E.38}$$

Notice that

$$\frac{1}{s_l(x)} \|U^l(x)^{-1}\partial U(x)\| \le K, \tag{E.39}$$

where K depends on x_* but does not depend on l.

Let $u_l : J_l \to \mathrm{so}(2,\mathbb{R})$, be the orthogonal projection of $\frac{1}{s_l(x)} U^l(x)^{-1}\partial U^l(x)$ onto $\mathrm{so}(2,\mathbb{R})$, which is continuous except at x_* and bounded (by K). Let

$$\eta_l' = \sup_{x \in J_l} \left\| \frac{1}{s_l(x)} U^l(x)^{-1}\partial U(x) - u_l(x) \right\|. \tag{E.40}$$

By Proposition E.7, $\lim \eta_l' = 0$.

Let also

$$\phi_l = \frac{1}{|J_l|}\int_{J_l} s_l(y)dy, \tag{E.41}$$

($\phi_l = \frac{1}{|J_l|}$ by Kak's Lemma or by easy verification),

$$\psi_l = \frac{1}{|J_l|}\int_{J_l} s_l(y)u_l(y)dy \in \mathrm{so}(2,\mathbb{R}). \tag{E.42}$$

Notice that $\frac{q_{k_l}}{2} < \phi_l < q_{k_l}$.

Let $x \in Y_{x_l,k_l,\eta_l}$, and let $n > 0$ be very large and such that $x + n\alpha \in J_l$. Then $n = s_l^t(x)$ for some $t > 0$ very large. We notice that (by unique ergodicity of irrational rotations) by taking n large we can assure that

$$\left| \frac{1}{t}\sum_{j=0}^{t-1} s_l(T_l^j(x)) - \phi_l \right| < \eta_l, \tag{E.43}$$

and

$$\left| \frac{1}{t}\sum_{j=0}^{t-1} s_l(T_l^j(x))u(T_l^j(x)) - \psi_l \right| < \eta_l. \tag{E.44}$$

We have

$$A_n^{x_l}(x)^{-1}\partial A_n^{x_l}(x) = \sum_{j=0}^{t-1} U_j^l(x)^{-1}U^l(T_l^j(x))^{-1}\partial U^l(T_l^j(x))U_j^l(x). \tag{E.45}$$

Let us estimate

$$\left\| A_n^{x_l}(x)^{-1}\partial A_n^{x_l}(x) - \sum_{j=0}^{t-1} s_l(T_l^j(x))u_l(T_l^j(x)) \right\| \le \sum_{j=0}^{t-1} s_l(T_l^j(x)) \left\| \frac{U_j^l(x)^{-1}U^l(T_l^j(x))^{-1}\partial U^l(T_l^j(x))U_j^l(x)}{s_l(T_l^j(x))} - u_l(T_l^j(x)) \right\|. \tag{E.46}$$

Let us split $\{0,...,t-1\} = I \cup I'$ where I is the set of j such that $\|U_j^l(x)\| \le 1 + \eta_l$ and I' is the complementary set. For $j \in I$,

$$\left\| \frac{U_j^l(x)^{-1}U^l(T_l^j(x))^{-1}\partial U^l(T_l^j(x))U_j^l(x)}{s_l(T_l^j(x))} - u_l(T_l^j(x)) \right\| \le 10\max\{\eta_l, \eta_l'\}. \tag{E.47}$$

We can estimate

$$\begin{aligned} \sum_{j\in I'} s_l(T_l^j(x)) \left\| \frac{U_j^l(x)^{-1}U^l(T_l^j(x))^{-1}\partial U^l(T_l^j(x))U_j^l(x)}{s_l(T_l^j(x))} - u_l(T_l^j(x)) \right\| &\le \sum_{j\in I'} 10K s_l^j(x)\|U_j^l(x)\|^2 \\ &\le \sum_{j\in I'} 10K q_{k_l}\phi_{x_l}(x)\phi_{x_l}(T_l^j(x)). \end{aligned} \tag{E.48}$$

If $j \in I'$, we must have $\phi_{x_l}(x)\phi_{x_l}(T_l^j(x)) \geq 1+\eta_l$, and since $x \in Y_{x_l,k_l,\eta_l}$ we have

$$\sum_{j\in I'} s_l(T_l^j(x)) \left\| \frac{U_j^l(x)^{-1}U^l(T_l^j(x))^{-1}\partial U^l(T_l^j(x))U_j^l(x)}{s_l(T_l^j(x))} - u_l(T_l^j(x)) \right\| \leq 10Kq_{k_l}t\eta_l. \tag{E.49}$$

Notice that $q_{k_l}t < 3n$, thus

$$\|A_n^{x_l}(x)^{-1}\partial A_n^{x_l}(x) - \sum_{j=0}^{t-1} s_l(T^j(x))u(T^j(x))\| \leq (30K+10)\max\{\eta_l,\eta_l'\}n, \tag{E.50}$$

which implies

$$\left\| \frac{1}{n}A_n^{x_l}(x)^{-1}\partial A_n^{x_l}(x) - \frac{\psi_l}{\phi_l} \right\| \leq 30(K+1)\max\{\eta_l,\eta_l'\}. \tag{E.51}$$

Since $\|B(x_*)-B(x_l)\| \leq \eta_l$, we conclude that for n sufficiently large such that $x \in J_l$, $x+n\alpha \in J_l$ we have

$$\left\| \frac{1}{n}A_n^{x_*}(x)^{-1}\partial A_n^{x_*}(x) - \frac{\psi_l}{\phi_l} \right\| \leq \epsilon_l \tag{E.52}$$

with $\lim \epsilon_l = 0$.

Since x_* is a density point of Y_{x_l,k_l,η_l}, any $x \in \mathbb{R}/\mathbb{Z}$ such that $|x-x_*| < \frac{d}{n}$ is at distance at most $\frac{\epsilon_{l,n}'}{n}$ from some $x_0 \in Y_{x_l,k_l,\eta_l}$, where $\lim_{n\to\infty}\epsilon_{l,n}' = 0$. By Lemma E.4, if n is such that $\|n\alpha\| \leq \frac{d}{n}$ this implies that

$$\left\| \frac{1}{n}A_n^{x_*}(x)^{-1}\partial A_n^{x_*}(x) - \frac{1}{n}A_n^{x_*}(x_0)^{-1}\partial A_n^{x_*}(x_0) \right\| \leq \delta_{l,n}. \tag{E.53}$$

where $\lim_{n\to\infty}\delta_{l,n} = 0$. In particular, if n is sufficiently large,

$$\left\| \frac{1}{n}A_n^{x_*}(x)^{-1}\partial A_n^{x_*}(x) - \frac{\psi_l}{\phi_l} \right\| \leq \epsilon_l + \delta_{l,n} \leq 2\epsilon_l, \tag{E.54}$$

holds for every $x \in \mathbb{R}/\mathbb{Z}$ such that $|x-x_*| \leq \frac{d}{n}$. To conclude, we notice that this implies that

$$\left\| \frac{\psi_l}{\phi_l} - \frac{\psi_m}{\phi_m} \right\| \leq 2(\epsilon_l+\epsilon_m), \tag{E.55}$$

so that $\frac{\psi_l}{\phi_l}$ converges to some $\omega \in \mathbb{R}/\mathbb{Z}$, and we can rewrite (E.54) as

$$\left\| \frac{1}{n}A_n^{x_*}(x)^{-1}\partial A_n^{x_*}(x) - \omega \right\| \leq 4\epsilon_l \tag{E.56}$$

as required.

Appendix F. More on premonotonic cocycles

For simplicity, we shall restrict ourselves to the case of cocycles which are at least C^1.

It will be convenient to identify $\mathbb{R}/\mathbb{Z}$ with $\partial\mathbb{D}$ through $h : \mathbb{R}/\mathbb{Z} \to \partial\mathbb{D}$ given by $h(y) = e^{2\pi i y}$. Let $\Pi_1, \Pi_2 : \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the coordinate projections.

Given a cocycle $(\alpha, A) \in \mathbb{R} \times C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, we let $F \equiv F_{\alpha,A} : \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$ be its projective action: $F(x,y) = (x+\alpha, h^{-1}(A(x)\cdot h(y)))$.

Let us consider a foliation $\mathcal{F}$ of $\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$ which is transvers to the vertical foliation $\{\{x\}\times\mathbb{R}/\mathbb{Z}\}_{x\in\mathbb{R}/\mathbb{Z}}$. We will say that $\mathcal{F}$ is trivial if its holonomy for horizontal loops is trivial: in other words, its leaves are circles projecting homeomorphically on the first coordinate.

We will say that a foliation is projective if it can be obtained locally as the image of the horizontal foliation by $F_{0,B}$ for some $B \in C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. A trivial projective foliation is globally the image of the horizontal foliation by some $B \in C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$.

We will say that a foliation $\mathcal{F}$ is transverse (with respect to $F = F_{\alpha,A}$) if $F(\mathcal{F})$ is transverse with respect to F. We may distinguish between positively transverse foliations and negatively transverse foliations according to whether the slope of $F(\mathcal{F})$ is bigger or smaller than the slope of $\mathcal{F}$.

In this language, monotonicity and premonotonicity can be defined as follows: a cocycle is monotonic if the horizontal foliation is transverse, and it is premonotonic if there exists a trivial projective foliation.

This suggests several possible extensions of the concept of monotonicity based on dropping the requirements on the transverse foliation. The weakest is to ask just for the existence of a transverse foliation, and we will call this notion weak monotonicity.

Associated to a foliation $\mathcal{F}$ which is transverse to the vertical foliation is a vector field of the form $(1, u)$ where $u \in C^0(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}, \mathbb{R})$ which is tangent to the foliation. Following this link, we can defined transverse, trivial and projective functions $u \in C^0(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}, \mathbb{R})$. For instance, u is transverse if

$$\Pi_2 DF(x, y)(1, u(x, y)) \neq u(F(x, y)). \tag{F.1}$$

It is often more convenient to work with u than with $\mathcal{F}$. The class of transverse (respectively, projective) functions is an open convex cone (respectively, closed subspace) in $C^0(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}, \mathbb{R})$.

F.1. **Behavior under renormalization.** We say that an action Φ is weakly monotonic if its frequency module is non-trivial and there exists a foliation $\mathcal{F}$ of $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ which is transverse to the vertical foliation and which is positively transverse for $\Phi(1, 0)$ and negatively transverse for $\Phi(0, 1)$. Obviously weak monotonicity is C^1-open.

Proposition F.1. *A normalized action Φ is weakly monotonic if and only if $\Phi(1, 0)$ is a weakly monotonic cocycle. Moreover, weak monotonicity is invariant by conjugation, dilatation, translation and base change. In particular it is also invariant under renormalization.*

Proof. If (α, A) is a weakly monotonic cocycle then there exists a 1-periodic foliation $\mathcal{F}$ on $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ which is, say, positively transverse for $\Phi_{\alpha,A}(1, 0)$. This foliation is associated to a one-periodic function $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z}$. Let us consider an increasing function $u' : \mathbb{R} \to \mathbb{R}$ such that $|u'| < 1$. Then $u''(x, y) = u(x, y) + \epsilon u'(x)$ is negatively transverse for $\Phi_{\alpha,A}(0, 1)$ for $\epsilon > 0$. Moreover, if ϵ is small u'' is close to u so it is positively transverse for $\Phi_{\alpha,A}(1, 0)$. Thus $\Phi_{\alpha,A}$ is weakly monotonic.

If $\Phi_{\alpha,A}$ is a weakly monotonic action then there exists $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ which is, say, positively transverse for $\Phi_{\alpha,A}(1, 0)$ and such that $u(x + 1, y) > u(x, y)$. Let $u' : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be given by $u'(x, y) = u'(x - [x], y)$ which is one-periodic and L^∞, but not continuous. Define $u''(x, y) = \int K(t)u'(x + \epsilon t, y)dt$ where $K(t)$ is a C^∞ function with compact support which is non-negative and $\int K(t)dt = 1$. Then $u''(x, y)$ is C^∞, one-periodic, and it is easy to see that it is positively transverse for $\Phi_{\alpha,A}(1, 0)$. It follows that (α, A) is weakly monotonic.

The only remaining non-trivial part is to show invariance of weak monotonicity by base change. Thus let $\Phi_{\alpha,A}$ be a weakly monotonic normalized action and let $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a one-periodic C^∞ function which is, say, positively transverse for $\Phi_{\alpha,A}(1, 0)$. Let $\Psi(1, 0) = \Phi(a, b)$, $\Psi(0, 1) = \Phi(c, d)$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$. We can reduce to the case when $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is either $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and all those are obvious. □

Corollary F.2. *Let $(\alpha, A) \in C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ be non-homotopic to the identity and L^2-conjugated to a cocycle of rotations. Then (α, A) is weakly monotonic.*

Remark F.1. Actually it is possible to show a stronger result: there exists $B \in C^\omega(\mathbb{R}/2\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ such that $B(x + \alpha) \circ A(x) \circ B(x)^{-1}$ is monotonic.

F.2. **Examples of non-premonotonic cocycles.** Let $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be C^1 of non-zero degree, and assume that $D\gamma$ changes sign. Let $A(\theta) = R_{\gamma(\theta)}$. Then $(0, A)$ is not homotopic to the identity and not weakly monotonic, and indeed if $B \in C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ is C^1-close to A, the cocycle $(0, B)$ is not weakly monotonic. More generally, it is easy to construct, for each $\alpha \in \mathbb{Q}$, C^1-open subsets of cocycles which are not-homotopic to the identity and not weakly monotonic with frequency α.

It is harder to give examples of non-premonotonic, non-homotopic to the identity cocycles with irrational frequencies. For instance, any cocycle of rotations with irrational frequencies is premonotonic, so the example above does not work. Indeed, it could be expected that cocycles with irrational frequency tend eventually to start turning in the direction of the degree. However, the growth of the degree is only linear, so it might not be enough to overcome the exponential behavior of cocycles with a positive Lyapunov exponent. To make this idea work, we will need an extra ingredient: in non-uniformly hyperbolic settings, there are frequently points in the phase space where there is coincidence of stable and unstable directions. Our contruction is based on the following result.

Theorem F.3 (Young, [Y]). *Let $B_t \in C^1(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be a one-parameter family (defined on a neighborhood of $t=0$), such that $(t,x) \mapsto B_t(x)$ is C^1. Let $\beta(x,t) = B_t(x)^{-1}(-1)$ and $C = \{x,\ \beta(x,0)=1\}$. Assume that $\partial_x \beta(x,t) \neq 0$, $x \in C$ and $\{\frac{\partial_t \beta(x,t)}{\partial_t \beta(x,t)}\}_{x \in C}$ are all distinct. Let $A_{\lambda,t} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} B_t$.*

Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be a Brjuno number. Then there exists $\epsilon_0 \equiv \epsilon_0(\alpha) > 0$ (small) and $\lambda_0 \equiv \lambda_0(\alpha, \epsilon_0) > 0$ (large) such that for every $\epsilon < \epsilon_0$ and for every $\lambda > \lambda_0$ there exists $\delta \equiv \delta(\alpha,\epsilon,\lambda) > 0$, $\eta \equiv \eta(\alpha,\lambda) > 0$, such that $\lim_{\epsilon\to 0, \lambda\to\infty} \delta = 0$ and $\lim_{\lambda\to\infty} \eta = 0$, and a set $X \equiv X(\alpha,\epsilon,\lambda) \subset (-\epsilon,\epsilon)$ such that $|X| > 2\epsilon - \eta$ with the following property. If $t \in X$ then for every $x \in C$, there exists $c \equiv c(\alpha,\lambda,t,x) \in \mathbb{R}/\mathbb{Z}$, $z \equiv z(\alpha,\lambda,t,x) \in \partial\mathbb{D}$ such that $|c-x| \le \delta$, $|z-1| \le \delta$ and for $n \ge 0$ we have

$$\left\| \left(\prod_{k=n-1}^{0} A_{\lambda,t}(x+k\alpha) \right) \cdot \begin{pmatrix} z_M & 1 \end{pmatrix} \right\| \le \lambda^{-2n/3}, \tag{F.2}$$

$$\left\| \left(\prod_{k=-n}^{-1} A_{\lambda,t}(x+k\alpha)^{-1} \right) \times \begin{pmatrix} z_M & 1 \end{pmatrix} \right\| \le \lambda^{-2n/3}. \tag{F.3}$$

This result uses an inductive construction inspired by the work of Benedicks-Carleson [BC] on Hénon maps. The points c that appear in the description are 'critical points". In the case of Hénon maps the critical set is a Cantor set of tangencies between stable and unstable manifolds [WY]. In the case discussed here the critical set is a finite set of points displaying coincidence between the stable and unstable directions. Besides the qualitative aspects of the result, it will be very important that those results localize very precisely the critical points near the easily defined set C, and gives quantitative estimates for the behavior of the orbit of the critical points.

Theorem F.4. *In the same setting of the previous lemma, assume that $\{\Im \partial_x \beta(x,t)\}_{x\in C}$ do not have all the same sign. Then if ϵ is sufficiently small and λ is sufficiently big, if $t \in X(\alpha,\epsilon,\lambda)$ then $A_{\lambda,t}$ is not premonotonic.*

Proof. Let $h : \mathbb{R}/\mathbb{Z} \to \partial\mathbb{D}$ be given by $h(x) = e^{2\pi i x}$. Let $F, G, H : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ be given by $F(x,y) = (x+\alpha, h^{-1}(A_{\lambda,t}(x)\cdot h(x)))$, $G(x,y) = (x+\alpha, h^{-1}(B_t(x)\cdot h(x)))$, and $H(x,y) = (x, h^{-1}(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \cdot h(x)))$, so that $F = H \circ G$. Let also $\Pi_2 : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the projection on the second coordinate. We have

$$\Pi_2 \partial_1 F(x,y) = (\Pi_2 \partial_2 H(G(x,y)))(\Pi_2 \partial_1 G(x,y)) = (\Pi_2 \partial_2 F(x,y))(\Pi_2 \partial_2 G(x,y))^{-1}(\Pi_2 \partial_1 G(x,y)), \tag{F.4}$$

$$\Pi_2 \partial_1 F^{-1}(F(x,y)) = \Pi_2 \partial_1 G^{-1}(G(x,y)) = -\frac{\Pi_2 \partial_1 G(x,y)}{\Pi_2 \partial_2 G(x,y)}. \tag{F.5}$$

By the hypothesis, there exists $x \in C$ such that $\Pi_2 \partial_1 G(x,0) > 0$. Let $c = c(\alpha,\lambda,t,x)$ and let $z = z(\alpha,\lambda,t,x)$. Let $d = h^{-1}(z)$. Then $\Pi_2 \partial_1 G(c,d) > 0$.

Let us consider a continuous function $u : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$. We have

(F.6)
$$\Pi_2 DF^n(F^{-n}(c,d))(1, u(F^{-n}(c,d))) = \sum_{k=0}^{n-1} (\Pi_2 \partial_2 F^k(F^{-k}(c,d)))(\Pi_2 \partial_1 F(F^{-k-1}(c,d))) + \Pi_2 \partial_2 F^n((F^{-n}(c,d)) u(F^{-n}(c,d))$$

so we have

$$\gamma^-(c,d) \equiv \lim_{n\to\infty} \Pi_2 DF^n(F^{-n}(c,d))(1,u(F^{-n}(c,d))) = \sum_{k=0}^{\infty}(\Pi_2\partial_2 F^k(F^{-k}(c,d)))(\Pi_2\partial_1 F(F^{-k-1}(c,d))), \tag{F.7}$$

and the series in the right side converges exponentially fast. We may rewrite

$$\gamma^-(c,d) = \sum_{k=0}^{\infty}(\Pi_2\partial_2 F^k(F^{-k-1}(c,d)))(\Pi_2\partial_2 G(F^{-k-1}(c,d)))^{-1}(\Pi_2\partial_1 G(F^{-k-1}(c,d))), \tag{F.8}$$

which gives

$$|\gamma^-(c,d)| = O(\lambda^{-4/3}). \tag{F.9}$$

On the other hand, we have

$$\Pi_2 DF^{-n}(F^n(c,d))(1,u(F^n(c,d))) = \sum_{k=0}^{n-1}(\Pi_2\partial_2 F^{-k}(F^k(c,d)))(\Pi_2\partial_1 F^{-1}(F^{k+1}(c,d))) + \Pi_2\partial_2 F^{-n}(F^n(c,d))u(F^n(c,d)) \tag{F.10}$$

so that

$$\gamma^+(c,d) \equiv \lim_{n\to\infty} DF^{-n}(F^n(c,d))(1,u(F^n(c,d))) = \sum_{k=0}^{\infty}(\Pi_2\partial_2 F^{-k}(F^k(c,d)))(\Pi_2\partial_1 F^{-1}(F^{k+1}(c,d))), \tag{F.11}$$

which we can rewrite

$$\gamma^+(c,d) = -\sum_{k=0}^{\infty}(\Pi_2\partial_2 F^{-k}(F^k(c,d)))\frac{\Pi_2\partial_1 G(F^k(c,d))}{\Pi_2\partial_2 G(F^k(c,d))}, \tag{F.12}$$

so that

$$\left|\gamma^+(c,d) + \frac{\Pi_2\partial_1 G(c,d)}{\Pi_2\partial_2 G(c,d)}\right| = O(\lambda^{-4/3}). \tag{F.13}$$

Those estimates together imply that $\gamma^+(c,d) < \gamma^-(c,d)$. In particular, if n is large,

$$\Pi_2 DF^n(F^{-n}(c,d))(1,u(F^{-n}(c,d))) > \Pi_2 DF^{-n}(F^n(c,d))(1,u(F^n(c,d))), \tag{F.14}$$

which implies $\Pi_2 DF^{2n}(F^{-n}(c,d))(1,u(F^{-n}(c,d))) > u(F^n(c,d))$. Thus u can not be negatively transverse.

An analogous argument (considering a different critical point) shows that u can not be positively transverse. Thus u cannot be transverse at all. □

Remark F.2. Let us define the sign of a critical point as the sign of $\Pi_2\partial_1 G(c,d)$ in the notation of the proof of the previous lemma. In the argument above, we could have used only one critical point with sign opposite to the degree (the case of degree 0 being trivial). But this does not give a result stronger then what is stated above: indeed, there are always critical points with the same sign of the degree. Actually there are always $2\deg$ critical points more with the sign of the degree than with opposite sign (since $x \mapsto \beta(x,0)$ has degree $2d$).

It would be interesting to know if (in the case of non-zero degree) the absence of critical points with sign opposite to the degree implies premonotonicity of the cocycles coming from Young's construction (for λ sufficiently large and ϵ sufficiently small).

The examples discussed above show that absence of premonotonicity is non-negligeable in the measure-theoretical sense. We believe that premonotonicity is not even dense, but the method above does not answer this question.

Problem F.1. Let $(\alpha, A) \in (\mathbb{R}\setminus\mathbb{Q}) \times C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $r \geq \mathrm{Lip}$. Does $L(\alpha, A) = 0$ imply that (α, A) is premonotonic?

A positive answer to this problem would show that, for irrational frequencies, the only obstruction to non-uniform hyperbolicity (in C^r, $r > 2$) is to be smoothly conjugated to a cocycle of rotations.

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