You first plow in the dynamical plane and
then harvest in the parameter plane.
Adrien Douady

DYNAMICS OF QUADRATIC POLYNOMIALS, III PARAPUZZLE AND SBR MEASURES.

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June 13, 1997

1. Introduction

This is a continuation of notes on dynamics of quadratic polynomials. In this part we transfer the geometric result of [L3] to the parameter plane. To any parameter value $c \in M$ in the Mandelbrot set (which lies outside of the main cardioid and satellite Mandelbrot sets attached to it) we associate a "principal nest of parapuzzle pieces"

$$\Delta^0(c) \supset \Delta^1(c) \supset \dots$$

corresponding to the generalized renormalization type of c. Then we prove:

Theorem A. *The moduli of the parameter annuli* $\mathrm{mod}(\Delta^l(c) \smallsetminus \Delta^{l+1}(c))$ *grow at least linearly (see §4 for a more precise formulation).*

This result was announced at the Colloquium in honor of Adrien Douady (July 1995), and in the survey [L4], Theorem 4.8. The main motivation for this work was to prove the following:

Theorem B (joint with Martens and Nowicki). *Lebesgue almost every real quadratic* $P_c : z \mapsto z^2 + c$ *which is non-hyperbolic and at most finitely renormalizable has a finite absolutely continuous invariant measure.*

More specifically, Martens and Nowicki [MN] have given a geometric criterion for existence of a finite absolutely continuous invariant measure (acim) in terms of the "scaling factors". Together with the result of [L2] on the exponential decay of the scaling factors in the quasi-quadratic case this yields existence of the acim once "the principal nest is eventually free from the central cascades". Theorem A above implies that this condition is satisfied for almost all real quadratics which are non-hyperbolic and at most finitely renormalizable (see Theorem 5.1). Note that Theorem A also implies that this condition is satisfied on a set of positive measure, which yields a new proof of Jacobson's Theorem [J] (see also Benedicks & Carleson [BC]).

This is a revised version of Preprint IMS at Stony Brook # 1996/5.

A measure μ will be called SBR (Sinai-Bowen-Ruelle) if

$$\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x} \to \mu \tag{1.1}$$

for a set of x of positive *Lebesgue* measure. It is known that if an SBR measure exists for a real quadratic map $f = P_c$, $c \in [-2, 1/4]$, on its invariant interval I_c, then it is unique and (1.1) is satisfied for Lebesgue almost all $x \in I_c$ (see Introduction of [MN] for a more detailed discussion and references). Theorem B yields

Corollary. *For almost all $c \in [-2, 1/4]$, the quadratic polynomial P_c has a unique SBR measure on its invariant interval I_c.*

Another consequence of our geometric results is concerned with the shapes of little Mandelbrot copies (see [L3], §2.5, for a discussion of little Mandelbrot copies). Let us say that a Mandelbrot set M' has a (K, ϵ)-*a bounded shape* if the straightening $\chi : M' \to M$ admits a K-quasi-conformal extension to an $(\epsilon \operatorname{diam} M')$-neighborhood of M'. We say that the little Mandelbrot sets of some family *have bounded shape* if a bound (K, ϵ) can be selected uniform over the family.

A Mandelbrot copy M' is called *maximal* if it is not contained in any other copy except M itself. It is called *real* if it is centered at the real line.

A little Mandelbrot copy encodes the combinatorial type of the corresponding renormalization. In [L3] we dealt with diverse numerical functions of the combinatorial type. For real copies a crucial information is encoded by the *essential period* $p_e(M')$ (see [L3, §8.1], [LY]).

For a definition of Misiurewicz wakes see §3.3 of this paper.

Theorem C. *For any Misiurewicz wake O, the maximal Mandelbrot copies contained in O have bounded shape. In particular, all maximal real Mandelbrot copies, except the doubling one, have a bounded shape. Moreover, a real copy M' has a (K, ϵ)-bounded shape, where $K \to 1$ and $\epsilon \to \infty$ as $p_e(M') \to \infty$.*

In §6 we will refine this statement and will comment on its connection with the MLC problem and the renormalization theory.

Let us now take a closer look at Theorem A. It nicely fits to the general philosophy of correspondence between the dynamical and parameter plane. This philosophy was introduced to holomorphic dynamics by Douady and Hubbard [DH1]. Since then, there have been many beautiful results in this spirit, see Tan Lei [TL], Rees [R], Shishikura [Sh], Branner-Hubbard [BH], Yoccoz (see [H]).

In the last work, special tilings into "parapuzzle pieces"of the parameter plane are introduced. Its main geometric result is that the tiles around at most finitely renormalizable points shrink. It was done by transferring, in an ingenious way, the corresponding dynamical information into the parameter plane.

In [L3] we studied the rate at which the dynamical tiles shrink. Our main geometric result is that the moduli of the principal nest of dynamical annuli grow at least linearly. Let us note that the way we transfer this result to the parameter plane (Theorem A) is substantially different from that of Yoccoz. Our main conceptual tool is provided by holomorphic motions whose transversal quasi-conformality is responsible for commensurability between the dynamical and parameter pictures (though to make it work we exploit existence of uniform quasi-conformal pseudo-conjugacy between the generalized renormalizations [L3]). It is not the first time when holomorphic motions are used for comparing dynamical and parameter planes (compare Shishikura [Sh]) but we believe they were never used before in the puzzle context.

The properties of holomorphic motions are discussed in §2. In §3 we describe the principal parameter tilings according to the generalized renormalization types of the maps. In §4 we prove Theorem A. In §5, we derive the consequence for the real quadratic family (Theorem B). In the last section, §6, we prove Theorem C on the shapes of Mandelbrot copies.

Let us finally draw the reader's attention to the work of LeRoy Wenstrom [W] which studies in detail parapuzzle geometry near the Fibonacci parameter value.

Remark. We have recently proven that the set of infinitely renormalizable real parameter values has zero linear measure. Together with Theorem B this implies that almost every real quadratic has either an attracting cycle or an absolutely continuous invariant measure [L7].

2. Background

2.1. Notations and terminology. $\mathbb{D}_r(p) = \{z : |z-p| < r\}$,
$\mathbb{D}_r \equiv \mathbb{D}_r(0)$, $\mathbb{D} \equiv \mathbb{D}_1$;
$\mathbb{T}_r = \{z : |z| = r\}$;
$\mathbb{A}(r, R) = \{r < |z| < R\}$. The closed and semi-closed annuli are denoted accordingly: $\mathbb{A}[r, R]$, $\mathbb{A}(r, R]$, $\mathbb{A}[r, R]$.

By a *topological disc* we will mean a simply connected domain $D \subset \mathbb{C}$ whose boundary is a Jordan curve.

Let π_1 and π_2 denote the coordinate projections $\mathbb{C}^2 \to \mathbb{C}$. Given a set $\mathbb{X} \subset \mathbb{C}^2$, we denote by $X_\lambda = \pi_1^{-1}\{\lambda\}$ its vertical cross-section through λ (the "fiber" over λ). Vice versa, given a family of sets $X_\lambda \subset \mathbb{C}$, $\lambda \in D$, we will use the notation: $\mathbb{X} = \cup_{\lambda \in D} X_\lambda = \{(\lambda, z) \in \mathbb{C}^2 : \lambda \in D, z \in X_\lambda\}$.

Let us have a discs fibration $\pi_1 : \mathbb{U} \to D$ over a topological disc $D \subset \mathbb{C}$ (so that the sections U_λ are topological discs, and the closure of U in $D \times \mathbb{C}$ is homeomorphic to $D \times \bar{\mathbb{D}}$ over D). In this situation we call $\mathbb{U}$ an (open) *topological bidisc* over D. We say that this fibration admits an extension to the boundary ∂D if the closure $\bar{U}$ of $\mathbb{U}$ in $\mathbb{C}^2$ is homeomorphic over $\bar{D}$ to $\bar{D} \times \bar{\mathbb{D}}$. The set $\bar{\mathbb{U}}$ is called a (closed) bidisc. We keep the notation $\mathbb{U}$ for the fibration of *open* discs over the closed disc $\bar{D}$ (it will be clear from the context over which set the fibration is considered).

If $U_\lambda \ni 0$, $\lambda \in D$, we denote by $\mathbf{0}$ the zero section of the fibration.

Given a domain $\Delta \subset D$, let $\mathbb{U}|\Delta = \mathbb{U} \cap \pi_1^{-1}\Delta$. This is a bidisc over Δ.

If the fibration π_1 admits an extension over the boundary ∂D, we define the *frame* $\delta\mathbb{U}$ as the topological torus $\cup_{\lambda\in\partial D}\partial U_\lambda$. A section $\Phi : D \to \mathbb{U}$ is called *proper* if it is continuous up to the boundary and $\Phi(\partial D) \subset \delta U$.

We assume that the reader is familiar with the theory of quasi-conformal maps (see e.g., [A]). We will use a common abbreviation K-qc for "K-quasi-conformal". Dilatation of a qc map h will be denoted as $\mathrm{Dil}(h)$.

Notation $a_n \asymp b_n$ means, as usual, that the ratio a_n/b_n is positive and bounded away from 0 and ∞.

2.2. Holomorphic motions. Given a domain $D \subset \mathbb{C}$ with a base point $*$ and a set $X_* \subset \mathbb{C}$, a *holomorphic motion* $\mathbf{h}$ of X_* over D is a family of injections $h_\lambda : X_* \to \mathbb{C}$, $\lambda \in D$, such that $h_* = \mathrm{id}$ and $h_\lambda(z)$ is holomorphic in λ for any $z \in X_*$. We denote $X_\lambda = h_\lambda X_*$. The restriction of $\mathbf{h}$ to a parameter domain $\Delta \subset D$ will be denoted as $\mathbf{h}|\Delta$.

Let us summarize fundamental properties of holomorphic motions which are usually referred to as the λ-*lemma*. It consists of two parts: extension of the motion and transversal quasi-conformality, which will be stated separately. The consecutively improving versions of the Extension Lemma appeared in [L1] and [MSS], [ST],[BR],[Sl]. The final result, which will be actually exploited below, is due to Slodkowsky:

Extension Lemma. *A holomorphic motion $h_\lambda : X_* \to X_\lambda$ of a set $X_* \subset \mathbb{C}$ over a topological disc D admits an extension to a holomorphic motion $H_\lambda : \mathbb{C} \to \mathbb{C}$ of the whole complex plane over D.*

Quasi-Conformality Lemma [MSS]. *Let $h_\lambda : U_* \to U_\lambda$ be a holomorphic motion of a domain $U_* \subset \mathbb{C}$ over a hyperbolic domain $D \subset \mathbb{C}$. Then the maps h_λ are $K(r)$-quasi-conformal, where r is the hyperbolic distance between $*$ and λ in D.*

Let us define the dilatation of the holomorphic motion as

$$\mathrm{Dil}(\mathbf{h}) = \sup_{\lambda\in D} \mathrm{Dil}(h_\lambda).$$

It can be equal to ∞ over the whole domain D but becomes finite ($\le K(r)$) over the hyperbolic disk of radius r.

A holomorphic motion $h_\lambda : U_* \to U_\lambda$ over D can be viewed as a complex one-dimensional foliation of the domain $\mathbb{U} = \cup_{\lambda\in D}U_\lambda$, whose leaves are graphs of the functions $\lambda \mapsto h_\lambda(z)$, $z \in U_*$. A *transversal* to the motion is a complex one dimensional submanifold of $\mathbb{C}^2$ which transversally intersects every leaf at one point (so that "transversal" will mean a *global* transversal). Given two transversals X and Y, we thus have a well-defined holonomy map $H : X \to Y$, $H(p) = q$ iff p and q belong to the same leaf.

A map $H : X \to Y$ is called *locally qc* at $p \in X$ if it is qc in some neighborhood of p. In this case the local dilatation of H at p is defined as the limit of $\mathrm{Dil}(H \,|\, \mathbb{D}_\epsilon(p))$, as $\epsilon \to 0$.

Corollary 2.1 (Transversal qc structure). *Any holomorphic motion* $\mathbf{h}$ *over* D *is locally transversally quasi-conformal. More precisely, for any two transversals* X *and* Y*, the holonomy map* $H : X \to Y$ *is locally quasi-conformal. If* $H(p) = q$ *then* H *is locally qc at* p*, and the local dilatation of* H *at* p *depends only on the hyperbolic distance between the* $\pi_1(p)$ *and* $\pi_1(q)$ *in* D*. If* $\mathrm{Dil}(\mathbf{h}) < \infty$ *then the holonomy* H *is globally qc with* $\mathrm{Dil}(H) \le 2\,\mathrm{Dil}(\mathbf{h})$.

Proof. Let $p = (\lambda, \alpha)$, $q = (\mu, \beta)$. By the λ-Lemma, the map $G = h_\mu \circ h_\lambda^{-1} : U_\lambda \to U_\mu$ is quasi-conformal, with dilatation depending only on the hyperbolic distance between λ and μ in D and bounded by $2\,\mathrm{Dil}(\mathbf{h})$. Hence a little disc $D(\alpha, \epsilon) \subset U_\lambda$ is mapped by G onto an ellipse $Q_\epsilon \subset U_\mu$ with bounded eccentricity about β (where the bound depends only on the hyperbolic distance between α and β, and does not exceed $2\,\mathrm{Dil}(\mathbf{h})$).

But the holonomy $U_\lambda \to X$ is asymptotically conformal near p. To see this, let us select a holomorphic coordinates (θ, z) near p in such a way that $p = 0$ and the leaf via p becomes the parameter axis. Let $z = \psi(\theta) = \epsilon + \dots$ parametrizes a nearby leaf of the foliation, while $\theta = g(z) = bz + \dots$ parametrizes the transversal X.

Let us do the rescaling $z = \epsilon\zeta, \theta = \epsilon\nu$. In these new coordinates, the above leaf is parametrized by the function $\Psi(\nu) = \epsilon^{-1}\psi(\epsilon\nu)$, $|\nu| < R$, where R is a fixed parameter. Then $\Psi'(\nu) = \psi'(\epsilon\nu)$ and $\Psi''(\nu) = \epsilon\psi''(\epsilon\nu)$. Since the family of functions $\{\psi(\nu)\}$ is normal, $\Psi''(\nu) = O(\epsilon)$. Moreover, ψ uniformly goes to 0 as $\psi(0) \to 0$. Hence $|\Psi'(0)| = |\psi'(0)| \le \delta_0(\epsilon)$, where $\delta_0(\epsilon) \to 0$ as $\epsilon \to 0$. Thus $\Psi'(\nu) = \delta_0(\epsilon) + O(\epsilon) \le \delta(\epsilon) \to 0$ as $\epsilon \to 0$ uniformly for all $|\nu| < R$. It follows that $\Psi(\nu) = 1 + O(\delta(\epsilon)) = 1 + o(1)$ as $\epsilon \to 0$.

On the other hand, the manifold X is parametrized in the rescaled coordinates by a function $\nu = b\zeta + 0(1)$. Since the transversal intersection persists, X intersects the leaf at the point $(\nu_0, \zeta_0) = (1, b)(1 + o(1))$ (so that R should be selected bigger than b). In the old coordinates the intersection point is $(\theta_0, z_0) = (\epsilon, b\epsilon)(1 + o(1))$.

Thus the holonomy from U_λ to X transforms the disc of radius $|\epsilon|$ to an ellipse with small eccentricity, which means that this holonomy is asymptotically conformal. As the holonomy from $U(\mu)$ to Y is also asymptotically conformal, the holonomy $H : (X, p) \to (Y, q)$ is locally qc at p, and its local dilatation at p is the same as the local dilatation of $G : (U(\nu), p) \to (U(\mu, q)$. Thus it depends only on the hyperbolic distance between p and q, and is bounded by $2\,\mathrm{Dil}(\mathbf{h})$.

To conclude the proof, one should just remark that a map is globally qc if and only if it is locally qc with uniformly bounded local dilatations, and then the global dilatation is equal to the essential supremum of the local ones. $\square$

2.3. Winding number. Given two curves $\psi_1, \psi_2 : \partial D \to \mathbb{C}$ such that $\psi_1(\lambda) \neq \psi_2(\lambda)$, $\lambda \in \partial D$, the winding number of the former about the latter is defined as the increment of $\frac{1}{2\pi}\arg(\psi_1(\lambda) - \psi_2(\lambda))$ as λ wraps once around ∂D.

Let us have a bidisc $\mathbb{U}$ over $\bar{D}$. Given a proper section $\Phi : D \to \mathbb{U}$ let us define its *winding number* as follows. Let us mark on the torus $\delta\mathbb{U}$ the homology basis $\{[\partial D], [\partial \mathbb{U}_*]\}$. Then the winding number $w(\Phi)$ is the second coordinate of the curve $\Phi : \partial D \to \delta\mathbb{U}$ with respect to this basis.

Argument Principle. *Let us have a bidisc $\mathbb{U}$ over $\bar{D}$ and a proper holomorphic section $\Phi : D \to \mathbb{U}$, $\phi = \pi_2 \circ \Phi$. Let $\Psi : \bar{D} \to \mathbb{U}$ be another continuous section holomorphic in D, $\psi = \pi_2 \circ \Psi$. Then the number of solutions of the equation $\phi(\lambda) = \psi(\lambda)$ counted with multiplicity is equal to the winding number $w(\Phi)$.*

Proof. Indeed, $w(\Phi)$ is equal to the winding number of ϕ around ψ, which is equal, by the standard Argument Principle, to the number of roots of the equation

$$\phi(\lambda) = \psi(\lambda).$$

□

3. Parapuzzle combinatorics

3.1. Holomorphic families of generalized quadratic-like maps. Let us consider a topological disc $D \subset \mathbb{C}$ with a base point $* \in D$, and a family of topological bidiscs $\mathbb{V}_i \subset \mathbb{U} \subset \mathbb{C}^2$ over D (*tubes*), such that the $\mathbb{V}_i$ are pairwise disjoint. We assume that $V_{0,\lambda} \ni 0$.

Let

$$\mathbf{g} : \cup \mathbb{V}_i \to \mathbb{U} \tag{3.1}$$

be a fiberwise map, which admits a holomorphic extension to some neighborhoods of the $\mathbb{V}_i$ (warning: these extensions don't fit), and whose fiber restrictions

$$\mathbf{g}(\lambda, \cdot) \equiv g_\lambda : \bigcup_i V_{i,\lambda} \to U_\lambda, \quad \lambda \in D,$$

are generalized quadratic-like maps with the critical point at $0 \in V_\lambda \equiv V_{0,\lambda}$ (see [L3], §3.7 for the definition). We will assume that the discs U_λ and $V_{i,\lambda}$ are bounded by piecewise smooth quasi-circles.

Let us also assume that there is a holomorphic motion $\mathbf{h}$ over $(D, *)$,

$$h_\lambda : (\bar{U}_*, \bigcup_i \partial V_{i,*}) \to (\bar{U}_\lambda, \bigcup_i \partial V_{i,\lambda}), \tag{3.2}$$

which respects the boundary dynamics:

$$h_\lambda \circ g_*(z) = g_\lambda \circ h_\lambda(z) \quad \mathit{for} \quad z \in \cup \partial V_{i,*}. \tag{3.3}$$

A *holomorphic family* $(\mathbf{g}, \mathbf{h})$ *of (generalized) quadratic-like maps over* D is a map (3.1) together with a holomorphic motion (3.2) satisfying (3.3). We will sometimes reduce the notation to $\mathbf{g}$. In case when the domain of $\mathbf{g}$ consists of only one tube $\mathbb{V}_0$, we refer to $\mathbf{g}$ as *DH quadratic-like family* (for "Douady and Hubbard", compare [DH2]).

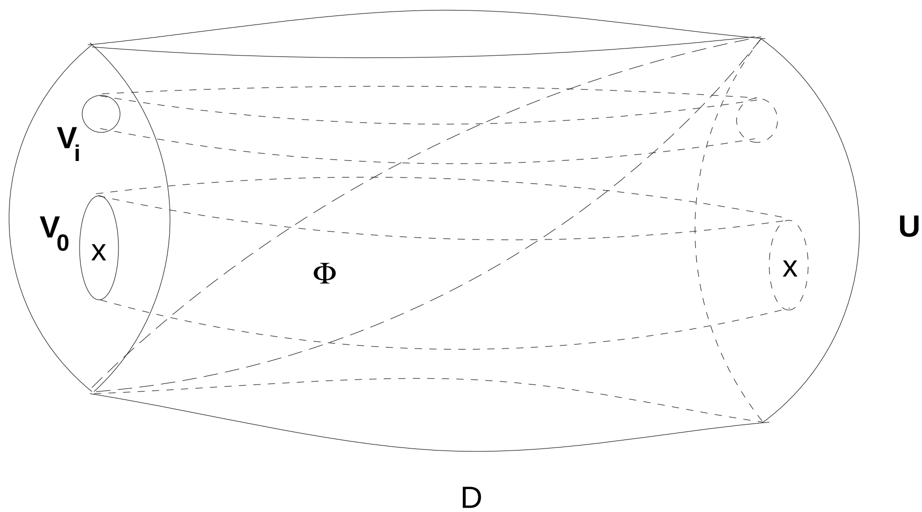

Figure 1: Generalized quadratic-like family.

Remark. It would be more consistent to call just $\mathbf{g}$ a holomorphic family, while the pair $(\mathbf{g}, \mathbf{h})$, say, an *equipped* holomorphic family. However, in this paper we will assume that the families are equipped, unless otherwise is explicitly stated.

Let us now consider the critical value function $\phi(\lambda) \equiv \phi_{\mathbf{g}}(\lambda) = g_\lambda(0)$, $\Phi(\lambda) \equiv \Phi_{\mathbf{g}}(\lambda) = \mathbf{g}(\lambda, 0) \equiv (\lambda, \phi(\lambda))$. Let us say that $\mathbf{g}$ is a *proper* (or *full*) holomorphic family if the fibration $\pi_1 : \mathbb{U} \to D$ admits an extension to the boundary $\bar{D}$, $\bar{\mathbb{V}}_i \subset \mathbb{U}$, and $\Phi : D \to \mathbb{U}$ is a proper section. Note that the fibration $\pi_1 : V_0 \to D$ cannot be extended to $\bar{D}$, as the domains $V_{\lambda,0}$ pinch to figure eights as $\lambda \to \partial D$.

Given a proper holomorphic family $\mathbf{g}$ of generalized quadratic-like maps, let us define its *winding number* $w(\mathbf{g})$ as the winding number of the critical value $\phi(\lambda)$ about the critical point 0. By the Argument Principle, it is equal to the winding number of the critical value about any section $\bar{D} \to \mathbb{U}$.

We will also face the situation when $\mathbf{g}$ does not map every tube $\mathbb{V}_i$ onto the whole tube $\mathbb{U}$ but still satisfies the following Markov property: $\mathbf{g}\mathbb{V}_i$ either contains $\mathbb{V}_j$ or disjoint from it (and all the rest properties listed above are still valid, see §3.3). Then we call $\mathbf{g}$ a holomorphic family of Markov maps.

Let $\operatorname{mod}(\mathbf{g}) = \inf_{\lambda \in D} \operatorname{mod}(U_\lambda \setminus V_{0,\lambda})$.

3.2. Douady & Hubbard quadratic-like families. Let us have a proper holomorphic family $\mathbf{f} : \mathbb{V} \to \mathbb{U}$ of DH quadratic-like maps, with winding number 1. The Mandelbrot set $M(\mathbf{f})$ is defined as the set of $\lambda \in D$ such that the Julia set $J(f_\lambda)$ is connected. We will assume that $* \in M(\mathbf{f})$.

Since the U_λ and $V_{i,\lambda}$ are bounded by quasi-circles, there is a qc straightening $\omega_* : \mathrm{cl}(U_* \setminus V_*) \to \mathbb{A}[2,4]$ conjugating $f_* : \partial V_* \to \partial U_*$ to $z \mapsto z^2$ on $\mathbb{T}_2$. The holomorphic motion $\mathbf{h}$ on the "condensator" $\mathbb{U} \setminus \mathbb{V}$ spreads this straightening over the whole parameter region D. We obtain a family of quasi-conformal homeomorphisms

$$\omega_\lambda : \mathrm{cl}(U_\lambda \setminus V_\lambda) \to \mathbb{A}[2,4] \tag{3.4}$$

conjugating $f_\lambda | U_\lambda$ to $z \mapsto z^2$ on $\mathbb{T}_2$. Pulling them back, we obtain for every f_λ the straightening $\omega_\lambda : \Omega_\lambda \to \mathbb{A}(\rho_\lambda, 4)$ well-defined up to the critical point level $\rho_\lambda = |\omega_\lambda(0)|$ (so that for $\lambda \in M(\mathbf{f})$ it is well-defined on the whole complement of the Julia set). This determines *external coordinates* of points $z \in \Omega_\lambda$, equipotential radius r and angle θ, defined as the polar coordinates of $\omega_\lambda(z)$.

Note that if $\mathrm{Dil}(\mathbf{h}) < \infty$ then the straightenings ω_λ are uniformly L-qc with $L = \mathrm{Dil}(\mathbf{h}) + \mathrm{Dil}(\omega_*)$. Note also that $\mathrm{Dil}(\omega_*)$ depends only on the qc dilatation of the quasi-circle ∂U_*, ∂V_* and on $\mathrm{mod}(U_* \setminus V_*)$.

By the *geometry* of $(\mathbf{f}, \mathbf{h})$ we will mean a triple of parameters: $(\mathrm{mod}(\mathbf{f}))^{-1}, \mathrm{Dil}(\mathbf{h})$, and the best dilatation of ω_*. If $\mathrm{mod}(\mathbf{f}) \to \infty$, while the $\mathrm{Dil}(\mathbf{h})$ and $\mathrm{Dil}(\omega_*)$ go to 1 (over some directed set of quadratic-like families), then we say that the geometry of $(\mathbf{f}, \mathbf{h})$ vanishes.

By an *adjustment* of a DH quadratic-like family we will mean replacement the domains U_*, V_* with some other domains $\tilde{U}_* \subset U_*$, $\tilde{V}_* = f_*^{-1} U_*$, spreading them around by $\mathbf{h}$ ($\tilde{U}_\lambda = h_\lambda \tilde{U}_*$, $\tilde{V}_\lambda = h_\lambda \tilde{V}_*$), and the corresponding shrinking of the parameter domain: $\tilde{D} = \Phi^{-1} \tilde{\mathbb{U}}$. It provides us with an adjusted family $(\mathbf{f} : \tilde{\mathbb{V}} \to \tilde{\mathbb{U}},\ \mathbf{h})$ over $\tilde{D}$.

We will use the following standard adjustment. Select $\tilde{U}_*$ to be bounded by the hyperbolic geodesic Γ in the annulus $U_* \setminus V_*$. Then $\tilde{V}_*$ is bounded by the hyperbolic geodesic Γ' in $V_* \setminus f_*^{-1} V_*$. By the Koebe Theorem, the geometry of these geodesics (i.e., their qc dilatation) depends only on $\mathrm{mod}(U_* \setminus V_*)$. Thus after this adjustment, $\mathrm{Dil}(\tilde{\omega}_*)$ depends only on $\mathrm{mod}(U_* \setminus V_*)$. Moreover, $\mathrm{mod}(\tilde{U}_* \setminus \tilde{V}_*) \geq (3/4)\,\mathrm{mod}(U_* \setminus V_*)$. Thus the geometry of the adjusted family depends only on $\mathrm{mod}(U_* \setminus V_*)$ and $\mathrm{Dil}(\mathbf{h})$.

Moreover, if we fix $\mathrm{Dil}(\mathbf{h})$ and let $\mathrm{mod}(U_* \setminus V_*) \to \infty$, then $\mathrm{mod}(\tilde{U}_\lambda \setminus \tilde{V}_\lambda)) \to \infty$, $\mathrm{Dil}(h | \tilde{D}) \to 1$ (by λ-lemma), and $\mathrm{Dil}(w_*) \to 1$, so that the geometry of the adjusted family vanishes.

In what follows we will not change notations when we adjust quadratic-like families.

Let us now define a map $\xi : D \setminus M(\mathbf{f}) \to \mathbb{A}(1,4)$ in the following way:

$$\xi(\lambda) = \omega_\lambda(f_\lambda 0). \tag{3.5}$$

Lemma 3.1. *Let $(\mathbf{f}, \mathbf{h})$ be a DH quadratic-like family with winding number 1. Then formula (3.5) determines a homeomorphism $\xi : D \setminus M(\mathbf{f}) \to \mathbb{A}(1,4)$. If* $\mathrm{Dil}(\mathbf{h}) < \infty$

then ξ is L-qc with L depending only on the geometry of $(\mathbf{f},\mathbf{h})$. *Moreover, $L \to 1$ as the geometry of* $(\mathbf{f},\mathbf{h})$ *vanishes.*

Proof. Let us consider the critical value graph $C = \Phi(\lambda) \equiv \{(\lambda, f_\lambda 0),\ \lambda \in D\}$. By the Argument Principle, it intersects at a singe point each leaf of the holomorphic motion $\mathbf{h}$ on $\mathbb{U} \setminus \mathbb{V}$, so that the holonomy $\gamma : U_* \setminus V_* \to X$ is a homeomorphism onto the image R_1. Hence $A_1 \equiv \pi_1 R_1 \subset D$ is a topological annulus, and the map

$$\xi^{-1} = \pi_1 \circ \gamma \circ \omega_*^{-1} : \mathbb{A}[2,4) \to A_1$$

is a homeomorphism.

Let Γ_1 be the inner boundary of A_1, and D_1 be the topological disc bounded by Γ_1. Since the critical value $f_\lambda(0)$, $\lambda \in D_1$, does not land at the leaves of holomorphic motion $\mathbf{h}|D_1$, it can be lifted by $\mathbf{f}$ to a holomorphic motion $\mathbf{h}_1$ of the annulus $V_*^1 \setminus V_*^2$ over D_1. Since the graph C intersects every leaf belonging to $\partial\mathbb{V}^1$ at a single point, the family $(\mathbf{f} : \mathbb{V}^2 \to \mathbb{V}^1, \mathbf{h})$ is proper over D_1 and has winding number 1. Let $A_2 = \Phi^{-1}(\mathbb{V}^1 \setminus \mathbb{V}^2)$. Then the same argument as above shows that the map $\xi^{-1} : \mathbb{A}[\sqrt{2}, 2] \to A_2$ is also a homeomorphism.

Continuing in the same way, we will inductively construct a sequence of holomorphic motions $\mathbf{h}_n$ over nested discs D_n, and a nest of adjoint annuli $A_n = D_{n-1} \setminus D_n$ which are homeomorphically mapped by ξ onto the round annuli $\mathbb{A}[2^{1/(n-1)}, 2^{1/(n-2)}]$. Altogether this shows that ξ is a homeomorphism.

Finally, assume $\mathbf{h}$ is K-qc. Since all further motions $\mathbf{h}_n$ are holomorphic lifts of $\mathbf{h}$ over D_n by $\mathbf{f}^n$, they are K-qc over their domains of definition as well. By Corollary 2.1, they are transversally K-qc. Moreover, the straightening $\omega_* : U_* \setminus J(f_*) \to \mathbb{A}(1,4)$ is qc, while the projection $\pi_1 : X \to D$ is conformal. Since ξ is the composition of the straightening, the holonomy and the projection, it is L-qc with $L = K + \mathrm{Dil}(\omega_*)$. In particular, $L \to 1$ as K and $\mathrm{Dil}(\omega_*)$ go to 1. $\square$

Note that the motions $\mathbf{h}_n$ over domains D_n constructed in the above proof preserve the external coordinates: $\omega_\lambda(h_\lambda z) = \omega_*(z)$, $z \in U_* \setminus f_*^{-n} V_*$. We will refer to this property by saying that $\mathbf{h}$ *respects the external marking*, or that $\mathbf{h}$ is *marked.*

Example (see [DH1]). Let us consider the Mandelbrot set M of the quadratic family $P_c : z \mapsto z^2 + c$. Let $R : \mathbb{C} \setminus M \to \mathbb{C} \setminus \mathbb{D}$ be the Riemann mapping tangent to id at ∞. Recall that parameter equipotentials and external rays are defined as the R-preimages of the round circles and radial rays. Let Ω_r be the topological disc bounded by the equipotential $R^{-1}\{re^{i\theta} : 0 \le \theta \le 2\pi\}$ of level $r > 1$.

For every $c \in \Omega_4$, let us consider the quadratic-like map $P_c : V_c \to U_c$ where V_c and U_c are topological discs bounded by the dynamical equipotentials of level 2 and 4 correspondingly. Then the conformal map $\omega_c : U_c \setminus V_c \to \mathbb{A}(2,4)$ conjugates $P_c|\partial V_c$ to $z \mapsto z^2$ on $\mathbb{T}_2$, so that it can serve as a straightening (3.4). With this choice of straightening, the parameter map $\xi : D \setminus M \to \mathbb{A}(1,4)$ constructed in Lemma 3.1 just coincides with the Riemann map R. $\square$

With Lemma 3.1, we can extend the notion of parameter rays and equipotentials to quadratic-like families as the ξ-preimages of the polar coordinate curves in $\mathbb{A}(1,4)$. If $\xi(\lambda) = re^{i\theta}$ then r and θ are called the *external radius* and the *external angle* of the parameter value λ. Note that ∂D becomes the equipotential of radius 4.

Before going further, let us state a general lemma about qc maps:

Gluing Lemma. *Let us have a compact set $Q \subset \mathbb{C}$ and two its neighborhoods U and V. Let us have two qc maps $\phi : U \to \mathbb{C}$ and $\psi : V \setminus Q \to \mathbb{C}$. Assume that these maps match on ∂Q, i.e., the map $f : V \to \mathbb{C}$ defined as ϕ on Q and as ψ on $V \setminus Q$ is continuous. Then f is quasi-conformal and* $\mathrm{Dil}(f) = \max(\mathrm{Dil}(\phi|Q), \mathrm{Dil}(\psi))$.

Proof. See e.g., [DH2, Lemma 2, p. 303]. □

Recall now that every quadratic-like map $f : V \to U$ is hybrid equivalent to a quadratic polynomial $P_c : z \mapsto z^2 + c$ (The Straightening Theorem [DH2]). It is constructed by gluing $f|U$ to $z \mapsto z^2$ on $\mathbb{C} \setminus \mathbb{D}_2$, and pulling the standard conformal structure on $\mathbb{C} \setminus \mathbb{D}$ back to $U \setminus K(f)$ by iterates of f. The construction depends on the choice of a qc straightening $\omega : \mathrm{cl}(U \setminus V) \to \mathbb{A}[2,4]$ conjugating $f|\partial V$ to $z \mapsto z^2$ on $\mathbb{T}_2$. However, if the Julia set $J(f)$ is connected, the parameter value $c \equiv \chi(f)$ is determined uniquely.

Given a quadratic-like family $f_\lambda : V_\lambda \to U_\lambda$ over D with winding number 1, let us consider a family of straightenings (3.4) and the corresponding family of quadratic polynomials $P_{\chi(\lambda)} : z \mapsto z^2 + \chi(\lambda)$.

Note that

$$\xi = R \circ \chi, \tag{3.6}$$

where ξ is defined in (3.5), and R is the Riemann mapping on the complement of the Mandelbrot set. This formula follows from the definitions of ξ and χ and the description of R given in the above Example.

Lemma 3.2. *Under the circumstances just described, the straightening $\chi : (D, M(\mathbf{f})) \to (\Omega_4, M)$ is a K-qc homeomorphism of the disc D onto a neighborhood Ω_4 of the Mandelbrot set M bounded by the parameter equipotential of radius 4. The dilatation K depends only on the geometry of* $(\mathbf{f}, \mathbf{h})$.

After adjusting the family $(\mathbf{f}, \mathbf{h})$, $\mathrm{Dil}(\chi)$ *will depend only on* $\mathrm{mod}(U_* \setminus V_*)$ *and* $\mathrm{Dil}(\mathbf{h})$. *Moreover,* $\mathrm{Dil}(\chi) \to 1$ *and* $\mathrm{mod}(D \setminus M(\mathbf{f})) \to \infty$ *as* $\mathrm{mod}(U_* \setminus V_*) \to \infty$ *(with a fixed* $\mathrm{Dil}(\mathbf{h})$*).*

Proof. By [DH2], χ is a homeomorphism. By [L5], $\chi|M(\mathbf{f})$ admits a local qc extension χ_λ to a neighborhood U_λ of any point $\lambda \in M(\mathbf{f})$, with dilatation depending only on $\mathrm{mod}(\mathbf{f})$. Let us select neighborhoods $V_\lambda \Subset U_\lambda$. Then let us select finitely many

$\chi_i \equiv \chi_{\lambda_i}$ such that the corresponding neighborhoods $V_i \equiv V_{\lambda_i}$ cover $M(\mathbf{f})$. By the Gluing Lemma,

$$\mathrm{Dil}(\chi|(D \smallsetminus M(\mathbf{f})) \cup V_i) \leq \max(\mathrm{Dil}(\chi|D \smallsetminus M(\mathbf{f})),\ \mathrm{Dil}(\chi_i)).$$

Taking into account Lemma 3.1, we conclude that

$$\mathrm{Dil}(\chi) = \sup_i \mathrm{Dil}(\chi|(D \smallsetminus M(\mathbf{f})) \cup V_i)$$

depends only on the geometry of $(\mathbf{f}, \mathbf{h})$.

Moreover, by [L5], the $\mathrm{Dil}(\chi) \to 1$ as $\mathrm{mod}(\mathbf{f}) \to \infty$. By Lemma 3.2, after the adjustment of $(\mathbf{f}, \mathbf{h})$), $\mathrm{Dil}(\chi|D \smallsetminus M(\mathbf{f})) \to 1$ as $\mathrm{mod}(f_*) \to \infty$ keeping $K \equiv \mathrm{Dil}(\mathbf{h})$ fixed. Hence by the Gluing Lemma, $\mathrm{Dil}(\chi) \to 1$ as $\mathrm{mod}(f_*) \to \infty$.

Finally, by transversal quasi-conformality of holomorphic motions, $\mathrm{mod}(D \smallsetminus M(\mathbf{f})) \geq K^{-1} \mathrm{mod}(U_* \smallsetminus V_*) \to \infty$. □

We will mostly deal with equipotentials of radius $4^{1/2^n}$, the preimages of the outermost equipotential of radius 4. Let us say that the equipotential of radius $4^{1/2^n}$ has *level* n, so that the outermost equipotential has level 0, the equipotential of radius 2 has level 1, etc.

3.3. Wakes and initial Markov families. Lemma 3.2 shows that the landing properties of the parameter rays in a quadratic-like family coincide with the corresponding properties in the quadratic family. This allows us to extend the notions of the parabolic and Misiurewicz wakes from the quadratic to the quadratic-like case. Namely, the *q/p-parabolic wake* $P_{q/p} = P_{q/p}(\mathbf{f})$ is the parameter region in D bounded by the external rays landing at the q/p-bifurcation point $b_{q/p}$ on the main cardioid of $M(\mathbf{f})$ and the appropriate arc of ∂D. Dynamically it is specified by the property that for λ in this wake there are p rays landing at the α-fixed point α_λ of $J(f_\lambda)$, and they form a cycle with rotation number q/p.

The maps

$$f_\lambda^p : V_\lambda \to U_\lambda \tag{3.7}$$

restricted to appropriate domains form a (non-equipped) quadratic-like family over the wake (see [D], [L3], §§2.5,3.2). (The domain V_λ is a thickening of the puzzle piece $Y_\lambda^{(1+p)}$ bounded by two pairs of rays landing at the α-fixed and co-fixed points and two equipotential arcs. The domain U_λ is a thickening of the puzzle piece $Y_\lambda^{(1)}$ bounded by two rays landing at the α-fixed point and an equipotential arc of level 2.) Note however that this family fails to be proper as the domains U_λ don't admit continuous extension at the root.

Proposition 3.3 (see [D]). *Let $\mathbf{f}$ be a DH quadratic-like family with winding number 1. Then the winding number of the critical value $\lambda \mapsto f_\lambda^p(0)$ about 0 when λ wraps once about the boundary of the parabolic wake $\partial P_{q/p}$ is also equal to 1.*

By [DH2, D], the quadratic-like family (3.7) generates a homeomorphic copy $M_{q/p} = M_{q/p}(\mathbf{f})$ of the Mandelbrot set attached to the bifurcation point $b_{q/p}$. Its complement $M \setminus M_{q/p}$ consists of a component containing the main cardioid and infinitely many *decorations* (using terminology of Dierk Schleicher [Sch]) $D^{\sigma,i}_{q/p}$, where σ is a dyadic sequence of length $|\sigma| = t-1$, $t = 1, 2, \dots,$ $i = 1, \dots, p-1$. The decoration $D^{\sigma,i}_{q/p}$ touches $M_{q/p}$ at a Misiurewicz point $\mu = \mu^{\sigma}_{q/p}$ for which

$$f^{pk}_{\mu}(0) \in Y^{(1+p)}_{\mu},\ k = 0, \dots, t-1, \ \text{ while } \ f^{pt}_{\mu}(0) = \alpha'_{\mu},$$

where α'_{μ} is the α-co-fixed point (i.e., the f_{μ}-preimage of the fixed point α_{μ}). (Such Misiurewicz points are naturally labeled by the dyadic sequences).

Every decoration $D^{\sigma,i}_{q/p}$ belongs to the *Misiurewicz wake* $\hat{O}^{\sigma,i}_{q/p}$ *of level* t bounded by two parameter rays landing at $\mu^{\sigma}_{q/p}$ (there are p rays landing at this point). Let us truncate such a wake by the equipotential of level $1+p(t-1)$. We will obtain the initial puzzle pieces $O^{\sigma,i}_{q/p}$ which sometimes will also be called "Misiurewicz wakes". They can be dynamically specified in terms of the initial puzzle (see [L3], §3.2). Namely, there are $p-1$ puzzle pieces $Z^{(1)}_i$, $i = 1, \dots, p-1$, attached to the co-fixed point α'. Pulling them back by $(t-1)$-st iterate of the double covering $f^p : Y^{(1+p)} \to Y^{(1)}$, we obtain 2^{t-1} puzzle pieces $Z^{(1+(t-1)p)}_{\sigma,i}$ labeled by the dyadic sequences. The wake $O^{\sigma,i}_{q/p}$ is specified by the property that $f^p 0 \in Z^{(1+(t-1)p)}_{\sigma,i}$.

The wake $O^{\sigma,i}_{q/p}$ containing a point λ will also be denoted by $O(\lambda)$.

By *tiling* we mean a family of topological discs with disjoint interiors. Let us consider the initial tiling constructed in [L3], §3.2:

$$Y^{(0)}_{\lambda} \supset V^0_{\lambda} \cup \bigcup_{k>0} \bigcup_{i} X^k_{i,\lambda} \cup \bigcup_{k \geq 0} \bigcup_{j} Z^{(1+kp)}_{j,\lambda}, \tag{3.8}$$

where $V^0_{\lambda} \equiv X^0_{0,\lambda}$.

Lemma 3.4. *The family of puzzle pieces* $Y^{(1+kp)}_{\lambda}$ *and* $Z^{(1+kp)}_{j,\lambda}$, $k \leq t-1$, *moves holomorphically in the region inside the parabolic wake* $P_{q/p}$ *bounded by the parameter equipotential of level* $1+(t-1)p$ *with all the Misiurewicz wakes of level* $\leq t-1$ *removed.*

Let us recall that $\mathbb{Z}^{(1)}_j$ means $\cup_{\lambda} Z^{(1)}_{j,\lambda}$.

Lemma 3.5. *Let* $\mathbf{f}$ *be a DH quadratic-like family with winding number 1. The initial tiling (3.8) moves holomorphically within the Misiurewicz wake* $O = O^{\sigma,i}_{p/q}$. *The critical value of the return map,* $\Phi : O \to \mathbb{Z}^{(1)}_j$, $\Phi(\lambda) = f^{pt}_{\lambda} 0$, *is a proper map with winding number 1 (where* $t = |\sigma| + 1$*).*

Proof. Indeed, all puzzle pieces of this initial tiling are the pullbacks of $Z^{(1)}_{j,\lambda}$. As λ ranges over the wake $O \equiv O^{\sigma,i}_{q/p}$, the corresponding iterates of O don't cross the boundary of $Z^{(1)}_{j,\lambda}$. It follows that the boundary of the initial tiling moves holomorphically.

Moreover, the torus $\delta\mathbb{Z}^{(1)}_j$ is foliated by the curves with the same external coordinates, and one curve corresponding to the motion of the α-co-fixed point. By definition of the Misiurewicz wake, the critical value $\Phi(\lambda)$ intersects once every leaf of this foliation when λ wraps once around ∂O. Hence $\Phi : O \to \mathbb{Z}^{(1)}_j$ is a proper map with winding number 1. □

Recall for further reference that there are two puzzle pieces $Q_{1,\lambda}$ and $Q_{2,\lambda}$ in $Y^{(1+(t-1)p)}_\lambda$ which are univalently mapped by f^p_λ onto $Y^{(1+(t-1)p)}_\lambda$ (see Figure 3 of [L3]). The pieces $X^k_{i,\lambda}$ are the pull-backs of $V^0_\lambda \equiv X^0_{0,\lambda}$ under the k-fold iterate of a Bernoulli map

$$f^p_\lambda : Q_{1,\lambda} \cup Q_{2,\lambda} \to Y^{(1+(t-1)p)}_\lambda.$$

Let $\Omega^k_{i,\lambda} \supset X^k_{i,\lambda}$ denote the domain in $Y^{(1+(t-1)p)}_\lambda$ which is mapped under f^{kp}_λ onto $Y^{(1+(t-1)p)}_\lambda$; in particular, $\Omega^k_{0,\lambda} \equiv Y^{(1+(t-1)p)}_\lambda$.

3.4. First generalized quadratic-like family. Let us consider a proper DH quadratic-like family $\mathbf{f} = \{f_\lambda\}$ over D with winding number 1. Fix a Misiurewicz wake O of this family. The first generalized quadratic-like map $g_{1,\lambda} : \cup V^1_{i,\lambda} \to V^0_\lambda$ is defined as the first return map to V^0_λ (see [L3], §3.5). The itinerary of the critical point via the elements P_i of the initial tiling (3.8) determines the parameter tiling $\mathcal{D}^0$ of a Misiurewicz wake O by the corresponding puzzle pieces. Let $\Delta^1(\lambda)$ stand for such parapuzzle piece containing λ.

More precisely, for any $\lambda \in O$, let us consider the first landing map to $T_\lambda : \cup L_{\bar{i},\lambda} \to V_{0,\lambda}$ (see [L3], §11.3). The puzzle piece $L_{\bar{i},\lambda}$ is specified by its itinerary $\bar{i} = (i_0, \dots, i_{t-1})$ under iterates of f^p_λ through non-central pieces P_i of the initial tiling until the first landing at $V_{0,\lambda}$:

$$L_{\bar{i},\lambda} = \{z : f^{pk}_\lambda z \in P_{i_k},\ k = 0, \dots, s-1,\ \ f^{ps}_\lambda z \in V_{0,\lambda}\}. \tag{3.9}$$

These tiles are organized in tubes $\mathbb{L}_{\bar{i}}$ with holomorphically moving boundary. Moreover, the first landing map induces a diffeomorphism $\mathbf{T} : \mathbb{L}_{\bar{i}} \to \mathbb{V}_0$ fibered over id.

Let $\bar{i}_\lambda$ stand for the itinerary of the critical value $\phi_0(\lambda) = f^p_\lambda 0$ through the initial tiling, so that $f^p_\lambda 0 \in L_{\bar{i}_\lambda,\lambda}$. Let $\mathbb{L}_* \equiv \mathbb{L}_{i_*}$ and $\Phi_0(\lambda) = (\lambda, \phi_0(\lambda))$. Then the parapuzzle pieces of the tiling $\mathcal{D}^0$ are defined as follows:

$$\Delta^1(*) = \Phi_0^{-1}\mathbb{L}_* = \{\lambda \in O : f^p_\lambda 0 \in L_{*,\lambda}\}.$$

Let $\mathbb{V}^1_j$ denote the components of $\mathbf{f}^{-p}(\mathbb{L}_{\bar{i}}|\Delta^1(*))$. Since the critical value $\Phi(\lambda)$ lands at the tube $\mathbb{L}_* = \mathbf{f}^p\mathbb{V}^1_0$ as λ ranges over $\Delta^1(*)$, the annulus $V^0_\lambda \setminus V^1_\lambda$ holomorphically moves over $\Delta^1(*)$. Let us extend this motion by the λ-lemma to V^1_λ.

Since the winding number of Φ_0 about the tubes of the initial tiling (3.8) over O is equal to 1 (by Lemma 3.5), the function $\Phi_0 : \Delta^1(*) \to \mathbb{L}_*$ is proper with winding number 1. Since the first landing map is a fiberwise diffeomorphism of every tube $\mathbb{L}_{\bar{i}}$ onto $\mathbb{V}_0$, it induces a homeomorphism between the marked tori $\delta\mathbb{L}_{\bar{j}} \to \delta\mathbb{V}_0$. Hence the function $\Phi_1(\lambda) = (\lambda, T_\lambda \circ \phi(\lambda))$, $\Delta^1(\lambda) \to \mathbb{V}_0$, is also proper with winding number 1. Thus we have:

Lemma 3.6. *Let* $\mathbf{f}$ *be a DH quadratic-like family with winding number 1. Then the first generalized renormalization* $\mathbf{g}_1 : \cup\mathbb{V}^1_{\bar{j}} \to \mathbb{V}^0 \equiv \mathbb{U}^1$ *is a proper family with winding number 1 over* $\Delta^0(*)$.

Together with the tubes (3.9) let us also consider bigger tubes $\mathbb{W}_{\bar{i}}$ over O with the following fibers:

$$W_{\bar{i},\lambda} = \{z : f_\lambda^{kp} z \in P_{i_k},\ k = 0, \dots, s-1,\ \ f_\lambda^{sp} z \in Y_\lambda^{(1+(t-1)p)}\}. \tag{3.10}$$

In other words, if r is the first moment when $f_\lambda^{rp} z$ belongs to some piece $X^k_{j,\lambda}$ of the initial tiling (3.8), then

$$f_\lambda^{rp} W_{\bar{i},\lambda} = \Omega^k_{j,\lambda}, \tag{3.11}$$

where $\Omega^k_{j,\lambda}$ is the domains defined at the end of §3.3. The discussion in §3.3 shows that f_λ^{rp} univalently maps $\Omega^k_{j,\lambda}$ onto $Y_\lambda^{(1+(t-1)p)}$. Moreover, $f_\lambda^{p(s-r)}$ univalently maps $W_{\bar{i},\lambda}$ onto $\Omega^k_{j,\lambda}$ (it follows by considering the Bernoulli map (3.5) from [L3]). Thus $\mathbf{f}^{sp} : \mathbb{W}_{\bar{i}} \to \mathbb{Y}_\lambda^{(1+(t-1)p)}$ is a fiberwise conformal bundle diffeomorphism fibered over id.

Hence the holomorphic motion of $\mathbb{Y}^{(1+(t-1)p)}$ (see Lemma 3.4) can be lifted to holomorphic motions of the $\mathbb{W}_{\bar{i}}$. Let $\bar{i}_*$ stand for the itinerary of the critical value $f_*^p 0$ through the initial tiling, and $\mathbb{W}_* \equiv \mathbb{W}_{\bar{i}_*}$. Let us now define the following parameter domains in O:

$$\Lambda^1(*) = \Phi^{-1}\mathbb{W}_* = \{\lambda : f_\lambda^p(0) \in W_{*,\lambda}\} \supset \Delta^1(*). \tag{3.12}$$

Thus for $\lambda \in \Lambda^1(*)$, the critical point of f_λ has the same itinerary through the initial tiling as the critical point of f_*, except for the last moment sp, when it lands somewhere in $Y_\lambda^{(1+(t-1)p)}$. This extension of $\Delta^1(*)$ will be used for a priori bounds on the parameter geometry (see §4).

3.5. Renormalization of holomorphic families. Let us now have a generalized quadratic-like family $(\mathbf{g} : \cup\mathbb{V}_i \to \mathbb{U}, \mathbf{h})$ over $(D, *)$. Let $\mathcal{I}$ stand for the labeling set of tubes $\mathbb{V}_i$. Remember that $\mathcal{I} \ni 0$ and $\mathbb{V}_0 \ni \mathbf{0}$. Let $\mathcal{I}_\#$ stand for the set of all finite sequences $\bar{i} = (i_0, \dots, i_{t-1})$ of non-zero symbols $i_k \in \mathcal{I} \setminus \{0\}$. For any $\bar{i} \in \mathcal{I}_\#$, there is a tube $\mathbb{V}_{\bar{i}}$ such that

$$\mathbf{g}^k \mathbb{V}_{\bar{i}} \subset \mathbb{V}_{i_k},\ k = 0, \dots, t-1 \quad \textit{and} \quad \mathbf{g}^t \mathbb{V}_{\bar{i}} = \mathbb{U}.$$

We call $t = |\bar{i}|$ the rank of this tube. The map $\mathbf{g}^t : V_{\bar{i}} \to \mathbb{U}$ is a holomorphic diffeomorphism which fibers over id, that is, $g^t_\lambda V_{\bar{i},\lambda} = U_\lambda$, $\lambda \in D$.

Let us lift the holomorphic motion $\mathbf{h}$ of $\mathbb{U}$ to a holomorphic motion $\hat{\mathbf{h}}$ of the $\mathbb{V}_{\bar{i}}$:

$$g_\lambda^t \circ \hat{h}_{\bar{i},\lambda}(z) = h_\lambda(g_*^t z), \; z \in V_{\bar{i},*}.$$

Note that by (3.3) it coincides with $\mathbf{h}$ on the $\partial \mathbb{V}_i$.

Let $\mathbb{L}_{\bar{i}} \subset \mathbb{V}_{\bar{i}}$ be such a tube that $\mathbf{g}^{|\bar{i}|}\mathbb{L}_{\bar{i}} = \mathbb{V}_0$. The first landing map $\mathbf{T} : \cup \mathbb{L}_{\bar{i}} \to \mathbb{V}_0$ is defined as $\mathbf{T}|L_{\bar{i}} = \mathbf{g}^{|\bar{i}|}$ (see [L3,]). It is a holomorphic diffeomorphism fibered over id. Extend the holomorphic motion $\hat{h}_\lambda$ to the tubes $\mathbb{L}_{\bar{i}}$ by pulling it back from $\mathbb{V}_0$ by $\mathbf{T}$. Then extend it by the λ-lemma to the whole tube $\mathbb{U}$ keeping it unchanged on the boundaries $\partial\mathbb{U}, \cup\partial\mathbb{V}_{\bar{i}}$.

Let $\phi(\lambda) = g_\lambda 0$ and $\Phi(\lambda) = (\lambda, \phi(\lambda))$. Let $\bar{i}_*$ be the itinerary of the critical value $\phi(*)$ under iterates of g_* through the domains $V_{i,*}$, until its first return to $V_{0,*}$. In other words, let $g_*(0) \in \mathbb{L}_{\bar{i}_*} \equiv \mathbb{L}_*$.

Let us now consider the following parameter region around $*$:

$$D' \equiv D'(*) = \Phi^{-1}\mathbb{L}_*.$$

For $\lambda \in D'$, the itinerary of the critical value under iterates of g_λ until the first return back to $V_{0,\lambda}$ is the same as for g_* (that is, $\bar{i}_*$). Let us define new tubes $\mathbb{V}'_j \subset \mathbb{V}_0$ as the components of $(\mathbf{g}|\mathbb{V}_0)^{-1}(\mathbb{L}_{\bar{i}}|D')$. Let

$$\mathbf{g}' : \cup \mathbb{V}'_j \to \mathbb{V}_0|D' \equiv \mathbb{U}' \tag{3.13}$$

be the first return map of the union of these tubes to $\mathbb{V}_0$.

For $\lambda \in D'$, the critical value $\Phi(\lambda)$ does not intersect the boundaries of the the tubes $\mathbb{L}_{\bar{i}}$. Hence we can lift the holomorphic motion on $\mathbb{U} \setminus \mathbb{L}_*$ to a holomorphic motion $\mathbf{h}'$ on $\mathbb{U}' \setminus \mathbb{V}_0$ over D' and extend it by the λ-lemma to the whole tube $\mathbb{U}'$. Thus we obtain a generalized quadratic-like family $(\mathbf{g}', \mathbf{h}')$ over D' which will be called the *generalized renormalization* of the family $(\mathbf{g}, \mathbf{h})$ (with base point $*$).

If $\mathbf{g}$ is a proper family then $\mathbf{g}'$ is clearly proper as well. Moreover, $w(\mathbf{g}') = 1$ if $w(\mathbf{g}) = 1$. Indeed, by the Argument Principle the curve $\Phi|D'$ intersects once every leave of $\partial\mathbb{L}_*$. Hence it has winding number 1 about this tube. As the first landing map $\mathbf{T} : \mathbb{L}_* \to \mathbb{V}_0$ is a fiber bundles diffeomorphism, it preserves the winding number. Thus the new critical value $\Phi' : D' \to \mathbb{U}'$, $\Phi' = \mathbf{T} \circ \Phi$, has also winding number 1.

Let us summarize the above discussion:

Lemma 3.7. *Let* $\mathbf{g} : \cup\mathbb{V}_i \to \mathbb{U}$ *be a generalized quadratic-like family over* $(D, *)$. *Assume it is proper and has winding number 1. Then its generalized renormalization* $\mathbf{g}' : \cup\mathbb{V}'_j \to \mathbb{U}'$ *over* D' *is also proper and has winding number 1.*

3.6. Central cascades. In this section we will describe the renormalization of a generalized quadratic-like family through a central cascade, which will be then treated as a single step in the procedure of parameter subdivisions. Let us have a holomorphic family $(\mathbf{g} : \cup\mathbb{V}_i \to \mathbb{U}, \mathbf{h})$ of generalized quadratic-like maps over $(\Delta, *)$. We will now subdivide D according to the combinatorics of the central cascades of maps g_λ (see

[L3], §§3.1, 3.6). To this end let us first stratify the parameter values according to the length of their central cascade. This yields a nest of parapuzzle pieces

$$\Delta \equiv D \supset D' \supset \cdots \supset D^{(N)} \supset \ldots$$

For $\lambda \in D^{(N)}$, the map g_λ has a central cascade

$$V_\lambda^{(0)} \equiv U_\lambda \supset V_\lambda \equiv V_\lambda^{(1)} \supset \cdots \supset V_\lambda^{(N)} \tag{3.14}$$

of length N, so that $g_\lambda 0 \in V_\lambda^{(N-1)} \setminus V_\lambda^{(N)}$. Note that the puzzle pieces $V_\lambda^{(k)}$ are organized into the tubes $\mathbb{V}^{(k)}$ over $D^{(k-1)}$ (with the convention that $D^{(-1)} \equiv D$).

The intersection of these puzzle pieces, $\cap D^{(N)}$, is the little Mandelbrot set $M(\mathbf{g})$ centered at the superattracting parameter value $c = c(\mathbf{g})$ such that $g_c(0) = 0$. Let us call c the *center* of D.

Let $* \in D^{(N-1)} \setminus D^{(N)}$. Let us consider the Bernoulli map

$$\mathbf{G} : \cup \mathbb{W}_j \to \mathbb{U} \tag{3.15}$$

associated with the cascade (3.14) (see [L3], §3.6). Here the tubes $\mathbb{W}_j$ over $D^{(N-1)}$ are the pull-backs of the tubes $\mathbb{V}_i | D^{(N-1)}$, $i \neq 0$, by the covering maps

$$\mathbf{g}^k : (\mathbb{V}^{(k)} \setminus \mathbb{V}^{(k+1)}) | D^{(N-1)} \to (\mathbb{U} \setminus \mathbb{V}) | D^{(N-1)}, \quad k = 0, 1 \ldots, N-1. \tag{3.16}$$

In the same way as in §3.5, to any string $\bar{j} = (j_0, \ldots, j_{t-1})$ corresponds the tube over $D^{(N-1)}$,

$$\mathbb{W}_{\bar{j}} = \{p \in \mathbb{U} | D^{(N-1)} : \mathbf{G}^n p \in \mathbb{W}_{j_n},\ n = 0, \ldots, t-1\}.$$

Note that $\mathbf{G}^t$ univalently maps each $\mathbb{W}_{\bar{j}}$ onto $\mathbb{U}|D^{(N-1)}$. Thus $\mathbb{W}_{\bar{j}}$ contains a tube $\mathbb{L}_{\bar{j}}$ which is univalently mapped by $\mathbf{G}^t$ onto the central tube $\mathbb{V}^{(N)}$. These maps altogether form the first landing map to $\mathbb{V}^{(N)}$,

$$\mathbf{T} : \cup \mathbb{L}_{\bar{j}} \to \mathbb{V}^{(N)}. \tag{3.17}$$

Remark. Note that

$$\mathrm{mod}(W_{\bar{j},\lambda} \setminus L_{\bar{j},\lambda}) = \mathrm{mod}(U_\lambda \setminus V_\lambda^N) \geq \mathrm{mod}(U_\lambda \setminus V_\lambda), \tag{3.18}$$

since G_λ^t univalently maps the annulus $W_{\bar{j},\lambda} \setminus L_{\bar{j},\lambda}$ onto $U_\lambda \setminus V_\lambda^N$.

Let us now consider the itinerary $\bar{j}_*$ of the critical value $\phi(*) \equiv g_*(0)$ through the tubes W_j until its first return to $V^{(N)}$, so that $\Phi(*) \in \mathbb{L}_{\bar{j}_*} \equiv \mathbb{L}_*$. Let $\mathbb{W}_* \equiv \mathbb{W}_{\bar{j}_*}$ and

$$\Delta^\diamond(*) = \Phi^{-1}\mathbb{L}_*, \quad \Lambda^\diamond(*) = \Phi^{-1}\mathbb{W}_*. \tag{3.19}$$

Thus the annuli $D^{(N-1)} \setminus D^{(N)}$ are tiled by the parapuzzle pieces $\Delta^\diamond(\lambda)$ according as the itinerary of the critical point through the Bernoulli scheme (3.15) until the first return to $V_\lambda^{(N)}$. Altogether these tilings form the desired new subdivision of Δ. (Note however that the new tiles don't cover the whole domain Δ: the residual set consists of the Mandelbrot set $M(\mathbf{g})$ and of the parameter values $\lambda \in D^{(N-1)} \setminus D^{(N)}$ for which the critical orbit never returns back to $V_\lambda^{(N)}$.)

The affiliated quadratic-like family over $\Delta^\diamond(*)$ is defined as the first return map to $V_\lambda^{(N)} \equiv U_\lambda^\diamond$. Its domain $\cup \mathbb{V}_i^\diamond$ is obtained by pulling back the tubes $\mathbb{L}_{\bar{j}}$ from (3.17) by the double branched covering $\mathbf{g} : \mathbb{V}^{(N)} \to \mathbb{V}^{(N-1)} | \Delta^\diamond(\lambda)$, and the return map itself is just $\mathbf{T} \circ \mathbf{g}$.

The affiliated holomorphic motion is also constructed naturally. Let us first lift the holomorphic motion $\mathbf{h}$ from the condensator $\mathbb{U} \setminus \mathbb{V}_0$ to the condensators $(\mathbb{V}^{(k)} \setminus \mathbb{V}^{(k+1)}) | D^{(N-1)}$ via the coverings (3.16). This provides us with a holomorphic motion of $(\mathbb{U} \setminus \mathbb{V}^{(N)}, \cup \mathbb{W}_j)$ over $D^{(N-1)}$. Extend it through $\mathbb{V}^{(N)}$ by the λ-lemma, lift it to the tubes $(\mathbb{W}_{\bar{j}}, \mathbb{L}_{\bar{j}})$ and then extended again by the λ-lemma to the whole domain $\mathbb{U}$ over D^{N-1}. Let us denote it by $\mathbf{H}$.

Lifting this motion via the fiberwise analytic double covering over $\Delta^\diamond(*)$,

$$\mathbf{g} : (\mathbb{U}^\diamond \setminus \mathbb{V}_0^\diamond, \bigcup_{i \neq 0} \mathbb{V}_i^\diamond) \to (\mathbb{V}^{(N-1)} \setminus \mathbb{L}_\lambda \, , \bigcup_{\bar{j} \neq \bar{j}_*} \mathbb{L}_{\bar{j}}),$$

we obtain the desired motion of $(\mathbb{U}^\diamond \setminus \mathbb{V}_0^\diamond, \bigcup_{i \neq 0} \mathbb{V}_i^\diamond)$ over $\Delta^\diamond(\lambda)$. By the λ-lemma it extends through $\mathbb{V}_0^\diamond$.

3.7. Principal parapuzzle nest. Let us now summarize the above discussion. Given a quadratic-like family $(\mathbf{f}, \mathbf{h})$ over $D \equiv \Delta^0$, we consider the first tiling $\mathcal{D}^1$ of a Misiurewicz wake O as described in §3.4. Each tile $\Delta \in \mathcal{D}^1$ comes together with a generalized quadratic-like family $(\mathbf{g}_\Delta, \mathbf{h}_\Delta)$ over Δ.

Now assume inductively that we have constructed the tiling $\mathcal{D}^l$ of level l. Then the tiling of the next level, $\mathcal{D}^{l+1}$ is obtained by partitioning each tile $\Delta \in \mathcal{D}^l$ by means of the cascade renormalization as described in §3.6.

Let $\Delta^l(\lambda)$ stand for the tile of $\mathcal{D}^l$ containing λ, while $\Delta^l(\lambda) \subset \Lambda^l(\lambda) \subset \Delta^{l-1}(\lambda)$ stand for another tile defined in (3.19). Each tile $\Delta = \Delta^l(\lambda)$ contains a *central subtile* $\Pi^l(\lambda) = \Phi_\Delta^{-1} \mathbb{V}_0$ corresponding to the central return of the critical point (here $\Phi_\Delta(\lambda) = (\lambda, \mathbf{g}_\Delta(\lambda)))$. Note that $\Pi^l(\lambda)$ may or may not contain λ itself.

Let us then consider the sequence of renormalized families $(\mathbf{g}_{l,\lambda}, \mathbf{h}_{l,\lambda})$ over topological discs $\Delta^l(\lambda)$. We call the nest of topological discs $\Delta^0(\lambda) \supset \Delta^1(\lambda) \supset \Delta^2(\lambda) \supset \dots$ (supplied with the corresponding families) *the principal parapuzzle nest* of λ. This nest is finite if and only if λ is renormalizable.

Let $c_{l,\lambda} \in \Delta^l(\lambda)$ be the centers of the corresponding parapuzzle pieces. Let us call them the *principal superattracting approximations* to λ. If λ is not renormalizable, then $c_{l,\lambda} \to \lambda$ as $l \to \infty$, as $\operatorname{diam} \Delta^l(\lambda) \to 0$ (see the next section).

The $\operatorname{mod}(\Delta^l(\lambda) \setminus \Delta^{l+1}(\lambda))$ are called the *principal parameter moduli* of $\lambda \in D$.

When we fix a base point $*$, we will usually skip label $*$ in the above notations, so that $\Delta^l \equiv \Delta^l(*)$, $\mathbf{g}_l \equiv \mathbf{g}_{l,*}$, $\mathbf{h}_l \equiv \mathbf{h}_{l,*}$ etc.

4. Parapuzzle geometry

The following is the main geometric result of this paper:

Theorem A. *Let us consider a proper DH quadratic-like family* $(\mathbf{f}, \mathbf{h})$ *with winding number 1 over* D*, and a Misiurewicz wake* $O \subset D$*. Then for any* $\lambda \in M(\mathbf{f}) \cap O$*,*

$$\operatorname{mod}(\Delta^l(\lambda) \smallsetminus \Delta^{l+1}(\lambda)) \geq Bl, \quad \textit{and} \quad \operatorname{mod}(\Delta^l(\lambda) \smallsetminus \Pi^l(\lambda)) \geq Bl,$$

where the constant $B > 0$ *depends only on* O *and* $\operatorname{mod}(\mathbf{f})$*.*

The rest of this section will be devoted to the proof of this theorem.

4.1. Initial parameter geometry. In this section we will give a bound on the geometry of the parapuzzle of zero level. Fix a quadratic-like family $(\mathbf{f}, \mathbf{h})$ and its Misiurewicz wake $O = O^{\sigma,i}_{q/p}$, $|\sigma| = t$, as in §3.3. In what follows we will use the notations of §3.3 and §3.4.

Lemma 4.1. *There is a marked holomorphic motion of any annulus* $\Omega^k_{i,\lambda} \smallsetminus X^k_{i,\lambda}$*, with dilatation depending on the geometry of* $(\mathbf{f}, \mathbf{h})$ *and the choice of* O *only.*

Proof. Indeed, by Lemma 3.4, the configuration

$$(\, \mathbb{Y}^0, \bigcup_{k \leq t-1} \partial \mathbb{Y}^{1+kp}, \bigcup_{k \leq t-1} \bigcup_j \mathbb{Z}^k_j \,)$$

moves holomorphically over the parabolic wake $P_{q/p}$ (truncated by the appropriate equipotential) with removed Misiurewicz wakes of level $\leq t-1$. Since O is compactly contained in this region, this holomorphic motion has a finite dilatation $K = K(O)$ over O (depending only on O and the geometry of $(\mathbf{f}, \mathbf{h})$).

Let us now lift this motion to the tubes $\mathbb{Z}^{(1+tp)}_j$ and $\mathbb{Q}_1$, $\mathbb{Q}_2$ by the map $\mathbf{f}^p$. Since this map is a fiberwise diffeomorphism over O, it preserves the dilatation of the motion (though the motion does not extend beyond O any more). Similarly we can lift the motion to all quadrilaterls between the equipotentials of level $1 + (t-1)p$ and $1 + tp$ left after removing Z-pieces of level $\leq t-1$ (see Figure 3 in [L3]). This provides us with a marked motion of the annulus $Y^{(1+(t-1)p)}_\lambda \smallsetminus V^0_\lambda$ over O with the same dilatation K, which handles the case $k = 0$.

For $k > 0$, lift the above holomorphic motion by the following fiberwise diffeomorphism over O:

$$f^{pk}_\lambda : (\Omega^k_{i,\lambda}, X^k_{i,\lambda}) \to (Y^{(1+(t-1)p)}_\lambda, V^0_\lambda).$$

□

Lemma 4.2. *All parapuzzle pieces of zero level are well inside the corresponding wake:* $\operatorname{mod}(O \smallsetminus \Delta^0(*)) \geq \nu > 0$*, with* ν *depending only on the geometry of* $(\mathbf{f}, \mathbf{h})$ *and the choice of* O*. Moreover, the holomorphic motion* $\mathbf{h}_0$ *of the condensator* $\mathbb{U}^0 \smallsetminus V^0$ *over* Δ^0 *is* K*-qc with* K *depending only on the geometry of* $(\mathbf{f}, \mathbf{h})$ *and the choice of* O*.*

Proof. Let us consider a tube $\mathbb{L}_* \subset \mathbb{W}_*$ over O constructed in §3.4. By means of the fiberwise conformal diffeomorphism (3.11)

$$f_\lambda^{rp} : W_{*,\lambda} \setminus L_{*,\lambda} \to \Omega^k_{i,\lambda} \setminus X^k_{i,\lambda}$$

we can lift the motion constructed in Lemma 4.1 to the condensator $\mathbb{W}_* \setminus \mathbb{L}_*$. Since the dilatation of the motion under such a lift is preserved, it depends only on the geometry of $(\mathbf{f}, \mathbf{h})$ and the choice of O.

Let us now consider the parameter annulus

$$\Lambda^1(*) \setminus \Delta^1(*) = \Phi^{-1}(\mathbb{W}_* \setminus \mathbb{L}_*) \subset O$$

(see (3.12)). By transversal quasi-conformality of holomorphic motions (Corollary 2.1),

$$\operatorname{mod}(\Lambda^1(*) \setminus \Delta^1(*)) \geq K^{-1} \operatorname{mod}(W_* \setminus L_*),$$

where $L_* \setminus W_*$ is the $*$-fiber of the condensator $\mathbb{W}_* \setminus \mathbb{L}_*$.

But f_*^{sp} univalently maps $W_* \setminus L_*$ onto $Y_*^{(1+(t-1)p)} \setminus V_*^0$ (see (3.10)). As by [L3], §4.1, the modulus of the latter annulus depends only on the geometry of $(\mathbf{f}, \mathbf{h})$ and the choice of O, the first statement of the lemma follows.

It implies by the λ-lemma that the holomorphic motion h_0 has a bounded dilatation over $\Delta^1(*)$ depending only on the geometry of $(\mathbf{f}, \mathbf{h})$ and the choice of O. But the motion $\mathbf{h}_1$ on $\mathbb{U}^1 \setminus \mathbb{V}^1$ is a double covering of $\mathbf{h}_0$ on $\mathbb{Y}^{(1+(t-1)p)} \setminus \mathbb{L}_*$ over Δ_1. Hence these motions have the same dilatation, and we are done. □

4.2. Inductive estimate of the parameter geometry.

Lemma 4.3. *Let us have a generalized quadratic-like family* $(\mathbf{g} : \cup \mathbb{V}_i \to \mathbb{U}, \mathbf{h})$ *over* Δ. *Assume that the dilatation of* $\mathbf{h}$ *on* $\mathbb{U} \setminus \mathbb{V}_0$ *is bounded by* K *and* $\operatorname{mod}(U_* \setminus V_{0,*}) \geq \mu > 0$, $\lambda \in D$. *Then the dilatation of the cascade renormalized motion* $\mathbf{h}^\diamond$ *on* $\mathbb{U}^\diamond \setminus \mathbb{V}_0^\diamond$ *over* $D^\diamond$ *is bounded by* $K^\diamond = K^\diamond(\mu, K)$.

Proof. We will use the notations of §3.5 and §3.6. We assume that f_* has a central cascade (3.14) of length N, so that $* \subset D^{(N-1)} \setminus D^{(N)}$. The holomorphic motion $\mathbf{h}$ on $\mathbb{U} \setminus \mathbb{V}_0$ can be lifted to a motion $\mathbf{H}$ on $\mathbb{W}_* \setminus \mathbb{L}_*$ by a fiberwise conformal diffeomorphism $\mathbf{T}$ (extension of first landing map (3.17)). Hence this motion has dilatation K on the tube $\mathbb{W}_* \setminus \mathbb{L}_*$. By transversal quasi-conformality of holomorphic motions,

$$\mu^\diamond \equiv \operatorname{mod}(\Lambda^\diamond \setminus \Delta^\diamond) \geq K^{-1}\mu.$$

By the λ-lemma, $\mathbf{H}$ has dilatation $K^\diamond = K^\diamond(\mu^\diamond)$ on the whole tube $\mathbb{W}_* | \Delta^\diamond$. But the motion $\mathbf{h}^\diamond$ on $\mathbb{U}^\diamond \setminus \mathbb{V}_0^\diamond$ is the lift of $\mathbf{H}$ on $\mathbb{U} \setminus \mathbb{L}_*$ over $\Delta^\diamond$ by the fiberwise conformal double covering $\mathbf{g}$. Hence it has the same dilatation $K^\diamond$. □

4.3. Inscribing rounds condensators. In this section we will show that the parameter annuli have definite moduli. Given a holomorphic motion h_λ and a holomorphic family of affine maps $g_\lambda : z \mapsto a_\lambda z + b_\lambda$, we can consider an "affinely equivalent"motion $g_\lambda \circ h_\lambda$. In this way the motion can be normalized such that any two points $z, \zeta \in U_*$ don't move (that is, $h_\lambda(z) \equiv z$ and $h_\lambda(\zeta) \equiv \zeta$ for $\lambda \in D$). Let us start with a technical lemma:

Lemma 4.4. *Let us have a holomorphic motion $h : (U_*, V_*, 0) \to (U_\lambda, V_\lambda, 0)$ of a pair of nested topological discs over a domain D. Assume that the maps $h_\lambda : (\partial U_*, \partial V_*) \to (\partial U_\lambda, \partial V_\lambda)$ admit K-qc extensions $H_\lambda : (\mathbb{C}, U_*) \to (\mathbb{C}, V_\lambda)$ (not necessarily holomorphic in λ but with uniform dilatation K). Then there exists an $M = M(K)$ such that if* $\mathrm{mod}(U_* \setminus V_*) > M$ *then after appropriate normalization of the motion, there exists a round cylinder $D \times \mathbb{A}(q, 2q)$ embedded into $\mathbb{U} \setminus \mathbb{V}$.*

Proof. Let z_* be a point on ∂U_* closest to 0. Normalize the motion in such a way that $z_* = 1$, and this point does not move either. With this normalization, $V_* \subset D(0, \epsilon)$ where $\epsilon = \epsilon(m) \to 0$ as $m \equiv \mathrm{mod}(U_* \setminus V_*) \to \infty$.

Since the space of normalized K-qc maps is compact, $|H_\lambda(\epsilon e^{i\theta}|) < \delta$, where $\delta = \delta(\epsilon, K) \to 0$ as $\epsilon \to 0$, K being fixed, and $|H_\lambda(e^{i\theta})| > r$ where $r = r(K) > 0$. It follows that the domain $\mathbb{U}$ contains the round cylinder $D \times \mathbb{A}(\delta, r)$, and we are done. $\square$

Corollary 4.5. *Under the circumstances of Lemma* 4.4*, let $\Phi : D \to \mathbb{U}$ be a proper analytic map with winding number 1. Let $D' = \Phi^{-1}\mathbb{V}$. If* $\mathrm{mod}(U_* \setminus V_*) > M = M(K)$ *then* $\mathrm{mod}(D \setminus D') \geq \log 2$.

Proof. By Lemma 4.4, $\mathbb{U} \setminus \mathbb{V} \supset D \times A$ where $A = \mathbb{A}(q, 2q)$. Let $Q = \Phi^{-1}(D \times A)$. By the Argument Principle, $\phi = \pi_2 \circ \Phi$ univalently maps Q onto A, so that $\mathrm{mod}(D \setminus D') \geq \mathrm{mod}\, Q = \mathrm{mod}\, A = \log 2$. $\square$

4.4. Puzzle geometry. Let us now recall for reader's convenience two key results of [L3], which will be used below.

Theorem 4.6 (Moduli growth [L3]**, Theorem III).** *Let f be a quadratic-like map whose straightening $c = \chi(f)$ belongs to a Misiurewicz wake O. Let $n(k)$ be the non-central levels of its principal nest $V^0 \supset V^1 \supset \dots$. Then*

$$\mathrm{mod}(V^{n(k)+1} \setminus V^{n(k)+2}) \geq Bk,$$

where B depends only on O and $\mathrm{mod}(f)$.

Remark. A related result on moduli growth for real parameter values was independently proven by Graczyk & Swiatek [GS]. Note in this respect that the proof of the corresponding parameter result (Theorem A) needs in a crucial way the above Theorem 4.6 with complex parameter values (even if one is ultimately interested in the real case).

Let us consider a quadratic-like family $(\mathbf{f}, \mathbf{h})$ and its parameter tilings. Let $\tilde{\lambda} \in \Delta^l(\lambda)$. Let us consider the corresponding l-fold generalized renormalizations of these two maps $g_l : \cup V_i \to U$ and $\tilde{g}_l : \cup \tilde{V}_i \to U$. Then the holomorphic motion transforms the domains of g_l to the corresponding domains of $\tilde{g}_l$ respecting the boundary marking (coming from the external coordinate system, see §3.2). In this sense f_λ and $f_{\tilde{\lambda}}$ have "the same combinatorics up to level" l.

Let us say that g_l and $\tilde{g}_l$ K-qc pseudo-conjugate if there is a K-qc homeomorphism

$$h : (U, \cup V_i) \to (\tilde{U}, \cup \tilde{V}_i),$$

respecting the boundary marking. Thus it matches with the boundary holomorphic motion, and hence respects the boundary dynamics: $h(g_l z) = \tilde{g}_l(hz)$ for $z \in \cup V_i$.

Theorem 4.7 (Unformly qc pseudo-conjugacies [L3], §**11).** *Assume that $\tilde{\lambda} \in \Lambda^{l+1}(\lambda)$, where the tile $\Lambda^{l+1}(\lambda)$ is defined by (*3.19*). Then the corresponding generalized renormalizations g and $\tilde{g}$ are K-qc pseudo-conjugate, with K depending only on the Misiurewicz wake $O(\lambda)$ and geometry of* $(\mathbf{f}, \mathbf{h})$.

Remarks: 1. Concerning the assumption $\tilde{\lambda} \in \Lambda^{l+1}(\lambda)$, see [L3], Remarks on pp. 89 and 94.

2. In [L3] (see §4) the initial choice of straightening of two maps f and $\tilde{f}$ is made independently and its dilatation depends only on the mod f and mod $\tilde{f}$. In the above formulation, the choice should be consistent with the holomorphic motion $\mathbf{h}$, so that its dilatation depends on the geometry of $(\mathbf{f}, \mathbf{h})$.

4.5. Uniform bound of dilatation.

Lemma 4.8. *Let $*$ belong to a Misiurewicz wake O. For any principal parapuzzle piece $\Delta = \Delta^{l+1}(*)$, the corresponding holomorphic motion $\mathbf{h}_\Delta$ of $\mathbb{U}^{l+1} \setminus \mathbb{V}_0^{l+1}$ over Δ has a uniformly bounded dilatation, depending only on the choice of the Misiurewicz wake O and the geometry of* $(\mathbf{f}, \mathbf{h})$.

Proof. Let K be a dilatation bound given by Theorem 4.7. Find an $M = M(K)$ by Corollary 4.5. By Theorem 4.6 and (3.18), there exists an l_0 such that $\operatorname{mod}(W^l_{j_*} \setminus L^l_{j_*}) \geq M$ for $l \geq l_0$.

For $l \leq l_0$, the desired dilatation bound is guaranteed by Lemmas 4.2 and 4.3.

Fix an $l \geq l_0$. Let us consider the generalized quadratic-like family $(\mathbf{g} : \cup \mathbb{V}_i \to \mathbb{U}, \mathbf{h})$ over $D \equiv \Delta^l(*)$. In what follows we will use the notations of §3.6. Let $* \in D^{(N-1)} \setminus D^{(N)}$.

By Theorem 4.7, for $\lambda \in \Lambda^{l+1}(*)$, there is a K-qc pseudo conjugacy $\psi_\lambda : (U_*, \cup V_{i,*}) \to (U_\lambda, \cup V_{i,\lambda})$, with K depending only on the choice of wake O and geometry of $(\mathbf{f}, \mathbf{h})$. As $\operatorname{mod}(W^l_{j_*} \setminus L^l_{j_*}) \geq M$, Corollary 4.5 can be applied. We conclude that

$$\operatorname{mod}(\Lambda^{l+1}(*) \setminus \Delta^{l+1}(*)) \geq \log 2 \tag{4.1}$$

for l sufficiently big (depending on O and geometry of $(\mathbf{f},\mathbf{h})$).

In §3.6 we have constructed a holomorphic motion $\mathbf{H}$ of $(\mathbb{U},\mathbb{W}_j,\mathbb{L}_{\bar{j}})$ over $D^{(N-1)}$. By the λ-lemma and (4.1), $\mathbf{H}$ is L-qc over $\Delta=\Delta^{l+1}(*)$, with an absolute L provided l is big enough. But the holomorphic motion $\mathbf{h}_\Delta$ on $\mathbb{U}^{l+1}\setminus\mathbb{V}_0^{l+1}$ is the lift of $\mathbf{H}$ on $\mathbb{V}^{(N-1)}\setminus\mathbb{L}_{\bar{j}*}$ over Δ via the fiberwise analytic double covering

$$\mathbf{g}:\mathbb{U}^{l+1}\setminus\mathbb{V}_0^{l+1}\to\mathbb{V}^{(N-1)}\setminus\mathbb{L}_{\bar{j}*}.$$

Hence $\mathbf{h}_\Delta$ on $\mathbb{U}^{l+1}\setminus\mathbb{V}_0^{l+1}$ is also L-qc. $\square$

4.6. Proof of Theorem A. We are now prepared to complete the proof:

$$\operatorname{mod}(\Delta^l\setminus\Delta^{l+1})\geq K^{-1}\operatorname{mod}(W_{\bar{i}_*}\setminus L_{\bar{i}_*})\geq Bl.$$

The first estimate in the above row follows from Lemma 4.8 and Corollary 2.1. The last estimate is due to Theorem 4.6.

For the same reason,

$$\operatorname{mod}(\Delta^l\setminus\Pi^l)\asymp\operatorname{mod}(U_*^l\setminus V_{0,*}^l)\geq Bl.$$

5. Application to the measure problem

In this section we will apply the previous results to the real quadratic family $P_c: z\mapsto z^2+c$, $c\in\mathbb{R}$. Let $\mathcal{NR}$ stand for the set of non-renormalizable real parameter values $c\in[-2,-3/4)$. Note that all periodic points of the $P_c: z\mapsto z^2+c$, $c\in\mathcal{NR}$, are repelling. Indeed, the interval $[-3/4,1/4]$ where P_c has a non-repelling fixed point is excluded, while maps with non-repelling cycles of higher period are renormalizable.

Let $\mathcal{NC}$ stand for the set of parameter values $c\in\mathcal{NR}$ such that the principal nest of P_c constants only finitely many non-trivial (i.e., of length >1) central cascades.

Theorem 5.1.
- *The set $\mathcal{NR}$ has positive measure;*
- *The set $\mathcal{NC}$ has full Lebesgue measure in $\mathcal{NR}$.*

Remarks. 1. The former (positive measure) result is known (see [BC], [J]). The latter (full measure) is new.

2. The corresponding statements concerning at most finitely renormalizable parameter values are derived from the above statements by considering quadratic-like families associated with little copies of the Mandelbrot set.

3. By the result of Martens & Nowicki [MN] together with [L2], P_c has an absolutely continuous invariant measure for any $c\in\mathcal{NC}$. Altogether these yield Theorem B stated in the Introduction.

Proof of Theorem 5.1. Let d stand for the real tip of the little Mandelbrot set attached to the main cardioid (i.e. $P_d^3(0)=\alpha$). As all parameter values $c\in[d,-3/4)$ are renormalizable, we can restrict ourselves to the interval $[-2,d)\supset\mathcal{NR}$. This interval belongs to the Misiurewicz wake O attached to d.

Given measurable sets $X, Y \subset \mathbb{R}$, with $\text{length}(Y) > 0$, let $\text{dens}(X|Y)$ stand for the $\text{length}(X \cap Y)/\,\text{length}(Y)$.

We will now restrict all tilings $\mathcal{D}^l$ constructed above to the real line, without change of notations. We will use the same notation, $\mathcal{D}^l$, for the union of all pieces of $\mathcal{D}^l$. For every $\Delta = \Delta^l(\lambda) \in \mathcal{D}^l$, let us consider the central piece $\Pi \subset \Delta$ corresponding to the central return of the critical point. By Theorem A, $\text{dens}(\Pi|\Delta) \leq Cq^l$ for absolute $C > 0$ and $q < 1$. Let Γ^l be the union of these central pieces. Summing up over all $\Delta \in \mathcal{D}^l$, we conclude that

$$\text{length}(\Gamma^l) \leq \text{dens}(\Gamma^l|\mathcal{D}^l) \leq Cq^l \tag{5.1}$$

(the whole interval is normalized so that its length is equal to 1).

It follows that for l sufficiently big,

$$\text{dens}(\bigcup_{k\geq 0} \Gamma^{l+k}|\mathcal{D}^l) \leq C_1 q^n < 1,$$

which means that with positive probability central returns will never occur again. This proves the first statement.

To prove the second one just notice that (5.1) together with the Borel-Cantelli Lemma yield that infinite number of central returns occurs with zero probability. □

6. Shapes of the Mandelbrot copies

In this section we will prove Theorem C stated in the Introduction. Let us fix a quadratic-like family $(\mathbf{f}, \mathbf{h})$ and a Misiurewicz wake O in it.

Lemma 6.1. *All maximal Mandelbrot copies in O have a bounded shape depending only on the geometry of $(\mathbf{f}, \mathbf{h})$ and the choice of O. In particular, in the quadratic family the shape depends only on the wake O.*

Proof. Take a maximal Mandelbrot copy $M' \subset O$ centered at $*$. Let $(\mathbf{f}_l : \mathbb{V}^l \to \mathbb{U}^l, \mathbf{h}_l)$ be the DH quadratic-like family in the principal nest of $*$ generating M'. By Lemma 4.8, the dilatation of $\mathbf{h}_l$ on $\mathbb{U}^l \smallsetminus \mathbb{V}^l$ is bounded by a constant K depending only on the geometry of $(\mathbf{f}, \mathbf{h})$ and the choice of O. By a weak form of Theorem 4.6, $\text{mod}(\mathbf{f}_l) \geq \epsilon > 0$, where ϵ depends on the same data only. Hence by Lemma 3.2, M' has a bounded shape depending on the same data only. □

Corollary 6.2. *All real maximal Mandelbrot copies in $(\mathbf{f}, \mathbf{h})$, except the doubling one, have a bounded shape depending only on the geometry of $(\mathbf{f}, \mathbf{h})$. In particular, all real maximal copies in the quadratic family, except the doubling one, have a bounded shape with an absolute bound.*

Let us consider a family $\mathcal{L}$ of maximal Mandelbrot copies supplied with a combinatorial parameter $\tau(M')$, $M' \in \mathcal{L}$, satisfying the following

Big Modulus Property. Let $M' \in \mathcal{L}$ be generated by a quadratic-like family over a domain Δ'. Then for $c \in \Delta'$, $\mathrm{mod}(RP_c) \geq \mu(\tau) \to \infty$ as $\tau \to \infty$.

For the family $\mathcal{L} = \mathcal{L}_O$ of maximal Mandelbrot copies contained in a wake O, several examples of such parameters are given in [L3] (see Theorems IV and IV′): the height, the return time, etc. For the family $\mathcal{L} = \mathcal{L}_{\mathbb{R}}$ of maximal real Mandelbrot sets, all these parameters can be unified in a single one called the *essential period* p_e (see [L3, §8], [LY]).

We say that the shapes of Mandelbrot copies M' *approach the shape of* M as $\tau(M') \to \infty$ is the M' have a (K, ϵ)-bounded shape with $K \to 1$, $\epsilon \to \infty$ as $\tau \to \infty$. Lemma 3.2 implies:

Lemma 6.3. *Assume that we have a family $\mathcal{L}$ of maximal Mandelbrot copies and a combinatorial parameter $\tau : \mathcal{L} \to \mathbb{R}$ satisfying the Big Modulus Property. Then the shapes of the Mandelbrot copies $M' \in \mathcal{L}$ approach the shape of M as $\tau(M') \to \infty$. In particular, the shapes of the $M' \in \mathcal{L}_{\mathbb{R}}$ approach M as $p_e(M') \to \infty$.*

Putting together the above statements, we obtain Theorem C.

Remark. If the essential period $p_e(M')$ stays bounded but the period $p(M')$ grows, we obtain Mandelbrot copies near a parabolic cusp. The shapes of such copies were analysed by Douady and Devaney [DD].

6.1. Relation to MLC. Let us consider a family $\mathcal{L}$ of Mandelbrot copies. Assume that any copy M' with combinatorial type $(M_0, \dots, M_n)$, $M_i \in \mathcal{L}$, has a bounded shape. Let us say that such an $\mathcal{L}$ is fine.

Given a family $\mathcal{L}$ of Mandelbrot copies, let $E_{\mathcal{L}}$ stand for the set of infinitely renormalizable parameter values with combinatorics $(M_0, M_1, \dots)$, where $M_n \in \mathcal{L}$.

Proposition 6.4. *Let $\mathcal{L}$ be a fine family of Mandelbrot copies. Then*

- *The Mandelbrot set is locally connected at any point $c \in E_{\mathcal{L}}$;*
- *The set $E_{\mathcal{L}}$ has zero Lebesgue measure.*

Proof. • Take a string $\tau = (M_0, M_1, \dots)$ with $M_i \in \mathcal{L}$. It determines a nest $M^0 \supset M^1 \supset \dots$ of little Mandelbrot copies shrinking to the combinatorial class $\mathcal{C}_\tau$ of infinitely renormalizable maps with combinatorics τ. To prove MLC at a point $c \in \mathcal{C}_\tau$ we need to show that c is a single point of $\mathcal{C}_\tau$.

Let c^n stand for the center of M^n, and $H^n \ni c^n$ stand for the corresponding hyperbolic component. Let r_n be the inner radius of H^n, i.e., the radius of maximal round disc centered at c^n and inscribed into H^n. As the domains H^n are pairwise disjoint, $r_n \to 0$. Since the sets M^n have a bounded shape, $r_n \asymp \mathrm{diam}\, M^n$. Hence $\mathrm{diam}\, M^n \to 0$, so that the combinatorial class C_τ consists of a single point.

• To prove the second statement, note that the hyperbolic components H^n do not belong to $E_{\mathcal{L}}$ as they are not infinitely renormalizable. Hence near any point $c \in E_{\mathcal{L}}$

there are gaps of definite relative size in arbitrary small scales. By the Lebesgue density points theorem, $\text{meas}(E_{\mathcal{L}}) = 0$. □

Examples of fine families.

a) Family of maximal Mandelbrot copies M' with big height: $\chi(M') \geq \bar{\chi}$.

b) Family of real maximal Mandelbrot copies with big period: $p(M') \geq \bar{p}$ (see [L7]).

Remarks: 1. We are confident that the whole family of real maximal Mandelbrot copies is fine (so that all real Mandelbrot copies have bounded shape). We are not sure whether this is still valid for the full family of complex maximal Mandelbrot copies. This would imply MLC but it may happen that MLC is still true, though there exist very distorted Mandelbrot copies.

2. In [L6] we have constructed a fine family $\mathcal{L}$ of Mandelbrot copies such that $\text{HD}(E_{\mathcal{L}}) \geq 1$. Thus the Hausdorff dimension of the set of infinitely renormalizable maps is at least 1.

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