Six Lectures on Real and Complex Dynamics

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Contents

Introduction

In these lecture notes we will present, starting from scratch, main recent advances in real and complex one-dimensional dynamics. They include three major themes: rigidity, renormalization, and measurable dynamics. All these phenomena will be discussed in the context of the quadratic family $f_c : z \mapsto z^2 + c$ which has proved to be a great model for chaotic dynamics.

For $c \in [-2, 1/4]$, the map f_c preserves an interval I_c. When c is "close" to 1/4, namely $c \in (-3/4, 1/4)$, the dynamics of f_c is very simple: all orbits except the endpoints of I_c, converge to a fixed point, the dynamics is "regular". On the other hand, for the endpoint, $c = -2$, the situation is very different: typical orbits behave like independent random variables, the dynamics is "chaotic".

For intermediate parameters, regular and chaotic regimes are intertwined in an intricate way. The dynamics is called *regular* if almost all orbits converge to an attracting cycle (the corresponding maps are also called *hyperbolic*). It is called *stochastic* if almost all orbits are equidistributed with respect to an invariant measure with positive entropy. The ultimate goal of these lecture notes is to present the following result, which provides us with a complete measure-theoretic picture of dynamics in the quadratic family:

Regular of Stochastic Theorem. *Almost all maps f_c, $c \in [-2, 1/4]$, are either regular or stochastic.*

Note that the set of regular maps is obviously open, while the set of stochastic maps has positive measure Lebesgue measure [**J, BC**]. Thus, non of these sets can be neglected in the measure-theoretic picture of dynamics.

Three themes are hidden behind the Regular or Stochastic Theorem.

The first theme is the Rigidity Phenomenon asserting that combinatorics of a non-hyperbolic map determines the map itself (Lecture 4). This phenomenon is reminiscent to the Mostow Rigidity: geometry of a (compact) hyperbolic manifold of dimension ≥ 3 is determined by its topology. And indeed, dynamical and geometric rigidity are intimately related.

Quasiconformal maps and "puzzle techniques" will provide us with the main tools to tackle the rigidity problem. The idea of the puzzle techniques introduced in the context of the quadratic family by Yoccoz is to break a dynamical picture (the Julia set) into pieces of different scales and to study these pieces recursively passing from scale to scale. Putting them back again, one gets an important topological and geometric information about the original picture.

In Lecture 3 we describe the combinatorics of the puzzle based on its "principal nest' and the "generalized renormalization". We then outline a proof of the crucial geometric property of the puzzle, linear growth of its "principal moduli" [**L4**].

The Rigidity theory yields density of hyperbolic maps in the real quadratic family.

The second theme is the Stochastic Phenomenon (Lecture 5). Only maps that are not "infinitely renormalizable" have a chance to be stochastic. In fact, only maps with sufficiently slow recurrence of the critical point have this chance. So, the game is to find such a condition on the recurrence which yields stochasticity, on the one hand, and which is satisfied for almost all maps which are neither hyperbolic nor infinitely renormalizable. Such a criterion can be formulated in terms of the principal nest of the puzzle (Martens & Nowicki [**MN**]). Applying the puzzle techniques to the parameter plane (by breaking the Mandelbrot set into "parapuzzle pieces"), one can prove that this criterion is indeed typically satisfied:

Theorem A [**L5**]**.** *Almost all quadratic polynomials f_c which are neither hyperbolic nor infinitely renormalizable are stochastic (satisfying the Martens-Nowicki criterion).*

The last theme is the Renormalization Theory (Lecture 6). It gives us an explanation of quantitative universalities observed in different families of dynamical systems. After its discovery in 1970's by Feigenbaum and independently by Coullet & Tresser, a major effort has been made in order to justify it mathematically. It was recently completed in the works of Sullivan [**S2**], McMullen [**McM2**], and the author [**L6**] consecutively dealing with different parts of the Conjecture. A characteristic feature of this development is that it is almost completely based upon the methods of holomorphic dynamics, though the final results can be formulated in purely real terms.

A generalization of the Renormalization Theory to all possible combinatorial types given in [**L7**] leads to the following assertion:

Theorem B [L7]. *The set of infinitely renormalizable real parameters has zero Lebesgue measure*

Theorems A and B together imply the Regular or Stochastic Theorem.

Theory of quadratic-like maps originated by Douady & Hubbard [**DH2**] has played a crucial role in all stages of the above development. The 2nd lecture is devoted to this theory.

The basic background in real and complex one-dimensional dynamics (in the context of the quadratic family) is given in the 1st lecture.

These notes are based on the European Lectures given by the author in May - June, 1999, in Barcelona, Copenhagen, and St Petersburg. (Similar lecture series were given in Kyoto (2000), Trieste (2001), and at the Dynamics seminar in UCLA (1998).) Of course, the notes do not exactly correspond to the lectures, even as the number of lectures is concerned. However, we tried to follow the spirit of the lectures, by focusing on the conceptual background and basic constructions at expense of technical work-out, and by making various informal comments throughout the text.

For a preview of these lectures, the reader can look at author's article in the Notices of the AMS (October 2000) "The quadratic family as a qualitatively solvable model of chaos".

Notations and terminology

As usual, $\mathbb{C}$ is the complex plane; $\mathbb{R}$ is the real line;
$\mathbb{N} = \{1, \dots\}$ is the set of natural numbers; $\mathbb{Z}$ is the set of integers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$;

$\mathbb{D}(a, r) = \{z : |z - a| < r\}$ is the open round disk of radius r,
$\mathbb{D}_r \equiv \mathbb{D}(0, r)$, $\mathbb{D} \equiv D_1$; $\mathbb{D}^* = \mathbb{D} \smallsetminus \{0\}$;
$\mathbb{T}_r = \partial\mathbb{D}_r$ is the circle of radius r, $\mathbb{T} \equiv \mathbb{T}_1$;
$\mathbb{A}(r, R) = \{z : r < |z| < R\}$;

$\operatorname{cl} X$ denotes the closure of a set X;
$U \Subset V$ means that U is *compactly contained* in V, that is, $\operatorname{cl} U$ is compact and is contained in V.
A topological disk means a simply connected domain in $\mathbb{R}^2$;
An (open/closed) *Jordan disk* in $\mathbb{R}^2$ is a (domain/closure of a domain) bounded by a Jordan curve.
A topological annulus means a doubly connected domain in $\mathbb{C}$.
A *tiling* of a set $X \subset \mathbb{R}^2$ is a covering of X with closed Jordan disks (*tiles*) $D_i \subset X$ with disjoint interiors.

Given a measurable set $X \subset \mathbb{R}$, $|X|$ will stand for its Lebesgue measure.

A *dynamical system* is a self-map f of some topological space M, the "phase space". Usually f is assumed to be continuous; however, we will also deal with discontinuous and even partially defined maps. The n-fold iterate of f, wherever it is defined, is denoted by f^n.
The *orbit* ($\equiv$ *trajectory*) of a point $x \in M$, $\mathrm{orb}_f(x) \equiv \mathrm{orb}(x)$, is $\{f^n x\}_{n\in\mathbb{Z}_+}$.
The limit set of the orb(x) is denoted $\omega_f(x) \equiv \omega(x)$.

A point α is called *fixed* if $f\alpha = \alpha$.
A point α is called *periodic* if $f^p\alpha = \alpha$ for some $p \in \mathbb{N}$. The smallest p with this property is called the *period* of α. The orbit of a periodic point, $\{f^n\alpha\}_{n=0}^{p-1}$, is naturally called a *periodic orbit*, or a *cycle*.
A point x is called *preperiodic* if $f^n x$ is periodic for some $n \in \mathbb{N}$, but x itself is not periodic.

Assuming f is smooth, a point x is called *precritical* if $f^N x$ is critical for some $N|in\mathbb{Z}_+$.

A set X is called *invariant* if $fX \subset X$; an invariant set is called *completely invariant* if $f^{-1}(X) \subset X$.

Given two dynamical systems, f on space S and $\tilde{f}$ on space $\tilde{S}$, a map $h : S \to \tilde{S}$ (maybe partially defined) is called $(f, \tilde{f})$-*equivariant* on a set $X \subset S$ if $h(fx) = \tilde{f}(hx)$ for $x \in X$. We will skip the reference to $(f, \tilde{f})$ as long as it is clear from the context.

If h above is a continuous surjection then it is called a *topological semi-conjugacy* between f and $\tilde{f}$. If h is a homeomorphism then it is called a *topological conjugacy*, and the maps f and $\tilde{f}$ are called *topologically conjugate*. This notion can be further refined to smooth/quasiconformal/conformal etc. conjugacy depending on the regularity of h.

In these notes, a "proof" will usually mean an idea or an outline of the proof. We will rarely supply full proofs of the results. The reader interested in the full technical proofs should either reconstruct them him/herself or consult the sources referred to in the Bibliographical Notes at the end of every lecture.

We will use the following reference system. A reference to, say, §3.1 from the 2nd lecture, will look like "see Theorem II.2.1". However, if the reference is given in the same (2nd) lecture, then "II" will be skipped: "see §2.1". In the references to mathematical statements (Theorems, Propositions, etc.) the first digit stands for the chapter where the statement appeared.

LECTURE 1

Julia sets and the Mandelbrot set

Let $f = f_c : z \mapsto z^2 + c$. Dependence of a certain object on f will be also marked as dependence on the parameter c, e.g., $J(f) \equiv J_c$, $B_f \equiv B_c$.

1. Julia sets

1.1. Looking from infinity. Extend f to an endomorphism of the Riemann sphere $\bar{\mathbb{C}}$. This extension has a critical point at ∞ fixed under f. We will start exploring the dynamics of f from there. The first observation is that $\bar{\mathbb{C}} \smallsetminus \mathbb{D}_R$ is f-invariant for a sufficiently big R, and moreover $f^n z \to \infty$ as $n \to \infty$ for $z \in \bar{\mathbb{C}} \smallsetminus \mathbb{D}_R$. This can be expressed by saying that $\mathbb{C} \smallsetminus \mathbb{D}_R$ belongs to the *basin of infinity* defined as the set of all escaping points:

$$D_f(\infty) = \{z : f^n z \to \infty,\ n \to \infty\} = \bigcup_{n=0}^{\infty} f^{-n}(\mathbb{C} \smallsetminus \mathbb{D}_R).$$

PROPOSITION 1.1. *The basin of infinity $D_f(\infty)$ is a completely invariant domain containing ∞.*

1.2. Basic Dichotomy. We can now introduce the fundamental dynamical object, the *filled Julia set* $K(f) = \bar{\mathbb{C}} \smallsetminus D_f(\infty)$. Proposition 1.1 implies that $K(f)$ is a completely invariant compact subset of $\mathbb{C}$. Moreover, it is *full*, i.e., it does not separate the plane (since $D_f(\infty)$ is connected).

The filled Julia set and the basin of infinity have a common boundary, which is called the *Julia set*, $J(f) = \partial K(f) = \partial D_f(\infty)$. Figures 1.1–1.2 show several Julia sets $J_c \equiv J(f_c)$ for different parameter values c. Generally, topology and geometry of the Julia set is very complicated, and it is hard to put a hold on it. However, there is the following rough classification:

THEOREM 1.2. *The Julia set (and the filled Julia set) is either connected or Cantor. The latter happens if and only if the critical point escapes to infinity: $f^n(0) \to \infty$ as $n \to \infty$ (see Figure 1.3).*

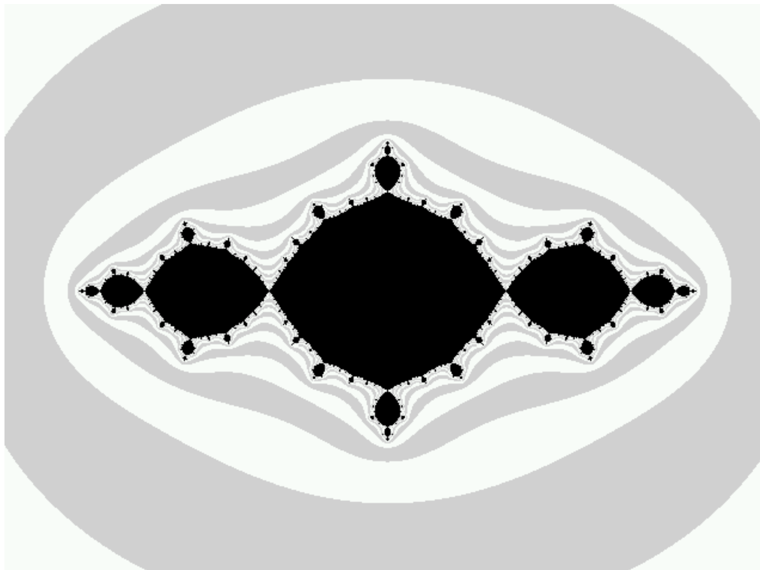

FIGURE 1.1. The Julia set of $z \mapsto z^2 - 1$. This map has a superattracting cycle of period 2.

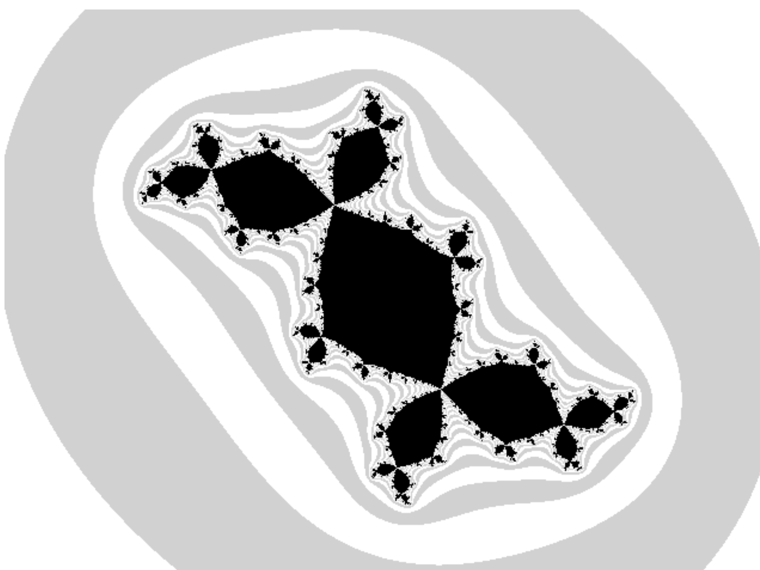

FIGURE 1.2. The "Douady rabbit". This map has a superattracting cycle of period 3.

This Basic Dichotomy is the first example of how the behavior of the critical point influences the global dynamics. In fact, at least on the philosophical level, the dynamics is completely determined by the

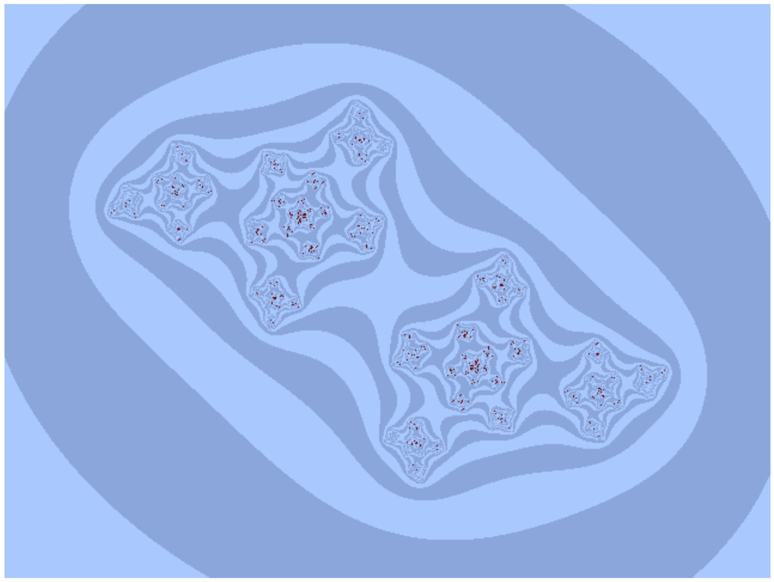

FIGURE 1.3. A Cantor Julia set.

behavior of this single point. We will see many confirmations of this principle.

EXERCISE 1.3. Assuming c is real, let β stand for the positive fixed point of f_c. Show that J_c is connected if and only if $c \in [-\beta, \beta]$. Moreover, in the connected case, $J_c \cap \mathbb{R} = [-\beta, \beta]$.

2. View from inside

According to Poincaré (in free translation), *periodic orbits give us the only opening to otherwise unaccessible corners of dynamics*. So, let us try to look through this opening at the phase space of the quadratic maps.

2.1. Rough classification of periodic points. Consider a periodic point α of period p. The local dynamics near its cycle depends first of all on its *multiplier* $\lambda = (f^p)'(z)$.

If $|\lambda| < 1$ then α is called *attracting*. The orbits of all nearby points exponentially fast converge to the cycle $\boldsymbol{\alpha} = \{f^k\alpha\}_{k=0}^{p-1}$ and, in particular, are bounded. Thus, attracting cycles belong to $F(f)$.

A particular case of an attracting cycle is a *superattracting* one when $|\lambda| = 0$. Nearby points converge to a superattracting cycle at a superexponential rate.

The *basin of attraction* of an attracting cycle $\boldsymbol{\alpha}$ is the set of all points whose orbits converge to $\boldsymbol{\alpha}$:

$$D_f(\boldsymbol{\alpha}) = \{z : \ f^n z \to \boldsymbol{\alpha} \text{ as } n \to \infty.\}$$

The union of components of $D_f(\boldsymbol{\alpha})$ containing the points of $\boldsymbol{\alpha}$ is called the *immediate basin* of attraction of the cycle $\boldsymbol{\alpha}$. We will now state one of the most important facts of the classical holomorphic dynamics:

Theorem 1.4. *The immediate basin of attraction of an attracting cycle contains the critical point 0.*

It follows that a quadratic polynomial can have at most one attracting cycle. Of course, the period of this cycle can be arbitrary big. If a quadratic polynomial does indeed have an attracting cycle, it is called *hyperbolic*. For instance, polynomials $z \mapsto z^2$, $z \mapsto z^2 - 1$ (see Figure 1.1) are hyperbolic. Though dynamically non-trivial, it is a well understood class of quadratic polynomials:

Theorem 1.5. *If a quadratic polynomial f has an attracting cycle $\boldsymbol{\alpha}$, then $D_f(\boldsymbol{\alpha}) = \operatorname{int} K(f)$ and $J(f)$ has zero Lebesgue measure.*

Note that quadratic polynomials with Cantor Julia set are also called hyperbolic. A reason is that in this case the orbit of the critical point still converges to an attracting fixed point (at ∞). Quadratic polynomials with connected Julia set but without attracting periodic points are not hyperbolic (by definition).

If $|\lambda| > 1$ then α (and its cycle) is called *repelling*. Nearby points exponentially fast escape from a neighborhood of a repelling cycle. Repelling periodic points belong to the Julia set and, in fact, they are dense in the Julia set, so that the Julia set can be alternatively defined as the closure of repelling cycles. It gives us a view of the Julia set "from inside".

If $|\lambda| = 1$ then α (and its cycle) is called *neutral*. Local dynamics near a neutral point delicately depends on the arithmetic of the rotation number $\theta = \frac{1}{2\pi} \arg \lambda$. If θ is rational then α is called *parabolic*; otherwise it is called *irrational*.

Parabolic points belong to the Julia set. The basin of attraction of a parabolic cycle $\boldsymbol{\alpha}$ is defined as follows:

$$D_f(\boldsymbol{\alpha}) = \{z : f^n z \to \boldsymbol{\alpha} \text{ as } n \to \infty \text{ but } f^n z \notin \boldsymbol{\alpha} \text{ for any } n \in \mathbb{Z}_+\}.$$

(By excluding the orbits landing at $\boldsymbol{\alpha}$ one makes the basin open.) As in the attracting case, the basin of a parabolic cycle also must contain the critical point. Hence a quadratic polynomial can have at most one parabolic cycle. A polynomial with a parabolic cycle is naturally called *parabolic.*

Irrational periodic points may or may not belong to the Julia set (depending on the Diophantine properties of its rotation number). Irrational periodic points lying in the Fatou set are called *Siegel*, and those lying in the Julia set are called *Cremer*. The component of $F(f)$ containing a Siegel point is called a *Siegel disk*. Local dynamics on a Siegel disk is quite simple: If U is a Siegel disk of period p containing a periodic point α with rotation number θ, then $f^p|U$ is conformally conjugate to the rotation of $\mathbb{D}$ by θ.

THEOREM 1.6. *A quadratic polynomial can have at most one non-repelling cycle*

If it has one, it can be non-contradictory classified as either hyperbolic, or parabolic, or Siegel, or Cremer. We will refer to a parameter value $c \in \mathbb{C}$ as hyperbolic, parabolic, etc, if the corresponding map f_c is such. We will also say that f_c (and the corresponding parameter c) is *purely repelling* if all periodic points of f_c (except ∞, of course) are periodic.

A polynomial f_c (and the corresponding parameter value c) is called *Misiurewicz* if the critical point 0 is preperiodic. In this case, the orbit of 0 lands at some *repelling* cycle.

3. External rays and equipotentials

3.1. Böttcher coordinate. The fact that ∞ is a superattracting fixed point leads to a precise dynamical model for any polynomial near ∞:

THEOREM 1.7. *Any quadratic polynomial f_c near ∞ is conformally conjugate to $z \mapsto z^2$. The conjugacy $B_c : U_c \to \mathbb{C} \smallsetminus \bar{\mathbb{D}}_R$ is unique and is given by the following explicit formula:*

$$B_c(z) = \lim (f_c^n z)^{1/2^n}, \tag{1.1}$$

where the root in the right-hand side is selected so that $(f_c^n z)^{1/2^n} \sim z$ as $z \to \infty$. Moreover, $B_c(z) \sim z$ near ∞.

The function B_c is called the *Böttcher coordinate* near ∞, or the Böttcher function associated with f_c. By definition, it satisfies the equation

$$B_c(f_c z) = (B_c(z))^2 \tag{1.2}$$

called the *Böttcher equation.*

3.2. Analytic extension. The first question asked in classical analysis about an analytic function is what is its natural domain of definition. It turns out that in the case of the Böttcher function, this question can be fully addressed by means of the Böttcher equation. If the domain U_c does not contain the critical value c then by means of (1.2) B_c can be lifted to the preimage $f_c^{-1}U_c$. This give the analytic extension of f_c to this bigger domain. If it does not contain c either then we can lift B_c to the next preimage $f_c^{-2}U_c$, etc.

In the case of disconnected Julia set, this process can be continued to infinity given the analytic extension of B_c to the whole basin of ∞, $\mathbb{C} \smallsetminus K(f_c)$. In the disconnected case, we can proceed until we hit the critical value and then carry one more lift so that the boundary of the domain will contain 0. Summarizing this consideration, we obtain:

PROPOSITION 1.8. *The Böttcher function admits the analytic extension to a univalent map $B_c : \Omega_c \to \mathbb{C} \smallsetminus \mathbb{D}_{R_c}$ on an invariant domain Ω_c. If $J(f_c)$ is connected then $\Omega_c = \mathbb{C} \smallsetminus K(f_c)$ and $R_c = 1$. Otherwise Ω_c is bounded by a "figure eight" centered at the origin, and $R_c > 1$. picture*

Notice that in the case of disconnected Julia set, $c \in \Omega_c$, so that the Böttcher function is well defined at c. This function, $B_c(c)$, will play a very important role.

In the connected case, we obtain an explicit dynamical formula (1.1) for the Riemann map $C \smallsetminus K(f_c) \to \mathbb{C} \smallsetminus \mathbb{D}$. Given very complicated fractal structure of the Julia set, this is quite a surprise! In fact, the logic can be reversed, and the Böttcher function can be produced by means of the Riemann Mapping Theorem:

EXERCISE 1.9. If the Julia set $J(f_c)$ is connected, then the Riemann mapping $\mathbb{C} \smallsetminus K(f_c) \to \mathbb{C} \smallsetminus \bar{\mathbb{D}}$ tangent to z at ∞ conjugates f_c to $f_0 : z \mapsto z^2$.

3.3. Two invariant foliations. The map $f_0 : z \mapsto z^2$ has two invariant foliations on $\mathbb{C} \smallsetminus \bar{\mathbb{D}}$, the foliation by straight rays $\{re^{2\pi i\theta},\ 0 < r < \infty\}$ and the foliation by round circles $\{re^{2\pi i\theta},\ 0 \le \theta < 2\pi\}$. By means of the Böttcher function, these two foliations can be transferred to the domain Ω_c providing us with two invariant foliations for f_c. The leaves of these foliations are called *external rays* and *equipotentials* respectively. An external ray is specified by its external angle θ, while an equipotential is specified by its radius r. Under the dynamics, the external angle is doubled, $\theta \mapsto 2\theta \mod 1$, while the radius is squared, $r \mapsto r^2$. It follows that the rays with rational external angles $\theta = q/p$ are either periodic or preperiodic depending on whether the denominator p is is odd or even.

In the disconnected case, the Böttcher function starts to branch if we try to extend it further to preimages of Ω_c. However, the *Green function*, $G_c(z) = \log|B_c(z)|$, does not branch. Indeed, by means of equation $G(fz) = 2G(z)$ it can be harmonically extended to the whole complement of $K(f_c)$. This allows us to extend two external foliations to the whole complement of $K(f)$. However, these foliations will have singular points at the origin and all its preimages under iterates of f_c.

3.4. Landing rays. This provides us with a good dynamical picture outside the filled Julia set (particularly, in the connected case). The next idea is to try to understand the Julia set by exploring how these foliations degenerate near it. This leads us to the problem of *landing* of external rays: is it true that any ray lands at some particular point of the Julia set? If the function $B_c^{-1} : \mathbb{C} \smallsetminus \bar{\mathbb{D}} \to \mathbb{C} \smallsetminus K(f_c)$ admitted a continuous extension to the unit circle $\mathbb{T}$, the answer would certainly be "yes". But continuity of B_c^{-1} depends on fine topological properties of the Julia set:

Carathéodory Theorem. *Let K be a full compact subset of the complex plan and $\psi : \mathbb{C} \smallsetminus \bar{\mathbb{D}} \to \mathbb{C} \smallsetminus K$ be the corresponding Riemann mapping. Then ψ admits a continuous extension to $\mathbb{T}$ if and only if the set K is locally connected.*

It turns out that unfortunately the Julia set is not necessarily locally connected. Still, there are always some landing rays:

THEOREM 1.10. *Assume that the Julia set $J(f)$ is connected.*

Any ray with rational external angle with odd denominator lands at some repelling or parabolic periodic point of the Julia set. Vice versa, any repelling or parabolic periodic point is the landing point of at least one but at most finitely many external rays. All these rays have external angles with odd denominators.

Similar statements hold for rays with rational external angles with even denominators if to replace periodic points with preperiodic ones.

3.5. Fixed points. In particular, the ray with external angle $\theta = 0$ (the "zero-ray") lands at some fixed point of f. Moreover, this point is either repelling or parabolic with multiplier 1. This fixed point is called $\beta = \beta_f$. It turns out that the zero-ray is the only ray landing at β.

Another fixed point is called $\alpha = \alpha_f$. Note that if $\alpha = \beta$ then this is a parabolic fixed point with multiplier 1. This happens only for one quadratic map, $f : z \mapsto z^2 + 1/4$.

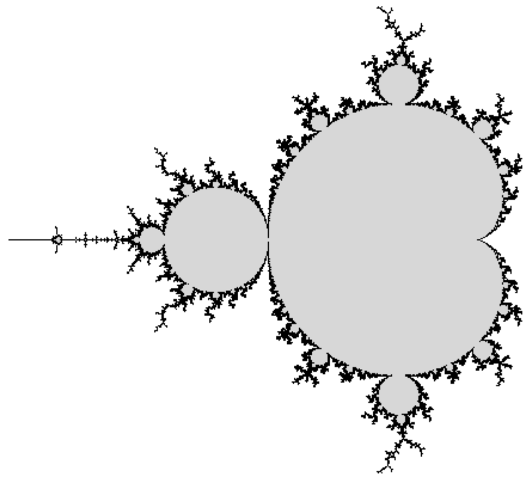

FIGURE 1.4. The Mandelbrot set

PROPOSITION 1.11. *Let f be a quadratic polynomial with connected Julia set. If the fixed point α is repelling then it is the landing point of $p > 1$ rays which are cyclically permuted by the dynamics.*

The rays landing at the α-fixed point are called α-rays.

Thus, in the case of connected Julia set with both fixed points repelling, these points can by dynamically distinguished: one of them, β, is the landing point of a single invariant ray, another one, α, is a landing point of several cyclically permuted α-rays. One can also characterize β as a *non-dividing* fixed point, in the sense that $K(f) \setminus \{\beta\}$ is connected, while point α is characterized as *dividing*.

In the case of disconnected Julia set, there is no dynamical difference between the two fixed points.

4. The Mandelbrot set

4.1. Definition. The Mandelbrot set depicted on Figure 1.4 is the bifurcation diagram of the quadratic family. This single picture encodes all metamorphoses of the Julia sets $J(f_c)$ as c varies, from simplest to most intricate forms. The set itself is very complicated but its formal definition motivated by the Basic Dichotomy is very simple:

The *Mandelbrot set* $\mathbb{M}$ is the set of parameter values $c \in \mathbb{C}$ such that the corresponding Julia set $J(f_c)$ is connected. Equivalently,

$$\mathbb{C} \setminus \mathbb{M} = \{c \in \mathbb{C} : \ f_c^n(0) \to \infty \text{ as } n \to \infty\}.$$

Note that $p_n(c) = f_c^n(0)$ is the polynomial in c of degree 2^{n-1}. Here is an initial piece of this sequence of polynomials:

$$0 \mapsto c \mapsto c^2 + c \mapsto (c^2 + c) + c \mapsto ((c^2 + c)^2 + c)^2 + c \mapsto \dots .$$

EXERCISE 1.12. Show that if $|p_n(c)| > 2$ then $c \in \mathbb{C} \setminus \mathbb{M}$. Moreover, $\mathbb{M} \subset \mathbb{D}_2$ and $\mathbb{M} \cap \mathbb{R} = [-2, 1/4]$.

The first part of this exercise gives one a simple algorithm of plotting the Mandelbrot set on the computer screen. It also easily implies:

PROPOSITION 1.13. *The Mandelbrot set is a full compact subset of the complex plane.*

Since $\mathbb{M}$ is full, the connected components of $\operatorname{int} \mathbb{M}$ are topological disks. Since attracting cycles persist under perturbations, any hyperbolic parameter value c belongs to $\operatorname{int} \mathbb{M}$. A component H of $\operatorname{int} \mathbb{M}$ is called *hyperbolic* if all parameters $c \in H$ are hyperbolic. Otherwise, H is called *queer*. If H is queer then non of the parameters $c \in H$ is hyperbolic. (Conjecturally, queer components do not exist at all.)

PROPOSITION 1.14. *Any point $c \in \partial \mathbb{M}$ can be approximated by hyperbolic parameter values.*

COROLLARY 1.15. *If there are no queer components, then hyperbolic maps are dense in* $\mathbb{M}$.

Given a hyperbolic component H, let $\lambda(c)$ denote the multiplier of the attracting cycle of f_c. By the Implicit Function Theorem, λ is holomorphic. Moreover, λ extends continuously to the closure of H and maps ∂H to the unit circle. Hence $\lambda : H \to \mathbb{D}$ is a finite degree branched covering. In fact, much deeper fact is valid (see Theorem 4.13 in 4th lecture):

THEOREM 1.16. *The multiplier function λ univalently maps H onto the unit disk D.*

Hence λ has a unique zero c_H in H called the *center* of H. At this point the attracting cycle becomes superattracting. Moreover, λ extends to a homeomorphism $\partial H \to \mathbb{T}$, so that there exists a unique parameter $c \in \partial H$ with $\lambda(c) = 1$. This parameter is called the *root* of H. Furthermore, there is a dense set of parabolic points on ∂H corresponding to values $\lambda = e^{2\pi i \theta}$ with rational rotation numbers $q/p \in [0, 1)$.

Siegel and Cremer parameters on ∂H correspond to irrational rotation numbers θ. Which case, Siegel or Cremer, actually occurs depends on the Diophantine properties of θ.

Let us finish this section with the following easy result:

PROPOSITION 1.17. *All neutral parameters lie on $\partial\mathbb{M}$.*

4.2. Cascades of bifurcations. Let us now take a little walk around $\mathbb{M}$. Our departing point is the origin, $c = 0$, which is the beginning of the genealogy of the quadratic family. The map $f_0(z) = z^2$ has a superattracting fixed point 0 which attracts all points of the unit disk $\mathbb{D}$. The corresponding Julia set $J(f_0)$ is just the round circle $\mathbb{T}$.

If we perturb the parameter a little bit then this picture will topologically persist: any nearby map $f_\varepsilon : z \mapsto z^2 + \varepsilon$ has an attracting fixed point α_ε whose basin $D(\alpha)$ is a Jordan disk. The dynamics on the Julia set $J(f_\varepsilon) = \partial D(\alpha)$ is topologically conjugate to the dynamics of z^2 on the unit circle.

EXERCISE 1.18. Let H_0 be the component of $\operatorname{int}\mathbb{M}$ containing 0. Show that H_0 is a Jordan disk bounded by the cardioid

$$c = \frac{e^{2\pi i\theta}}{2} - \frac{e^{4\pi i\theta}}{4}, \quad 0 \le \theta < 1.$$

The above cardioid is called the *main cardioid* of $\mathbb{M}$. Call it C. On C, the attracting fixed point α_c becomes neutral with multiplier $\lambda = e^{2\pi i\theta}$. The cardioid has a root at $c = 1/4$ which is called the *cusp* of the Mandelbrot set.

Parabolic points on C play a distinguished role since they are points of bifurcations of attracting cycles. These bifurcations are easily visible on the picture. For instance, if we cross the main cardioid at the real point $c = -3/4$ (corresponding to $\theta = 1/2$), we observe the *doubling bifurcation*: the fixed point α_c will become repelling and will pass its attractiveness to a cycle $\boldsymbol{\gamma}_c$ of period 2. If we cross $-3/4$ moving along the real line and watching only the real slice of the picture, then it appears that $\boldsymbol{\gamma}_c$ is "born" from α_c at the moment of bifurcation. (In fact, the cycle $\boldsymbol{\gamma}_c$ was complex before the bifurcation, collapses into the fixed point α_c at the moment of bifurcation, and afterwards appears on the real line as an attractive cycle.)

Parameter values c for which the cycle $\boldsymbol{\gamma}$ preserves its attractiveness fill a hyperbolic component H touching H_0 at $-3/4$. At the center of this disk, $c = -1$, $\boldsymbol{\gamma}$ becomes superattracting. The corresponding Julia set is depicted on Figure 1.1

Similarly, if we cross the main cardioid at points with rotation number $\theta = 1/3$ or $2/3$, we observe the tripling bifurcation and the corresponding hyperbolic components attached to the main cardioid at the bifurcation points. The corresponding filled Julia sets are usually referred to as *Douady rabbits.*

In general, if we cross the cardioid at a point with rotation number $\theta = q/p$, we observe a bifurcation of birth of an attracting cycle of period p. The domain where this cycle preserves its attractiveness is a hyperbolic domain attached to the main cardioid at the bifurcation point. These components are called *satellite hyperbolic components* of $\operatorname{int} \mathbb{M}$.

Similarly, the boundary of any satellite component contains a dense set of parabolic bifurcation points where higher order attracting cycles are born, etc. In this way we obtain an infinite tree of hyperbolic components which are born through a series of the above bifurcations. They, however, do not exhaust all hyperbolic components. For instance, one can see on Figure 1.4 a distinguished component of period 3 which does not appear in this way. This component gives birth to infinitely many new satellite components.

A hyperbolic component is called *primitive* if it is not born via bifurcation from another hyperbolic component. There are infinitely many primitive hyperbolic components in $\mathbb{M}$. On the picture, it is easy to distinguish primitive components from the satellite ones: the former have cusps at the corresponding root points, while the latter are bounded by smooth curves.

4.3. Uniformization of $\mathbb{C} \smallsetminus \mathbb{M}$.

THEOREM 1.19. *The Mandelbrot set M is connected. The Riemann mapping $\Phi : \mathbb{C} \smallsetminus \mathbb{M} \to \mathbb{C} \smallsetminus \bar{\mathbb{D}}$ tangent to* id *at ∞ is explicitly given by the formula*

$$\Phi_{\mathbb{M}}(c) = B_c(c), \tag{1.3}$$

where B_c is the Böttcher function of f_c.

PROOF. First, recall from §3.2 that the expression $B_c(c)$ makes sense for $c \in \mathbb{C} \smallsetminus \mathbb{M}$. Second, by (1.1), it admits the following explicit formula:

$$B_c(c) = \lim_{n\to\infty} p_n(c)^{1/2^n},$$

where $p_n(c) = f_c^n(0)$. This formula implies that $B_c(c)$ is a proper holomorphic map from $\mathbb{C} \smallsetminus \mathbb{M}$ onto $\mathbb{C} \smallsetminus \bar{\mathbb{D}}$. Moreover, it is tangent to id at ∞ and hence has degree 1. Hence it is a univalent. □

Formula (1.3) gives a remarkable dynamical meaning to the Riemann mapping $\Phi_{\mathbb{M}}$. Given a quadratic map f_c, $c \in \mathbb{C} \smallsetminus \mathbb{M}$, we can interpret the point c in two different ways: as the parameter of f_c and as the critical value of f_c. Accordingly, we can calculate the uniformizing coordinate of this point by means of the parameter uniformization $\Phi_{\mathbb{M}}$ and by means of the dynamical uniformization B_c. Formula (1.3) shows that both ways lead to the same result!

4.4. Parameter rays and equipotentials. The uniformization $\Phi_{\mathbb{M}}$ allows us to construct two foliations in the complement of the Mandelbrot set. Indeed, consider the foliations by straight rays and round circles on $\mathbb{C} \smallsetminus \bar{\mathbb{D}}$ and transfer them by means of $\Phi_{\mathbb{M}}^{-1}$ to $\mathbb{C} \smallsetminus \mathbb{M}$. The leaves of these foliations are called *parameter rays* and *equipotentials* respectively. As in the dynamical settings, the rays and equipotentials are specified by their external angles and radii respectively. A ray is called *rational* if it has a rational external angle $\theta \in \mathbb{Q}/\mathbb{Z}$.

The dynamical meaning of the parameter rays and equipotentials is not obvious at first glance but formula (1.3) provides it. For instance, a *parameter* $c \in \mathbb{C} \smallsetminus \mathbb{M}$ belongs to a parameter ray of angle θ when the *critical value* c belongs to the dynamical ray of the same angle θ.

As in the case of Julia sets (§3.4), we can now try to understand the structure of the Mandelbrot set by describing how the above foliations degenerate near it. This leads us to the problem of landing of the parameter rays and, via the Carathéodory theorem, to the problem of local connectivity of $\mathbb{M}$. The latter problem turned out to be in the very heart of holomorphic dynamics (see §4.7 and §IV.1.2). Though it is still open, nice landing properties of rational parameter rays are available (compare §3.4):

THEOREM 1.20. *The main cusp $c = 1/4$ is the landing point of the zero parameter ray.*

Any non-zero parameter ray with odd denominator lands at some bifurcation parabolic point of the Mandelbrot set. Moreover, any such point is the landing point of exactly two rational rays with odd denominator.

Any non-zero parameter ray with even denominator lands at some Misiurewicz point.

The two rays landing at some bifurcation parabolic point c divide the parameter plane into two regions. The region W_c that does not contain the origin is called the *wake* of c. The set

$$L_c = \mathrm{cl}(\mathbb{M} \cap W_c) = (\mathbb{M} \cap W_c) \cup \{c\}$$

is called the *limb* of $\mathbb{M}$ at c.

4.5. Ray portraits. Let $\boldsymbol{\alpha}_c$ be a repelling or parabolic cycle of a map f_c. If $J(f_c)$ is connected then according to Theorem 1.10, there are finitely many rational rays landing at this cycle. The configuration of these rays (with external angles attached to them) up to a homeomorphism of the complex plane is called the *ray portrait* of $\boldsymbol{\alpha}_c$. picture We say that this portrait is *non-trivial* if there are different rays in the portrait landing at the same point.

If $J(f_c)$ is disconnected then the foliation of external rays has singularities. Still, we can consider the portraits of rays landing at $\boldsymbol{\alpha}_c$. Note that this portrait can consist of infinitely many rays.

LEMMA 1.21. *If the cycle $\boldsymbol{\alpha}_c$ is repelling and all rays landing at $\boldsymbol{\alpha}_c$ are non-singular, then the ray portrait of $\boldsymbol{\alpha}_c$ is stable under a small perturbation of c.*

Starting with this lemma, one can proceed to a full description of the parameter regions with a given ray portrait:

THEOREM 1.22. *Let $c_0 \in \mathbb{M}$ be a bifurcation parameter and $\boldsymbol{\alpha}_{c_0}$ be the corresponding parabolic cycle. Then there is a unique repelling cycle $\boldsymbol{\alpha}_c$, $c \in W_{c_0}$, holomorphically depending on c such that $\boldsymbol{\alpha}_c \to \boldsymbol{\alpha}_{c_0}$ as $c \to c_0$.*

The ray portrait of $\boldsymbol{\alpha}_c$ is the same as the ray portrait of $\boldsymbol{\alpha}_{c_0}$.

If $c \in \mathbb{M}$ and $\boldsymbol{\alpha}'_c$ is a repelling cycle of f_c with a non-trivial ray portrait, then there is a unique bifurcation point c_0 such that $c \in L_{c_0}$ and $\boldsymbol{\alpha}'_c = \boldsymbol{\alpha}_c$.

In particular, let us consider bifurcation points on the main cardioid:

PROPOSITION 1.23. *Let c_0 be the bifurcation point with rotation number $\theta = q/p$ on the main cardioid C, and let $\boldsymbol{\alpha}_c$, $c \in W_{c_0}$, be the corresponding repelling fixed point in the wake of c_0. Then there are exactly p external rays landing at $\boldsymbol{\alpha}_c$ and these rays are cyclically permuted by f_c with rotation number θ.*

The number θ in the above proposition is called the *combinatorial rotation number* of the fixed point $\boldsymbol{\alpha}_c$.

4.6. Combinatorics and local connectivity of Julia sets. Given a quadratic polynomial f_c with connected Julia set, consider the following equivalence relation $\underset{c}{\sim}$ on the rational circle $\mathbb{Q}/\mathbb{Z}$: $\theta \underset{c}{\sim} \theta'$ if the external rays with angles θ and θ' land at the same point.

Now consider two quadratic polynomials f_c and f_d with connected Julia set that do not have neutral cycles. They are called *combinatorially equivalent* if the corresponding equivalence relations $\underset{c}{\sim}$ and $\underset{d}{\sim}$ coincide.

The equivalence relation $\underset{c}{\sim}$ gives rise to a "combinatorial model" of the Julia set.

Consider the unit disk $\mathbb{D}$ as a model of the hyperbolic plane. Recall that the hyperbolic geodesics in this model are given by arcs of circles orthogonal to the unit circle $\mathbb{T}$.

Two pairs of points θ_1, θ_2 and t_1, t_2 on $\mathbb{T}$ are called *linked* if the hyperbolic geodesics $\gamma_{\theta_1,\theta_2}$ and γ_{t_1,t_2} intersect in $\mathbb{D}$. In this case *any two arcs* δ_1 and δ_2 in $\mathbb{D}$ ending respectively at θ_1, θ_2 and t_1, t_2 *must intersect.* Two subsets X and Y on the unit circle are called linked if they contain two pairs of linked points $\theta_1, \theta_2 \in X$ and $t_1, t_2 \in Y$.

Let us now extend the equivalence relation $\underset{c}{\sim}$ to the whole unit circle by declaring two angles θ and t equivalent if there are two sequences of pairwise equivalent rational angles $\theta_n \to \theta$ and $t_n \to t$. We will use the same notation $\underset{c}{\sim}$ for the extended equivalence relation. Note that the corresponding equivalence classes are closed. One can show that any two of these classes are unlinked. clear??

A *convex hull* $h(X)$ of a closed set $X \subset \mathbb{T}$ is the union of all geodesics joining various pairs of points of X. It is closed and $h(X) \cap \mathbb{T} = X$.

Consider now convex hulls $h(X)$ of all equivalence classes $X \subset \mathbb{T}$ of $\underset{c}{\sim}$. Since the equivalence classes are pairwise unlinked, the corresponding convex hulls are pairwise disjoint. Thus, the complex plane is partitioned into a disjoint union of these convex hulls and single points. Let us think of this partition as an equivalence relation on $\mathbb{C}$ that extends $\underset{c}{\sim}$. We will keep the same notation for the extended relation.

THEOREM 1.24. *The quotient* $\mathbb{C}/\underset{c}{\sim}$ *is homeomorphic to the plane* $\mathbb{R}^2$. *There exists a natural continuous surjection* $j_c : K_c \to \bar{\mathbb{D}}/\underset{c}{\sim}$. *Moreover,* j_c *is a homeomorphism if and only if* K_c *is locally connected.*

The set $\bar{\mathbb{D}}/\underset{c}{\sim}$ is called the *combinatorial model* for the filled Julia set K_c. Note that this set is always locally connected, so that the "only if" statement in the last theorem is obvious. Clearly, combinatorially equivalent quadratic maps have the same model.

It is instructive to understand fibers of the projection j_c in Theorem 1.24. To this end, let us introduce a couple of terms. A pair of rational external rays of f landing at the same (periodic or pre-periodic) point is called a *dividing pair of rays* (since it divides the plane into two regions). A *puzzle piece* of f is a closed Jordan disk bounded by finitely many dividing pairs of rays and arcs of equipotentials.

It is clear that two points z and ζ of J_c belong to different fibers of the projection j_c if there is a dividing pair of rays which separates z from ζ. This leads to the following statement:

LEMMA 1.25. *Assume* $c \in M$ *is purely repelling. Then the fiber of* j_c *containing* $z \in J_c$ *is the intersection of all puzzle pieces around* z.

Hence, local connectivity of the Julia set J_c at some point $z \in J_c$ is equivalent to shrinking to z of the puzzle pieces around z. This observation is a powerful key to the problem of local connectivity of Julia sets (see §III.3.5).

Let us finish with a statement which directly relates local connectivity to shrinking of puzzle pieces:

LEMMA 1.26. *For any puzzle piece P for f_c ($c \in M$), the intersection $P \cap K_c$ is connected.*

PROOF. The intersection $P \cap K_c$ consists of finitely many points α_k. Moreover, for any connected component X of $K_c \setminus P$, the closure $\bar{X}$ touches P at a single point α_k.

Assume $P \cap K_c$ can be decomposed into two disjoint closed subsets Q^1 and Q^2. Let K^i be the union of Q^i and those components of $K_c \setminus P$ that touch Q^i. Then the K^i are disjoint closed sets such that $K_c = K^1 \cup K^2$ contradicting connectivity of K_c. □

4.7. MLC and the combinatorial model of $\mathbb{M}$. We will now carry a similar construction in the parameter plane. Starting with rational parameter rays, we can define an equivalence relation $\underset{M}{\sim}$ on the rational circle $\mathbb{Q}/\mathbb{Z}$, and then promote it to an equivalence relation on the whole parameter plane (denoted in the same way). Similarly to the dynamical situation, it gives us a combinatorial model for the Mandelbrot set:

THEOREM 1.27. *The quotient $\mathbb{C}/\underset{\mathbb{M}}{\sim}$ is homeomorphic to the plane $\mathbb{R}^2$. There exists a natural continuous surjection $j_{\mathbb{M}} : \mathbb{M} \to \bar{\mathbb{D}}/\underset{M}{\sim}$. Moreover, the fiber F^c of $j_{\mathbb{M}}$ at c is a single point if and only if $\mathbb{M}$ is locally connected at c. Thus, $j_{\mathbb{M}}$ is a homeomorphism if and only if $\mathbb{M}$ is locally connected.*

Thus, if the Mandelbrot set was locally connected, we would have an explicit topological model of it. There is a general belief that this is indeed the case:

MLC Conjecture. *The Mandelbrot set is locally connected.*

We can proceed further with the definition of *parameter puzzle pieces* (or briefly, *parapuzzle* pieces) which is completely analogous to the above definition of dynamical puzzle pieces but uses rational parameter rays instead of the dynamical rays.

LEMMA 1.28. *Assume $c \in \mathbb{M}$ is purely repelling. Then the fiber F^c of $j_{\mathbb{M}}$ is the intersection of all puzzle pieces containing c.*

Hence, local connectivity of the Mandelbrot $\mathbb{M}$ at some point $c \in \partial\mathbb{M}$ which does not belong to the boundary of a hyperbolic component is equivalent to shrinking to c of the parameter puzzle pieces around c. Also, similarly to Lemma 1.26 we have:

LEMMA 1.29. *For any parameter puzzle piece P, the intersection $P \cap \mathbb{M}$ is connected.*

Let us finally mention that in the above discussion (in both dynamical and parameter settings) it is sufficient to work only with rational rays with odd or even denominators.

5. Real quadratic family

5.1. Real bifurcations. According to Exercise 1.12, the real slice of the Mandelbrot set is the interval $[-2, 1/4]$. Moreover, for $c \in [-2, 1/4]$, the map f_c has an invariant interval I_c, and we will be much interested in the dynamics on this interval. It is illustrated on Figure 1.5

Let us consider the open set of hyperbolic parameters $c \in [-2, 1/4]$. Connected components of this set are called *hyperbolic windows*. On Figure 1.5 one can see quite a few hyperbolic windows but of course there are infinitely many invisible ones. The first hyperbolic window is the interval $W_0 = (-3/4, 1/4)$ where there exists an attracting fixed point α_c (in other words, this window is the real slice of the hyperbolic component H_0 bounded by the main cardioid, see §4.2).

After the first doubling bifurcation at point $c = -3/4$ the attracting cycle of period 2 is born and we observe the corresponding hyperbolic window $[-5/4, -3/4]$. This is the beginning of the cascade of doubling bifurcations and the corresponding sequence of satellite hyperbolic windows of periods 2^n. These windows converge to the *Feigenbaum point* $c_F = -1.401\ldots$.

Another distinguished window visible on the picture corresponds to period 3. It originates its own cascade of doubling bifurcations and satellite windows. In fact. on the real line only doubling bifurcations can occur, so that other satellite components are not presented over there.

If we remove all hyperbolic windows then we are left with a closed set of non-hyperbolic maps. Its connected components which are not singletons are called *queer intervals*. We will show in §IV.4.4 that in fact they do not exist, so that the hyperbolic windows are dense in $[-2, 1/4]$.

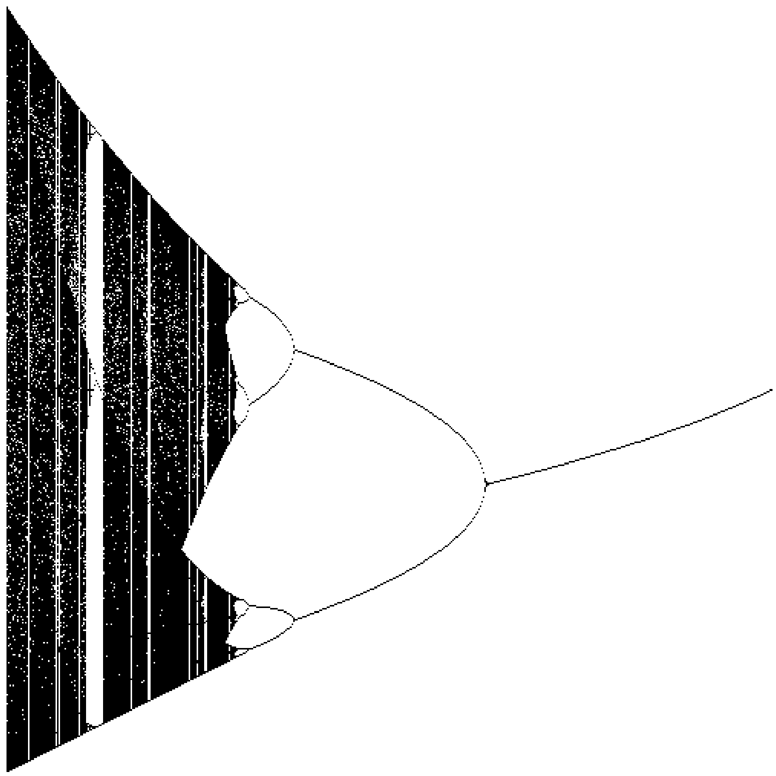

FIGURE 1.5. Real quadratic family. This picture presents how the limit set of the orbit $\{f_c^n(0)\}_{n=0}^{\infty}$ bifurcates as the parameter c changes from $1/4$ on the right to -2 on the left. Gaps in the black regions represent hyperbolic windows. In the beginning (on the right) you can see the cascade of doubling bifurcations. This picture became symbolic for one-dimensional dynamics

5.2. S-unimodal maps. Since the iterates of quadratic polynomials are not quadratic any more, we need to consider a bigger class of interval maps.

Let $I = [-\beta, \beta]$ be a 0-symmetric interval and $f : (I, \partial I) \to (I, \partial I)$ be a continuous even map. Assume that f is strictly monotone on each component $I^{\pm}$ of $I \smallsetminus \{0\}$ and has a single extremum at 0. Such a map is called *unimodal*. Assume for definiteness that 0 is the minimum, and hence β is a fixed point of f. Let $c = f(0)$.

The theory of unimodal maps is particularly complete under some regularity assumptions. A unimodal map is called *S-unimodal* if it is three times differentiable, does not have critical points except 0, and has negative Schwarzian derivative:

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 < 0.$$

We will also include into the definition of "S-unimodal" the assumptions that 0 is non-degenerate (i.e., $f''(0) \neq 0$) and that the fixed point β is not attracting: $f'(\beta) \geq 1$ (the last assumption allows one to avoid some boring subtleties).

Note that all the above conditions are satisfied for quadratic maps $f_c : I_c \to I_c$, $c \in [-2, 1/4]$.

The Schwarzian derivative satisfies the chain rule

$$S(f \circ g) = (Sf \circ g)(g')^2 + Sg,$$

which shows that the condition of negative Schwarzian derivative is preserved under iterates.

Remark. Of course, iterates f^n of unimodal maps are not unimodal any more. However, by restricting f^n to an appropriate interval we sometimes can recover unimodality. This observation is the beginning of the Renormalization Theory which will be a central theme of these lectures.

In many respects, S-unimodal maps are similar to quadratic polynomials:

THEOREM 1.30. *If an S-unimodal map $f : I \to I$ has a non-repelling cycle $\boldsymbol{\alpha}$ then* 0 *belongs to its immediate basin. Hence f can have at most one non-repelling cycle.*

An open interval $J \subset I$ is called *homterval* if all iterates f^n, $n = 0, 1, \ldots$ are homeomorphisms on J. An interval J is called *wandering* if all intervals $f^n J$, $n = 0, 1, \ldots$ are pairwise disjoint. A wandering interval (resp., homterval) is considered to be *trivial* if its orbit converges to a non-repelling cycle.

EXERCISE 1.31. If J is a wandering interval then $f^N J$ is a homterval for some $N \in \mathbb{Z}_+$. Vice versa, any non-trivial homterval is wandering.

THEOREM 1.32. *S-unimodal maps do not have non-trivial homtervals/wandering intervals.*

5.3. Kneading Theory. The Kneading Theory provides a complete combinatorial classification of unimodal maps.

Let $\boldsymbol{\kappa} = \{\kappa_n\}_{n=1}^N$, $N \in \mathbb{Z}_+ \cup \infty$, be either an infinite sequence of ± 1's, or a finite sequence of ± 1's except for the last entry κ_N which is equal to 0. Let $\mathcal{S}$ be the space of all such sequences, and let σ be the *shift* which acts on a sequence $\boldsymbol{\kappa}$ (different from $\{0\}$) by forgetting the first symbol.

Consider now a unimodal map $f : I \to I$. To any point $x \in I$ we can associate its *itinerary* $\boldsymbol{\kappa}(x) \equiv \boldsymbol{\kappa}_f(x) \in \mathcal{S}$ in the following way. If x is not precritical then $\boldsymbol{\kappa}$ is an infinite sequence of ± 1's defined according to the rule $f^n(x) \in I^{\kappa_n}$. If $f^N x = 0$ for some $N \in \mathbb{Z}_+$, then select the smallest N with this property and let $\boldsymbol{\kappa}$ be the sequence of length N such that $\kappa_N(x) = 0$, while the other entries of $\boldsymbol{\kappa}$ are defined according to the same rule as in the first case.

EXERCISE 1.33. Show that $\boldsymbol{\kappa}(x) = \boldsymbol{\kappa}(y)$, $x \neq y$, if and only if x and y belong to the same homterval.

The order on the interval I translates into a natural order on $\mathcal{S}$ which is a modification of the lexicographic order taking into account that f is orientation-reversing on I^-. Given two sequences $\boldsymbol{\kappa}$ and $\tilde{\boldsymbol{\kappa}}$ in $\mathcal{S}$, let N be the first moment when their entries differ. Let l be the number of -1's among the previous entries. Then $\boldsymbol{\kappa} > \tilde{\boldsymbol{\kappa}}$ if either l is even and $\kappa_N > \tilde{\kappa}_N$, or l is odd and $\kappa_N < \tilde{\kappa}_N$.

EXERCISE 1.34. Show that $x \leq y$ if and only if $\boldsymbol{\kappa}(x) \leq \boldsymbol{\kappa}(y)$

The *kneading sequence* $\boldsymbol{\kappa}(f)$ of f is the itinerary of the *critical value*, i.e., $\boldsymbol{\kappa}(f) = \boldsymbol{\kappa}_f(c)$.

PROPOSITION 1.35. *The kneading sequence is periodic if and only if f has an attracting (but not superattracting) cycle. If p is the period of the cycle then the period of $\boldsymbol{\kappa}(f)$ is either p or $2p$ depending on whether the multiplier of the cycle is positive or negative.*

EXERCISE 1.36. Consider a one-parameter family of unimodal maps f_c which passes through a superattracting parameter $*$ in a generic way, so that the multiplier of the attracting cycle changes the sign. Compare the kneading sequences just before and after passing through $*$. Show that the kneading sequence corresponding to the negative multiplier is bigger than the other one.

EXERCISE 1.37. Consider a one-parameter family of unimodal maps f_c which does not contain superattracting parameters on some parameter interval J. Then the kneading sequence $\boldsymbol{\kappa}(c) \equiv \boldsymbol{\kappa}(f_c)$ is constant over J.

In other words, kneading sequence can change only at superattracting parameter values (in particular, it does not change at the moments of doubling bifurcation!):

Proposition 1.38. *Let f_c be a smooth family of S-unimodal maps. If $\boldsymbol{\kappa}(c_1) \neq \boldsymbol{\kappa}(c_2)$ for some parameters c_1 and c_2 then there is a superattracting parameter $a \in [c_1, c_2]$.*

This result can also be deduced from the Implicit Function Theorem (stated below) and density of finite kneading sequences:

Exercise 1.39. For any two kneading sequences $\boldsymbol{\kappa} < \tilde{\boldsymbol{\kappa}}$, there is a finite kneading sequences $\boldsymbol{\mu}$ such that $\boldsymbol{\kappa} \leq \boldsymbol{\mu} \leq \tilde{\boldsymbol{\kappa}}$.

The following result shows that the kneading sequence provides an essentially complete topological invariant for S-unimnodal maps:

Theorem 1.40. *Let f and $\tilde{f}$ be two S-unimodal maps with the same kneading sequence $\boldsymbol{\kappa}$.*
(i) If $\boldsymbol{\kappa}$ is aperiodic (finite or infinite) then f and $\tilde{f}$ are topologically equivalent.
(ii) Assume $\boldsymbol{\kappa}$ is periodic. If the non-repelling periodic points of f and $\tilde{f}$ have the same period and are either both attracting or both parabolic, then f and $\tilde{f}$ are topologically conjugate.

Proof. The idea is that the kneading sequence of f allows one to build up a *combinatorial model* of the map. This model is provided by the shift σ acting on the interval $[\boldsymbol{\kappa}(f), \sigma(\boldsymbol{\kappa}(f))] \subset \mathcal{S}$. One can show that the map $\pi : x \mapsto \boldsymbol{\kappa}(x)$ is a (*surjective*) semi-conjugacy between f on $[c, f(c)]$ and the model. By Exercise 1.33, homtervals make the difference between the map and its model. But the No Wandering Intervals Theorem (1.32) implies that in case (i), π is in fact injective. (Case (ii) requires a more careful treatment.) □

A sequence $\kappa \in \mathcal{S}$ is called *admissible* if $\kappa \leq \sigma^n(\kappa) \leq \sigma(\kappa)$ for all $n \in \mathbb{N}$. Let $\mathcal{K}$ stand for the space of admissible sequences.

Proposition 1.41. *A sequence κ is admissible if and only if it is the kneading sequence of some unimodal map.*

Note that one direction of this statement is obvious: Since the interval $[c, f(c)]$ is f-invariant, $c \leq f^n c \leq f(c)$, $n = 2, 3, \ldots$, which implies admissibility of $\kappa(f)$.

In what follows, we will also refer to admissible sequences as "kneading sequences".

Exercise 1.42. Consider a finite sequence $\boldsymbol{\kappa} \in \mathcal{S}$ of length N. Mark some points $c_1, \ldots, c_{N-1}, c_N = 0$ on $\mathbb{R}$ which are ordered on $\mathbb{R}$ in the same way as the sequences $\boldsymbol{\kappa}, \sigma(\boldsymbol{\kappa}), \ldots, \sigma^{N-1}(\boldsymbol{\kappa}) = \{0\}$. Consider a piecewise linear function T on the convex hull L of these points such that $T(c_k) = c_{k+1}$ (where k is considered mod N). Show that $\boldsymbol{\kappa}$ is admissible if and only if 0 is a minimum point which is the only extremum of T in $\operatorname{int} L$. Prove Proposition 1.41 for finite sequences.

In general case, Proposition 1.41 follows from the following main result of the kneading theory:

Intermediate Value Theorem. *Let f_c be a smooth family of S-unimodal maps. If two kneading sequences $\kappa_1 < \kappa_2$ are realizable in this family, then any intermediate kneading sequence $\kappa \in [\kappa_1, \kappa_2]$ is realizable as well.*

EXERCISE 1.43. Show that the kneading sequence

$$\boldsymbol{\kappa}_{\min} = (-1, -1, -1, \dots)$$

(corresponding to the parabolic map $f_{1/4} : x \mapsto x^2 + 1/4$) is minimal among all kneading sequences. The kneading sequence

$$\boldsymbol{\kappa}_{\max} = (-1, 1, 1, \dots)$$

(corresponding to the Ulam-Neumann map $f_{-2} : x \mapsto x^2 - 2$) is maximal.

A smooth family f_c of S-unimodal maps is called *full* if *all* admissible kneading sequences are realized in this family. By the Intermediate Value Theorem, any family containing $\boldsymbol{\kappa}_{\min}$ and $\boldsymbol{\kappa}_{\max}$ is full. In particular, we have:

THEOREM 1.44. *The real quadratic family f_c, $c \in [-2, 1/4]$, is full.*

By Theorem 1.40, any S-unimodal map is topologically equivalent to some quadratic polynomial, so that the quadratic family is "representative" in this topological sense. Remarkably, the quadratic family is in fact representative in much stronger geometric sense. This will be one of most important lessons of this course.

For quadratic polynomials with real parameters, the kneading sequence contains in a concise way the same amount of information as the combinatorial model described in §4.6.

PROPOSITION 1.45. *Consider two complex quadratic polynomials f_c and $f_{\tilde{c}}$ with real parameter values $c, \tilde{c} \in [-2, 1/4]$ and aperiodic kneading sequence (maybe, finite). Then f_c and $f_{\tilde{c}}$ are combinatorially equivalent if and only if they have the same kneading invariant.*

EXERCISE 1.46. Study the relation between combinatorial equivalence and kneading invariant in the periodic case.

5.4. Structural stability. A real quadratic map f_* (and the corresponding parameter $* \in [-2, 1.4]$) is called *structurally stable* (in the real quadratic family) if it is topologically conjugate to all nearby maps is this family. By Theorem 1.40 (ii), hyperbolic but not superattracting maps are structurally stable. In fact, the kneading theory yields a much better result:

THEOREM 1.47. *Structurally stable maps are dense in the real quadratic family.*

PROOF. Let us show that any parameter interval $J \subset [-2, 1/4]$ contains a structurally stable parameter. If J contains a hyperbolic parameter then it contains a non-superattracting hyperbolic parameter as well, and we are done. If it does not, then $\boldsymbol{\kappa}(c)$ is constant on J by Proposition 1.38. Then all maps on J are structurally stable by Theorem 1.40. ☐

6. Bibliographical notes

A number of introductory sources in holomorphic dynamics (in the generality of rational endomorphisms of the Riemann sphere) are now available: see books by Milnor [**M1**] and Carleson & Gamelin [**CG**], or an earlier survey [**L1**].

Classical material of §§1-2 is mostly due to Fatou and Julia, preceded by Königs, Schröder, Leau, and Böttcher and complemented by Pfeiffer, Cremer, and Siegel (see the above sources for precise references and [**Al**] for a curious account of the early history of the field). The sharp Diophantine condition that distinguish Siegel points from Cremer ones was established by Bruno and Yoccoz (see [**D3**]). The sharp estimate on the number of non-repelling cycles (Theorem 1.6) was given in [**D1**] (see also Shishikura [**Sh1**] for a more general result).

A rough picture of the "Mandelbrot set" first appeared in [**BM**]. A global interest to this set was sparked by [**Ma**].

Study of the Julia sets and the Mandelbrot set (§§3-2) by means of external rays and equipotentials was originated by Douady and Hubbard in the fundamental but unpublished Orsay Notes [**DH1**]. This study was further refined in many papers, see in particular [**GM, Ki, M4, Sc2**].

Connectivity of the Mandelbrot set (§4.3) was proven by Douady & Hubbard [**DH1**]. The MLC Conjecture was formulated by the same authors [**DH1**]. Combinatorial models for Julia sets and the Mandelbrot set in terms of geodesic laminations (§§4.6-4.7) was given by Thurston [**Th1**].

Basic introductory sources in the dynamics of real unimodal maps (§5) are the books by Collet & Eckmann [**CE**] and de Melo & van Strien [**MS**].

A preliminary version of the kneading theory (§5.3) appeared in [**MeSS**]. It was systematically developed by Milnor & Thurston [**MT**].

Cascade of doubling bifurcations was discovered by Myrberg [**My**].

LECTURE 2

Quadratic-like maps and renormalization

One of the most fascinating feature of the Mandelbrot set is that it contains a lot of little copies of itself. Similarly, the parameter interval $[-2, 1/4]$ contains a lot of subintervals with the same dynamical structure as the original interval. These phenomenona can be explained by means of complex and real renormalization theory.

1. Quadratic-like maps

1.1. First definitions. To build up the renormalization theory, it is important to enlarge the one-parameter quadratic family. The real quadratic family is naturally enlarged to a space of unimodal maps. Quadratic-like maps are complex analogues of unimodal maps.

DEFINITION 2.1. A *quadratic-like map* is a holomorphic branched double covering $f : U \to U'$ between two topological disks such that $U \Subset U'$. The point of this definition is that the domain U is not invariant under the map, for otherwise the dynamics would be quite trivial. For technical reasons, we will assume that the domains U and U' are bounded by piecewise smooth curves without cusps.

The topological annulus $U' \setminus \bar{U}$ is called the *fundamental annulus* of f. The *filled Julia set* of f is the set of non-escaping points:

$$K(f) = \{z : \ f^n z \in U, \ n = 0, 1, 2, \dots\}.$$

The *Julia set* of f, $J(f)$, is the boundary of $K(f)$. (Sometimes, when it cannot lead to a confusion, we call $K(f)$ the "Julia set" as well.)

Any quadratic-like map has a single critical point. Unless otherwise is specified, we will assume that the map is normalized so that this point is located at the origin.

Restricted quadratic polynomials, $f_c : \ f^{-1}(\mathbb{D}_r) \to \mathbb{D}_r$, where $|c| < r$, provide examples of quadratic-like maps. Such quadratic-like maps are briefly called "quadratic polynomials".

Similarly to the polynomial case, we have:

THEOREM 2.1 (Basic Dichotomy). *The Julia set (resp., filled Julia set) of a quadratic-like map f is either connected or Cantor. It is connected if and only if the critical point is non-escaping: $0 \in K(f)$.*

A quadratic-like map f is called *real* if the domains U and U' are symmetric with respect to the real line and $f(U\cap\mathbb{R})\subset\mathbb{R}$. The restriction of such an f to the real line is unimodal.

At this point we can ask ourselves whether the topological dynamics of quadratic-like maps can be at all different from that of quadratic polynomials. It turns out that it is not the case: any quadratic-like map is topologically conjugate to a restricted quadratic polynomial. Moreover, this conjugacy has nice regularity properties. To state this result, we need a brief account of the theory of quasiconformal maps.

2. Quasiconformal maps

It is a remarkable class of maps fine enough to be a subject of analysis and at the same time rough enough to describe fractal objects. The formal definition follows.

A homeomorphism $h: U\to V$ between open subsets of the complex plane is called *quasiconformal* (abbreviated as "qc") if
- It has distributional partial derivatives ∂h and $\bar\partial h$ of class L^1_{loc} (and hence it is differentiable a.e. in the classical sense);
- There exists $k\in[0,1)$ such that $|\bar\partial h(z)|\le k|\partial h(z)|$ for a.e. $z\in U$.

The differential (-1,1)-form

$$\mu=\frac{\bar\partial h(z)}{\partial h(z)}\frac{d\bar z}{dz} \tag{2.1}$$

is called the *Beltrami differential* associated with the map h. To see its geometric meaning, let us consider the family of infinitesimal circles $C_\zeta\subset T_\zeta V$ over V. Their pullbacks $E_z=h^{-1}(C_{hz})$ form a measurable family of infinitesimal ellipses defined almost everywhere on U. The eccentricity of E_z is equal to $(1+|\mu(z)|)/(1-|\mu(z)|)$, while the direction of the small axis of E_z is equal to $\arg\mu(z)/2 \mod\pi$. Thus, the second property on the definition of qc maps means that the ellipses E_z have essentially bounded dilatation.

The *dilatation* of a qc map, $\operatorname{Dil}(h)$, is the essential supremum of the eccentricities of the associated infinitesimal ellipses.

A measurable field of infinitesimal ellipses on $U\subset\mathbb{C}$ considered up to similarity is called a *conformal structure* on U. The *dilatation* of the structure is the essential supremum of the eccentricities of the ellipses. To any conformal structure one can associate a measurable Beltrami differential $\mu(z)\bar\partial z/\partial z$ whose modulus and argument are related to the eccentricity and orientation of the infinitesimal ellipses by the rules described above. A conformal structure has bounded dilatation if and only if the L^∞-norm of the corresponding Beltrami differential is less than 1.

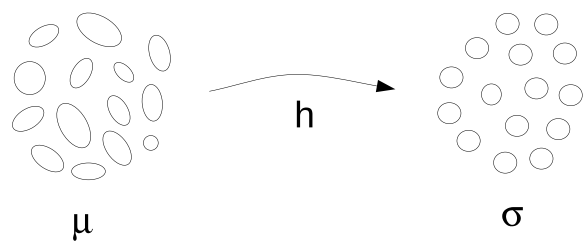

FIGURE 2.1. Pullback conformal structure.

The *standard* conformal structure on U is given by the field of infinitesimal circles. It corresponds to the vanishing Beltrami differential and will be denoted by σ.

This discussion of qc maps, Beltrami differentials and conformal structures can be generalized to the case of Riemann surfaces, in particular, to the Riemann sphere $\bar{\mathbb{C}}$. Conformal structures and measurable Beltrami differentials give us respectively geometric and analytic description of the same object. We will freely pass from one language to the other.

Quasiconformal maps act on conformal structures by pullbacks h^*. In particular, any qc map $h : U \to V$ generates a conformal structure $\mu = h^*\sigma$ on U with bounded dilatation satisfying the Beltrami equation (2.1) (see Figure 2.1). One of the most remarkable facts of analysis is that this statement can be reversed:

Measurable Riemann Mapping Theorem. *For any conformal structure μ with bounded dilatations on the Riemann sphere $\bar{\mathbb{C}}$, there exists a qc map $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that $\mu = h^*\sigma$. This map is unique up to post-composition with a Möbius transformation $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$.*

In the analytic language, μ is a Beltrami differential on $\bar{\mathbb{C}}$ in the unit ball of L^∞, and h is a solution of the Beltrami equation (2.1).

The uniqueness part of the theorem is the consequence of the following result:

Weyl's Lemma. *If a qc map h satisfies almost everywhere the Cauchy-Riemann equation $\bar{\partial} h = 0$, then h is conformal.*

Note that the notion of a conformal structure with bounded dilatation makes perfect sense on a *quasiconformal surface* (in particular, on a compact smooth surface). In this context the Measurable Riemann

Mapping Theorem can be formulated as follows: *Any conformal structure μ with bounded dilatation on a quasiconformal surface S admits a compatible complex structure on S.*

A map f is called *quasiregular* if it is a composition of a holomorphic map and a qc map. Quasiregular maps also act on conformal structures by means of pull-backs.

Let us now formulate two more fundamental properties of qc maps which play a crucial role in dynamics:

Compactness Lemma. *The space of K-qc maps $h : \mathbb{C} \to \mathbb{C}$ normalized by $h(0) = 0$ and $h(1) = 1$ is compact in the uniform topology on the Riemann sphere.*

A *quasiarc/quasicircle* is the image of an interval/circle under a qc map.

Gluing Lemma. *Let D_1 and D_2 be two disjoint domains. Let γ be a quasiarc such that $\gamma = \partial D_1 \cap \partial D_2 \cap U$ for some domain U . Let $D = D_1 \cup D_2 \cup \gamma$. If $h : D \to \mathbb{C}$ is a homeomorphism such that $h|D_i$ is K-qc, then h is K-qc.*

3. Straightening and hybrid classes

3.1. Hybrid equivalence. In accordance with general terminology introduced in the preamble, two quadratic-like maps $f : U \to U'$ and $g : V \to V'$ are called *topologically equivalent*, or *topologically conjugate*, if there exists a homeomorphism $h : U' \to V'$ which makes commutative the following diagram:

$$\begin{array}{ccc} U & \xrightarrow[f]{} & U' \\ h\downarrow & & \downarrow h \\ V & \xrightarrow[g]{} & V' \end{array}$$

The classes of topologically equivalent maps are naturally called topological classes. If h satisfies an additional regularity property (say, quasiconformality/smoothness/conformality), then we say that f and g are *qc/smooth/conformally equivalent* or *conjugate*. The corresponding equivalence classes are called qc/smooth/conformal classes.

The notion of qc equivalence admits the following useful refinement. Two quadratic-like maps are called *hybrid equivalent* if they are qc conjugate by a map h satisfying the Cauchy-Riemann equation $\bar\partial h(z) = 0$ a.e. on the filled Julia set $K(f)$. Note that by Weyl's Lemma, such an h is in fact conformal on $\operatorname{int} K(f)$. On the Julia set $J(f)$ this

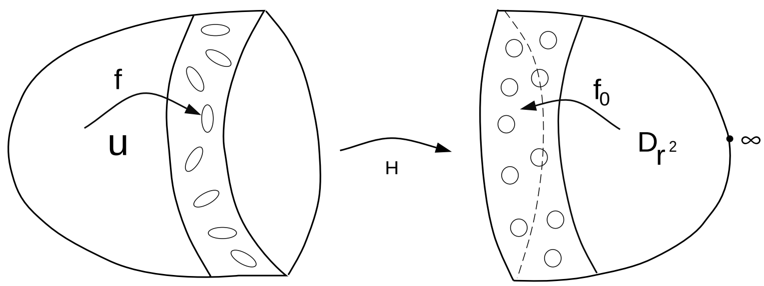

FIGURE 2.2. Gluing two hemi-spheres.

condition is automatically satisfied provided $J(f)$ has zero Lebesgue measure (and until now there are no examples of Julia sets of positive measure).

The hybrid classes in the connectedness locus can be endowed with the following *Sullivan's Teichmüller metric*:

$$\operatorname{dist}_T(f,g) = \inf_h \log \operatorname{Dil}(h), \tag{2.2}$$

where $h : \mathbb{C} \to \mathbb{C}$ runs over all qc maps which are hybrid conjugacies between f and g near their filled Julia sets.

3.2. Straightening Theorem.

THEOREM 2.2. *Any quadratic-like map $f : U \to U'$ is hybrid equivalent to a restricted quadratic polynomial f_c. If the Julia set $J(f)$ is connected then $c = \chi(f) \in \mathbb{M}$ is uniquely determined by f.*

PROOF. We will give an idea of the proof of the existence part of the theorem which is a nice illustration of the efficiency of the Measurable Riemann Mapping Theorem. Take any $r > 1$. View the topological disk $\bar{U}'$ and the round disk $\mathbb{C} \smallsetminus \mathbb{D}_r$ as two hemi-spheres endowed with dynamics $f : \bar{U} \to \bar{U}'$ and $f_0 : \mathbb{C} \smallsetminus \mathbb{D}_r \to \mathbb{C} \smallsetminus \mathbb{D}_{r^2}$ respectively (see Figure 2.2). Let us glue them by means of a qc map

$$H : \bar{U}' \smallsetminus U \to \bar{\mathbb{D}}_{r^2} \smallsetminus \mathbb{D}_r \tag{2.3}$$

which is equivariant on the boundary of the annuli, i.e. such that $H(fz) = f_0(Hz)$ for $z \in \partial U$. We obtain a qc sphere S endowed with a double covering F such that $F^{-1}(\infty) = \infty$, which can be viewed as a "topological polynomial".

Now we wish to turn this topological polynomial into a genuine one. To this end we will endow S with an F-invariant conformal structure μ. On the hemi-sphere $\mathbb{C} \smallsetminus \mathbb{D}_r$ let $\mu = \sigma$ be the standard structure. Transfer it to the fundamental annulus $U \smallsetminus \bar{U}'$ by means of H, and then pull it back to the preimages of this annulus by the iterates of

f. We obtain an f-invariant conformal structure on $U' \setminus K(f)$. Since conformal pull-backs preserve the dilatation of the structure, it has a bounded dilatation. Extend this structure to $K(f)$ as the standard one.

It provides us with an F-invariant conformal structure μ on S with bounded dilatation. By the Measurable Riemann Mapping Theorem, there exists a qc map $h : (S, \infty) \to (\bar{\mathbb{C}}, \infty)$ such that $\mu = h^*\sigma$. Then the map $P = h \circ F \circ h^{-1}$ is a quasi-regular double covering of the Riemann sphere preserving the standard conformal structure. By Weil's Lemma, it is holomorphic and hence is a quadratic polynomial. Normalizing h appropriately, we can bring P to a normal form $z \mapsto z^2 + c$.

Note that this construction proceeds canonically, once we have selected the qc map H (2.3) uniformizing the fundamental annulus. The point of the uniqueness part of the theorem is that in the connected case this choice H is irrelevant. (In the disconnected case it is quite relevant since all quadratic polynomials with disconnected Julia set are in fact hybrid equivalent.) □

The map H (2.3) is called a *tubing* of f. Note that the dilatation of the map h conjugating f to P is equal to the dilatation of the tubing.

Definition 2.2. Given a quadratic-like map f with connected Julia set, the quadratic-like map $\chi(f) \equiv f_{\chi(c)}$, $c \in \mathbb{M}$, which is hybrid equivalent to f is called the *straightening* of f.

Since quadratic-like maps are topologically equivalent to quadratic polynomials, they have a similar classification of the fixed points (see §I.3.5). If the Julia set $J(f)$ is connected then one of these points, called β, is a non-dividing repelling or parabolic with multiplier 1. Another one, called α, is either non-repelling or a dividing repelling point.

3.3. Distortion. To study further the geometry of quadratic-like maps, we need to recap some fundamental properties of univalent functions.

Let ϕ be a conformal diffeomorphism $U \to U'$ between two domains in the complex plane, and let X be a subset of its domain. The *distortion* of ϕ on X is defined as follows:

$$\mathrm{Dist}(\phi|X) = \sup_{x,y \in X} \frac{|D\phi(x)|}{|D\phi(y)|}.$$

The *nonlinearity* of ϕ on X is:

$$n(\phi|X) = \sup_{x \in X} \frac{|\phi''(x)|}{|\phi'(x)|}.$$

The nonlinearity can be bounded in terms of distortion on slightly bigger set, and vice versa.

Given a pointed topological disk (D, a) in the complex plane, let $R_{D,a}$ and $r_{D,a}$ stand respectively for the *outer* and *inner radii* of D relative a (i.e.. the radii of the disks centered at a which are respectively circumscribed around and inscribed in D).

Koebe Distortion Theorem. *Let $\phi : (\mathbb{D}, 0) \to (D, a)$ be a conformal map, and let $r \in (0, 1)$, $D_r = \phi(\mathbb{D}_r)$. Then there exist constants $K = K(r)$ and $L = L(r)$ (independent of a particular ϕ!) such that*

$$\mathrm{Dist}(\phi|\mathbb{D}_r) \leq K(r)$$

and

$$L(r)^{-1}|\phi'(0)| \leq r_{D_r,a} \leq R_{D_r,a} \leq L(r)\,|\phi'(0)|.$$

3.4. Geometry of quadratic-like maps. There are two ways to control the geometry of a quadratic-like map: by the dilatation of the map h conjugating f to its straightening f_c and by distortion the "diffeo part" of the map. It turns out that both methods depend on the modulus of the fundamental annulus of f, $\mathrm{mod}(U' \smallsetminus U)$.

Any conformal annulus A can be uniformized by one of the following standard models: a round annulus $\mathbb{A}(1, r)$, $r > 1$, the punctured disk $\mathbb{D}^*$, or the punctured plane $\mathbb{C}^*$. In the first case one let $\mathrm{mod}\, A = \log r$, while in the two other cases one let $\mathrm{mod}\, A = \infty$. In the first case, the modulus is the only conformal invariant of the annulus.

Any quadratic-like map $f : U \to U'$ can be decomposed as $\phi \circ f_0$, where $f_0(z) = z^2$ and $\phi : f(U) \to U'$ is a conformal diffeomorphism (called the "diffeo part" of f).

To control the geometry of a quadratic-like map, we should allow to shrink its domain a bit (without touching its Julia set). Let us say that a quadratic-like map $g : V \to V'$ is an *adjustment* of a quadratic-like map $f : U \to U'$ if $V \subset U$, $\partial V' \subset \bar{U}' \smallsetminus U$, and $f|V = g$. Obviously, $K(g) = K(f)$.

We say that *the geometry of a quadratic-like map $f : V \to V'$ is (ν, K)-bounded* if $\mathrm{mod}(V' \smallsetminus V) \geq \nu$ and

(i) The distortion of the diffeo part of $f|V$ is bounded by K;

(ii) $f|V$ can be straightened by means of a K-qc homeomorphism.

Proposition 2.3. *Let $f : U \to U'$ be a quadratic-like map such that $f(0) \in U$. Let $\mu = \mathrm{mod}(U' \smallsetminus \bar{U})$. Then f can be adjusted to a quadratic-like map $f : V \to V'$ with $(\mu/2, K(\mu))$-bounded geometry.*

Moreover, $K(\mu) \to 1$ as $\mu \to \infty$.

PROOF. Let V' be the domain bounded by the hyperbolic geodesic in $U' \setminus \bar{U}$, and let $V = f^{-1}V'$. Since $f(0) \notin V'$, the restriction $f : V \to V'$ is a quadratic-like map. It satisfies (i) by the Koebe Distortion Theorem.

One can show that the fundamental annulus of this adjustment, $A = V' \setminus \bar{V}$, can be mapped onto a round annulus $\mathbb{A}(\sqrt{r}, r)$ with $\log r = \mu$ by means of a qc map H which is equivariant on ∂A and whose dilatation K depends only on μ (and moreover, $K \to 1$ as $\mu \to \infty$). The proof of the Straightening Theorem shows that this map H can be turned into a qc map h with the same dilatation which straightens f. □

This proposition shows that a quadratic-like maps with a definite modulus is "purely quadratic up to bounded distortion". The last assertion of the proposition shows that a quadratic-like map with a big modulus is "close" to being purely quadratic.

4. Hybrid lamination in the space of quadratic-like germs

4.1. Quadratic-like germs.

4.1.1. *Notion.* The need to adjust the domain of a quadratic-like map leads us to the notion of a quadratic-like germ. We say that two quadratic-like maps f and g represent the same *quadratic-like germ* if one is obtained from the other through a finite sequence of adjustments or reverse operations. Any quadratic-like germ has the well-defined filled Julia set. Quadratic-like germs are considered up to affine conjugacy. Such a germ can be normalized so that at the origin it has an expansion

$$f(z) = z^2 + c + O(z^4). \tag{2.4}$$

Let $\mathcal{Q}$ be the space of quadratic-like germs (up to affine conjugacy). It contains the one-parameter quadratic family $\mathcal{QP} = \{f_c\}$ with the Mandelbrot set $\mathbb{M}$ inside. The *connectedness locus* $\mathcal{C} \subset \mathcal{Q}$ is the set of quadratic-like germs with connected Julia set (by definition, $\mathbb{M}$ is the slice of $\mathcal{C}$ by the $\mathcal{QP}$).

4.1.2. *Banach slices.* Let $\mathbf{U}$ be the set of topological discs $U \ni 0$ with piecewise smooth boundary symmetric with respect to the origin. For $U \in \mathbf{U}$, let $\mathcal{B}_U$ be the affine space of even holomorphic functions on U continuous on $\bar{U}$ normalized by (2.4) at the origin (endowed with the uniform norm $\|\cdot\|_U$). We will identify the affine space $\mathcal{B}_U$ with its tangent linear space by putting the origin at the point $f(z) = z^2$.

Let $\mathcal{B}_U(f, \varepsilon)$ denote the ball in $\mathcal{B}_U$ of radius ε centered at f.

Given a quadratic-like map $f : U \to U'$, all nearby maps $g \in \mathcal{B}_U(f, \varepsilon)$ are quadratic-like on slightly smaller domains. Thus, we have a natural inclusion j_U of the ball $\mathcal{B}_U(f, \varepsilon)$ into $\mathcal{Q}$. We will call it a *Banach ball* or a *Banach slice* of $\mathcal{Q}$. Somewhat loosely, we will also use notation $\mathcal{Q}_U$ for such a Banach slice (without specifying f and ε). More generally, given a set $\mathcal{X} \subset \mathcal{Q}$, the intersections $\mathcal{X} \cap \mathcal{B}_U(f, \varepsilon)$ are called the Banach slices of $\mathcal{X}$.

Endow $\mathcal{Q}$ with the finest topology that makes all the inclusions j_U continuous. In this topology, a set $\mathcal{U} \subset \mathcal{Q}$ is open if all its Banach slices are open in the corresponding Banach spaces. (Note that the intrinsic topology of the Banach slices is finer than the topology induced from $\mathcal{Q}$.)

4.1.3. *Compactness criteria.* Let $\mu > 0$ and $C > 0$. Let $\mathcal{QM}(\mu, C)$ stand for the union of the quadratic family $\mathcal{QP}$ and the space of normalized quadratic-like maps $f : U \to U'$ such that

$$\operatorname{mod}(U' \smallsetminus U) \geq \mu, \quad f(0) \in U, \quad |f(0)| \leq C,$$

and let $\mathcal{Q}(\mu, C)$ be the corresponding space of quadratic-like germs. Similarly, let $\mathcal{C}(\mu)$ be the set of quadratic-like germs that can be represented by normalized quadratic-like maps $f : U \to U'$ with connected Julia set such that $\operatorname{mod}(U' \smallsetminus U) \geq \mu$.

Exercise 2.4. Show that $\mathcal{C}(\mu) \subset \mathcal{Q}(\mu, C)$ for some C.

The following important fact easily follows from Proposition 2.3:

Proposition 2.5. *The sets $\mathcal{Q}(\mu, C)$ and $\mathcal{C}(\mu)$ are compact in $\mathcal{Q}$.*

Propositions 2.3 and 2.5 explain why it is so important to control the fundamental annuli of quadratic-like maps in question.

It is easy to check that if $f : V \to V'$ is a normalized quadratic-like map from $\mathcal{Q}(\mu, C)$ then $V \Supset \mathbb{D}_r \equiv U$ for some $r = r(\mu, C) > 0$. Hence the space $\mathcal{Q}(\mu, C)$ can be endowed with the metric induced from the Banach space $\mathcal{B}_U$. We call it a *Montel metric* $\operatorname{dist}_{Mon}$ on $\mathcal{Q}(\mu, C)$. One can show (using Hadamard's Three Circle Theorem) that all Montel's metrics are Hölder equivalent. In what follows, the particular choice of a Montel metric will not matter, so we will refer to this metric without further specifications.

Starting with the space of real quadratic-like maps, we can define a real quadratic-like germ. Let $\mathcal{Q}_{\mathbb{R}}$ stand for the space of real quadratic-like germs (the "real slice" of $\mathcal{Q}$).

4.2. Banach manifolds and laminations.

4.2.1. *First definitions.* Basic analytic function theory in Banach spaces is a well established subject mostly analogous to the classical theory. Below we will recall for convenience some initial facts. By default, *all Banach spaces in this course will be assumed complex*, unless otherwise is explicitly said.

Let $\mathcal{B}$ and $\mathcal{D}$ be two Banach spaces, and let $\mathcal{U}$ be an open set in $\mathcal{B}$. A continuous map $R : \mathcal{U} \to \mathcal{D}$ is called *holomorphic ($\equiv$ complex analytic)* if for any complex line $L = \{a + \lambda v \mid \lambda \in \mathbb{C}\}$ in $\mathcal{B}$, $a \in \mathcal{U}$, and for any linear functional $\phi \in \mathcal{D}^*$, the function $\phi \circ R \mid L$ is holomorphic in λ near the origin. It turns out that such a map is smooth, i.e., for any $a \in \mathcal{U}$ there is a bounded linear operator $DR(a) : \mathcal{B} \to \mathcal{D}$ such that

$$R(a + v) = R(a) + DR(a)v + o(\|v\|), \quad \text{as} \quad \|v\| \to 0,$$

and the differential $DR(a)$ depends continuously on a.

Now the notions of holomorphic ($\equiv$ complex analytic) Banach manifold and submanifold, embedding, submersion, and biholomprhic diffeomorphism can be introduced by repeating the standard finite dimensional definitions.

We can now formulate an important adding to the Measurable Riemann Mapping Theorem.

THEOREM 2.6. *Let $h_\mu : (\mathbb{C}, 0, 1) \to (\mathbb{C}, 0, 1)$ be the normalized solution of the Beltrami equation with differential μ. Then h_μ depends holomorphically on μ.*

In this statement μ is considered as a point of a complex Banach space $L^\infty(C)$, while h_μ is considered as an element of a complex Banach space of continuous functions $\mathbb{C} \to \mathbb{C}$.

4.2.2. *Laminations.* Consider a complex analytic Banach manifold $\mathcal{B}$ and a closed subset $\mathcal{A} \subset \mathcal{B}$. Assume that $\mathcal{A}$ is decomposed into disjoint connected holomorphic submanifolds (*leaves*) of the same (co-)dimension. Such a decomposition is called a *holomorphic lamination* $\mathcal{L}$ in $\mathcal{B}$ (supported on $\mathcal{A}$) if it has the following *local product structure.* For any point $a \in \mathcal{A}$, there is a neighborhood $\mathcal{U} \ni a$ in $\mathcal{B}$, a neighborhood $\mathcal{W} = \mathcal{W}^h \times \mathcal{W}^v \ni 0$ in the direct product $\mathcal{D}^h \times \mathcal{D}^v$ of two complex Banach spaces $\mathcal{D}^h$ and $\mathcal{D}^v$, and a homeomorphism $\phi : \mathcal{U} \to \mathcal{W}$ ("straightening") such that for any leaf L intersecting $\mathcal{U}$ and for any component $L_{\rm loc}$ of $L \cap \mathcal{U}$ we have: $\phi(L_{\rm loc}) = \mathcal{W}^h \times \{t\}$ for some $t \in \mathcal{W}^v$ and the restriction $\phi \mid L_{\rm loc}$ is holomorphic. The *(co-)dimension* of $\mathcal{L}$ is the (co-)dimension of the leaves.

Laminations with full support (i.e., $\mathcal{A} = \mathcal{B}$ are called *foliations.*

A *transversal* to $\mathcal{L}$ is a holomorphic submanifold $\mathcal{S} \subset \mathcal{B}$ transversally intersecting the leaves of $\mathcal{L}$ at isolated points (so that $\dim \mathcal{S} = \operatorname{codim} \mathcal{L}$). If $\mathcal{S}$ intersects all the leaves, it is called a *global transversal.* A global transversal intersecting every leaf at a single point is called *unfolded.*

Consider now two transversals $\mathcal{S}_1$ and $\mathcal{S}_2$ intersecting some leaf L at points a_1 and a_2 respectively. Then there exist relative neighborhoods $\mathcal{U}_1 \subset \mathcal{S}_1$ and $\mathcal{U}_2 \subset \mathcal{S}_2$ of these points and a homeomorphism $\phi : \mathcal{U}_1 \cap \mathcal{A} \to \mathcal{U}_2 \cap \mathcal{A}$ such that $\phi(a_1) = a_2$ and points $x \in \mathcal{U}_1 \cap \mathcal{A}$ and $\phi(x)$ belong to the same leaf of the lamination. Such a map is called a *holonomy* along the lamination.

If every holonomy map as above admits a quasiconformal (respectively, smooth, holomorphic, etc.) extension to $\mathcal{U}_1$ (at least, for a sufficiently small neighborhood $\mathcal{U}_1$), then the lamination is called *transversally quasiconformal* (respectively, smooth, holomorphic, etc.).

Codimension-one laminations are also known as "holomorphic motions".

4.3. Holomorphic motions.

DEFINITION 2.3. Let $X \subset \mathbb{C}$ and let $(\Lambda, *)$ be a pointed domain in some Banach manifold $\mathcal{B}$. A *holomorphic motion* of X over Λ is a family of maps $h_\lambda : X \to \mathbb{C}$, $\lambda \in \Lambda$, such that:

- $h_* = \mathrm{id}$;
- For any $\lambda \in \Lambda$, h_λ is injective;
- For any $z \in X$, the orbit $\lambda \mapsto f_\lambda(z)$ is holomorphic in λ.

The following two main properties of holomorphic motions are usually referred to as the λ-*lemma.*

Extension Theorem. *There exists a neighborhood $\Lambda' \subset \Lambda$ of the base point $*$ such that any holomorphic motion h_λ of a set $X \subset \mathbb{C}$ over Λ admits an extension to a holomorphic motion of the whole complex plane $\mathbb{C}$ over Λ'.*

Consider a Banach ball $\mathcal{B}_r = \{x : \|x\| < r\}$ in a Banach space $\mathcal{B}$. Let us define the hyperbolic distance between $x \in \mathcal{B}_r$ and the origin as the hyperbolic distance between these two points in the one-dimensional slice $\{\lambda x : |\lambda| < r/\|x\|\}$. (In fact, there is a "Kobayashi metric" in the whole ball coinciding with the hyperbolic metric on the above slices.)

Quasiconformality Lemma. *Let $\{h_\lambda\}$ be a holomorphic motion over a Banach ball $(\Lambda, *) = (\mathcal{B}_r, 0)$. Then all maps h_λ, $\lambda \in \Lambda$, are quasiconformal. Moreover, the dilatation of h_λ depends only on the hyperbolic distance between λ and $*$ in Λ.*

By definition, the orbits $\lambda \mapsto (\lambda, h_\lambda z)$, $\lambda \in \Lambda$, of a holomorphic motion are codimension-one holomorphic submanifolds in $\Lambda \times \mathbb{C}$. The continuity of the maps h_λ tells us that the decomposition into these submanifolds has a product structure with the straightening map $(\lambda, z) \mapsto (\lambda, h_\lambda^{-1} z)$. Thus, if the set X is closed then we obtain a holomorphic codimension-one lamination in $\Lambda \times \mathbb{C}$. Vice versa, any holomorphic codimension-one lamination can be locally represented as a holomorphic motion.

The Extension Lemma tells us that any such lamination locally extends to a foliation. The Quasiconformality Lemma tells us that this lamination is *transversally quasiconformal*. It is a remarkable "free" regularity property of codimension-one holomorphic laminations.

4.4. Complex structure in $\mathcal{Q}$. The inclusions j_U of the Banach balls can be viewed as local charts in the space $\mathcal{Q}$ which endow $\mathcal{Q}$ with a natural complex analytic structure (though they do not turn $\mathcal{Q}$ into a Banach manifold).

For $U \subset V$, let $j_{U,V} : \mathcal{B}_V \to \mathcal{B}_U$ stand for the restriction operator. They satisfy the following properties:

LEMMA 2.7. P1: *countable base and compactness. There exists a countable family of Banach slices $\mathcal{Q}_n \equiv \mathcal{Q}_{V_n}$ with the following property. For any $f \in \mathcal{Q}_V$, there is a $\delta > 0$ and a Banach slice $\mathcal{Q}_n$ such that $V_n \Subset V$, and the Banach ball $\mathcal{B}_V(f, \delta) \subset \mathcal{Q}$ is compactly embedded into $\mathcal{Q}_n$.*

P2: *lifting of analyticity. For $W \subset V$, the inclusion $j_{W,V} : \mathcal{Q}_V \to \mathcal{B}_W$ is complex analytic. Moreover, let $U \Subset V$. Let us consider a continuous map $\phi : \mathcal{V} \to \mathcal{B}_V$ defined on a domain $\mathcal{V}$ in some Banach space. Assume that the map $j_{W,V} \circ \phi : \mathcal{V} \to \mathcal{B}_W$ is analytic. Then the map $j_{U,V} \circ \phi : \mathcal{V} \to \mathcal{B}_U$ is analytic as well.*

P3: *density. If $W \subset V$, then the space $\mathcal{B}_V$ is dense in $\mathcal{B}_W$.*

We say that the family of local charts j_V endows $\mathcal{Q}$ with *complex analytic structure* modeled on the "sheaf" of Banach spaces $\mathcal{B}_V$. More generally, if we have a set $\mathcal{S}$ and a family of inclusions $j_V : \mathcal{S}_V \to \mathcal{S}$, where $\mathcal{S}_V$ is an open set in $\mathcal{B}_V$, satisfying properties P1-P3, we say that $\mathcal{S}$ is endowed with complex analytic structure modeled on the sheaf of Banach spaces $\mathcal{B}_V$. In what follows we will say briefly that $\mathcal{S}$ is a *complex space.* For instance, the hybrid class of z^2,

$$\mathcal{H}_0 = \{f \in \mathcal{Q} : \ f(0) = 0\}, \tag{2.5}$$

is clearly a complex space.

Remark. Since the transit maps $j_{U,V}$ in $\mathcal{Q}$ are affine, $\mathcal{Q}$ is actually endowed with a *complex affine* structure. Then the hybrid class $\mathcal{H}_0$ becomes a codimension-one affine subspace in $\mathcal{Q}$.

Let us consider two complex spaces $\mathcal{S}^1$ and $\mathcal{S}^2$. A map $\phi : \mathcal{S}^1 \to \mathcal{S}^2$ is called *holomorphic* if for any $f \in \mathcal{S}^1$ and any Banach slice $\mathcal{S}^1_U \ni f$, there is an $\varepsilon > 0$ and a Banach slice $\mathcal{S}^2_V$ such that

$$\phi(\mathcal{B}_U(f, \varepsilon)) \subset \mathcal{S}^2_V, \tag{2.6}$$

and the restriction $\phi : \mathcal{B}_U(f, \varepsilon) \to \mathcal{S}^2_V$ is holomorphic in the Banach sense. Note that by P2, this property is independent of the choice of slice $\mathcal{S}^2_V$ satisfying (2.6) if to allow a little shrinking of V. In the case when Ω is a domain in $\mathbb{C}$, a holomorphic map $\gamma : \Omega \to \mathcal{S}$ is called a *holomorphic curve* in $\mathcal{S}$.

Let us consider a complex space $\mathcal{S}$ and a point $f \in \mathcal{S}$. Let $\mathbf{U}_f = \{U \in \mathbf{U} : \ f \in \mathcal{Q}_U\}$. Let us call a point f of a complex space $\mathcal{S}$ *regular* if $\mathbf{U}_f$ is a directed set, i.e., for any U and V in $\mathbf{U}_f$, there exists a $W \in \mathbf{U}_f$ contained in $U \cap V$. At such a point we can define the tangent space $\mathrm{T}_f\mathcal{S}$ as the inductive limit of the Banach spaces $\mathcal{B}_V$, $V \in \mathbf{U}_f$. If all points of a space $\mathcal{S}$ are regular, we call it a complex *manifold* modeled on the sheaf of Banach spaces.

In the case of $\mathcal{S} = \mathcal{Q}$, $\mathbf{U}_f$ is the set of topological disks V on which f is quadratic-like. All points of the connectedness locus $\mathcal{C}$ are regular (in particular, all points of the space $\mathcal{H}_0$ are regular). The tangent space $\mathrm{T}_f\mathcal{Q}$, $f \in \mathcal{C}$, is identified with the space of germs of holomorphic vector fields $v(z)$ near the filled Julia set $K(f)$.

If $\phi : \mathcal{S}^1 \to \mathcal{S}^2$ is a holomorphic map between complex spaces and f, $\phi(f)$ are regular points in the corresponding spaces, then we can naturally define the differential $D\phi(f) : \mathrm{T}_f\mathcal{S}^1 \to \mathrm{T}_{\phi(f)}\mathcal{S}^2$ by restricting ϕ to the Banach slices.

Let us now discuss a notion of a submanifold in a complex space $\mathcal{S}$. We will deal with two situations.

1) *Finite dimensional submanifold* (more generally, a Banach submanifold) is a subset in $\mathcal{S}$ which locally sits in some Banach slice $\mathcal{B}_U$ and is submanifold therein. By P2, this definition is independent of the choice of the slice $\mathcal{B}_U$ (up to a slight shrinking of U).

2) *Regular parametrized submanifold.* Let $\mathcal{M}$ be a complex manifold modeled on a family of Banach spaces. A holomorphic map $i : \mathcal{M} \to \mathcal{S}$

into a regular part of $\mathcal{S}$ is called *immersion* if for any $m \in \mathcal{M}$ the differential $Di(m)$ is a linear homeomorphism onto its image. The image $\mathcal{X}$ of an injective immersion i is called an *immersed submanifold*. It is called an *(embedded) submanifold* if additionally i is a homeomorphism onto $\mathcal{X}$ supplied with the induced topology. For example, if there is an analytic projection $\pi : \mathcal{S} \to \mathcal{M}$ such that $\pi \circ i = \mathrm{id}$ then $\mathcal{X}$ is a submanifold in $\mathcal{M}$.

If $i : (\mathcal{M}, m) \to (\mathcal{X}, f) \subset (\mathcal{S}, f)$ is an embedding, then the tangent space $\mathrm{T}_f\mathcal{X}$ is defined as the image of the differential $Di(m)$. It is a closed linear subspace in $\mathrm{T}_f\mathcal{Q}$. Its codimension is called the codimension of $\mathcal{X}$ at f. We say that a submanifold $\mathcal{X}$ has codimension d if it has codimension d at all its points.

Two submanifolds $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{S}$ are called *transverse* at a point $g \in \mathcal{X} \cap \mathcal{Y}$ if $\mathrm{T}_g\mathcal{X} \oplus \mathrm{T}_g\mathcal{Y} = \mathrm{T}_g\mathcal{S}$.

A family of disjoint Banach submanifolds in $\mathcal{S}$ which partition some closed subset $\mathcal{A}$ of $\mathcal{S}$ form a *lamination* if for any $f \in \mathcal{A}$ there exists a Banach ball $\mathcal{B}_U(f, \varepsilon)$ such that the slices of the manifolds by this ball form a Banach lamination.

4.5. External maps.

4.5.1. *Construction.* Next, we will construct a natural projection π from the space $\mathcal{Q}$ to a space $\mathcal{E}$ of circle expanding maps.

Let $g : \mathbb{T} \to \mathbb{T}$ be a degree two real analytic endomorphism of the unit circle $\mathbb{T}$. It can be also viewed as a complex analytic germ near the circle. We call g *expanding* if it admits an analytic extension to a double covering $g : V \to V'$ between annular neighborhoods of $\mathbb{T}$ such that $V \Subset V'$. We consider such a map up to conjugacy by rotation, which is equivalent to normalizing it in such a way that $1 \in \mathbb{T}$ is a fixed point. Let $\mathcal{E}$ stand for the space of such circle endomorphisms (up to rotation).

There is a projection $\pi : \mathcal{Q} \to \mathcal{E}$ which associates to $f \in \mathcal{Q}$ its *external map* $g \in \mathcal{E}$. In the case when $f \in \mathcal{C}$, the construction is easily provided by the Riemann Mapping Theorem. Namely, let $f : U \to U'$ be a quadratic-like representative of the germ. Let us conjugate $f : U \smallsetminus K(f) \to U' \smallsetminus K(f)$ by the Riemann mapping

$$\phi = \phi_f : \mathbb{C} \smallsetminus K(f) \to \mathbb{C} \smallsetminus \bar{\mathbb{D}}$$

to a double covering $g : V \to V'$ between annuli with inner boundary $\mathbb{T}$. By the Reflection Principle, g extends to a circle endomorphism of class $\mathcal{E}$. Since the Riemann mapping ϕ is well-defined up to post-composition with rotation, g is well-defined up to conjugacy by rotation.

In the case of disconnected Julia set the construction is more subtle. Take a fundamental annulus $A = U' \setminus U$ with real analytic boundary curves $E = \partial U'$ and $I = \partial U$. Then $f : I \to E$ is a real analytic double covering.

Let $\mu = \operatorname{mod} A$. Let us consider an abstract double covering $\xi_1 : A_1 \to A$ of an annulus A_1 of modulus $\mu/2$ over A. Let I_1 and E_1 be the "inner" and "outer" boundary of A_1, i.e., ξ_1 maps I_1 onto I and E_1 onto E. Then there is a real analytic diffeomorphism $\theta_1 : E_1 \to I$ such that $\xi_1 = f \circ \theta_1$. This allows us to stick the annulus A_1 to the disk $\mathbb{C} \setminus U$ bounded by I. We obtain a Riemann surface $T_1 = (\mathbb{C} \setminus U) \cup_{\theta_1} A_1$. Moreover, the maps f on A and ξ_1 on A_1 match to form an analytic double covering $f_1 : A_1 \to A$.

This map f_1 restricts to a real analytic double covering of the inner boundary of A_1 onto its outer boundary. This allows us to repeat this procedure: we can attach to the inner boundary of T_1 an annulus A_2 of modulus $\frac{1}{4}\mu$, and extend f_1 to the new annulus T_2. Proceeding in this way, we will construct a Riemann surface

$$T^A \equiv T^A(f) = \lim T_n = (\mathbb{C} \setminus U) \cup_{\theta_1} A_1 \cup_{\theta_2} A_2 \dots \qquad (2.7)$$

and an analytic double covering $F : \cup_{n \geq 1} A_n \to \cup_{n \geq 0} A_n$ extending f.

Since the trajectories of F do not converge to the ""inner" ideal boundary of T^A, it is a punctured (at ∞) disk which can be conformally mapped onto $\mathbb{C} \setminus \bar{\mathbb{D}}$. Now by the reflection principle, this conformal representation of F can be extended to an analytic expanding endomorphism $g_A : \mathbb{T} \to \mathbb{T}$.

It is not hard to check that the map $g_A : V \to V'$ is well-defined up to rotation and that the corresponding circle endomorphism $g_A | \mathbb{T}$ does not actually depend on A. Thus, we have a well defined projection $\pi : \mathcal{Q} \to \mathcal{E}$ that associates to a quadratic-like germ f its external map $g = \pi(f)$.

LEMMA 2.8. *A quadratic-like map f is a quadratic polynomial if and only if its external map is $f_0 : z \mapsto z^2$. Thus, $\mathcal{QP} = \pi^{-1}(z^2)$.*

4.5.2. *Topology on $\mathcal{E}$.* Analogously to $\mathcal{Q}$, the space $\mathcal{E}$ can be endowed with an inductive limit topology based on a family of (real) Banach spaces. Namely, let us represent $\mathbb{T}$ as $\mathbb{R}/(\gamma : x \mapsto x + 1)$ so that $1 \in \mathbb{T}$ corresponds to $0 \in \mathbb{R}$. Let V be a γ-invariant $\mathbb{R}$-symmetric neighborhood of $\mathbb{R}$, and let $\mathcal{D}_V$ stand for the Banach space of functions f analytic on V, real on $\mathbb{R}$, normalized as $f(0) = 0$, and satisfying the following equation: $f(z+1) = f(z)+2$. (This corresponds to the space of degree two circle maps analytic in a given neighborhood of $\mathbb{T}$ and fixing 1.)

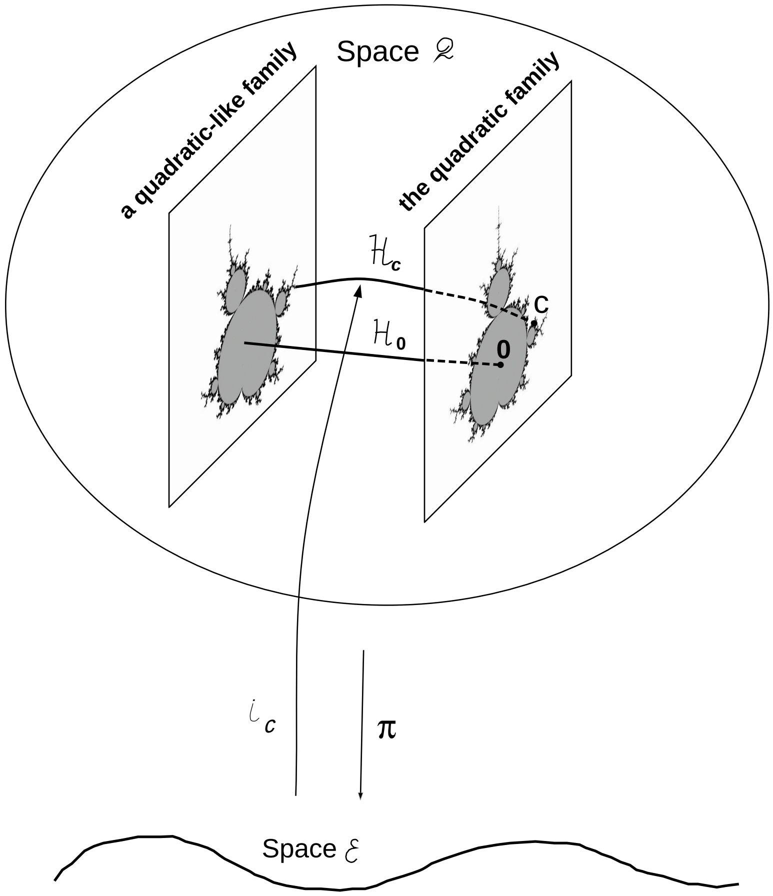

FIGURE 2.3. Space of quadratic-like germs.

Let $\mathcal{E}_V$ be the set of expanding circle maps $f \in \mathcal{E}$ which belong to $\mathcal{D}_V$. It is clearly an open subset of $\mathcal{B}_V$. Thus, we have a natural representation of $\mathcal{E}$ as the inductive limit of real Banach manifolds $\mathcal{E}_V$. This endows $\mathcal{E}$ with the inductive limit topology.

LEMMA 2.9. *The projection $\pi : \mathcal{Q} \to \mathcal{E}$ is continuous.*

As the space $\mathcal{E}$ is $\mathbb{R}$-symmetric, it is not at all obvious that it can be also endowed with a natural complex analytic structure. However, as we will see next, it can actually be done.

4.6. Hybrid lamination and complex structure on $\mathcal{E}$. By the Straightening Theorem, there is the straightening map $\chi : \mathcal{C} \to \mathbb{M}$ whose fibers $\mathcal{H}_c = \chi^{-1}(c)$ are the hybrid classes of quadratic-like germs (see Figure 2.3). In fact, the projection $\pi : \mathcal{H}_c \to \mathcal{E}$ is a homeomorphism:

LEMMA 2.10. *Any parameter $c \in M$ can be "mated" with any circle map $g \in \mathcal{E}$ to obtain a unique (up to affine conjugacy) map $f \equiv i_c(g) \in \mathcal{Q}$ such that $\chi(f) = c$ and $\pi(f) = g$. Moreover, the map $i_c : \mathcal{E} \to \mathcal{H}_c$ is continuous.*

Combining this lemma with Lemma 2.9, we conclude that for any $c \in \mathbb{M}$, the maps $\pi : \mathcal{H}_c \to \mathcal{E}$ and $i_c : \mathcal{E} \to \mathcal{H}_c$ are inverse homeomorphisms.

Consider now the hybrid class $\mathcal{H}_0$ of $z \mapsto z^2$ (2.5). As we have already mentioned, it is a complex space. *The promised complex structure on $\mathcal{E}$ is obtained by transferring the complex structure from $\mathcal{H}_0$ to $\mathcal{E}$ by means of the homeomorphism $i_0 : \mathcal{E} \to \mathcal{H}_0$.* Then we have:

Using holomorphic dependence in the Measurable Riemann Mapping Theorem (2.6), one can now show:

LEMMA 2.11. *For any $c \in \mathbb{M}$, both maps $\pi : \mathcal{Q} \to \mathcal{E}$ and $i_c : \mathcal{E} \to \mathcal{Q}$ are holomorphic.*

Since $\pi \circ i_c = \mathrm{id}$, the projection π is a holomorphic submersion onto $\mathcal{E}$, while i_c is a holomorphic embedding of $\mathcal{E}$ into $\mathcal{Q}$. In this sense, the hybrid classes $\mathcal{H}_c = i_c(\mathcal{E})$ are parametrized holomorphic submanifolds in $\mathcal{Q}$. However, to make this structure useful, we need to consider Banach slices of these manifolds.

Let $\mathcal{F}$ denote the partition of the connectedness locus $\mathcal{C}$ into hybrid classes $\mathcal{H}_c$.

LEMMA 2.12. *For any $f_0 \in \mathcal{C}$, there exists a domain $V_0 \in \mathbf{U}_f$ such that for any $V \subset V_0$, $V \in \mathbf{U}_f$, the slice $\mathcal{F}_V$ of the partition $\mathcal{F}$ near f_0 is a codimension-one holomorphic lamination in $\mathcal{B}_V$.*

In this sense, the hybrid classes form a lamination $\mathcal{F}$ of the connectedness locus. Moreover, since by Lemma 2.8 the quadratic family $\mathcal{QP}$ is a fiber of π, it is transverse to this lamination. Since by the Straightening Theorem, it intersects every leaf of $\mathcal{F}$ at a single point, we conclude that *the quadratic family $\mathcal{QP}$ is a global unfolded transversal to the lamination $\mathcal{F}$.*

From this point of view, the straightening χ is interpreted as the *holonomy* to the quadratic family along the lamination $\mathcal{F}$. Applying the Quasiconformality Lemma from §4.3, we obtain:

THEOREM 2.13. *The lamination $\mathcal{F}$ is transversally quasiconformal.*

5. Quadratic-like families

5.1. Definitions. A *quadratic-like family* over a domain $\Lambda \subset \mathbb{C}$ is a holomorphic family $\mathbf{f} = \{f_\lambda : U_\lambda \to U'_\lambda\}_{\lambda \in \Lambda}$ of quadratic-like maps.

Thus, the domains U_λ are vertical fibers of a domain $\mathbb{U} \subset \Lambda \times \mathbb{C}$ and $f_\lambda(z)$ is a holomorphic function on $\mathbb{U}$. Any quadratic-like family can be viewed as a holomorphic curve in $\mathcal{Q}$.

The M-set $\mathbb{M}_\mathbf{f}$ is defined as $\{\lambda \in \Lambda : J(f_\lambda) \text{ is connected}\}$. By means of the straightening (or rather, the holonomy along $\mathcal{F}$), the M-set $\mathbb{M}_\mathbf{f}$ can be projected to $\mathbb{M}$. If $\mathbf{f}$ is a global transversal to $\mathcal{F}$ then one can expect that this projection is a homeomorphism from $\mathbb{M}_\mathbf{f}$ onto $\mathbb{M}$. To make this statement precise we need a few more definitions.

A quadratic-like family $f_\lambda : U_\lambda \to U'_\lambda$ is called *equipped* if it is supplied with the holomorphic motion h_λ of the fundamental annulus $\bar{U}'_\lambda \setminus U_\lambda$ equivariant on the boundary. In what follows we will consider only equipped quadratic-like families.

Assume that the family $\mathbf{f}$ admits an analytic extension to a neighborhood of $\bar{\mathbb{U}} \subset \mathbb{C}^2$. Such a family $\mathbf{f}$ is called *proper* if $f_\lambda(0) \in \partial U'_\lambda$ for $\lambda \in \partial\Lambda$. Note that in this case, $\mathbb{M}_\mathbf{f} \Subset \Lambda$ (that is, the family $\mathbf{f}$ "overflows" the connectedness locus $\mathcal{C}$).

A proper quadratic-like family is called *unfolded* if the winding number of the curve $\lambda \mapsto f_\lambda(0)$, $\lambda \in \partial\Lambda$, around the origin is equal to 1. (This property ensures that $\mathbf{f}$ is transverse to the foliation $\mathcal{F}$.)

5.2. Straightening of families. The following result is fundamental in the renormalization theory:

THEOREM 2.14. *Let* $\mathbf{f}$ *be a proper unfolded quadratic-like family. Then the straightening* $\chi : \mathbb{M}_\mathbf{f} \to \mathbb{M}$ *is a homeomorphism.*

In fact, the straightening χ admits an extension to a homeomorphism from Λ onto some neighborhood Ω of $\mathbb{M}$. To see it, select some base point $* \in \Lambda$ (the most natural choice is the "origin" $* \in \mathcal{H}_0$). Then select a tubing of f_* (2.3). By means if the holomorphic motion h_λ it can be spread around to the whole family. As we noted in §3.2, the choice of the tubing determines uniquely the straightening of a quadratic-like map. This gives us a desired extension of χ.

5.3. Space of quadratic-like families.

5.3.1. *Convergence.* Let $\mathcal{G}$ stand for the class of proper unfolded equipped quadratic-like families up to affine change of variable in λ. We will normalize such a family so that the superattracting parameter value $*$ sits at the origin and $\operatorname{diam} \mathbb{M}_\mathbf{f} = 1$. We will impose the following convergence structure on $\mathcal{G}$: A sequence of normalized families $(\mathbf{f}_n : \mathbb{U}_n \to \mathbb{U}'_n, \mathbf{h}_n)$ over $(\Lambda_n, *)$ is declared to converge to a family $(\mathbf{f} : \mathbb{U} \to \mathbb{U}', \mathbf{h})$ over $(\Lambda, *)$ if:

(i) Parameter domains $(\Lambda_n, *)$ Carathéodory converge to $(\Lambda, *)$; define

(ii) Holomorphic motions $\mathbf{h}_n$ converge to $\mathbf{h}$ uniformly over any domain $\Omega \Subset \Lambda$; i.e., $h_{n,\lambda}(z) \to h_\lambda(z)$ uniformly for $(\lambda, z) \in \Omega \times \bar{\mathbb{C}}$ (where $\bar{\mathbb{C}}$ is endowed with the spherical metric);

(iii) The maps $\mathbf{f}_n$ converge to $\mathbf{f}$ uniformly on compact subsets of $\mathbb{U}$.

Note that the convergence of quadratic-like families yields uniform on compact sets convergence of the corresponding holomorphic curves in $\mathcal{Q}$.

5.3.2. *Geometry of a quadratic-like family.* Given an equipped quadratic-like family $(\mathbf{f}, \mathbf{h})$ over $(\Lambda, *)$, let

$$\operatorname{mod}(\mathbf{f}) = \inf_{\lambda \in \Lambda} \operatorname{mod}(U'_\lambda \setminus \bar{U}_\lambda), \quad \operatorname{Dil}(\mathbf{h}) = \sup_{\lambda \in \Lambda} \operatorname{Dil}(h_\lambda).$$

For $C, \mu > 0$, let

$$\mathcal{G}_{C,\mu} = \{(\mathbf{f}, \mathbf{h}) \in \mathcal{G} : \operatorname{diam} \mathbb{U}' \le C,\ f_* \in \mathcal{QM}(\mu, C), \\ \operatorname{Dil}(H_*) \le C,\ \operatorname{Dil}(\mathbf{h}) \le C\}, \tag{2.8}$$

where H_* is the tubing of f_* (see §3.2).

We will say that a quadratic-like family (taken from some collection under consideration) has a "bounded geometry" if it belongs to a certain class $\mathcal{G}_{C,\mu}$ with C and μ being uniform over the collection. A statement that certain bound "depends only on the geometry" of a quadratic-like family means that this bound is uniform over any class $\mathcal{G}_{C,\mu}$.

5.3.3. *Shape of M-sets.* Theorem 2.13 provides a control of the shape of the M-sets in quadratic-like families:

LEMMA 2.15. *Let us consider a quadratic-like family* $(\mathbf{f}, \mathbf{h})$ *over* $(\Lambda, *)$ *of class* $\mathcal{G}_{C,\mu}$. *Then the straightening* $\chi_{\mathbf{f}} : (\Lambda, \mathbb{M}_{\mathbf{f}}) \to (\Delta, \mathbb{M})$ *is a* $K(C, \mu)$*-qc map onto an appropriate neighborhood* Δ *of the Mandelbrot set* $\mathbb{M}$.

We will briefly say that the sets $\mathbb{M}_{\mathbf{f}}$ have a $K(C, \mu)$*-standard shape.* If we do not need to specify dilatation K, we say that the sets have *quasistandard shape.*

5.3.4. *Compactness criterion.* Similarly to the situation with a single quadratic-like map (see Proposition 2.5), uniform geometric bounds on a quadratic-like family yield compactness:

LEMMA 2.16. *For any* $C, \mu > 0$*, the space* $\mathcal{G}_{C,\mu}$ *is compact.*

5.4. Vertical tubes. Along with the projection $\pi : \mathcal{Q} \to \mathcal{E}$ introduced in §4.5, let us consider a projection $\Pi = i_0 \circ \pi : \mathcal{Q} \to \mathcal{H}_0$. The fibers

$$\mathcal{Z}(G) = \Pi^{-1}(G) = \pi^{-1}(g), \quad g = \pi(G),$$

of these projections are called *vertical fibers*. For a set $\mathcal{P} \subset \mathcal{Q}$, let $\mathcal{Z}_{\mathcal{P}}(G) = \mathcal{P} \cap \mathcal{Z}(G)$ stand for the vertical fibers in $\mathcal{P}$.

Let us say that $\mathcal{P}$ is a *vertical tube* over a Banach neighborhood $\mathcal{V} \subset \mathcal{H}_0$ if its vertical fibers are topological disks and it has a topological product structure over $\mathcal{V}$ (i.e., there exists a topological disk $W \subset \mathbb{C}$ such that $\mathcal{P}$ and $\mathcal{V} \times W$ are homeomorphic over $\mathcal{V}$). Let us say that a vertical tube $\mathcal{P}$ is *equipped* if

- There is a base map $G_* \in \mathcal{V}$ equipped with a tubing H_* (2.3);
- There is an equivariant holomorphic motion of the fundamental annulus A_f,

$$h_f : (\mathbb{C}, A_*) \to (\mathbb{C}, A_f),\ f \in \mathcal{P};$$

- The vertical fibers $\mathcal{Z}_{\mathcal{P}}(G)$ equipped with the above motion are proper unfolded quadratic-like families. (In particular, these fibers are holomorphic curves in $\mathcal{Q}$.)

By §5.2, for any equipped tube $\mathcal{P}$, there is a well defined straightening

$$\chi_{\mathcal{P}} : \mathcal{P} \to \mathbb{C}. \tag{2.9}$$

LEMMA 2.17. *Any $G_* \in \mathcal{H}_0$ belongs to an equipped vertical tube $\mathcal{P}$ over a Banach neighborhood $\mathcal{V} \subset \mathcal{H}_0$. The straightening $\chi_{\mathcal{P}}$ is a trivial fibration over some domain $\Delta \supset \mathbb{M}$ whose fibers are holomorphic leaves $\mathcal{L}_{\mathcal{P}}(f)$, $f \in \mathcal{P}$, parametrized by $\mathcal{V}$.*

Denote the above foliation by $\mathcal{F}_{\mathcal{P}}$. It will be naturally called the *horizontal foliation* in $\mathcal{P}$.

For $f \in \mathcal{C}$, let E^h_f stand for the tangent space to the hybrid class $\mathcal{H}(f)$ at f (the *horizontal space*), and let E^v_f stand for the complementary *vertical line* tangent to the vertical fiber $\mathcal{Z}(f)$.

For $f \in \mathcal{C} \cap \mathcal{P}$, we have the horizontal-vertical decomposition

$$\mathrm{T}_f\mathcal{P} = E^h_{f,\mathcal{P}} \oplus E^v_f, \tag{2.10}$$

where $E^h_{f,\mathcal{P}} = E^h_f \cap \mathrm{T}\mathcal{P}$. These two distributions admit an extension to the whole tube $\mathcal{P}$ as the tangent distributions respectively to the horizontal and the vertical foliations in $\mathcal{P}$. To simplify notations, we will often suppress the label "$\mathcal{P}$" in the notation for the horizontal spaces in $\mathcal{P}$.

Vertical tubes can be endowed with a Banach manifold structure. A local chart of this structure near a point $f \in \mathcal{P}$ is

$$g \mapsto (\Pi(g), \pi^v_f(g)) \in \mathcal{V} \times E^v_f,$$

where π^v is the linear projection of g onto E^v_f parallel to E^h_f.

5.5. Uniform transversality. Though the Montel metric on compact sets (see §4.1.3) is well defined up to Hölder equivalence only, it induces a Lipschitz structure in the vertical direction:

LEMMA 2.18. *Given a vertical tube $\mathcal{P}$, there exists a $\nu > 0$ and a Riemannian metric $\|\cdot\|_f$ on the vertical distribution $\{E^v_f\}_{f\in\mathcal{P}}$ such that $\|\cdot\|_f$ is uniformly Lipschitz equivalent to the Banach norm on any slice $\mathcal{B}_U$, provided $U \in \mathbf{U}_f$ and the geometry of the quadratic-like map $f : U \to U'$ is ν-bounded (see §3.4).*

This vertical metric will also be called "Montel".

Thus, for any tangent vector $u \in \mathrm{T}\mathcal{P}$ we can measure the angle $\alpha \in (0, \pi/2)$ between u and E^h_f by letting:

$$\operatorname{tg}\alpha = \frac{\|\pi^v_f(g)\|}{\|D\Pi(u)\|}.$$

We say that a collection $\mathcal{X}$ of quadratic-like families is *uniformly transverse* to the foliation $\mathcal{F}$ if there exist finitely many vertical tubes $\mathcal{P}_i$ and an $\alpha > 0$ such that any curve $\mathbf{f} = \{f_\lambda\}_{\lambda\in\Lambda} \in \mathcal{X}$ belongs to $\cup\mathcal{P}_i$ and if $f_\lambda \in \mathcal{P}_i$ then the angle between f_λ and the horizontal space $E^h_{f_\lambda,\mathcal{P}_i}$ is greater than α.

Compactness Lemma 2.16 yields:

LEMMA 2.19. *Given $C > 0$ and $\mu > 0$, the holomorphic curves $\mathbf{f} \in \mathcal{G}_{C,\mu}$ are uniformly transverse to the foliation $\mathcal{F}$.*

6. Real and complex renormalization

6.1. Real renormalization.

6.1.1. *Definition.* Consider an S-unimodal map $f : I \to I$ with critical point at 0. Assume there is an interval $J \ni 0$ and $p > 1$ such that the intervals $J_k \equiv f^k J$, $k = 0, 1, \dots, p-1$, have disjoint interiors, and $f^p(J, \partial J) \subset (J, \partial J)$. Then the map f is called *renormalizable* with period p, and the restriction $PRf = f^p|J$ good notation? is called the (real) *pre-renormalization* of f. This map considered up to rescaling of J is called the (real) *renormalization* Rf of f. It can be naturally normalized by rescaling J to $[-1, 1]$, i.e., by letting

$$Rf = T \circ (f^p|J) \circ T^{-1},$$

where $T : J \to [-1, 1]$ is the dilation which preserves or reverses orientation depending on whether 0 is the minimum of f^p or otherwise. We will call J the *central interval* of the renormalization.

EXERCISE 2.20. If f has an attracting cycle of period $p > 1$ then f is renormalizable with period p. If f has a parabolic cycle of period

$p > 1$ with multiplier $\lambda = 1$ (resp. $\lambda = -1$) then f is renormalizable with period p (resp. $2p$).

Note that though f^p is a multi-modal map, the renormalization Rf is still unimodal, so that it is a (partially defined) operator in the space of unimodal maps.

Among possible renormalization periods there is the smallest one corresponding to the "first" renormalization. In what follows we will use the notation R for the first renormalization, unless otherwise is explicitly specified.

Note that period 2 has a special feature: in this case the renormalization intervals $J = [\alpha, -\alpha]$ and $f(J)$ *touch* at the fixed point α with negative multiplier. Because of this subtlety the doubling renormalization often requires a special treatment.

EXERCISE 2.21. Show that the central renormalization interval J is the *biggest* f^p-invariant interval such that $f^p|J$ is unimodal. Show that $\tilde{J} = [f^p(0), f^{2p}(0)]$ is the minimal such an interval. It will be called the *smallest central interval* of the renormalization Rf.

6.1.2. *Combinatorics. Combinatorial type* of the renormalization is the order of intervals J_k, $k = 0, 1, \dots, p-1$, on the real line. Equivalently it can be described in terms of the kneading theory in the following way. Let $c_1 \in J_1$ be the preimage of 0 under $f^{p-1} : J_1 \to J$. Then the itinerary of c_1 is a finite kneading sequence $\boldsymbol{\kappa}$ of length p. Combinatorics of the renormalization is determined by $\boldsymbol{\kappa}$.

EXERCISE 2.22. Show that any finite kneading sequence of length greater than 1 can be realized as combinatorics of some renormalizable map (compare Exercise I.1.42). All periods, except 2 and 3, admit several renormalization combinatorics. Find all renormalization combinatorics of period 4 and 5.

If, in turn, Rf is renormalizable, then f is called *twice renormalizable* with the second renormalization $R^2 f$. If it is renormalizable again, then f is *three times renormalizable* with the third renormalization $R^3 f$, and so on. In this way, we can classify all unimodal maps according to the number of times they can be renormalized. In particular, a map can be *infinitely renormalizable.*

EXERCISE 2.23. Let f be renormalizable with combinatorics $\boldsymbol{\kappa}$ and let Rf be renormalizable with combinatorics $\boldsymbol{\mu}$. Describe combinatorics of $R^2 f$.

An S-unimodal map f is called (real) *Yoccoz* if is not infinitely renormalizable, neither hyperbolic nor parabolic.

6.1.3. *Feigenbaum attractor.* Infinitely renormalizable maps have a very nice topological structure. Let $p_1 < p_2 < \dots$ be the consecutive periods of the renormalizations and let $J^1 \supset J^2 \supset \dots$ be the corresponding central intervals. The intervals $J^n_k = f^k J^n$, $k = 0, 1, \dots p_n - 1$, are cyclically permuted by f forming a nest of cycles of intervals. Let us also consider the smallest renormalization intervals $\tilde{J}^n$ (see Exercise 2.21) and the corresponding cycles of intervals $\tilde{J}^n_k = f^k \tilde{J}^n$, $k = 0, 1, \dots p_n - 1$.

THEOREM 2.24. *Let f be an infinitely renormalizable S-unimodal map f. Then*

$$O_f = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{p_n-1} J^n_k = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{p_n-1} \tilde{J}^n_k \tag{2.11}$$

is a Cantor set equal to the postcritical set $\omega(0)$. It has a natural group structure (projective limit of cyclic groups of order p_n) such that the map f acts on O_f as a group translation ("adding machine").

Note that the property that the set O_f is Cantor depends on the No Wandering Intervals Theorem (I.1.32). This set is also called the *Feigenbaum attractor.*

The ratios $q_n = p_{n-1}/p_n$ are called the *relative renormalization periods.* Every interval of level $n-1$ contains exactly q_n intervals of the next level n. The most famous example of infinitely renormalizable map is the Feigenbaum polynomial $z \mapsto z^2 + c_F$ (see §I.5.1) whose relative periods are all equal to 2. In this case, the dynamics on the postcritical set is the adding machine on the dyadic group,

EXERCISE 2.25. Find the kneading sequence of the Feigenbaum map.

One says that f has *bounded combinatorics* if the relative periods are bounded.

We will see in §6.4 that for any string of admissible combinatorial types, $(\boldsymbol{\kappa}_0, \boldsymbol{\kappa}_1, \dots)$, there is an infinitely renormalizable quadratic map with this combinatorics.

6.1.4. *Injectivity.* Let us finish with a result saying that the renormalization operator is injective in the space of real analytic maps:

LEMMA 2.26. *Let f and g be two real analytic unimodal renormalizable maps (maybe with different combinatorics). If $Rf = Rg$ then $f = g$.*

6.2. Complex renormalization. To define the complex renormalization, we will replace in the above "real definition" the intervals by topological disks and unimodal maps by quadratic-like maps. Here is a precise notion.

A quadratic-like map $f : U \to U'$ is called *renormalizable* of period p is there is a topological disk $V \ni 0$ such that

- The map $g = f^p : V \to f^p V \equiv V'$ is quadratic-like;
- The *little Julia set* $K(g)$ is connected;
- The little Julia sets $K_k = f^k K(g)$, $k = 0, 1, \dots, p-1$, do not intersect except perhaps touching at their β-fixed points (see §§I.3.5 and 3.2).

picture

Remarks. a) Notice that the K_k are the (filled) Julia sets of quadratic-like maps $g_p = f^p : f^k V \to f^{k+p} V$ which are not normalized for $k = 1, \dots, p-1$: the critical point of g_p is not 0 but rather $f^k 0$. (Neither they are symmetric with respect to the critical point.)

b) In most cases, the little Julia sets J_k are actually pairwise disjoint and the domains $f^k V$, $k = p, \dots, p-1$, can be selected pairwise disjoint as well. Such a renormalization is called *primitive*. Otherwise it is called *satellite*. For real maps, only the renormalization of period 2 is satellite.

c) In the satellite case, the little Julia sets are organized in *bouquets* of l sets (for some l dividing p) touching at their common β-fixed point. For the first renormalization, there is only one bouquet of little Julia sets touched at the α-fixed point of f.

d) Finally, we should remark that the notions of real and complex renormalizations do not exactly match. If a real map is renormalizable in the complex sense then it is renormalizable in the real sense as well, and its renormalization intervals are just the real slices of the little Julia sets. However, the real map $x \mapsto x^2 - 3/4$ (corresponding to the first doubling bifurcation) is renormalizable in the real sense but is not renormalizable in the complex one. As the first renormalization is concerned, this doubling situation is the only one that makes the difference.

As in the real case, the *first* complex renormalization corresponds to the smallest period for which a map is renormalizable. For this period, the map $f^p : V \to V'$ is called (complex) *pre-renormalization* PRf of f. This map considered up to rescaling is called the (complex) *renormalization* Rf of f. It can be normalized at the origin as (normalization at 0). (We use the same notations for the real and complex (pre-)renormalizations: it will always be clear which one is considered.)

As in the real case, the complex renormalizations are distinguished not only by the period but also by their combinatorial types. Roughly speaking, this is the combinatorics of the little Julia sets K_k in the big set $K(f)$.

Let us give a precise definition. Consider two renormalizable maps f and $\tilde{f}$. Assume first that the corresponding renormalization is primitive. Then the renormalization domains V and $\tilde{V}$ can be selected in such a way that the domains $f^k V$ (respectively, $\tilde{f}^k \tilde{V}$) , $k = p, \dots, 2p-1$, are pairwise disjoint. The maps f and $\tilde{f}$ are *combinatorially equivalent* if there exist homeomorphisms h and h' of the complex plane such that $hV_k = \tilde{V}_k$, $k = 0, 1, \dots, p-1$, h is homotopic to h' rel $\cup_{0\le k\le p-1} \partial V_k$, and $h \circ f = \tilde{f} \circ h'$.

If the first renormalization happens to be satellite , then its combinatorics is specified by the combinatorial rotation number of the α-fixed point of f.

Obviously, if two renormalizable quadratic-like germs f and g are hybrid equivalent then their renormalizations Rf and Rg are hybrid equivalent as well (since an appropriate restriction of the conjugacy between f and g gives a hybrid conjugacy between their pre-renormalizations). Hence R preservers the hybrid foliation $\mathcal{F}$ and, moreover, contracts Sullivan's Teichmüller metric (2.2).

Note finally that similarly to the real case we can classify complex quadratic-like maps according to the number of times (from 0 to ∞) it can be renormalized. This makes sense of notation $R^k f$. As in the real case, a quadratic-like map is called *Yoccoz* if it is purely repelling and is not infinitely renormalizable.

6.3. Little Mandelbrot copies. One of the most fascinating features of the Mandelbrot set easily observable on computer pictures is the presence inside of it little copies of itself, "M-copies", (see Figure 2.4. This phenomenon can be completely understood by means of the complex renormalization.

Consider some primitive hyperbolic component H of int $\mathbb{M}$ of period $p > 0$ (i.e., H is not the component bounded by the main cardioid). It turns out that one can find a domain $\Lambda \Supset H$ and a holomorphically moving domains $0 \in V_c \Subset V'_c$, $c \in \Lambda$, such that:

- the maps $g_c = f_c^p : V_c \to V'_c$ are quadratic-like and, moreover, form a proper unfolded quadratic-like family $\mathbf{g}$ over Λ;
- for any $c \in \Lambda$, the domains $f_c^k V'_c$, $k = 0, 1, \dots, p-1$, are pairwise disjoint.

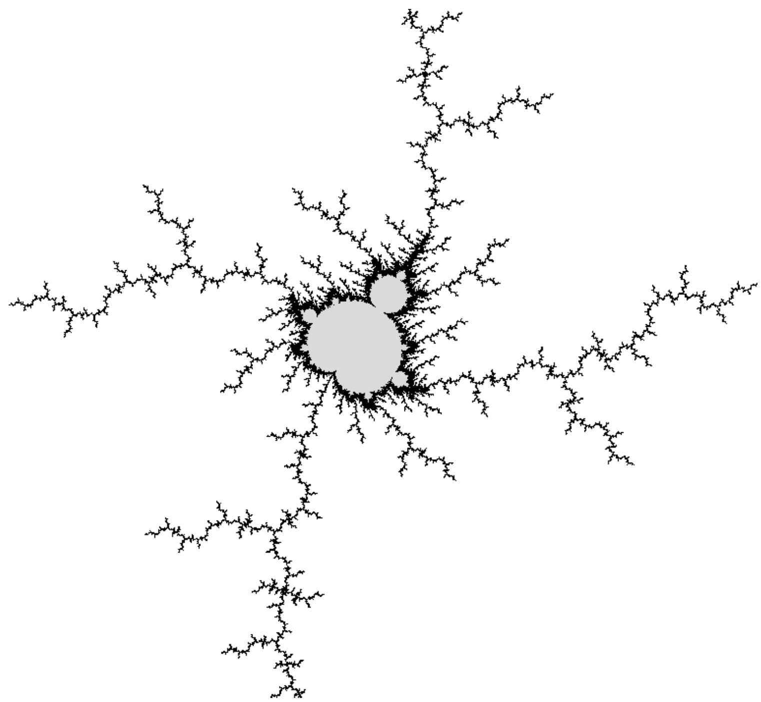

FIGURE 2.4. A little copy of the Mandelbrot set.

By Theorem 2.14, the Mandelbrot set $M_H \equiv M_{\mathbf{g}}$ of this family is homeomorphic, by means of the straightening χ, to the standard Mandelbrot set $\mathbb{M}$. This is how little M-copies appear.

Note that according to the definition of §6.2, the maps f_c are renormalizable for $c \in M_{\mathbf{g}}$ and $g_c = R(f_c)$ is the corresponding renormalization. However, even for $c \in \Lambda \smallsetminus M$ we view g_c as the renormalization of f_c, the *analytic continuation* of $R|M_H$ to the domain Λ. In terms of this renormalization and the picture described in §4.6, we have the following description of the homeomorphism $\sigma : M_H \to M$. Consider the renormalization operator $R : \Lambda \to \mathcal{Q}$ which associates to $c \in \Lambda$ the quadratic-like germ g_c. It maps Λ to a quadratic-like family $\mathcal{S}$ in $\mathcal{Q}$. The straightening χ maps $\mathcal{S}$ back to the quadratic family $\mathcal{QP}$. Then

$$\sigma = \chi \circ R \,|\, M_H. \tag{2.12}$$

We call σ the *stretching homeomorphism*.

In the case of a satellite hyperbolic component $H \subset \operatorname{int} M$, there is still a domain $\Lambda \supset H$ endowed with a quadratic-like family $g_c : f_c^p : V_c \to V_c'$. However, Λ does not contain $\bar{H}$ but instead $\partial\Lambda$ touches ∂H at the root of H. Because of that, the family $\mathbf{g}$ over Λ is not proper. Despite that, the Mandelbrot set $M_{\mathbf{g}}$ is homeomorphic to $\mathbb{M} \smallsetminus 1/4$

and this homeomorphism extends to a homeomorphism $\sigma : \bar{M}_{\mathbf{g}} \to \mathbb{M}$. Thus, we have:

THEOREM 2.27. *Any hyperbolic component H of* $\operatorname{int}\mathbb{M}$*, except the component H_0 bounded by the main cardioid, is contained in a canonically defined copy M_H of the Mandelbrot set* $\mathbb{M}$.

The little M-copies corresponding to the first renormalization are *maximal* in the sense that they are not contained in any other M-copy except $\mathbb{M}$ itself. Let $\mathcal{M}$ stand for the family of maximal Mandelbrot copies. Each copy $M \in \mathcal{M}$ corresponds to a certain combinators of the renormalization: the maps in the same M-copy are renormalizable with the came combinatorics while the maps in different M-copies have different combinatorics of the renormalization. Thus, we can label different combinatorial types of the first renormalization by symbols $M \in \mathcal{M}$.

Moreover, this correspondence between the copies and combinatorial types implies that different copies $M \in \mathcal{M}$ are disjoint. Hence we can define the *stretching map*

$$\sigma : M^1 \equiv \bigcup_{M \in \mathcal{M}} M \to \mathbb{M}$$

on the whole set M^1 of renormalizable maps.

Note that the maximal satellite M-copies are attached to the main cardioid C of the Mandelbrot set at the bifurcation points.

The set M^2 of twice renormalizable maps is equal to $M^1 \cap \sigma^{-1} M^1$ and consists of infinitely many copies of second order. In general,

$$M^n = M^1 \cap \sigma^{-1} M^1 \cap \cdots \cap \sigma^{-(n-1)} M^1$$

is the set of n times renormalizable maps. It consists of infinitely many pairwise disjoint copies $M_i^n \in \mathcal{M}^n$ corresponding to different combinatorics of the n-fold renormalization.

Their intersection,

$$M^\infty = \bigcap_{n=0}^{\infty} M^n$$

is the set of infinitely renormalizable maps.

PROPOSITION 2.28. *The connected components of M^∞ are equal to the combinatorial classes of infinitely renormalizable maps (see §I.4.6).*

Hence any infinitely renormalizable combinatorics can be encoded by an infinite string $M_{n_0}, M_{n_1}, M_{n_2}, \ldots$ of maximal M-copies M_{n_i}. In terms of the stretching map σ, this string is determined as follows: for any $c \in \mathcal{C}$, $\sigma^i c \in M_{n_i}$, $i = 0, 1, 2, \dots$, i.e.,

$$\mathcal{C} = \bigcap_{i=0}^{\infty} \sigma^{-i} M_{n_i}.$$

It follows that to any string $(M_{n_0}, M_{n_1}, M_{n_2}, \dots)$ of maximal M-copies corresponds a *non-empty compact connected* combinatorial class.

6.4. Renormalization windows. Let us now consider the M-copies centered at the real line. Such copies and the corresponding renormalization combinatorics will be called *real*. Each real copy M intersects the real line along a closed interval $W \subset [-2, 1/4]$ called a *renormalization window*. It consists of parameter values which are renormalizable with a certain real combinatorics as described in §6.1.

Remark. Note that unreal parameters c in a real M-copy are also considered to be renormalizable with a *real* combinatorics.

Let $\mathcal{M}_{\mathbb{R}}$ stand for the family of maximal real M-copies. Their real slices are *maximal* renormalization windows J_i corresponding to different combinatorics of the *first* renormalization. This family of intervals will also be denoted by $\mathcal{M}_{\mathbb{R}}$.

EXERCISE 2.29. Find the window corresponding to the doubling renormalization.

Each window is homeomorphic ally mapped onto $[-2, 1/4]$ by the stretching homeomorphism $\sigma : J_i \to [-2, 1/4]$. Altogether they form a set $\mathcal{J}^1 = \cup W_i$ of once renormalizable real maps. The set $\mathcal{J}^2 = \mathcal{J}^1 \cap \sigma^{-1} \mathcal{J}^1$ consists of twice renormalizable real maps. In general, the set

$$\mathcal{J}^n = \mathcal{J}^1 \cap \sigma^{-1} \mathcal{J}^1 \cap \dots \cap \sigma^{-(n-1)} \mathcal{J}^1$$

consists of n times renormalizable real maps. Its connected components are renormalization windows of order n corresponding to different combinatorial types. Their intersection,

$$\mathcal{I} \equiv \mathcal{J}^{\infty} = \cap \mathcal{J}^n$$

is the set of infinitely renormalizable maps. It is now obvious that any string $(\boldsymbol{\kappa}_{i_0}, \boldsymbol{\kappa}_{i_1}, \boldsymbol{\kappa}_{i_2}, \dots)$ of real combinatorial types is represented by a closed interval of infinitely renormalizable maps of this type (compare §6.1).

6.5. Structure of the parameter interval. Let us say that a map f_c (or the corresponding parameter value c) is *regular* if it is either hyperbolic or parabolic. Let $\mathcal{R} \subset [-2, 1/4]$ stand for the set of regular parameter values.

The above discussion leads to the following structure of the parameter interval:

$$[-2, 1/4] = (-3/4, 1/4] \cup \mathcal{J}^1 \cup \mathcal{N},$$

where $(-3/4, 1/4]$ is the initial window of regular parameters where the α-fixed point is either attracting or parabolic with multiplier 1, and $\mathcal{N}$ is the set of non-renormalizable irregular parameters.

Consider now a maximal renormalization window J of period p. Stretching it onto $[-2, 1/4]$ by means of σ, we find a similar structure inside of it:

$$J = H \cup (\mathcal{J}^2 \cap J) \cup (\mathcal{N}^1 \cap J),$$

where H is the semi-closed interval of regular parameters for which one of periodic points of period p is attracting or parabolic with multiplier 1 (called the *initial regular window in H*), and $\mathcal{N}^1 \cap J$ is the set exactly once renormalizable irregular parameters in J.

Putting these decompositions together, we obtain:

$$\mathcal{J}^1 = \mathcal{R}^1 \cup \mathcal{J}^2 \cup \mathcal{N}^1$$

with the obvious meaning of the notations. Proceeding to the deeper renormalization levels, we obtain:

$$\mathcal{J}^n = \mathcal{R}^n \cup \mathcal{J}^{n+1} \cup \mathcal{N}^n,$$

where $\mathcal{R}^n$ is the union of initial regular subwindows in all renormalization windows of level n (consisting of exactly n times renormalizable regular maps), and $\mathcal{N}^n$ is the set of exactly n times renormalizable irregular maps.

Note that $\mathcal{R} = \cup \mathcal{R}^n$. Furthermore, $\mathcal{Y} = \cup \mathcal{N}^n$ is the set of real Yoccoz maps (recall §6.1). Altogether, we obtain the following decomposition of the parameter interval into three disjoint subsets:

$$[-2, 1/4] = \mathcal{R} \cup \mathcal{Y} \cup \mathcal{I}. \tag{2.13}$$

Note that, in fact, this decomposition is valid formally by the definitions of the sets $\mathcal{R}$, $\mathcal{F}$, and $\mathcal{I}$. However, it is very important to see the structure of these sets as described above.

7. Bibliographical notes

Theory of quadratic-like maps was originated by Douady & Hubbard [**DH2**]. Straightening Theorems for maps and quadratic-like families (§3.2 and §5.2) are high points of the theory.

Complex structure in the space of quadratic-like germs and laminar structure of the hybrid partition was introduced in [**L6**]. For further study of the hybrid lamination in the space of real analytic unimodal maps, see [**ALM, AM3**].

Real renormalization was introduced by Feigenbaum [**F1, F2**] and independently by Coullet & Tresser [**TC**]. The structure of the Feigenbaum attractor (Theorem 2.11) was described by Misiurewicz [**Mi**]. Injectivity Lemma (2.26) is due to de Melo & van Strien [**MS**].

Complex renormalization was defined by Douady & Hubbard [**DH2**]. It was designed to explain the phenomenon of little Mandelbrot copies (§6.3).

For the background in the theory of quasiconformal maps see [**A, LV, GL**].

LECTURE 3

Puzzle and a priori bounds

1. Combinatorics of the puzzle

Kids know well the "puzzle game" of cutting a picture into small pieces and then trying to put them back together. Such a game can be played with dynamical pictures like Julia sets and the Mandelbrot set as well. It turned out to be a very efficient way to describe the combinatorics of the corresponding dynamical systems and to control their geometry.

Our standing assumption will be that *both fixed points of a map f are repelling.* We will first assume that f is a quadratic polynomial and will later explain how to generalize the construction to the quadratic-like case.

1.1. Description of the puzzle. The puzzle game starts by cutting the complex plane with the α-rays R_i, $i = 1, \dots p$, landing at the α-fixed point. These rays are cyclically permuted by the dynamics (see Proposition 1.11). The important feature of this initial configuration is that it is forward invariant under the dynamics.

Let us also select some equipotential $E = E_r$ and consider p closed Jordan disks bounded by this equipotential and two consecutive rays R_i. picture Denote these disks by $P_i^{(-1)}$ and call them *puzzle pieces of depth* -1.

Consider now the preimage of this configuration under f. It consists of the equipotential $E^{(0)} = E_{\sqrt{r}}$ and $2p$ external rays landing at the points α and $\alpha' = -\alpha$. These curves bound $2p - 1$ closed topological disks called *puzzle pieces of depth 0*, $P_i^{(0)}$. One of these puzzle pieces, $V^0 \equiv P_0^{(0)}$, contains the critical point. It is called *critical.* Under f, the puzzle pieces of depth 0 are mapped onto appropriate puzzle pieces of depth 1. This map is univalent for off-critical pieces and is a double covering for the critical one.

Let us keep taking preimages of this configuration of curves. The configuration of depth n consists of the equipotential $E^{(n)} = E_{r^{1/2^n}}$ and 2^{n+1} external rays landing at different points of $f^{-n}\alpha$. They tile the Jordan disk $\Omega^{(n)}$ bounded by $E^{(n)}$ into closed Jordan disks called *puzzle pieces of depth* n, $P_j^{(n)}$. Among these puzzle pieces there is one, $P_0^{(n)}$, containing the critical point. It is called critical. Under f, every puzzle piece $P_j^{(n)}$ of level n is mapped onto some puzzle piece $P_i^{(n-1)}$ of level $n-1$. This map is univalent if $P_j^{(n)}$ is off-critical, and is a double covering if $P_j^{(n)}$ is critical (i.e., if $j=0$) .

Moreover, given some puzzle piece P_i^{n-1} of depth $n-1$, consider those puzzle pieces $P_i^{(n)}$ of depth n whose interior intersects P_i^{n-1}. These puzzle pieces form a tiling of the disk $P_i^{(n-1)} \cap \Omega^{(n)}$. It follows that the family of puzzle pieces of level n satisfies the following *Markov property*:

- If $fP_j^{(n)}$ intersects the interior of $P_i^{(n)}$ then $fP_j^{(n)} \supset P_i^{(n)}$. In fact,

the whole family of puzzle pieces satisfies the following property:

- *Property N.* Any two puzzle pieces are either nested or have disjoint interiors.

Note also that the boundary of each puzzle piece P is a piecewise analytic Jordan curve. The analytic pieces are arcs of equipotentials and external rays. Moreover, ∂P intersects the filled Julia set at finitely many points, iterated preimages of α.

Thus, we obtain tilings of finer and finer neighborhoods $\Omega^{(n)}$ of $K(f)$ by more and more puzzle pieces which nicely behave under the dynamics. We will describe next how these tilings capture the recurrence of the critical orbit.

1.2. Principal nest. Consider a puzzle piece P of depth n and a point z such that $f^m z \in \operatorname{int} P$ for some $n \geq 0$. The puzzle piece Q of depth $n+m$ containing z is called the *pull-back* of P along the orbit $\{f^k z\}_{k=0}^m$. Clearly, the map $f^m : Q \to P$ is a branched covering of degree 2^l, where l is the number of critical puzzle pieces among $f^k Q$, $k = 0, 1, \ldots, m-1$. In particular, if there are no critical puzzle pieces among them, then $f^m : Q \to P$ is univalent. This yields:

LEMMA 3.1. *Let P be a critical puzzle piece and let Q be the pull-back of P along $\{f^k z\}_{k=0}^m$.*

If $f^m z$ is the first landing of the orb z *at* int P*, $m \geq 0$, then $f^m : Q \to P$ is univalent.*

If $z \in \operatorname{int} P$ and $f^m z$ is the first return of the orb z *to* int P*, $m > 0$, then $f^m : Q \to P$ is univalent or a double covering depending on whether Q is off-critical or otherwise.*

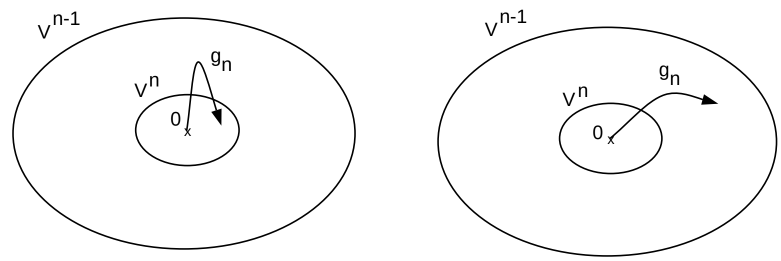

FIGURE 3.1. Central and non-central returns.

We are now ready to introduce the *principal nest* of critical puzzle pieces,

$$V^0 \supset V^1 \supset V^2 \supset \cdots \ni 0, \tag{3.1}$$

and associated double coverings $g_n : V^n \to V^{n-1}$.

Let $V^0 = P_0^{(0)}$. good? Assume inductively that we have defined the nest up to V^{n-1}. If the orb(0) never returns to $\operatorname{int} V^{n-1}$ then the construction stops here. Otherwise consider the first return $f^{l_n}0$ of the critical point back to V^{n-1}. Let V^n be the pull-back of V^{n-1} along this orbit and let $g_n = f^{l_n} : V^n \to V^{n-1}$. By Lemma 3.1, this map is a double covering. This completes the construction.

We call V^n the principal puzzle piece of *level* n (pay attention to the difference between the "level" and the "depth").

A map f is called *combinatorially recurrent* if the critical orbit visits all critical puzzle pieces. In this (and only this) case, the principal nest is infinite.

1.3. Central returns and renormalization. There are two different combinatorial possibilities on every level which are important to distinguish. The return of the critical point to level $n-1$ (and the level itself) is called *central* if $g_n 0 \in V^n$ (see Figure 3.1). In this case, the critical orbit returns to level $n-1$ at the same time as to level n, so that $l_n = l_{n+1}$ and $g_{n+1} : V^{n+1} \to V^n$ is just the restriction of g_n to V^{n+1}. Central returns indicate the *fast* recurrence of the critical orbit.

If N consecutive levels, $m-1$, m, ... , $m+N-2$, are central then the nest

$$V^{m-1} \supset V^m \supset \cdots \supset V^{m+N-1} \tag{3.2}$$

is called a *central cascade* of length $N+1$. In this case, $g^{l_m}0 \in V^{m+N-1}$ and the maps

$$g_{m+k} : V^{m+k} \to V^{m+k-1},\ k = 1, \ldots, N,$$

are just the restrictions of g_m to the corresponding puzzle pieces.

If this cascade is maximal then the levels $m-2$ and $m+N-1$ are non-central. In this case, the length $N+1$ is equal to the *escaping time* it takes for the critical orbit to escape V^m under the iterates of g_m.

If the return to level $m-1$ is non-central, we will formally consider $\{V^{m-1}\}$ to be a "central cascade" of length 1. With this convention, the whole principal nest is decomposed into consecutive maximal central cascades. In fact, one of these cascades, the last one, can have an infinite length:

PROPOSITION 3.2. *A map f is renormalizable if and only if its principal nest ends up with an infinite central cascade $V^{m-1} \supset V^m \supset \dots$. Moreover, in this case the map $g_m : V^m \to V^{m-1}$ is the renormalization of f.*

PROOF. We will explain the "if" direction of this assertion.

Assume that we immediately observe an infinite central cascade $V^0 \supset V^1 \supset \dots$. In this case we say that f is *immediately renormalizable*. One can show that this corresponds to parameters in the satellite M-copies attached to the main cardioid (compare §II.6.2 and §II.6.3).

In the immediately renormalizable case, the critical orbit never escapes V^1 under the iterates of $g_1 = f^p : V^1 \to V^0$ (where p is the number of α-rays). The map g_1 is a double covering of a smaller domain onto a bigger one but it is not a quadratic-like map, since the domains V^1 and V^0 have a common boundary (consisting of four external arcs). To turn this map into a quadratic-like, one should "thicken" the domains V^0 and V^1 a little bit (see Figure ...).

Assume that f is not immediately renormalizable. One can show that in this case, $V^m \Subset V^{m-1}$, so that $g_m : V^m \to V^{m-1}$ is a quadratic-like map with non-escaping critical point, which can be identified with the first renormalization of f. □

Let us define the *height* of f as the number of the maximal central cascades in the principal nest. We see that f *is renormalizable if and only if it has a finite height.*

Thus, the principal nest provides an algorithm to decide whether the map in question is renormalizable, whether this renormalization is of satellite type or otherwise, and to capture this renormalization.

On the negative side, the puzzle provides us with dynamical information only up to the first renormalization level. If we wish to penetrate deeper, we need to cut the Julia set of the renormalization into pieces and to go through its principal nest. Since the renormalization is a quadratic-like map rather than a quadratic polynomial, this motivates the need of the puzzle for quadratic-like maps. It will be discussed in §1.7.

1.4. The first return maps and generalized renormalizations.

DEFINITION 3.1. Let $\{V_i\}$ be a family of Jordan disks with disjoint closures compactly contained in a disk U, $V_0 \ni 0$. A holomorphic map $g : \cup V_i \to U$ is called a *generalized quadratic-like map* if the restriction $g : V_0 \to U$ is a double branched covering (with critical point at the origin), while all other restrictions $g : V_i \to U$ are diffeomorphisms. We will also assume that the central domain V_0 is symmetric with respect to the origin and the restriction $g|V_0$ is even. The continuous extension of g to a map $\cup \bar{V}_i \to \bar{U}$ will also be called a generalized quadratic-like which will not be distinguished from the original one.

Main examples of generalized quadratic-like maps are provided by the "first return maps" $g_n : \cup V_i^n \to V^{n-1}$ to the principal puzzle pieces. They are defined as follows. Take a point $z \in \operatorname{int} V^{n-1}$ which returns back to $\operatorname{int} V^{n-1}$ and consider this first return $f^{l(z)}z$. Let $V(z)$ be the pull-back of V^{n-1} along $\{f^k z\}_{k=0}^{l(z)}$ and let $g_n|V(z) = f^{l(z)}$. Property N and Lemma 3.1 easily imply:

PROPOSITION 3.3. *For two points z and ζ, the puzzle pieces $V(z)$ and $V(\zeta)$ either coincide or have disjoint interiors, so that they can be labelled as V_i^n starting with the critical puzzle piece $V^n \equiv V_0^n \equiv V^n(0)$ (provided 0 returns to $\operatorname{int} V^{n-1}$). The map $g_n : V_0^n \to V^{n-1}$ is a double covering while the maps $g_n : V_i^n \to V^{n-1}$ are univalent for $i > 0$. Moreover, if $V^{n-1} \subset \operatorname{int} V^{n-2}$, then $V_i^n \subset \operatorname{int} V^{n-1}$ and all these pieces are pairwise disjoint.*

To keep track of the recurrence of the critical orbit, consider the *itinerary* $(i_1, \dots, i_{r-1})$ of it under iterates of g_n until it returns back to V^n. This itinerary is defined by the following rule:

$$g_n^k 0 \in V_{i_k}^n, \quad k = 1, \dots, r-1,$$

where $g_n^r 0$ is the first return of the critical point back to V^n. These itineraries contain the most basic combinatorial information about f.

It is often sufficient to consider only the puzzle pieces V_i^n intersecting the critical orbit. The restriction of g_n to those puzzle pieces (considered up to rescaling) is called the *generalized renormalization* of f on V^{n-1}. In many interesting cases the generalized renormalizations are defined on finitely many puzzle pieces V_i^n:

PROPOSITION 3.4. *Let f be a quadratic map with an infinite principal nest. The property that all levels $n-1$ contain only finitely many puzzle pieces V_i^n intersecting $\operatorname{orb}(0)$ is equivalent to one of the following conditions:*

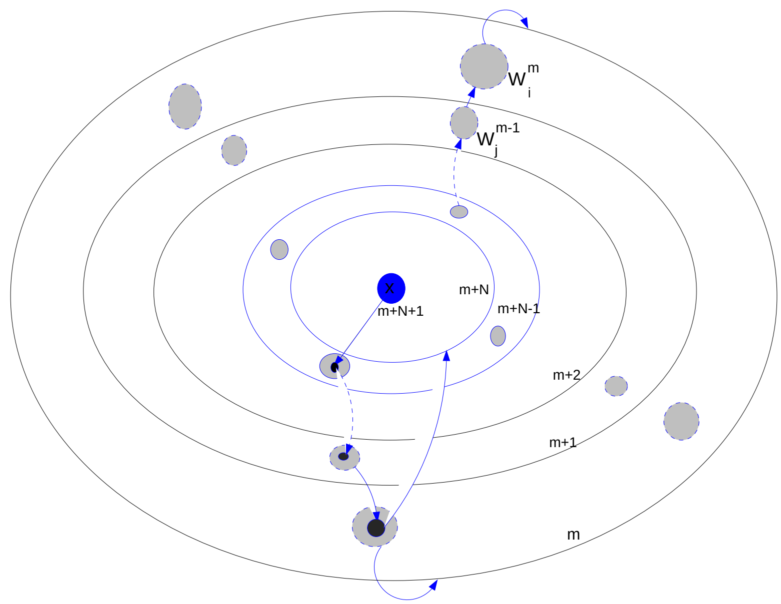

FIGURE 3.2. Solar system.

- *f is renormalizable,*
- *or f is non-renormalizable and the postcritical set $\omega(f)$ is minimal (i.e., all orbits of $x \in \omega(f)$ are dense in $\omega(f)$).*

1.5. Bernoulli scheme. We will now describe a Bernoulli scheme which performs a transit from the bottom to the top of a central cascade. Such transits will be treated as single steps of the generalized renormalization procedure ("cascade renormalization").

Consider a central cascade (3.2) and let $g = g_m : V^m \to V^{m-1}$. Then the restrictions

$$g : V^k \smallsetminus V^{k+1} \to V^{k-1} \smallsetminus V^k, \quad k = m, \dots, m+N-1,$$

are double branched coverings. Pull the non-central puzzle pieces $V_i^m \subset V^{m-1} \smallsetminus V^m$ from the top annulus to the consecutive annuli $V^k \smallsetminus V^{k+1}$. We obtain a family of puzzle pieces $W_j^k \subset V^k \smallsetminus V^{k+1}$ such that g^{k-m} univalently maps W_j^k onto some puzzle piece $V_i^m \equiv W_i^m$ (see Figure 3.2).

change m to m-1

Let us consider the following map

$$G_m : \cup W_j^k \to V^{m-1}, \quad G|W_j^k = g_m \circ g^{k-m} \tag{3.3}$$

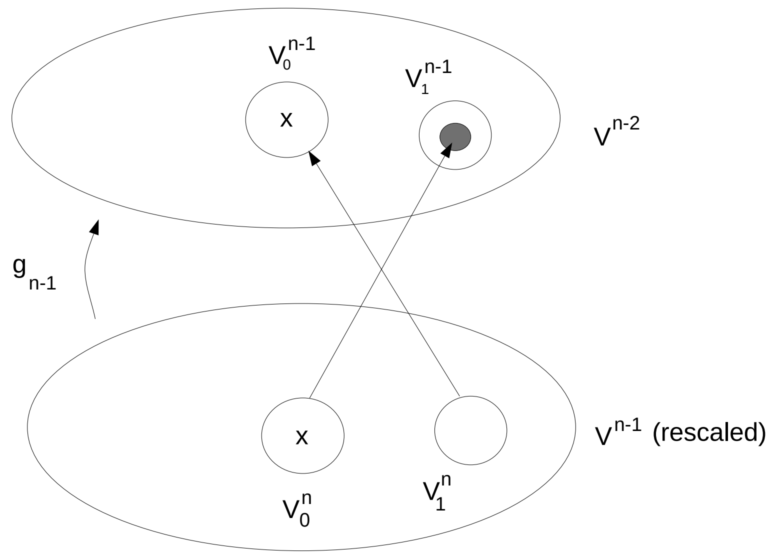

FIGURE 3.3. Fibonacci renormalization scheme.

This map is *(unbranched) Bernoulli* in the sense that it univalently maps each domain W_j^n onto V^{m-1}. This yields the following decomposition of the next renormalization, g_{m+N+1}:

PROPOSITION 3.5. *The map* $g_{m+N+1} : V^{m+N+1} \to V^{m+N}$ *can be represented as* $h_{m+N+1} \circ f$, *where* h_{m+N+1} *is a univalent map with range* V^{m-1}.

1.6. The Fibonacci map. There is a remarkable non-renormalizable real quadratic map $f_{\text{fib}} : z \mapsto z^2 + c_{\text{fib}}$ called *Fibonacci*. It is combinatorially determined by the property that the closest returns of the critical point to itself occur at the Fibonacci moments $1, 2, 3, 5, 8, \ldots$. Many interesting phenomena of the quadratic dynamics can be tested on this example.

The most efficient way of understanding combinatorics of the Fibonacci map is provided by the generalized renormalization discussed above. For this map, the domain of g_n consists of only *two* puzzle pieces, so that

$$g_n : V_0^n \cup V_1^n \to V_0^{n-1}, \tag{3.4}$$

where $g_n : V_0^n \to V_0^{n-1}$ is a double branched covering, while $g_n : V_1^n \to V_0^{n-1}$ is univalent. Moreover, $g_n(0) \in V_1^n$, while $g_n(g_n 0) \in V_0^n$, i.e., after landing at the non-central piece V_1^n, the critical point immediately returns back to the central piece V_0^n. Thus, the Fibonacci map is the most recurrent among maps without central cascades.

Figure 3.3 n -¿ n+1 shows how to pass from one level of the Fibonacci map to the next under the generalized renormalization. Notice that V^{n+1} is mapped under g_n *strictly inside* V_1^n, and that the annulus $V_1^n \setminus g_n(V^{n+1})$ is conformally equivalent (by means of g_n) to $V^{n-1} \setminus V^n$. This annulus provides a comfortable extra space securing nice properties of the generalized renormalization.

Even if you never heard about Fibonacci map, the generalized renormalization analysis would inevitably lead you to this extreme renormalization scheme.

1.7. Quadratic-like case. In the case of a quadratic-like map $f : U \to U'$, there are no canonically defined external rays and equipotentials. However, one can make a choice which will suit to all purposes. Namely conjugate f to its straightening $f_c : \Omega_{\sqrt{r}} \to \Omega_r$ by means of some qc map $h : U' \to \Omega_r$, where Ω_r is the domain bounded by the equipotential of radius r. Then external rays and equipotential of f can be defined as the pull-backs of the corresponding curves of f_c.

With this choice we can start the puzzle by cutting U' with the α-rays of f. We obtain p puzzle pieces $P_i^{(-1)}$. Taking the preimages of these puzzle pieces, we obtain a tiling of U by $2p-1$ puzzle pieces $P_i^{(0)}$, etc.: the whole above discussion can be carried on without changes.

To control the geometry of this puzzle, we will need to control the qc dilatation of h. According to Proposition II.2.3, h can be selected in such a way that its dilatation depends only on the modulus of the fundamental annulus, $\mathrm{mod}(U' \setminus U)$ (after some adjustment of f). In what follows we will always assume that h is selected in this way.

1.8. Real principal nest. If $f : I \to I$ is a real quadratic map then taking the real traces of puzzle pieces V_k^n we obtain the *real principal nest* of puzzle pieces,

$$[-\alpha, \alpha] \equiv I^0 \supset I^1 \supset I^2 \supset \ldots,$$

and corresponding first return maps $g_n : \cup I_k^n \to I^{n-1}$. The restriction of g_n to the central interval, $g_n : (I^n, \partial I^n) \to (I^{n-1}, \partial I^{n-1})$, is unimodal, while its restrictions to the non-central intervals I_k^n, $k \neq 0$, are diffeomorphisms onto I^{n-1}.

In the real case, non-central returns to I^{n-1} are naturally classified as *high* and *low* depending on whether $g_n(I^n) \ni 0$ or otherwise. Accordingly, there are two types of central cascades

$$I^{m-1} \supset I^m \supset \cdots \supset I^{m+N-1} \tag{3.5}$$

depending on whether the return $g_m : I^m \to I^{m-1}$ is high or low. In the high case, the central cascade is called *Ulam-Neumann* or *Chebyshev.*

In the low case, it is called *saddle-node* or *parabolic*. The reason is that in the high case the map $g_m : I^m \to I^{m-1}$ is combinatorially close to the Ulam-Neumann map $f_{-2} : x \mapsto x^2 - 2$, while in the low case it is close to the parabolic map $f_{1/4} : x \mapsto x^2 + 1/4$.

The principal nest ends with an infinite central cascade if and only if the map f is renormalizable. In this case, the intersection $\cap I^n$ is the maximal domain J of the first pre-renormalization.

The above discussion admits a straightforward generalization to all S-unimodal maps. In this case, we let $I^0 = [-\alpha, \alpha]$ and define I^n inductively as the pullback of I^{n-1} corresponding to the first return of the critical orbit to I^{n-1} (that is, I^n is the component of $f^{-l}(I^{n-1})$ containing 0, where l is the first return time of 0 to I^{n-1}).

The notions of generalized renormalization,

$$g_n : I^n_k \to I^{n-1}, \tag{3.6}$$

Bernoulli scheme, etc. also extend readily to the S-unimodal setting.

1.9. Essentially bounded combinatorics. We will now introduce a combinatorial parameter, "essential period", which will control geometry of the puzzle.

Let us consider a maximal saddle-node cascade (3.5). Take a point $x \in \omega(0) \cap (I^{m-1} \setminus I^m)$ and assume that $g_m(x) \in I^j \setminus I^{j+1}$. Then let

$$d(x) = \max(j - m,\ m + N - j).$$

(If $g_m(x) \in I^{m+N+1}$ then let $d(x) = 0$.) This parameter, the *depth of return*, shows how deep inside the cascade the point x lands under the return map. Let $d = \max d(x)$ over x as above. Then the levels $l \in (m + d, m + N - d)$ of the cascade are called *neglectable*.

Let now f be a renormalizable unimodal map of period p. Consider the orbit $\{f^n 0\}_{n=0}^{\infty}$, and and remove from it all "neglectable" points, i.e., such that $f^n 0 \in I^l \setminus I^{l=1}$ for some neglectable level l in some saddle-node cascade. The number of intervals that are left is called the *essential period* of f, $\mathrm{per}_e(f)$. Roughly speaking, it is the renormalization period neglecting time spent near "ghost parabolic points", deep inside saddle-node cascades.

We say that an infinitely renormalizable map has *essentially bounded combinatorics* if $\sup_n \mathrm{per}_e(R^n f) < \infty$.

Note that a bound on the essential period is equivalent to a bound on the following combinatorial parameters:

(i) the height (see §1.3);

(ii) the return times of the intervals I^n_k (the domains of the generalized renormalizations) to I^{n-1} under iterates of g_{n-1};

(iii) the lengths of the Ulam-Neumann cascades;
(iv) the depths of landings at the saddle-node cascades.

2. Geometry of the puzzle

2.1. Principal moduli. The main geometric parameters that control the geometry of the puzzle are the moduli of the annuli in the principal nest:

$$\mu_n = \mathrm{mod}(V^{n-1} \setminus V^n),$$

called the *principal moduli* of the puzzle. If the principal moduli are *definite* (i.e., $\mu_n > \mu$ for some $\mu > 0$), then the situation is under control (see §II.3.4). Note, however, that $\mu_{n+1} = \mu_n/2$ if $n-1$ is a central level, so that the principal moduli decay exponentially within central cascades. This does not cause a big trouble, though, because the Bernoulli scheme of §1.5 allows us to control the whole cascade by the top principal modulus (Proposition 3.3). So, what is important is to have definite moduli on the top of the cascades.

It turns out that the actual situation is even much better: the principal moduli on those levels grow at linear rate, so that by Proposition 2.3 the corresponding quadratic-like maps are getting close (exponentially fast) to pure quadratic polynomials. This is the crucial geometric property of the puzzle:

Theorem 3.6. *Let $f : U \to U'$ be a quadratic-like map with*

$$\mathrm{mod}(U' \setminus \bar{U}) \geq \nu.$$

Assume that the critical orbit escapes $V^0 \cup_{i \neq 0} P_i^{(-1)}$ in no longer than N iterates. Let $\{n_k - 1\}$ stand for the sequence of non-central levels in the principal nest. Then

$$\mathrm{mod}(V^{n_k} \setminus V^{n_k+1}) \geq Ck,$$

where the constant C depends only on N and ν. In particular, the constant C is uniform over all real polynomials.

Exercise 3.7. Show that for a real map, which is not renormalizable with period 2, one can take $N = 2$.

Below we will outline key ideas of the proof of Theorem 3.6.

2.2. First modulus. The bound N on the escaping time allows us to control the geometry of the initial levels of the puzzle:

Lemma 3.8. *Let $V^0 \supset V^1 \supset V^{m-1}$ be the first central cascade of the puzzle. Then the principal modulus* $\mathrm{mod}(V^m \setminus V^{m+1})$ *is bounded away from 0 by some constant $\underline{\mu}_1$ depending only on N and ν.*

2.3. Asymmetric moduli. The principal moduli do not behave nicely under the generalized renormalization, most notably, they decay within central cascades. For this reason, we will consider other geometric parameters, "asymmetric moduli" σ_n, which monotonically grow under the generalized renormalization. They are made of some combinations of moduli of certain annuli in the puzzle.

In the Fibonacci case (see §1.6), the definition is quite simple. Let R_i^n denote the maximal annulus in $V^{n-1} \setminus (V_0^n \cup V_1^n)$ which goes around V_i^n but does not go around V_{1-i}^n. Then

$$\sigma_n = \operatorname{mod} R_0^n + \frac{1}{2} \operatorname{mod} R_1^n.$$

To estimate inductively the asymmetric moduli, we need the following classical inequality:

Grötzsch Inequality. *Let A be a conformal annulus which is divided by a homotopically non-trivial Jordan curve γ into two annuli, A_1 and A_2. Then*

$$\operatorname{mod} A \geq \operatorname{mod} A_1 + \operatorname{mod} A_2.$$

Now, the Fibonacci renormalization scheme depicted on Figure **??** yields:

$$\operatorname{mod} R_1^{n+1} \geq \operatorname{mod} R_0^n;$$

$$\operatorname{mod} R_0^{n+1} \geq \frac{1}{2} \operatorname{mod}(V^{n-1} \smallsetminus g_n(V_0^{n+1})) \geq \frac{1}{2}(\operatorname{mod} R_0^n + \operatorname{mod} R_1^n),$$

where the last line follows from the Grötzsch inequality and the fact that the annulus $V_1^n \smallsetminus g_n(V_0^{n+1})$ is conformally equivalent to $V^{n-1} \smallsetminus V^n$. Taking the combination of the above two estimates with coefficients $1/2$ and 1, we conclude that $\sigma_{n+1} \geq \sigma_n$.

The general estimate given below is much more involved.

Let us fix a level $n > 0$, and denote $V^{n-1} = \Delta$, $V_i = V_i^n$, $g = g_n$, $\mu = \mu_n$. Mark the objects of the next level $n+1$ with prime: $\Delta' \equiv V \equiv V_0$, and $g' : \cup V_i' \to \Delta'$. (However, we restore the index n whenever we need it).

Let $\{V_i\}_{i \in \mathcal{I}}$ be a finite family of disjoint puzzle pieces consisting of at least two pieces (that is $|\mathcal{I}| \geq 2$) and containing the critical puzzle piece V_0. Let us call such a family *admissible*. We will freely identify the label set $\mathcal{I}$ with the family itself.

Given a puzzle piece $D \subset \Delta$, let $\mathcal{I}|D$ denote the family of puzzle pieces of $\mathcal{I}$ contained in D. Let D be a puzzle piece containing at least two pieces of family $\mathcal{I}$. For $V_i \subset D$ let

$$R_i \equiv R_i(\mathcal{I}|D) \subset D \setminus \bigcup_{j \in \mathcal{I}|D} V_j$$

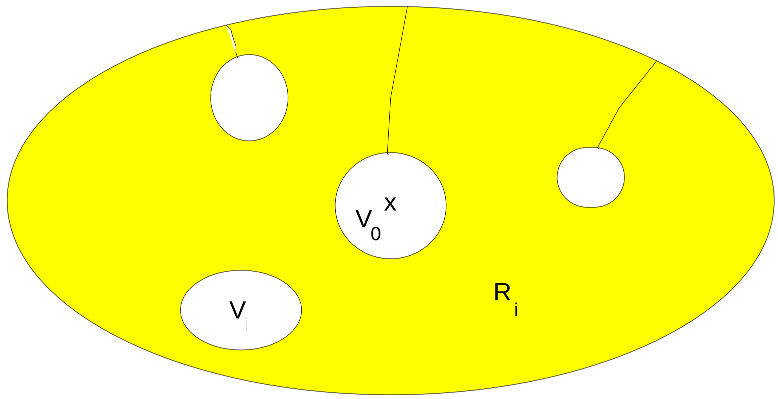

FIGURE 3.4. Annulus R_i

be an annulus of maximal modulus enclosing V_i but not enclosing other pieces of the family $\mathcal{I}$ (see Figure 3.4). (Such an annulus exists by Montel's Theorem).

Let us define *the asymmetric modulus of the family* $\mathcal{I}$ *in* D as

$$\sigma(\mathcal{I}|D) = \sum_{i\in\mathcal{I}} \frac{1}{2^{1-\delta_{i0}}} \operatorname{mod} R_i(\mathcal{I}|D),$$

where δ_{ji} is the Kronecker symbol. So the critical modulus is supplied with weight 1, while the off-critical moduli are supplied with weights 1/2 (if D itself is off-critical then all the weights are actually 1/2).

In the case $D = \Delta \equiv V^{n-1}$, let $\sigma_n(\mathcal{I}) \equiv \sigma(\mathcal{I}|V^{n-1})$. The asymmetric modulus of level n is defined as follows:

$$\sigma_n = \min_{\mathcal{I}} \sigma_n(\mathcal{I}),$$

where $\mathcal{I}$ runs over all admissible subfamilies of $\mathcal{V}^n$.

2.4. Non-decreasing of the moduli. Let $\{V_i'\}_{i\in\mathcal{I}'}$ be an admissible subfamily of $\mathcal{V}'$. Let us organize the pieces of this family in *isles* in the following way. A puzzle piece $D' \subset \Delta'$ is called an *island* (for family I') if

- D' contains at least two puzzle pieces of family $\mathcal{I}'$;
- There is a $t \geq 1$ such that $g^k D' \subset V_{i(k)}$, $k = 1, \dots t-1$, with $i(k) \neq 0$, while $g^t D = \Delta$.

Given an island D', let $\phi_{D'} = g^t : D' \to \Delta$. This map is either a double covering or a biholomorphic isomorphism depending on whether D' is critical or not. In the former case, $D' \supset V_0'$ (for otherwise $D' \subset V_0'$ contradicting the first part of the definition of isles).

We call a puzzle piece $V'_j \subset D'$ $\phi_{D'}$*-precritical* if $\phi_{D'}(V'_j) = V_0$. There are at most two precritical pieces in any D'. If there are actually two of them, then they are off-critical and symmetric with respect to the critical point 0. In this case D' must also contain the critical puzzle piece V'_0.

Let $\mathcal{D}' = \mathcal{D}(\mathcal{I}')$ be the family of isles associated with $\mathcal{I}'$.

Let us call an island D' *innermost* if it does not contain any other isles of the family $\mathcal{D}(\mathcal{I}')$. As this family is finite, innermost isles exist.

LEMMA 3.9. *Let $\mathcal{I}'$ be an admissible family of puzzle pieces. Let D' be an innermost island associated to the family $\mathcal{I}'$, and let $\mathcal{J}' = \mathcal{I}'|D$. For $j \in \mathcal{J}'$, let us define $i(j)$ by the property $\phi_{D'}(V'_j) \subset V_{i(j)}$, and let $\mathcal{I} = \{i(j) : j \in \mathcal{J}'\} \cup \{0\}$. Then $\{V_i\}_{i \in \mathcal{I}}$ is an admissible family of puzzle pieces, and*

$$\sigma(\mathcal{I}'|D') \geq \frac{1}{2}\left((|\mathcal{J}'| - s)\mu + s \mod R_0 + \sum_{j \in \mathcal{J}', i(j) \neq 0} \mod R_{i(j)}\right),$$

where $s = \#\{j : i(j) = 0\}$ is the number of $\phi_{D'}$-precritical pieces, and R_i are the maximal annuli enclosing V_i in Δ rel $\mathcal{I}$.

Since $\mu \geq \mod R_0$, we conclude:

COROLLARY 3.10. *For any island D' of the family $\mathcal{I}'$*

$$\sigma(\mathcal{I}'|D') \geq \frac{1}{2}\mu \quad \textit{and} \quad \sigma(\mathcal{I}'|D') \geq \sigma(\mathcal{I}) \geq \sigma.$$

Hence:

COROLLARY 3.11. *The asymmetric moduli σ_n do not decrease under the generalized renormalization: $\sigma_n \geq \sigma_{n-1} \geq \cdots \geq \sigma_2 \geq \mu_1/2$.*

This yields *a priori* bounds on the principal moduli:

THEOREM 3.12. *Under the circumstances of Theorem 3.6,*

$$\mathrm{mod}(V^{n_k} \setminus V^{n_k+1}) \geq \underline{\mu} > 0,$$

where the constant $\underline{\mu}$ depends only on N and ν.

PROOF. It is easy to see that if $n-1$ is a non-central level, then $\mu_{n+1} \geq \sigma_n/2$. With this, the assertion (with $\underline{\mu} = \underline{\mu}_1/4$) follows from Corollary 3.11 and Lemma 3.8. ☐

2.5. Linear growth of the moduli. To obtain the linear growth of the principal moduli, we need to exploit Lemma 3.9 more carefully. There are several circumstances that can give us some extra gain in Corollary 3.11 and to show that $\sigma' \geq \sigma + a$ with a definite $a > 0$, at least on every other level, except for the tails of long central cascades. Clearly, it is enough to show that

$$\sigma(\mathcal{I}'|D') \geq \sigma + a \tag{3.7}$$

for any innermost island D'.

Assume that level $n-1$ is not in the tail of a central cascade, so that the modulus $\mu > 0$ is definite by Theorem 3.12. We will use notation of Lemma 3.9 and refer to the estimate therein as the "key estimate".

• If the innermost island D contains at least three puzzle pieces, i.e., $|\mathcal{J}'| \geq 3$, then we can split off one μ in the key estimate to obtain:

$$\sigma(\mathcal{I}'|D') \geq \frac{1}{2}\mu + \sigma(I),$$

and we get that extra gain with $a = \mu/2$.

In what follows we assume that all innermost isles contain two puzzle pieces.

• Let us consider the disk Δ as the hyperbolic plane. Fix some big constant L. If the hyperbolic distance from any off-critical puzzle piece $V_{i(j)} \subset \Delta$ to 0 is bounded by L, then $\mu \geq \text{mod}(R_0) + a$, where $a = a(L) > 0$. This lower bound on the μ's in the key estimate, yields (3.7).

• Assume now that the hyperbolic distance from any off-critical puzzle piece $V_{i(j)}$ to 0 is at least L. The case when both puzzle pieces $V_{i(j)}$ are off-critical is easy to treat, so that we assume that one of these puzzle piece, $V_0 \equiv V_0^n$, is critical, and the other one, $V_1 \equiv V_1^n$, is off-critical. But for the same reason we can assume that the images of these two puzzle pieces under g_{n-1} satisfy the same properties, which brings us to one of the following situations:

1) *Fibonacci return*: $g_{n-1}V_0^n \subset V_1^{n-1}$ and $g_{n-1}V_1^n = V_0^{n-1}$ (see Figure 3.3);

2) *Central return*: $g_{n-1}V_0^n = V_0^{n-1}$ and $g_{n-1}V_1^n \subset V_1^{n-1}$.

The Fibonacci return is the most delicate case to analyze. In this case we use the last reserve in our disposal:

Definite Grötzsch Inequality. *Under the circumstances of the Grötzsch Inequality (§2.3), let K be the set of points in A which are separated by $A_1 \cup A_2$ from ∂A. Then*

$$\text{mod}(A) \geq \text{mod}(A_1) + \text{mod}(A_2) + \beta,$$

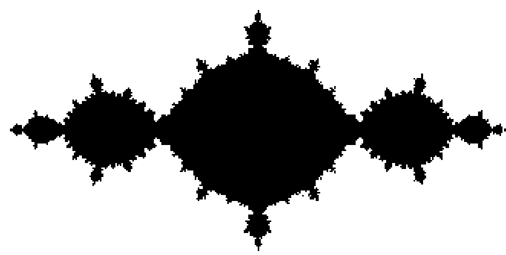

FIGURE 3.5. A puzzle piece for the Fibonacci map. There is a good reason why it resembles the Julia set of $z^2 - 1$.

where the constant $\beta > 0$ *depends only on the width of* K *in* A *as defined below.*

To define the *width* of K, uniformize A by a round annulus and inscribe K (in this round model) into the smallest concentric annulus R. Then $\text{width}(K) = \text{mod}(R)$.

One can now show that in the Fibonacci case, the boundary ∂V^{n-1} is *pinched* in between V_0^n and V_1^n (see Figure 3.5), and this pinching yields (3.7) due to the Definite Grötzsch Inequality.

In the central case, one should go all way from the top to the bottom of the central cascade and carry similar estimates for this cascade renormalization.

2.6. Scaling factors. In the real case, a natural replacement for the principal moduli μ_n is provided by *scaling factors*

$$\lambda_n = \frac{|I^n|}{|I^{n-1}|},$$

where $\{I^n\}$ is the real principal nest from §1.8. Since $\lambda_n = O(e^{-\mu_n})$, Theorem 3.6 implies exponential decay of the scaling factors:

THEOREM 3.13. *Let* $f : U \to U'$ *be a real quadratic-like map with* $\text{mod}(U' \setminus \bar{U}) \geq \nu$. *Let* $\{n_k - 1\}$ *stand for the sequence of non-central levels in the principal nest. Then*

$$\lambda_{n_k+1} \leq C\rho^k,$$

where the constants C *and* $\rho < 1$ *depends only on* ν. *In particular, the constants are uniform over all real quadratic polynomials.*

In the non-holomorphic case the situation is similar:

THEOREM 3.14. *Let f be a non-renormalizable S-unimodal map, and let $\{n_k - 1\}$ be the sequence of non-central levels of its principal nest. Then $\lambda_{n_k+1} \le C\rho^k$, where $\rho < 1$.*

3. A priori bounds

3.1. Distortion principles. The reason why Schwarzian derivative is so important for one-dimensional dynamics is because the classical Schwarz Lemma and Koebe Distortion Theorem (see §II.3.4) are valid for maps with positive Schwarzian derivative. This leads to distortion techniques which plays a fundamental role in the field.

Any interval $I = (a, b)$ can be viewed as a hyperbolic line endowed with the hyperbolic metric $dx/(x-a)(b-x)$. In the metric, the length of an interval $J = (c, d) \Subset I$ is given by the logarithm of the following cross-ratio:

$$|J|_{\text{hyp}} = \log \frac{(d-a)(b-c)}{(c-a)(b-d)} = \log \left(1 + \frac{|K|}{|L|}\right) + \log \left(1 + \frac{|K|}{|R|}\right),$$

where L and R are the left-hand and the right-hand components of $I \smallsetminus J$ respectively.

Recall that maps with vanishing Schwarzian derivative preserve cross-ratios (as such maps are actually Möbius). So, it is not surprising that maps with positive Schwarzian derivative must either contract or expand cross-ratios (depending on the particular choice of the cross-ratio). This can be expressed in the following geometric way:

Real Schwarz Lemma. *A diffeomorphism $\phi : I \to I'$ with positive Schwarzian derivative is contracting with respect to the hyperbolic metric. Hence, for any interval $J \Subset I$, if $|L| \ge \varepsilon |J|$ and $|R| \ge \varepsilon |J|$ then*

$$|\phi(L)| \ge \delta |\phi(J)| \quad \text{and} \quad |\phi(\mathrm{R})| \ge \delta |\phi(\mathrm{J})|,$$

where $\delta = \delta(\varepsilon) > 0$ depends only on $\varepsilon > 0$.

Distortion and nonlinearity for real diffeomorphisms are defined exactly in the same way as for conformal ones (see §II.3.4). An important consequence of the Schwarz Lemma is:

Real Koebe Distortion Theorem. *Let $\phi : I \to I'$ be a diffeomorphism with positive Schwarzian derivative, and let $K \Subset I$.*
If $|L| \ge \varepsilon |J|$ and $|L| \ge \varepsilon |J|$, then $n(\phi|J) \le \log K$, where $K = K(\varepsilon)$ depends only on $\varepsilon > 0$.

3.2. Real bounds. The results of this section constitute the first important step towards the Universality Phenomenon which will be discussed in the last lecture.

Let $f : I \to I$ be an infinitely renormalizable S-unimodal map. Recall from §II.6.1 that J^n stand for the central interval of the the n-fold renormalization, and $J_k^n = f^k J^n,\ k = 0, 1 \ldots, p_n - 1$.

Note that any even unimodal map can be decomposed as $\phi \circ f_0$, where $f_0 : x \mapsto x^2$ and $\phi : f_0(I) \to I$ is a diffeomorphism onto its image. If f is S-unimodal and $n(\phi) \le \log K$, we say that f is a *K-quasiquadratic map.* Since the bound on the non-linearity yields a bound on the second derivative, we have:

LEMMA 3.15. *The space of K-quasiquadratic maps $f : I \to I$ is compact in C^1-topology.*

not quite true: go to the Epstein class

We say that an interval J is *well inside* $T \Supset J$ if $|L| \ge \delta|I|$ and $|R| \ge \delta|I|$ for both connected component L and R of $T \smallsetminus J$, where $\delta > 0$ depends only on some specified quantifiers (e.g., on the distortion K in Theorem 3.16).

THEOREM 3.16. *Let f be an infinitely renormalizable K-quasiquadratic map. Then:*

(i) *There is a $\delta = \delta(K) > 0$ and intervals $T^n \supset S^n \supset J^n \ni 0$ such that $f^{p_n} : (S^n, \partial S^n) \to (T^n, \partial T^n)$ is a unimodal map, and $|T^n| \ge (1+\delta)|S^n|$;*

(ii) *The real renormalizations $R^n f$ are C-quasiquadratic maps, with C depending only on K. Moreover, they form a precompact family in C^1 topology.*

All the above bounds are eventually uniform, that is, they do not depend on K on sufficiently big (depending on f) renormalization levels.

We express property (i) by saying that the maps $f^{p_n} : J^n \to J^n$ admit a *definite extension*, or that there is a *definite space* around J^n.

PROOF. The argument whose idea is given below is called the *Shortest Interval Argument.* It is based on the observation that there is some "space" around the shortest interval J_k^n of level n which can be pulled back to produce space around the central interval J^n. Consider next to J_k^n intervals of level n and their midpoints. (There are two such intervals unless J_k^n is the first or the second iterate of J^n.) Let H^n be the convex hull of J_k^n and these midpoints. Since J_k^n is the shortest interval, $|L| \ge (1/2)|J|$ for any connected component L of $H^n \smallsetminus J_k^n$.

Let $T^n \ni 0$ be the pullback of J_k^n under f^k. It is a fun exercise to show that the map $f^k : T^n \to H^n$ is unimodal, i.e., that the map

$f^{k-1}: f(T^n) \to H^n$ is a diffeomorphism onto its image. Moreover, by the Real Schwarz Lemma, the interval J^n is well inside T^n.

Let now $S^n \ni 0$ be the pullback of T^n under f^{p_n}. For the same reason as above, the map $f^{p_n}: S^n \to T^n$ is unimodal and J^n is well inside T^n. This yields (i).

Bounded nonlinearity follows by the Real Koebe Distortion Theorem. It implies bounded second derivative for the renormalization, which in turn implies C^1-precompactness.

Let $f|J^n = \phi_n \circ f_0$. Notice that all the above bounds depend only on the non-linearity of ϕ_n. Since $n(\phi_n) \to 0$ as $n \to \infty$, the bounds are eventually uniform. □

Recall that in the doubling case the two renormalization intervals touch each other. To deal with this situation, let us replace the maximal central intervals J^n by the minimal central intervals $\tilde{J}^n$ (see Exercise 2.21). Then the intervals $\tilde{J}^n_k = f^n \tilde{J}^n$, $k = 0, 1, \dots, p_n - 1$, do not touch any more. The connected components of $\tilde{J}^{n-1}_k \setminus \cup_j \tilde{J}^n_j$ are called the *gaps* of level n. It is not hard to deduce from Theorem 3.16 that the gaps cannot be too small compared with the adjacent intervals $\tilde{J}^n_k$:

Lemma 3.17. *Let f be an infinitely renormalizable K-quasiquadratic map. Then there exists an $\varepsilon = \varepsilon(K) > 0$ such that for any gap G^n_k adjacent to an interval $\tilde{J}^n_k$ we have: $|G^n_k| \geq \varepsilon |\tilde{J}^n_k|$.*

Now the Lebesgue Density Points Theorem yields:

Corollary 3.18. *Let f be an infinitely renormalizable S-unimodal map. Then the postcritical set O_f has zero Lebesgue measure.*

One says that f has *bounded geometry* if for any interval J^{n-1}_k, all intervals and all gaps of level n in $\tilde{J}^{n-1}_k$ are commensurable with $\tilde{J}^{n-1}_k$, with a constant independent of level n and the interval J^{n-1}_k. If the commensurability constant becomes uniform on sufficiently big levels, then the geometry is called *beau* ("bounded and eventually uniform").

Proposition 3.19. *An infinitely renormalizable quasiquadratic map with bounded combinatorics has bounded geometry. Moreover, it is beau.*

Proof. It is easy to see that infinitely renormalizable maps with q-bounded combinatorics form a closed subset in the space of unimodal maps (with uniform topology). By Lemma 3.15, the space of infinitely renormalizable K-quasiquadratic maps with q-bounded combinatorics is compact. Hence they have uniformly bounded geometry. Now Theorem 3.16 (ii) implies the assertion. □

3.3. Essentially bounded geometry. Long saddle-node cascades make the geometry of a map unbounded (namely, the scaling factors λ_n become close to 1 in the middle of the cascade). But this unboundedness can occur only in a specific controlled way as formalized below.

Let f be an S-unimodal map. Recall that I^n_k stand for the domains of the real generalized renormalizations (3.6).

DEFINITION 3.2. Let us say that f has essentially *K-bounded geometry* (until the first renormalization level) if the scaling factors λ_n are bounded from below by K^{-1}, while the configurations $(I^{n-1} \smallsetminus I^n, I^n_k)$ have K-bounded geometry. The latter means that all the off-central intervals I^n_k, $k \neq 0$, and all the gaps (i.e., the components of $I^{n-1} \smallsetminus \cup_k I^n_k$) are K-commensurable.

THEOREM 3.20. *A real quadratic map has essentially bounded period if and only if it has an essentially bounded geometry. More precisely:*

• *For any $\varepsilon > 0$, there exists a p such that if* $\mathrm{per}_e(f) \geq p$ *then $\lambda_n < \varepsilon$ for some n;*

• *For any K there exists a p such that if* $\mathrm{per}_e(f) \leq p$ *then f has K-bounded geometry.*

The proof of this result consists of analyzing how different combinatorial parameters, (i)-(iv) from §1.9, influence the geometry of f. For instance, if the first parameter, height, is big, then the scaling factors become small by Theorem 3.13. It is relatively easy to see that once any of other parameters (ii)-(iv) becomes big, a small scaling factor is created.

On the other hand, if all parameters (i)-(iv) are bounded then analyzing long saddle-node cascades as perturbations of parabolic maps, one can see that the geometry of f stays essentially bounded.

3.4. Complex bounds. Given a quadratic-like germ f, let

$$\mathrm{mod}(f) = \sup \mathrm{mod}(U' \smallsetminus U),$$

where the supremum is taken over all quadratic-like representatives $f : U \to U'$ of f. Theorem 3.6 shows that the geometry of the puzzle is controlled by $\mathrm{mod}(f)$. To control the geometry on all renormalization levels, we need to control the moduli of all the renormalizations $R^n f$. This motivates the following definition:

Let f be an infinitely renormalizable map. One says that f has *a priori bounds* if there is a $\mu > 0$ such that $\mathrm{mod}(R^n f) \geq \mu$.

THEOREM 3.21. *Any infinitely renormalizable real quadratic-like map $f : U \to U'$ has a priori bounds depending only on* $\nu = \mathrm{mod}(U' \smallsetminus U)$

(hence these bounds are absolute for quadratic polynomials). Moreover, there exists an absolute μ *and* $N = N(\nu)$ *such that* $\mathrm{mod}(R^n f) \geq \mu$ *for* $n \geq N$.

This result establishes one of the basic features of real dynamics which is not generally valid in the complex case.

The proof of Theorem 3.21 is split into two cases depending on the essential period. The case of high combinatorics is treated as follows:

THEOREM 3.22. *There exists a* p *with the following property. If* $f : U \to U'$ *is a renormalizable real quadratic-like map with* $\mathrm{per}_e(f) \geq p$ *and* $\mathrm{mod}(U' \setminus U) \geq \nu > 0$*, then there exists a quadratic-like renormalization* $Rf : V \to V'$ *such that* $\mathrm{mod}(V' \setminus V) \geq \mu(\nu, p)$*, where* $\mu(\nu, p) \to \infty$ *as* $p \to \infty$ *(while* ν *being frozen).*

The idea is that by Theorem 3.20, a big essential period will create a small scaling factor λ_n. Taking a round disk $V' = D(I^n)$ based upon I^n as the diameter and pulling it back by the central branch of g_n, we obtain a domain V based upon I^n. The double branched covering $g_n : V \to V'$ is the desired quadratic-like map.

The case of bounded combinatorics is treated as follows:

THEOREM 3.23. *Let* $f : U \to U'$ *be* $N + 1$ *times renormalizable real quadratic polynomial with* $\mathrm{per}_e(R^n f) \leq p$, $n = 0, 1, \dots, N - 1$, *and* $\mathrm{mod}(U' \setminus U) \geq \nu$. *Then there exists an* $N = N(p, \nu)$ *and a quadratic-like renormalization* $R^N f : V \to V'$ *such that* $\mathrm{mod}(V' \setminus V) \geq \mu > 0$, *where* μ *is an absolute bound.*

In this case, consider the intervals S^N and T^N from Theorem 3.16. Consider the slit plane $\mathbb{C} \setminus (\mathbb{R} \setminus T^N)$ as the hyperbolic plane. By symmetry, the interval T^N represents a hyperbolic geodesic in this plane. Hyperbolic r-neighborhoods of this geodesic are bounded by two circle arcs (see Figure **??**). Take a sufficiently big such a neighborhood V' and pull it back by the N-fold pre-renormalization, f^{p_N}. We obtain a domain V based on S^N, and a branched double covering $f^{p_N} : V \to V'$. Once can show using essentially bounded geometry of f (Theorem 3.20) that V is "well inside" V', so that $f^{p_N} : V \to V'$ is the desired quadratic-like map.

Putting the last two theorems together, we obtain *a priori* bounds from Theorem 3.21.

3.5. Local connectivity of Julia sets. Notice the the Grötzsch Inequality implies:

LEMMA 3.24. *Let $A_n \subset \mathbb{C}$ be a nested sequence of pairwise disjoint annuli surrounding a compact set K. If $\sum \operatorname{mod} A_n = \infty$ then K is a single point.*

This yields an important conclusion about the structure of Julia sets (compare §I.4.6):

THEOREM 3.25. *If f is a non-renormalizable quadratic polynomial, then the Julia set $J(f)$ is locally connected.*

PROOF. We will restrict ourselves to a more interesting case when the principal nest is infinite. Since f is non-renormalizable, the principal nest contains infinitely many non-central levels (Proposition 3.2). By Theorem 3.12,

$$\sum \operatorname{mod}(V^{n-1} \setminus V^n) = \infty.$$

By Lemma 3.24, $\operatorname{diam} V^n \to 0$. By Lemma 1.26, the Julia set is locally connected at 0.

Using Lemma 3.1 and the Koebe Distortion Theorem, this property can be spread around the whole Julia set. □

From *a priori bounds* of Theorem 3.21 one can also derive:

THEOREM 3.26. *Any real quadratic polynomial f_c, $c \in [-2.1/4]$, has locally connected Julia set.*

PROOF. It is not hard to see that *a priori bounds* imply shrinking of little Julia sets J_n to the critical point. On the other hand, any little Julia set is the intersection of some puzzle pieces of f (in the sense of §I.4.6). Hence, there is a nest of puzzle pieces shrinking to 0. By Lemma 1.26, this implies local connectivity of $J(f)$ at 0. *A priori* bounds and the Koebe Distortion Theorem allow one to spread it around the whole Julia set. □

However, there exist complex quadratic polynomials whose Julia set is not locally connected.

4. Bibliographical notes

This lecture is mostly based upon [**L4**], part I. The central result here is Theorem 3.6 [**L4**]. A related result in the particular case of real polynomials was independently proven in [**GS1**].

For quadratic polynomials, puzzle was introduced by Yoccoz who proved Theorem 3.25 (see [**M2**]). Yoccoz's work was preceded by the work of Branner & Hubbard [**BH**] where the puzzle was introduced for cubic polynomials with one escaping critical point.

Real *a priori* bounds (§3.2) appeared in [**BL1, MS, S2**]. Theorem 3.21, and Theorem 3.26 as a consequence, were proven in [**LS, LY**]. The corresponding results in the case of bounded combinatorics had been earlier proven in [**S2, MS**] and [**HJ**] respectively. For other important results on *a priori* bounds and local connectivity of Julia sets see [**P, Y1, PZ**].

LECTURE 4

Rigidity phenomenon

1. Rigidity Conjecture

1.1. We will now discuss the classification of quadratic polynomials f_c up to various equivalence relations introduced above. Given a purely repelling point $c \in \mathbb{M}$, we have the following inclusions:

$$Comb(c) \supset Top(c) \supset Qc(c) \supset \mathcal{H}_c \supset Conf(c) = \{c\}, \tag{4.1}$$

where $Comb(c)$ stands for the combinatorial class of c, and all other notations have a similar self-explanatory meaning (except the notation $\mathcal{H}_c$ for the hybrid classes defined in §II.3.1). The corresponding classes in the real quadratic family will be labelled by $\mathbb{R}$: $Comb_{\mathbb{R}}(c)$, $Top_{\mathbb{R}}(c)$, etc.

Exercise 4.1. Two quadratic maps are *conformally equivalent* if they are conjugate by a Möbius transformation. Show that any quadratic map $z \mapsto \alpha z^2 + \beta z + \gamma$ is conformally equivalent to a unique map $f_c : z \mapsto z^2 + c$ (the uniqueness part is the last equality in (4.1)). Thus, the family $\{f_c\}$ can be identified with the quotient of the full 3-parameter family of quadratic polynomials modulo the conformal equivalence.

A central open conjecture in holomorphic dynamics asserts that in the purely repelling case, all these classes are actually reduced to a single point:

Rigidity Conjecture. *If a parameter $c \in \mathbb{M}$ is purely repelling then*

$$Comb(c) = \{c\}. \tag{4.2}$$

In other words, combinatorics of a quadratic map should uniquely determine the map itself! This phenomenon is intimately related to the rigidity phenomenon in hyperbolic geometry, particularly in dimension 3. The classical Mostow Rigidity theorem tells us that if two compact hyperbolic 3-manifolds are topologically equivalent then they are isometric. In the non-compact case, Thurston described the combinatorics of a 3-manifold (homotopically equivalent to a compact surface) in terms of "ending laminations" and conjectured that these laminations determine the manifold. This certainly sounds very similar to the above Rigidity Problem, and it turns out that the connection between the rigidity phenomena in hyperbolic geometry and holomorphic dynamics is very deep indeed.

A map f_c (and the corresponding parameter c) is called *combinatorially rigid* if (4.2) holds.

The above discussion can be naturally reduced to the real slice f_c, $c \in [-2, 1/4]$, of the quadratic family, which leads us to a notion of *really rigid maps* and the corresponding Real Rigidity Conjecture. In fact, this conjecture has been already proven:

Real Rigidity Theorem. *If a parameter $c \in [-2, 1/4]$ is purely repelling then*

$$Comb_{\mathbb{R}}(c) = \{c\}. \tag{4.3}$$

The main goal of this lecture is to outline a proof of this theorem.

1.2. Rigidity and MLC. There is a deep and surprising connection between the two conjectures stated above (§1.1 and §I.4.7):

PROPOSITION 4.2. *The Rigidity Conjecture is equivalent to the MLC Conjecture.*

PROOF. Recall from §I.4.6 that the combinatorics of a purely repelling quadratic polynomial is determined by the portrait of rational external rays landing at various repelling periodic points. By Theorem 1.22, the portrait of rays landing at a repelling point persists in a wake of a certain parabolic point and it bifurcates when the parameter exits the wake.

Let us consider *parapuzzle pieces* (in the sense of §I.4.7) bounded by several pairs of rays landing at parabolic points and several equipotentials. Let $\mathcal{P}(c)$ stand for the family of such parapuzzle pieces around c. We see that

$$Comb(c) = \bigcap_{P \in \mathcal{P}(c)} P \tag{4.4}$$

Thus c is combinatorially rigid if and only of $\bigcap_{P \in \mathcal{P}(c)} P = \{c\}$.

On the other hand, $\bigcap_{P \in \mathcal{P}(c)} P = F^c$ where F^c is the fiber of the projection j_M from Theorem 1.27. (We use the remark that the combinatorial model of $\mathbb{M}$ can be constructed by means of rational rays with odd denominators only.) By that theorem, $F^c = \{c\}$ if and only if $\mathbb{M}$ is locally connected at c.

Altogether, this yields the result. □

remarks on neutral case

1.3. Density of hyperbolic maps. There is yet one more important open conjecture in holomorphic dynamics:

Density Conjecture. *Hyperbolic parameters are dense in the Mandelbrot set.*

It turns out that this conjecture is intimately related to the previous ones:

PROPOSITION 4.3. *The Rigidity Conjecture (or equivalently, the MLC Conjecture) implies the Density Conjecture:*

PROOF. By Corollary 1.15, only existence of a queer component U of $\operatorname{int}\mathbb{M}$ can violate the Density Conjecture. But since the dividing pairs of rational rays cannot cut through the interior of $\mathbb{M}$, the whole queer component U would belong to the same combinatorial class, which would violate the Rigidity Conjecture. ☐

The situation is similar on the real line:

PROPOSITION 4.4. *The Real Rigidity Theorem implies density of hyperbolic parameters in* $[-2, 1/4]$.

PROOF. Take two non-hyperbolic parameters c_1 and c_2 in $[-2, 1/4]$. Since they are rigid, they have different kneading invariants. By Proposition 1.38, the interval (c_1, c_2) contains a hyperbolic parameter. ☐

1.4. Reduction to the rough geometry. Let us start with a quick outline of a classical approach to the Mostow rigidity theorem. Represent a 3-manifold M^3 as a quotient of the hyperbolic space $\mathbb{H}^3$ modulo an action of a Kleinian group Γ of hyperbolic motions. The approach is to prove first that topology of a manifold determines its *rough geometry*, i.e., determines the manifold up to quasi-isometry. This translates into quasi-conformal conjugacy between the corresponding group actions on the sphere at infinity, $\bar{\mathbb{C}} = \partial\mathbb{H}^3$. Then, by means of a certain ergodic argument, one can show that this conjugacy must me conformal (if M^3 is compact).

A similar approach proved to be quite efficient in the context of holomorphic dynamics as well: first prove that combinatorics determines rough geometry of the map (i.e., determines it up to qc conjugacy) and then prove that rough geometry determines the map itself. In fact, it turns out that in the situation under consideration the second step goes through automatically:

LEMMA 4.5. *Assume c is not hyperbolic. If $Comb(c) = Qc(c)$ then $Comb(c) = \{c\}$.*

This result is a consequence of the following two lemmas:

LEMMA 4.6. *Quasiconformal classes of quadratic polynomials are either domains or single points.*

PROOF. Let $c \in \mathbb{M}$ and $\tilde{c} \in Qc(c)$. Consider a qc map $h : \mathbb{C} \to \mathbb{C}$ conjugating $f \equiv f_c$ to $\tilde{f} \equiv f_{\tilde{c}}$. The Beltrami differential $\mu = \bar{\partial}h/\partial h$ of this map is invariant under the f-action. Then all Beltrami differentials $\mu_\lambda = \lambda\mu$, $|\lambda| < 1/\|\mu\|_\infty$, are also f-invariant. By the Measurable Riemann Mapping Theorem, there is a qc solution $h_\lambda : \mathbb{C} \to \mathbb{C}$ of the Beltrami equation $\bar{\partial}h_\lambda/\partial h_\lambda = \mu_\lambda$. If h_λ is appropriately normalized then the map $f_\lambda = h_\lambda \circ f \circ h_\lambda^{-1}$ is a quadratic polynomial $z \mapsto z^2 + c(\lambda)$. Moreover, $c(0) = c$, $c(1) = \tilde{c}$, and $c(\lambda)$ is holomorphic in λ (by Theorem 2.6). Since holomorphic maps are open, the range of $\lambda \mapsto c(\lambda)$ fills a domain containing both c and $\tilde{c}$. Since this range is contained in $Qc(c)$, the conclusion follows. □

On the other hand, the following statement easily follows from the definitions:

LEMMA 4.7. *Combinatorial classes of non-hyperbolic quadratic polynomials are closed.*

Proof of Lemma 4.5. Assume $Comb(c) = Qc(c) \equiv C$. If $C \neq \{c\}$ then by the last two lemmas C would be simultaneously closed and open.

2. Stability theory

2.1. Structural stability. The rigidity phenomenon is opposite to a structural stability phenomenon, which is very important in the general theory of dynamical systems and its applications. In our context, a quadratic map f_* (and the corresponding parameter value $*$) is called *structurally stable* if all nearby complex quadratic maps f_c are topologically conjugate to f_* (compare §I.5.4). By general methods of dynamical systems one can prove that all hyperbolic quadratic maps, except superattracting ones, are structurally stable (and thus, non-rigid). However, there is a better way of doing it based on the theory of holomorphic motions. Moreover, though we do not yet know whether hyperbolic maps are dense in the quadratic family, the theory of holomorphic motions allows us to show that structurally stable maps are in fact dense (compare with Theorem 1.47):

THEOREM 4.8. *Any non-superattracting quadratic map f_* with $* \in \mathbb{C} \setminus \partial M$ is structurally stable.*

The idea is to construct an equivariant holomorphic motion of the complex plane (which conjugates f_* to nearby maps). Let us start the construction with the Julia set:

LEMMA 4.9. *Let* $* \in \mathbb{C} \setminus \partial M$. *Then there exists a holomorphic motion* $h_c : J_* \to J_c$ *over a neighborhood* Λ *of* $*$ *conjugating* $f_*|J_*$ *to* $f_c|J_c$.

PROOF. Take a repelling periodic point a_* of f_* of period p. By the Implicit Function Theorem, nearby maps f_c have a repelling periodic point a_c moving holomorphically with c. Note that the condition of the IFT is satisfied as long as a_c stays repelling, so by Proposition 1.17 the function $c \mapsto a_c$ can be analytically extended to any simply connected domain $\Lambda \subset \mathbb{C} \setminus \partial M$ (in particular, to any component of $\operatorname{int} M$). So, *all* repelling periodic points move holomorphically over the *same* domain Λ. Moreover, these points do not collide as collisions can occur only at parabolic parameters. Thus, we obtain a holomorphic motion of the set of periodic points over Λ. Obviously, this motion is equivariant under the dynamics.

By the Extension Lemma of §4.3, this motion extends to an equivariant holomorphic motion of the Julia set. □

The next step is to construct an equivariant holomorphic motion in the basin of ∞. This is particularly simple when the Julia set is connected:

LEMMA 4.10. *Assume* $* \in \operatorname{int} M$ *and let* λ *be the component of* $\operatorname{int} \mathbb{M}$ *containing* c. *Then there is an equivariant holomorphic motion* $h_c : D_*(\infty) \to D_c(\infty)$ *such that* $h_c(z)$ *is a holomorphic function in two variables.*

PROOF. Consider the Böttcher function B_c of f_c, $c \in \Lambda$. Since the Julia set J_c is connected, B_c conformally maps $D_c(\infty)$ onto $\mathbb{C} \setminus \bar{D}$ (and conjugates f_c to $z \mapsto z^2$). Hence $h_c \equiv B_c^{-1} \circ B_*$ is an equivariant conformal isomorphism from $D_*(\infty)$ onto $D_c(\infty)$. Moreover, explicit formula (1.1) shows that $h_c(z)$ is holomorphic in c as well. □

To complete the proof, we need to take care of some other dynamical regions:

LEMMA 4.11. (i) *Let* $*$ *be a hyperbolic but not superattracting parameter value. Let* D_c *be the basin of the attracting cycle for* c *near* $*$. *Then there exists an equivariant holomorphic motion* $h_c : D_* \to D_c$ *over some neighborhood of* $*$.

(ii) *Let* $* \in \mathbb{C} \setminus \mathbb{M}$. *Then there exists an equivariant holomorphic motion* $h_c : D_*(\infty) \to D_c(\infty)$ *over a neighborhood of* $*$.

PROOF. The idea is to construct first an equivariant holomorphic motion of a neighborhood of the attracting point (in case (i)) or a neighborhood of ∞ (in case (ii)) and then pull it back equivariantly to the whole basin. The local motion h_c can be constructed by means of the linearizing coordinate or the Böttcher coordinate respectively. It should be done in such a way that $h_c(f_*^n(0)) = f_c^n(0)$ whenever $f_*^n(0)$ lands in the domain of h_c. This would allow one to extend h_c to the whole basin. □

Putting the above motions together, we obtain an equivariant holomorphic motion h_c of the whole complex plane over some neighborhood Λ of $*$. By the λ-lemma, the maps h_c are automatically continuous, which proves Theorem 4.8. In fact, by the λ-lemma, these maps are automatically quasiconformal, so that we come up with the following qs stability result:

THEOREM 4.12. *Two non-superattracting quadratic polynomials which belong to the same component of* $\mathbb{C} \setminus \partial M$ *are qc conjugate.*

2.2. Centers of hyperbolic components. We can now give an idea of the proof of the Multiplier Theorem from the 1st lecture (Theorem 1.16):

THEOREM 4.13. *Any hyperbolic component of* $\operatorname{int} \mathbb{M}$ *contains a unique superattracting parameter (its center). More generally, it contains a unique parameter with a given multiplier* $\lambda \in \mathbb{D}$.

PROOF. Assume there are two different parameters with the same multiplier, $\lambda(c) = \lambda(\tilde{c})$, in some hyperbolic component of period p. Combining Lemma 4.10 with the λ-lemma we see that the maps $f \equiv f_c$ and $\tilde{f} \equiv f_{\tilde{c}}$ are conjugate on the complements of the basins of their attracting cycles. On the other hand, since $\lambda(c) = \lambda(\tilde{c})$, f^p is conformally conjugate to $\tilde{f}^p$ on the immediate basin of the attracting cycle. One can show that these two conjugacies can be combined into a global qc conjugacy. Since this conjugacy is conformal outside the Julia set (which has zero measure by Theorem 1.5), it is globally conformal, and hence $c = \tilde{c}$. □

2.3. Qc classification. At this point we are ready to give a full qc classification of quadratic maps:

THEOREM 4.14. *The parameter plane* $\mathbb{C}$ *of the quadratic family is partitioned in the following qc classes:*

(i) *The complement of the Mandelbrot set,* $\mathbb{C} \setminus \mathbb{M}$*;*

(ii) *Punctured hyperbolic components* $H \smallsetminus c_H$, *where* c_H *is the center of* H*;*

(iii) *Queer components;*

(iv) *Single points on the boundary* ∂M *or the centers of hyperbolic components.*

Thus, parameters on ∂M and superattracting parameters are *qc rigid*, while all the rest are *qc flexible*.

PROOF. By Theorem 4.12, any set in the above list belongs a single qc class.

Let us show that two different sets on the list belong to different qc classes. Take two parameters c and $\tilde{c}$ from different sets but in the same qc class Q. Obviously, parameters in $\mathbb{C} \smallsetminus \mathbb{M}$ are not even topologically equivalent to parameters in $\mathbb{M}$, so we can assume that both c and $\tilde{c}$ belong to $\mathbb{M}$. It is also obvious that superattracting parameters are not topologically equivalent to attracting parameters, so that we can assume that c and $\tilde{c}$ do not belong to the same component of $\operatorname{int} \mathbb{M}$. So, either one of the parameters belongs to $\partial \mathbb{M}$ or they belong to different components of $\operatorname{int} \mathbb{M}$. In ether case, Q would intersect $\mathbb{C} \smallsetminus \mathbb{M}$ (since by Lemma 4.6, Q is a domain), which is certainly impossible. ☐

2.4. Monotonicity of the real quadratic family. We know from the kneading theory that the real quadratic family f_c is full, that is, its kneading invariant $\boldsymbol{\kappa}(c)$ assumes all admissible values from $\boldsymbol{\kappa}_{\min}$ to $\boldsymbol{\kappa}_{\max}$ as c moves from the cusp $1/4$ to the tip -2. We will now prove that this dependence is monotone. This was historically the first deep application of holomorphic dynamics to real dynamics.

THEOREM 4.15. *The kneading invariant* $c \mapsto \boldsymbol{\kappa}(c)$ *depends monotonically on* $c \in [-2, 1/4]$.

PROOF. Assume there are two parameter values $c_1 < c_2$ such that $\boldsymbol{\kappa}(c_1) < \boldsymbol{\kappa}(c_2)$. Then by Proposition 1.38, there exist a superattracting parameter values $a \in [c_1, c_2]$. By the Intermediate Value Theorem, there exist two distinct superattracting parameters $a_1 \in (-2, c_1]$ and $a_2 \in [c_2, 1/4)$ with the same kneading invariant as a. This contradicts Theorem 4.14 (iv). ☐

COROLLARY 4.16. *Combinatorial classes of real quadratic maps are either intervals or single points in the parameter interval* $[-2, 1/4]$.

One more monotonicity property is noteworthy:

THEOREM 4.17. *Let $J = (a, b) \subset [-2, 1/4]$ be a hyperbolic window in the real quadratic family. Let $\lambda(c)$ be the multiplier of the corresponding attracting cycle, $c \in J$. Then $\lambda'(c) > 0$, and $\lambda(c)$ monotonically decays from 1 to -1 as c moves from b to a.*

PROOF. Obviously, λ assumes values 1 or -1 at the endpoints of J. By the Multiplier Theorem (1.16), its derivative does not vanish in J. By the monotonicity of the kneading invariant and Exercise 1.36, $\lambda(c)$ changes sign from $+$ to $-$ as c passes through the center of J in the negative direction. Hence $\lambda(b) = 1$, $\lambda(a) = -1$. □

2.5. Invariant line fields. Let P_z be the projective line associated with the tangent plane $T_z\mathbb{C}$. Points of P_z are tangent lines $L \subset T_z\mathbb{C}$. They can be represented as points of the unit circle, $\mu = e^{2\pi\theta} \in \mathbb{T}$, where $\theta \in \mathbb{R}/\pi\mathbb{Z}$ is the direction of the line. These projective lines P_z are fibers of the projective bundle $\mathcal{P}$ over $\mathbb{C}$.

A *(measurable) line field* on a set $J \subset \mathbb{C}$ is a measurable section $X \to \mathcal{P}$, where X is a measurable set $X \subset J$ of *positive* Lebesgue measure. It can be represented as a measurable function $\mu : X \mapsto \mathbb{T}$ (or rather, a measurable Beltrami differential $\mu(z)d\bar{z}/dz$). We will always assume that μ is extended to the whole plane by 0.

If z is not a critical point of f then the differential of f induces a natural map $Df : P_z \to P_{fz}$. This makes an obvious sense of the notion of *invariant* line field. It is represented by an f-invariant Beltrami differential μ, $f^*\mu = \mu$.

The following result provides a remarkable connection between the Rigidity Problem and ergodic theory:

PROPOSITION 4.18. *A parameter $* \in \mathbb{C}$ belongs to a queer component if and only if f_* has a measurable invariant line field on its Julia set.*

PROOF. If $*$ belongs to a queer component then by Lemma 4.10 f_* is conjugate to any nearby map f_c by means of a qc map h_c which is conformal on the basin of ∞. The small axes of the field of infinitesimal ellipses associated with h_c (represented by the normalized Beltrami differential of h_c) form an f_*-invariant line field on the Julia set of f_*.

Vice versa, assume f_* has an invariant line field on its Julia set represented by a Beltrami differential μ. Since the Julia set of a hyperbolic map has zero Lebesgue measure (Theorem 1.5), f_* can only be of type (iii) or (iv) from Theorem 4.14. Consider a Beltrami differential $\lambda\mu$ with some $\lambda \in \mathbb{D}^*$ and the solution h of the corresponding Beltrami equations. If h is appropriately normalized then $h \circ f_* \circ h^{-1}$ is a quadratic polynomial f_c with some $c \neq *$. Hence $*$ is qc deformable which leaves us only with one possibility: $*$ is of type (iii). □

3. Pullback Argument

3.1. Besides holomorphic motions, there is one more powerful tool to construct a qc conjugacy between two combinatorially equivalent maps called the Pullback Argument. The idea, due to Thurston and Sullivan, is to start with a qc "pseudo-conjugacy" equivariant on some dynamically significant part of the plane, and then promote it, by pulling it back and passing to a limit, to a genuine qc conjugacy.

Consider two quadratic-like maps $f : U \to U'$ and $\tilde{f} : \tilde{U} \to \tilde{U}'$ with connected Julia set. Assume they are topologically conjugate by a homeomorphism $\phi : U' \to \tilde{U}'$. Let

$$O \equiv O_f = \operatorname{orb}_f(0) \quad \text{and} \quad \tilde{O} \equiv O_{\tilde{f}} = \operatorname{orb}_{\tilde{f}}(0).$$

We say that f and $\tilde{f}$ are *Thurston equivalent* if there is a qc map $h : (\mathbb{C}, O) \to (\mathbb{C}, \tilde{O})$ homotopic to ϕ rel O (in particular, h conjugates f to $\tilde{f}$ on the postcritical sets).

The quint-essence of the method is contained in the following lemma:

LEMMA 4.19. *Any Thurston equivalence promotes to a qc conjugacy with the same dilatation.*

PROOF. It is easy to turn h into a qc map $U' \to \tilde{U}'$ coinciding with h on O and homotopic to ϕ rel $O \cup \partial U$. We will keep the same notation h for the modified map.

Let $U^n = f^{-n}U'$ (so that $U^0 \equiv U'$, $U^1 \equiv U$). The corresponding objects for $\tilde{f}$ will be marked with "tilde". Let h has dilatation K.

Since $h(f(0)) = \tilde{f}(0)$, we can lift h to a K-qc map $h_1 : U^1 = \tilde{U}^1$ homotopic to ϕ rel $(O, \partial U^1, \partial U^2)$. Since the lift is holomorphic, the dilatation of h_1 is the same as the dilatation of h. Since $h_1 = h$ on ∂U^1, we can extend h_1 to $U \smallsetminus U^1$ as h (keeping the same notation h_1). By the Gluing Lemma from §II.2, this extension has the same dilatation K. Moreover, this map is homotopic to ϕ rel $(O \cup \partial U^1 \cup \partial U^2)$. Also, it is equivariant on the annulus $U^1 \smallsetminus U^2$ (notice that h_1 is equivariant on a bigger set than h).

Let us now replace h with h_1 and repeat the procedure. We will construct a K-qc map $h_2 : U' \to \tilde{U}'$ homotopic to ϕ rel $(O \cup \partial U^1 \cup \partial U^2 \cup \partial U^3)$ and equivariant on the annulus $U^1 \backslash U^3$.

Proceeding in this way we construct a sequence of K-qc maps h_n homotopic to ϕ rel $(O \cup \partial U^1 \cup \cdots \cup \partial U^{n+1})$ and equivariant on the annulus $U^1 \backslash U^{n+1}$. By the Compactness Lemma from §II.2, we can select a converging subsequence $h_{n(l)} \to h$. The limit map h is a desired qc conjugacy. □

Putting together Lemma 4.19 and Theorem 4.14, we conclude:

THEOREM 4.20. *A superattracting parameter value is uniquely determined by its Thurston type.*

Thus, hyperbolic components are labelled by Thurston types of their centers.

3.2. Removability. We will need some background on removable sets.

DEFINITION 4.1. A compact set $Q \subset \mathbb{C}$ is called *(qc) removable* if for any neighborhood $U \supset K$, any (quasi-)conformal embedding $h : U \setminus Q \to \mathbb{C}$ extends to a (quasi-)conformal embedding $\tilde{h} : U \to \mathbb{C}$.

EXERCISE 4.21. (i) Show that removability is equivalent to qc removability.
(ii) Show that it is sufficient to take $U = \mathbb{C}$ in the above definition of removability.

Assume that we have a family of disjoint annuli $A^n_k \subset \mathbb{C} \setminus Q$, $n \in \mathbb{N}$, such that:

- For any given n, the annuli A^n_k are not nested and $\cup_k A^n_k$ separates Q from ∞;
- For $n > 1$, any annulus A^n_k, is surrounded by some annulus A^{n-1}_j;
- Divergence property: For any $x \in Q$,

$$\sum_{n=1}^{\infty} \operatorname{mod} A^n(x) = \infty,$$

where $A^n(x)$ stands for the annulus $A^n_{k(n)}$ surrounding x.

Then we say that Q satisfies the *divergent property.* Such a set is necessarily Cantor.

THEOREM 4.22. *A compact set $Q \subset \mathbb{C}$ satisfying the divergent property is removable.*

In particular, Cantor sets with bounded geometry are removable.

3.3. Rigidity of the Feigenbaum map. Next, we will show that the Feigenbaum map (and, more generally, any infinitely renormalizable real map with bounded combinatorics) is really rigid.

LEMMA 4.23. *Consider two infinitely renormalizable real parameters $c, \tilde{c} \in [-2, 1/4]$ of bounded combinatorial type. If the maps $f = f_c$ and $\tilde{f} = f_{\tilde{c}}$ are combinatorially equivalent then there is an $\mathbb{R}$-symmetric qc homeomorphism $h : (\mathbb{C}, O_f \to (\mathbb{C}, O_{\tilde{f}})$ conjugating the maps on the postcritical sets.*

PROOF. The key is bounded geometry of the maps (Proposition 3.19). We will use the notations preceding that Proposition (Letting $I^0 \equiv I$ and using "tilde"to mark the corresponding objects for $\tilde{f}$).

Let D^n_k be the closed $\mathbb{R}$-symmetric round disk based upon the interval I^n_k as a diameter, and let $D^n \equiv D^n_0$, $P^n_k = D^n_k \setminus \cup D^{n+1}_i$. Each P^n_k is a "generalized pair of pants", i.e., a disk with finitely many disjoint smaller disks removed. Moreover, the geometry of the P^n_k is bounded in the sense that all the removed disks and the distances in between them are commensurable with the diameter of D^n_k. (It is called a "pair of pants decomposition with bounded geometry", see Figure **??**.)

It follows that there exist $\mathbb{R}$-symmetric diffeomorphisms $h_{n,k} : P^n_k \to \tilde{P}^n_k$ with bounded dilatation which are affine on the boundaries ∂P^n_k. Since these diffeomorphisms match on the boundaries of the pairs of pants, they glue together into a global homeomorphism $h : D^0 \setminus O_f \to \tilde{D}^0 \setminus O_{\tilde{f}}$. By the Gluing Lemma from §II.2, this homeomorphism is quasiconformal. Since the Cantor set O_f is removable (by Theorem 4.22), h admits a qc extension through it, which yields the assertion. □

THEOREM 4.24. *There is only one infinitely renormalizable real parameter c of bounded type with a given combinatorics.*

PROOF. Let us consider some real combinatorial class $\mathcal{C}_{\mathbb{R}} \subset [-2, 1/4]$ of infinitely renormalizable maps of bounded type. By Corollary 4.16, it is either an interval $[c, \tilde{c}]$ or a single point. Assume it is an interval. Let h be a qc homeomorphism given by Lemma 4.23. It is easy to see that it provides a Thurston equivalence between f and $\tilde{f}$. By the Pullback Argument (Lemma 4.19), f_c and $f_{\tilde{c}}$ are qc equivalent. But then by Theorem 4.14, c and $\tilde{c}$ belong to a queer component of $\operatorname{int} \mathbb{M}$, so that they cannot be the endpoints of the combinatorial class $\mathcal{C}_{\mathbb{R}}$.

We concludes that $\mathcal{C}_{\mathbb{R}}$ is reduced to a single point. □

As a byproduct of this result (together with Proposition 4.18) we obtain:

COROLLARY 4.25. *Let c be an infinitely renormalizable real parameter of bounded type. Then there are no invariant line fields on the Julia set $J(f_c)$.*

It is still unknown, though, whether such a Julia set can have positive Lebesgue measure.

3.4. Rigidity of the Fibonacci map. Let us consider a quadratic Fibonacci map f. For this map we have a sequence of generalized renormalizations $g_n : V_0^n \cup V_1^n \to V_0^{n-1}$ with linearly increasing moduli $\mathrm{mod}(V^{n-1} \setminus V^n)$ (by Theorem 3.6). So the pairs of pants $V_0^{n-1} \setminus (V_0^n \cup V_1^n)$ do not have bounded geometry. However we will check that the corresponding pairs of pants stay bounded "Teichmüller distance away", that is, they are K-quasi-conformal equivalent with a uniform K.

We will mark the objects corresponding to $\tilde{f}$ with tilde. Note that all puzzle pieces come together with the boundary parametrization, induced e.g., by the Böttcher coordinate in the complement of the Julia set. Let us have a K-quasiconformal map

$$h_n : (V^{n-1}, V_0^n, V_1^n) \to (\tilde{V}^{n-1}, \tilde{V}_0^n, \tilde{V}_1^n),$$

respecting the boundary parametrization of the pieces. We would like to lift this map to a quasiconformal map

$$h_{n+1} : (V^{n+1}, V_0^n, V_1^n) \to (\tilde{V}^{n+1}, \tilde{V}_0^n, \tilde{V}_1^n)$$

picture with the same property. What causes a problem is that h_n does not carry the critical values $v_n = g_n(0)$ to $\tilde{v}_n = \tilde{g}_n(0)$. However, as $\mathrm{mod}(V_0^{n-1} \setminus V_1^n)$ is linearly big, $h_n(v_n)$ is exponentially close to $\tilde{v}_n$ in the hyperbolic metric of $\tilde{V}^{n-1}$.

By lifting h_n to the off-central puzzle pieces $V_1^n \to \tilde{V}_1^n$ via the univalent maps $g_n : V_1^n \to V^{n-1}$ and $g_n : \tilde{V}_1^n \to \tilde{V}^{n-1}$, we obtain a K-quasiconformal map $\hat{h}_n : V^{n-1} \to \tilde{V}^{n-1}$ matching with h_n on $V^{n-1} \setminus V_1^n$, with even better property: $\hat{h}_n(v_n)$ is exponentially close to $\tilde{v}_n$ in the hyperbolic metric of $\tilde{V}_1^n$.

Now we can replace $\hat{h}_n$ by another map H_n matching with it on $V^{n-1} \setminus V_1^n$, respecting the critical values and having dilatation

$$K(1 + \exp \text{ small term}).$$

This map can be already lifted to V_0^{n+1}. It needs not yet respect boundary parametrization of V_i^{n+2} but one more repetition of the pull-back procedure will do the job.

Repeating this procedure we will construct a quasiconformal equivalence between the pairs of pants of all levels with uniformly bounded dilatation (as the dilatation increases by exponentially small amount on every step, it stays bounded). Spreading it around the postcritical set, we conclude that the two Fibonacci maps in question are Thurston equivalent. By the Pullback Argument, they are qc conjugate.

4. Main rigidity results

The Feigenbaum and the Fibonacci maps considered above represent two main phenomena: *bounded* geometry based on *a priori* bounds and *decaying* geometry. We have seen that both phenomena lead to rigidity. These phenomena will be the core of general rigidity results presented below.

4.1. Rigidity of non-renormalizable maps.

THEOREM 4.26. *Any non-renormalizable quadratic polynomial is combinatorially rigid.*

PROOF. By Lemma 4.5, it is enough to prove that if two quadratic polynomials f and $\tilde{f}$ are combinatorially equivalent then they are qc equivalent. The idea is to construct inductively a sequence of qc pseudo-conjugacies

$$h_n : (V^{n-1}, \bigcup_k V_k^n) \to (\tilde{V}^{n-1}, \bigcup_k \tilde{V}_k^n),$$

i.e., qc maps respecting the boundary marking of the puzzle pieces (and hence $(g_n, \tilde{g}_n)$-equivariant on the corresponding boundary). To this end, start with some qc map $H : (U', U) \to (\tilde{U}', \tilde{U})$ equivariant on ∂U which maps the configuration of α-rays for f to the configuration of α-rays of $\tilde{f}$ (respecting the natural parametrization of the rays). Call this map "initial" pseudo-conjugacy.

The initial map can be lifted to a pseudo-conjugacy h_1 on the first level of the principal nest. Then by means of the generalized (cascade) renormalization (similarly to the Fibonacci case outlined above), h_1 can be consecutively lifted from one non-central level of the principal nest to the next one. Every lift will spoil dilatation by factor $1 + O(\exp(-\mu_{n(k)}))$, where $\mu_{n(k)}$ is the principal modulus on the corresponding non-central level. Since the principal moduli grow linearly (by Theorem 3.6), the dilatation of these pseudo-conjugacies stay bounded.

Then one can spread these pseudo-conjugacies around to off-critical puzzle pieces without loss of dilatation, and pass to a limit to obtain a desired qc conjugacy. □

4.2. A priori bounds and invariant line fields.

THEOREM 4.27. *A infinitely renormalizable quadratic-like map with a priori bounds. does not have invariant line fields on its Julia set.*

4.3. A priori bounds and rigidity. Let $N = N_f$ stand for the escaping time from Theorem 3.6, that is, the number of iterates it takes for the critical point to escape $V^0 \cup P^{(-1)}$. Let us say that an infinitely renormalizable map f has a *bounded escaping times* if the escaping times $N_{R^m f}$ of all renormalizations are bounded by some N.

THEOREM 4.28. *Let f and $\tilde{f}$ be two infinitely renormalizable quadratic-like maps with a priori bounds and bounded escaping times. If f and $\tilde{f}$ are combinatorially equivalent then they are hybrid equivalent.*

Since a quadratic polynomial with connected Julia set is uniquely determined by its hybrid class, we conclude:

COROLLARY 4.29. *Let f_c and $f_{\tilde{c}}$ be two infinitely renormalizable quadratic polynomials with a priori bounds and bounded escaping times. If f_c and $f_{\tilde{c}}$ are combinatorially equivalent then $c = \tilde{c}$.*

PROOF. Since the principal nest does not carry any information beyond the first renormalization level, on every renormalization levels we have to start over again. The dilatation of the initial pseudo-conjugacy H_m between $R^m f$ and $R^m \tilde{f}$ depends only on the *a priori* bounds and the bounds on the escaping times. Then, when we go through the principal nest of $R^m f$ (as in §4.1), we spoil this dilatation only by bounded amount, so that we obtain pseudo-conjugacy with uniformly bounded dilatations on all levels. These pseudo-conjugacies can be glued together and spread around to obtain a Thurston equivalence between the maps.

By the Pullback Argument, this Thurston equivalence can be turned into a qc conjugacy h. By Theorem 4.27, $\bar{\partial} h = 0$ a.e. on the Julia set $J(f)$, so that h is a hybrid equivalence between f and $\tilde{f}$. □

4.4. Density of real hyperbolic maps. Since by Theorem 3.21 all real maps have *a priori* bounds, we conclude:

THEOREM 4.30. *Any non-hyperbolic parameter $c \in [-2, 1/4]$ is really rigid.*

By Proposition 4.4,

THEOREM 4.31. *Hyperbolic maps are dense in the real quadratic family.*

5. Bibliographical notes

This lecture is mostly based on [**L4**], part II. The Real Rigidity Theorem and its consequence, the Density Theorem (4.31), are the main results here.

Theorem 4.27 is due to McMullen [**McM1**]. Theorem 4.26 (with a different proof) is due to Yoccoz (see [**H**]). It is proven by Kahn [**K**] that the Julia set of any Yoccoz quadratic is removable (in a somewhat different sense than defined in §3.2). This result implies Theorem 4.26.

For general theory of structural stability of hyperbolic dynamical systems see, e.g., [**Shu**]. Structural stability theory in holomorphic dynamics was developed in [**L2, MSS**].

The Monotonicity Theorem (4.15) appeared in [**MT**]. On the Pull-back Argument, see [**DH3**] and [**MS**]. Rigidity Theorem 4.20 is due to Thurston (see [**DH3**]). Rigidity Theorem 4.24 is due to Sullivan (see [**MS**]).

On the Mostow Rigidity Theorem, see [**Mo**]. See [**Th2, Min**] on Thurston's Ending Lamination Conjecture.

LECTURE 5

Measurable dynamics and parapuzzle geometry

1. Measurable dynamics

1.1. Almost all orbits follow the critical one. Measurable dynamics studies the behavior of Lebesgue almost all orbits of a smooth dynamical system. For interval maps, it starts with a crucial observation that almost all orbits either follow the critical orbit or else densely fill some interval:

LEMMA 5.1. *Let $f : I \to I$ be an S-unimodal map. Then for almost all points $x \in I$, one of the following two (overlapping) possibilities occur:*
(i) *The map f is at most finitely renormalizable, and*

$$\omega(x) = \bigcup_{k=0}^{p-1} f^k \tilde{J},$$

where p is the period of the last renormalization and $\tilde{J}$ is the corresponding smallest central interval.
(ii) $\omega(x) = \omega(0)$.

COROLLARY 5.2. (i) *If f is hyperbolic (or parabolic) then almost all orbits converge to the attracting (or resp. parabolic) cycle.*
(ii) *If f is infinitely renormalizable then almost all orbits converge to the Feigenbaum attractor O_f (see §6.1 of Lecture 2).*

In mid 1980's Milnor posed a problem whether case (i) of Lemma 5.1 always occurs for a Yoccoz map f. It turned out that it is indeed the case:

THEOREM 5.3. *If $f : I \to I$ is a quasiquadratic Yoccoz map then for almost all $x \in I$,*

$$\omega(x) = \bigcup_{k=0}^{p-1} f^k \tilde{J}.$$

The crucial geometric quality of quasiquadratic Yoccoz maps responsible for Theorem 5.3 is the exponential decay of scaling factors

(Theorem 3.14 from Lecture 3). One can deduce from it that the domains of the first return maps $g_n : \cup I_k^n \to I^{n-1}$ have exponentially decaying measure relative to the lengths of I^{n-1}.

1.2. Elements of ergodic theory.

1.2.1. *Ergodicity.* Let (K, μ) be a measure space and $f : K \to K$ be a measurable map. The measure μ is called *quasi-invariant* if

$$\mu(X) = 0 \Rightarrow \mu(f^{-1}X) = 0,$$

and it is called *invariant* if $\mu(f^{-1}X) = \mu(X)$ for any measurable subset $X \subset K$.

A quasi-invariant measure is called *ergodic* if K cannot be decomposed into two disjoint invariant measurable subsets X_1 and X_2 of positive measure.

Birkhoff Ergodic Theorem. *Let $f : K \to K$ be a map preserving an ergodic probability measure μ, and let $\phi \in L^1(\mu)$. Then for almost all $x \in K$,*

$$\lim_{n\to\infty} \frac{\phi(x) + \phi(fx) + \cdots + \phi(f^{n-1}x)}{n} = \int \phi d\mu. \tag{5.1}$$

This theorem tells us that the time averages of an observable ϕ under ergodic evolution exist and coincide with its space averages.

Let now $f : K \to K$ be a continuous map on a compact metrizable space. By the Bogolyubov-Kryloff Theorem, f has at least one invariant Borel probability measure μ.

As usual, δ_x stands for the Dirac δ-measure of a point $x \in K$. Recall that a sequence of measures μ_n on K *weakly converges* to a measure μ if for any continuous function ϕ, $\int \phi d\mu_n \to \int \phi d\mu$. In what follows, the convergence of measures will always be understood in the weak sense.

A point $x \in K$ is called *μ-typical* if

$$\frac{1}{n}(\delta_x + \delta_{fx} + \cdots + \delta_{f^{n-1}x}) \to \mu \quad \text{as } n \to \infty.$$

In other words, the orbit of a μ-typical point is equidistributed with respect to μ.

Applying the Ergodic Theorem to a dense family of continuous functions and passing to a limit, we obtain:

Proposition. *Let μ be an invariant measure of a continuous map on a compact metrizable space. Then μ-almost all points are μ-typical.*

1.2.2. *Mixing and Bernoulli properties.* One of the main concerns of the classical ergodic theory (in 1950-60's) was the exploration of stochastic properties of dynamical systems with invariant measures. Recall that such a dynamical system (f, μ) is called *mixing* if for any two measurable subsets X and Y,

$$\mu(f^{-n}X \cap Y) \to \mu(X)\,\mu(Y) \quad \text{as } n \to \infty.$$

An example of a mixing dynamical system is provided by one-sided or two-sided *Bernoulli shift.* The one-sided Bernoulli shift σ_+ of degree d acts on the space Σ_d^+ of one-sided sequences $(\varepsilon_0, \varepsilon_1, \dots)$ in d symbols by forgetting the first symbol. The two-sided Bernoulli shift σ acts on the space Σ_d of two-sided sequences $(\dots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \dots)$ by shifting the sequence by 1 to the left. Either space is endowed with a stationary product measure with each coordinate distributed according to a probability distribution $(p_1, \dots, p_d)$ (so, for each degree we actually have a simplex of Bernoulli shifts).

The Bernoulli shifts are systems with strongest imaginable stochastic properties: its typical trajectories look like sequences of independent random variables. An invertible dynamical system is called *Bernoulli* if it is isomorphic (i.e., conjugate by a measure preserving map) to a two-sided Bernoulli shift. It turns out that the analogous property for non-invertible systems is too restrictive, so it is replaced by the following weaker notion based on the construction of "natural extension".

Given a surjective map $f : K \to K$, let us consider the space $\hat{K}$ of all possible backward orbits $\hat{z} = (\cdots \mapsto z_{-2} \mapsto z_{-1} \mapsto z_0)$. There is a natural surjective projection $\pi : \hat{K} \to K$, $\hat{z} \mapsto z_0$, and natural equivariant lift $\hat{f} : \hat{K} \to \hat{K}$, $\hat{f}(\hat{z}) = (\cdots \mapsto z_{-1} \mapsto z_0 \mapsto f z_0)$. This lift is invertible with the inverse given by forgetting z_0.

If f is a continuous map on a topological space then $\hat{K}$ can be endowed with the weak topology, and all the above maps become continuous. If f preserves a measure μ then it can be lifted to a unique measure $\hat{\mu}$ preserved by $\hat{f}$. On the cylindrical sets this measure is defined as follows:

$$\hat{\mu}(X_0 \times X_1 \times \cdots \times X_n) = \mu(X_0 \cap f^{-1}X_1 \cap \cdots \cap f^{-n}X_n).$$

An invariant measure μ of a map f is called *weakly Bernoulli* if the *natural extension* $(\hat{f}, \hat{\mu})$ is isomorphic to a two-sided Bernoulli shift.

1.3. Absolutely continuous invariant measures. For a continuous interval map f, the Bogolyubov-Kryloff Theorem is obvious since the δ-measure of a fixed point is invariant. In fact, an interval map f usually has a plenty of invariant measures, for instance, uniform measures supported on periodic cycles. However, most of invariant measures are singular with respect to the Lebesgue measure λ, so that the Ergodic Theorem tells us nothing about the behavior of Lebesgue typical orbits. And what about invariant measures which are absolutely continuous with respect to λ (such a measure will be abbreviated as a.c.i.m.)? It turns out that such a measure (if exists) governs the behavior of Lebesgue almost all orbits:

THEOREM 5.4. *Let μ be an a.c.i.m. for an S-unimodal map $f : I \to I$. Then the Birkhoff averages of Lebesgue almost all points $x \in I$ weakly converge to μ.*

Invariant measures that govern in this sense the behavior of Lebesgue almost all points are called *SRB measures*, after Sinai, Ruelle and Bowen who introduced such measures in the context of hyperbolic diffeomorphisms. (In fact, an important phenomenon they discovered is that such a measure may often be *singular* with respect to the Lebesgue measure, compare Corollary 5.2).

Theorem 5.4 follows from the Ergodic Theorem and the following result:

THEOREM 5.5. *Any non-hyperbolic S-unimodal map is ergodic with respect to the Lebesgue measure. Hence it has at most one a.c.i.m.*

Thus, any completely invariant set of positive Lebesgue measure has full Lebesgue measure. Applying it to the set of μ-typical points (for an a.c.i.m. μ), we conclude that Lebesgue almost all points are μ-typical, as Theorem 5.4 asserts.

The following result describes the support of an a.c.i.m.:

THEOREM 5.6. *If an S-unimodal map has an a.c.i.m. μ, then f is a Yoccoz map. Moreover,*

$$\operatorname{supp}(\mu) = \bigcup_{k=1}^{p} f^k \tilde{J},$$

where p is the period of the last renormalization of f, and $\tilde{J}$ is the smallest central interval for the last renormalization (see Exercise 2.21).

PROOF. Let us explain why only Yoccoz maps can have a.c.i.m. Indeed, by Corollary 5.2 (i), an a.c.i.m. of a hyperbolic or parabolic map would be supported on its limit cycle, which is certainly impossible. By (ii), an a.c.i.m. of an infinitely renormalizable map would be supported on the Feigenbaum attractor O_f, which is impossible by Theorem 3.18 of Lecture 3. □

Let us finish this section with two results on stochastic properties of a.c.i.m. that manifest strong chaotic nature of the dynamics.

The Ergodic Theorem implies that for any ergodic invariant measure μ, the following limit exists for μ almost all points:

$$\chi_\mu(f) = \lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)| = \int \log |Df(x)| d\mu.$$

It is called the *characteristic exponent* of μ. If $\chi_\mu(f) > 0$ then μ-typical orbits are exponentially unstable in the sense that nearby orbits are repelled away exponentially fast. It turns out that a.c.i.m.'s have this property:

THEOREM 5.7. *Any a.c.i.m. μ of an S-unimodal map f has positive characteristic exponent.*

Note that the characteristic exponent of an a.c.i.m. is equal to its entropy $h_\mu(f)$.

THEOREM 5.8. *Under the circumstances of Theorem 5.6, the measure $\mu|\tilde{J}$ is mixing, in fact weakly Bernoulli, under $f^p|\tilde{J}$.*

An S-unimodal map which has an absolutely continuous invariant measure is called *stochastic*. This terminology is justified by the last two theorems.

1.4. Existence Problem.

1.4.1. *Expansion versus contraction.* The first example of a stochastic unimodal map was studied by Ulam and Neumann (1947) by means of one of the first available computers. It was the "Chebyshev map" $g : x \mapsto 4x(1-x)$ on the interval $[0,1]$.

EXERCISE 5.9. Show that g is affinely conjugate to the map f_{-2} : $x \mapsto x^2 - 2$ on $[-2,2]$ and to $\phi : x \mapsto 2x^2 - 1$ on $[-1,1]$. Using the functional equation $\cos(2\theta) = \phi(\cos\theta)$ show that $dx/\sqrt{x(1-x)}$ is an a.c.i.m. of g. (In fact, the map ϕ is a classical "Chebyshev map".)

The basic phenomenon responsible for stochastisity properties of a map is the competition between expanding and contracting mechanisms. Expansion created by repelling cycles leads to stochastic regimes, in favor of existence of a.c.i.m. On the other hand, contraction near the critical point attempts to destroy it. The question is which phenomenon prevail.

Note that in the Ulam-Neumann example the critical point lands, under the second iterate of the map, at the repelling fixed point, so that the contraction near 0 is compensated by the expansion near the fixed point. It suggests that sufficiently strong expansion along the critical orbit should lead to existence of an a.c.i.m. Let us say that an S-unimodal map f satisfies the *Collet-Eckmann* condition if there exist constants $C > 0$ and $\rho > 1$ such that

$$|Df^n(c)| \geq C\rho^n,$$

where $c = f(0)$ is the critical value.

THEOREM 5.10. *Collet-Eckmann maps are stochastic.*

A softer condition was suggested by Nowicki and van Strien. An S-unimodal map satisfies the *summability condition* if

$$\sum \frac{1}{\sqrt{|Df^n(c)|}} < \infty.$$

THEOREM 5.11. *Let f be an S-unimodal map with non-degenerate critical point. If f satisfies the summability condition then it is stochastic.*

1.4.2. *Martens-Nowicki criterion.* It is intuitively natural to expect that the rate of expansion along the critical orbit should be related to the rate of recurrence of the orbit: more frequently the critical point returns back to itself, less expanding the dynamics is. An efficient way to formalize this intuition is provided by the principal nest and scaling factors (recall §III.2.6).

THEOREM 5.12. *Let f be an S-unimodal map with non-degenerate critical point. If $\sum \sqrt{\lambda_n} < \infty$ then f satisfies the summability condition and hence is stochastic.*

Together with Theorem 3.14, this implies:

COROLLARY 5.13. *Let f be an S-unimodal map with non-degenerate critical point. If all but finitely many returns in the principal nest are non-central then f is stochastic.*

Since central returns correspond to fast recurrence of the critical orbit, this result confirms our intuitive expectations.

1.4.3. *Typicality.* We are now ready to formulate one of the main results of this course (see Theorem A in the Introduction):

THEOREM 5.14. *Almost all parameters $c \in \mathcal{Y}$ are stochastic.*

Remark. Along the lines we will give a new proof that $\text{meas}(\mathcal{Y}) > 0$.

Let us give a heuristic proof of Theorem 5.14. Since $\mathcal{Y} = \cup \mathcal{N}^n$ (recall §II.6.5), it is enough to prove the result separately for each $\mathcal{N}^n$. We will restrict ourselves to the set $\mathcal{N}$ of non-renormalizable irregular

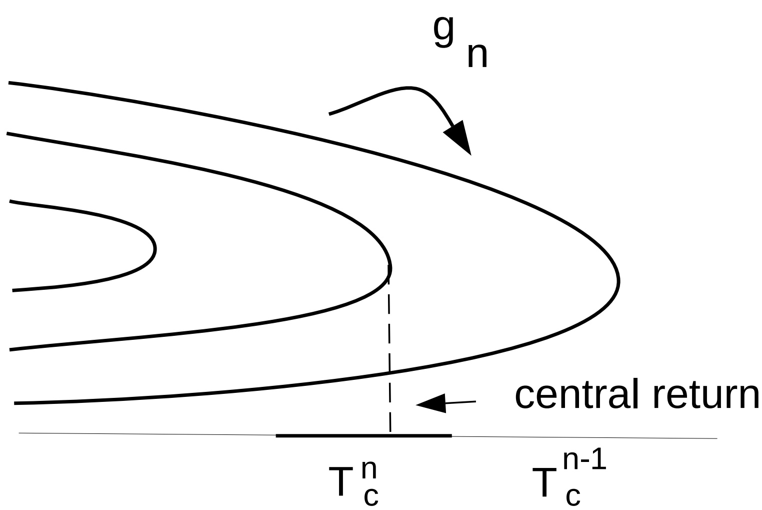

FIGURE 5.1. How the critical value $g_{n,c}(0)$ moves through the moving interval I_c^{n-1}.

parameters (the other sets are treated by renormalizing the family over renormalization windows).

Imagine a one parameter family of return maps $g_{n,c} : I_c^n \to I_c^{n-1}$ depending on $c \in J$. Imagine that when the parameter c runs over the interval J, the critical value $g_n(0)$ runs through I_c^{n-1} with a more or less uniform speed (see Figure 5.1). Then the probability that $g_n(0)$ lands at I_c^n (i.e., the probability of the central return) is comparable with $|I_c^n|/|I_c^{n-1}|$. But Theorem 3.13 tells us that the latter is exponentially small, provided that the previous level was not central.

Hence if we start on a sufficiently deep level of the principal nest then the probability to observe a central return on one of further levels will be exponential small as well. Hence the complementary set (which is contained in the set of non-renormalizable stochastic maps) has positive measure. This proves the first assertion of the theorem.

Furthermore, by Borel-Cantelli Lemma, the probability of infinitely many central returns is equal to zero, which proves the second assertion.

fix the picture

There is one big assumption in this heuristic argument, namely that the critical value moves with *uniform* speed through the interval I_c^{n-1} (which is also moving with c). To justify it, we need to prove transversality of two motions involved. With real methods only, it would be a desperate problem. However, one of the miracles of complex world is that transversality can be obtained for purely topological reasons (a là Argument Principle). In the following sections we will develop a complex machinery which will allow us to justify the above heuristic argument.

2. Parapuzzle combinatorics

Parapuzzle describes a hierarchical structure of the parameter domain of the quadratic family by splitting it into nests of (para-)puzzle pieces. These partitions explicitly correspond to the partitions of the dynamical plane into nests of puzzle pieces described in Lecture 3. *Renormalization and parapuzzle together provide a full combinatorial picture of the quadratic family.*

2.1. Parabolic limbs. Let us start with recalling the bifurcation picture described in the previous lectures. The quadratic family originates at 0 with a simple well-understood map $z \mapsto z^2$. The origin belongs to the hyperbolic component H_0 bounded by the main cardioid C (see Exercise 1.18), where the map f_c has an attracting fixed point α_c. For $c \in C$, the map f_c has a neutral fixed point with some rotation number $\theta \in [0, 1)$.

Denote this parameter by $c(\theta)$. When $\theta = p/q$ is rational, the parameter $c_{p/q}$ is a parabolic bifurcation point. At this point, another hyperbolic component, $H_{p/q}$, is attached to the main cardioid. For $c \in H_{p/q}$, the map f_c has an attracting cycle of period p. Moreover, there are two parameter external rays landing at $c_{p/q}$ that bound the parabolic wake $W_{p/q}$ (see Proposition 1.23). Within this wake, there are exactly p dynamical rays landing at the fixes point α_c that are cyclically permuted with rotation number q/p. Moreover, this ray portrait moves holomorphically as c ranges over the wake.

The set $L_{q/p} = \mathrm{cl}(\mathbb{M} \cap W_{p/q})$ is the q/p-limb of the Mandelbrot set. Thus we have the first decomposition of the parameter plane:

$$\mathbb{M} = \mathrm{cl}\, H_0 \cup \bigcup_{q/p \neq 0} L_{q/p}$$

according to the properties of the α-fixed point: its "attractiveness" and combinatorial rotation number.

2.2. Satellite M-copies. By Theorem 2.27, every hyperbolic component $H_{q/p}$ originates a satellite copy $M_{q/p}$ of the Mandelbrot set. These copies can be nicely specified in terms of the top levels of the puzzle described in §III.1.1. Recall that puzzle pieces $P_i^{(0)}$ of zero depth are bounded by an equipotential E^0 and external rays landing at points α and $-\alpha$. Let Ω^0 be the domain bounded by E^0

Remark. One should keep in mind that all the maps and the sets depend on c. To simplify the notations, we will often drop the label c.

The critical puzzle piece $P_0^{(0)}$ was also called V^0, the first piece in the principal nest. Let $Y_i = f_c^i(V^0) \cap \Omega^0$, $i = 1, \dots, p-1$, be the non-critical puzzle pieces of depth 0 attached to the α-fixed point, and let $Z_i = -Y_i$ be the symmetric puzzle pieces attached to $-\alpha$. Note that

$$f_c(Y_{p-1}) \cap \Omega^0 = V^0 \cup \bigcup_{i=1}^{p-1} Z_i. \tag{5.2}$$

Hence under f^p points from $V^0 \cap K(f)$ either return to V^0 or escape to one of puzzle pieces Z_i. It turns out that the satellite copy $M_{q/p}$ consists of the parameters for which the critical orbit never escapes V^0:

THEOREM 5.15. $M_{q/p} = \{c \in L_{q/p} : \ f_c^{np}(0) \in V^0, \ n = 0, 1, 2, \dots\}$. *In particular, the doubling renormalization window is described as follows (compare §II.6.4):*

$$M_{1/2} \cap \mathbb{R} = \{c \in [-2, 1/4] : \ f_c^{2p} \in [\alpha_c, -\alpha_c], \ n = 0, 1, 2, \dots\} = [d, b],$$

where $b = -3/4$ is the doubling bifurcation parameter and d is a root of the equation $f_d^2(0) = -\alpha$ (so that the renormalization

$$Rf_d = f_d^2 | [\alpha_d, -\alpha_d]$$

is a Chebyshev map).

2.3. Misiurewicz limbs. By logic, $L_{q/p} \smallsetminus M_{q/p}$ consists of those parameters in the limb $L_{q/p}$ for which $f^{pn}(0)$ escapes through one of the puzzle pieces Z_i attached to $-\alpha$. By specifying the escape time and the "escape route" we can decompose $L_{q/p} \smallsetminus M_{q/p}$ into the union of "Misiurewicz limbs" $\Lambda_{\sigma,i}^n$ described below.

We fix the rotation number q/p and will skip it from the notations for the Misiurewicz limbs. There are $p-1$ Misiurewicz limbs of level 1:

$$\Lambda_i^1 = \{c \in L_{q/p} : \ f_c^p(0) \in Z_i\}.$$

They are attached to $M_{q/p}$ at the Misiurewicz point $c = c^{q/p}$ for which $f_c^p(0) = -\alpha$ (so that the renormalization of f_c is the Chebyshev map). This point represents a distinguished tip of the satellite copy $M_{q/p}$ where the parabolic limb $L_{q/p}$ visibly bifurcates into $p-1$ branches (see Figure ...). Note that this gives us a way to figure out from the picture what are the rotation numbers q/p of different bifurcation points $c_{p/q}$ on the main cardioid.

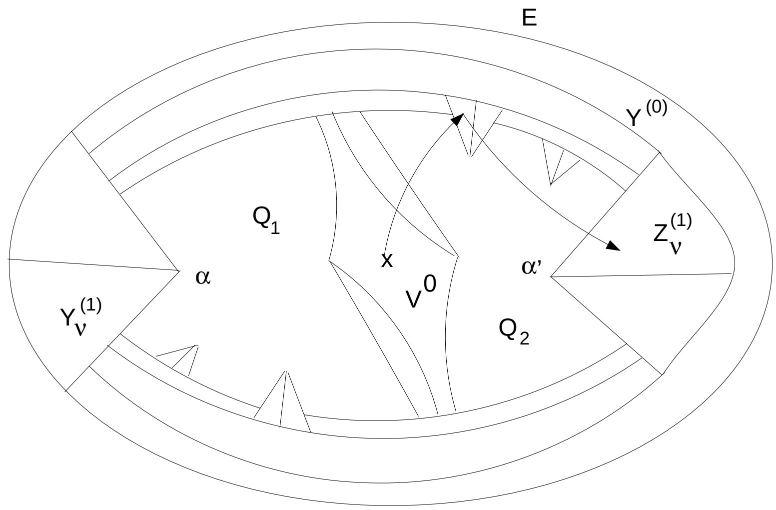

FIGURE 5.2. Initial combinatorics of the puzzle.

Next, consider the equipotential $E^1 = f^{-p}E^0$ and let $\Omega^1 \ni 0$ be the domain bounded by this equipotential and the appropriate pairs of rays landing at α and $-\alpha$. Then

$$f^p : \Omega^1 \to V^0 \cup Z_i$$

is a double branched covering. If $f^p(0) \in V^0$ then

$$\Omega^1 = V^1 \cup Z^1_{\sigma,i},$$

where $V^1 \ni 0$ and $f^p : V^1 \to V^0$ is a double branched covering, while the maps $f^p : Z^1_{\sigma,i} \to Z^0_i$ are univalent, $\sigma \in 1/2$.

If $f_c^{2p}(0)$ escapes through some Z_i then $f^p(0)$ belongs to some domain $Z^1_{\sigma,i}$ (see Figure 5.2). This specifies $2(p-1)$ Misiurewicz limbs

$$\Lambda^2_{\sigma,i} = \{c \in L_{q/p} \setminus \Lambda^1 : \ f_c^p(0) \in Z^1_{\sigma,i}\}$$

attached to $M_{q/p}$ at Misiurewicz points for which $f_c^{2p}(0) = -\alpha$.

Proceeding in this way, we see that the set of parameters $c \in L_{q/p}$ for which the critical orbit escapes V^0 under f^{pN} consists of $(p-1)2^N$ limbs $\Lambda^N_{\sigma,i}$ attached to $M_{q/p}$ at 2^N tips, appropriate Misiurewicz points. These limbs are called *Misiurewicz limbs*. For $c \in \Lambda^N_{\sigma,i}$, we have the following dynamical decomposition of the first central puzzle piece:

$$P^{(-1)} \cap K(f) = \left(V^N \cup \bigcup_{k=1}^{N-1} \bigcup_{\sigma,i} Z^k_{\sigma,i} \right) \cap K(f), \tag{5.3}$$

where for each k, the index σ runs through 2^{k-1} values while the index i runs independently through $p-1$ values. Moreover, the whole configuration moves holomorphically as c ranges over the Misiurewicz limb.

Note that for real parameters $c \in [-2, c^{1/2}] = (L_{1/2} \setminus M_{1/2}) \cap \mathbb{R}$, *the escaping time* N *is equal to 1, and the whole interval* $[-2, c^{1/2}]$ *is contained in a single Misiurewicz limb* Λ *attached to* $c^{1/2}$.

EXERCISE 5.16. Find $c^{1/2}$ and the external angles of the parameter rays landing at this point.

2.4. First decomposition of the Misiurewicz limbs. Let us fix a Misiurewicz limb $\Lambda = \Lambda^N_{\sigma,i}$. We will now decompose it according to the route of the critical orbit back to the puzzle piece V^{N-1}. Let l be the return time of the point $a = f^{pN}(0) \in Z_i$ back to V^{N-1}. According to decomposition (5.3), the orbit $\{f^n a\}_{n=0}^{l-1}$ goes through the puzzle pieces $Z^k_{\sigma,i}$. The itinerary of this orbit through these puzzle pieces specifies the parapuzzle piece $\Delta^0 = \Delta^0(c)$ of the decomposition.

Note that these puzzle pieces do not cover the whole limb Λ. Indeed, in the set $\cup Z^k_{\sigma,i}$ there are some points that *never return back to* V^{N-1}. They form an expanding Cantor set Q. If $a \equiv f^{Np}(0) \in Q$ then the critical orbit never returns back to V^{N-1}. This specifies a Cantor set of Misiurewicz parameters that are left over in Λ after tiling it with the parapuzzle pieces $\Delta^0(c)$. (These Misiurewicz parameters are specified by the itinerary of the infinite orbit $\{f^n(a)\}_{n=0}^{\infty}$ through the puzzle pieces $Z^k_{\sigma,i}$.)

By definition, the next puzzle piece V^{N+1} of the principal nest is the the pullback of V^N by f^l containing 0. This puzzle piece has an important virtue:

LEMMA 5.17. *The puzzle piece* V^N *is compactly contained in* V^{N-1}.

This belongs to Lecture 3?

In the real case, $N = 1$ and there is only one puzzle piece Z. Hence $a = f^2(0)$ and $f^n(a) \in Z$, $n = 0, 1, \dots, l-1$. Thus, the real parapuzzle piece $\Delta^0 \equiv \Delta^0_l$ is completely specified by the return time l. The real traces $\Delta^0_l \cap \mathbb{R}$ ("parapuzzle intervals") cover the parameter interval $(-2, c^{1/2})$ from the right to the left. The only left-over point is the Chebyshev parameter $c = -2$ for which the critical orbit never returns back to V^0.

Returning back to the complex situation, consider the full return maps $g_{N,c} : \cup V^N_{i,c} \to V^{N-1}_c$ (see §III.1.4). They form a "full unfolded generalized quadratic-like family" over the parapuzzle piece Δ^0. Next, we will define for you all these terms.

2.5. Generalized quadratic-like families. Let $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ be the projection to the first coordinate. Given a set $\mathbb{U} \subset \mathbb{C}^2$, we denote by $U_\lambda = \pi_1^{-1}\{\lambda\}$ its vertical cross-section through λ (the "fiber" over λ). Vice versa, given a family of sets $U_\lambda \subset \mathbb{C}$, $\lambda \in D$, we will use the notation:

$$\mathbb{U} = \bigcup_{\lambda \in D} U_\lambda = \{(\lambda, z) \in \mathbb{C}^2 : \lambda \in D, z \in U_\lambda\}.$$

Consider a topological disk $D \subset \mathbb{C}$ and a domain $\mathbb{U} \subset \pi^{-1}(D) \subset \mathbb{C}^2$. Let $\bar{\mathbb{U}}$ be the closure of U in $\pi^{-1}(D)$. If $\pi_1 : \bar{\mathbb{U}} \to D$ is a Jordan discs fibration over D, we call $\mathbb{U}$ (resp, $\bar{\mathbb{U}}$) an open (resp., closed) *topological bidisk* over D.

Let $\mathbb{V}_i \subset \mathbb{U} \subset \mathbb{C}^2$ be a family of topological bidisks over D ("tubes") with pairwise disjoint closures $\bar{\mathbb{V}}_i$ such that that $V_{0,\lambda} \ni 0$. Let

$$\mathbf{g} : \cup \mathbb{V}_i \to \mathbb{U} \tag{5.4}$$

be a fiberwise map whose fiber restrictions

$$\mathbf{g}(\lambda, \cdot) \equiv g_\lambda : \bigcup_i V_{i,\lambda} \to U_\lambda, \quad \lambda \in D,$$

are generalized quadratic-like maps with the critical point at $0 \in V_\lambda \equiv V_{0,\lambda}$ (see §III.1.4).

Pick further a base point $*$ in D and assume that there is a holomorphic motion $\mathbf{h}$ over $(D, *)$,

$$h_\lambda : (\bar{U}_*, \bigcup_i \partial V_{i,*}) \to (\bar{U}_\lambda, \bigcup_i \partial V_{i,\lambda}), \tag{5.5}$$

which respects the boundary dynamics:

$$h_\lambda \circ g_*(z) = g_\lambda \circ h_\lambda(z) \quad for \quad z \in \cup \partial V_{i,*}. \tag{5.6}$$

A *holomorphic family* $(\mathbf{g}, \mathbf{h})$ *of (generalized) quadratic-like maps over* D is a map (5.4) together with a holomorphic motion (5.5) satisfying (5.6). We will sometimes reduce the notation to $\mathbf{g}$. In case when the domain of $\mathbf{g}$ consists of only one tube $\mathbb{V}_0$, we obtain a quadratic-like family in the sense of §II.2.14.

Remark. It would be more consistent to call just $\mathbf{g}$ a holomorphic family, while to call the pair $(\mathbf{g}, \mathbf{h})$, say, an *equipped* holomorphic family. However, in this paper we will assume that the families are equipped, unless otherwise is explicitly stated.

Let us now consider the critical value function $\phi(\lambda) \equiv \phi_{\mathbf{g}}(\lambda) = g_\lambda(0)$, $\Phi(\lambda) \equiv \Phi_{\mathbf{g}}(\lambda) = \mathbf{g}(\lambda, 0) \equiv (\lambda, \phi(\lambda))$. Let us say that $\mathbf{g}$ is a *proper* (or *full*) holomorphic family if the fibration $\pi_1 : \mathbb{U} \to D$ admits an extension to the boundary $\bar{D}$, $\bar{\mathbb{V}}_i \subset \mathbb{U}$, and $\Phi : D \to \mathbb{U}$ is a proper

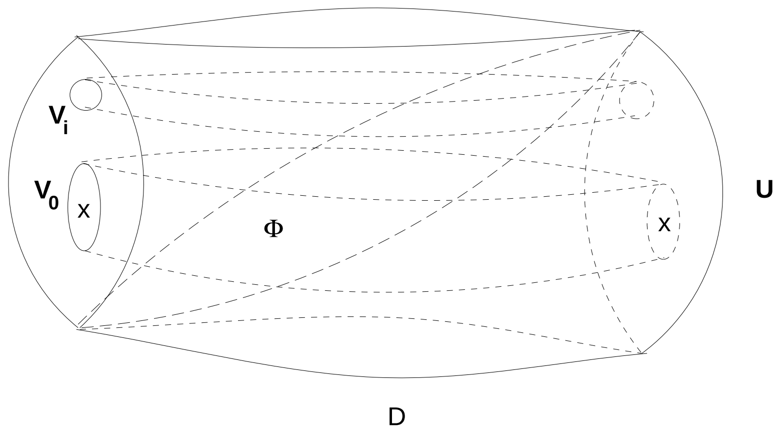

FIGURE 5.3. Generalized quadratic-like family.

section. Note that the fibration $\pi_1 : V_0 \to D$ cannot be extended to $\bar{D}$, as the domains $V_{\lambda,0}$ pinch to figure eights as $\lambda \to \partial D$.

Given a proper holomorphic family $\mathbf{g}$ of generalized quadratic-like maps, let us define its *winding number* $w(\mathbf{g})$ as the winding number of the critical value $\phi(\lambda)$ about the critical point 0. By the Argument Principle, it is equal to the winding number of the critical value about any section $\bar{D} \to \mathbb{U}$.

2.6. Generalized renormalization of holomorphic families. Let us now consider a generalized quadratic-like family ($\mathbf{g} : \cup \mathbb{V}_i \to \mathbb{U}, \mathbf{h}$) over $(D, *)$. Let $\mathcal{I}$ stand for the labeling set of tubes $\mathbb{V}_i$. Remember that $\mathcal{I} \ni 0$ and $\mathbb{V}_0 \ni \mathbf{0}$. Let $\mathcal{I}_\#$ stand for the set of all finite sequences $\bar{i} = (i_0, \dots, i_{t-1})$ of non-zero symbols $i_k \in \mathcal{I} \setminus \{0\}$. For any $\bar{i} \in \mathcal{I}_\#$, there is a tube $\mathbb{V}_{\bar{i}}$ such that

$$\mathbf{g}^k \mathbb{V}_{\bar{i}} \subset \mathbb{V}_{i_k},\ k = 0, \dots, t-1 \quad \textit{and} \quad \mathbf{g}^t \mathbb{V}_{\bar{i}} = \mathbb{U}.$$

We call $t = |\bar{i}|$ the rank of this tube. The map $\mathbf{g}^t : \mathbb{V}_{\bar{i}} \to \mathbb{U}$ is a holomorphic diffeomorphism which fibers over id, that is, $g^t_\lambda V_{\bar{i},\lambda} = U_\lambda,\ \lambda \in D$.

Let us lift the holomorphic motion $\mathbf{h}$ of $\mathbb{U}$ to a holomorphic motion $\hat{\mathbf{h}}$ of the $\mathbb{V}_{\bar{i}}$:

$$g^t_\lambda \circ \hat{h}_{\bar{i},\lambda}(z) = h_\lambda(g^t_* z),\ z \in V_{\bar{i},*}.$$

Note that by (5.6) it coincides with $\mathbf{h}$ on the $\partial \mathbb{V}_i$.

Let $\mathbb{L}_{\bar{i}} \subset \mathbb{V}_{\bar{i}}$ be such a tube that $\mathbf{g}^{|\bar{i}|} \mathbb{L}_{\bar{i}} = \mathbb{V}_0$. The first landing map $\mathbf{T} : \cup \mathbb{L}_{\bar{i}} \to \mathbb{V}_0$ is defined as $\mathbf{T}|L_{\bar{i}} = \mathbf{g}^{|\bar{i}|}$. It is a holomorphic diffeomorphism fibered over id. Extend the holomorphic motion $\hat{h}_\lambda$ to the tubes $\mathbb{L}_{\bar{i}}$ by pulling it back from $\mathbb{V}_0$ by $\mathbf{T}$. Then extend it by the λ-lemma to the whole tube $\mathbb{U}$ keeping it unchanged on the boundaries $\partial \mathbb{U}, \cup \partial \mathbb{V}_{\bar{i}}$.

Let $\phi(\lambda) = g_\lambda 0$ and $\Phi(\lambda) = (\lambda, \phi(\lambda))$. Let $\bar{i}_*$ be the itinerary of the critical value $\phi(*)$ under iterates of g_* through the domains $V_{i,*}$, until its first return to $V_{0,*}$. In other words, let $g_*(0) \in \mathbb{L}_{\bar{i}_*} \equiv \mathbb{L}_*$.

Let us now consider the following parameter region around $*$:

$$D' \equiv D'(*) = \Phi^{-1}\mathbb{L}_*.$$

For $\lambda \in D'$, the itinerary of the critical value under iterates of g_λ until the first return back to $V_{0,\lambda}$ is the same as for g_* (that is, $\bar{i}_*$). Let us define new tubes $\mathbb{V}'_j \subset \mathbb{V}_0$ as the components of $(\mathbf{g}|\mathbb{V}_0)^{-1}(\mathbb{L}_{\bar{i}}|D')$. Let

$$\mathbf{g}' : \cup\mathbb{V}'_j \to \mathbb{V}_0|D' \equiv \mathbb{U}' \tag{5.7}$$

be the first return map of the union of these tubes to $\mathbb{V}_0$.

For $\lambda \in D'$, the critical value $\Phi(\lambda)$ does not intersect the boundaries of the the tubes $\mathbb{L}_{\bar{i}}$. Hence we can lift the holomorphic motion on $\mathbb{U} \setminus \mathbb{L}_*$ to a holomorphic motion $\mathbf{h}'$ on $\mathbb{U}' \setminus \mathbb{V}_0$ over D' and extend it by the λ-lemma to the whole tube $\mathbb{U}'$. Thus we obtain a generalized quadratic-like family $(\mathbf{g}', \mathbf{h}')$ over D' which will be called the *generalized renormalization* of the family $(\mathbf{g}, \mathbf{h})$ (with base point $*$).

LEMMA 5.18. *Let* $\mathbf{g} : \cup\mathbb{V}_i \to \mathbb{U}$ *be a generalized quadratic-like family over* $(D, *)$. *Assume it is proper and has winding number 1. Then its generalized renormalization* $\mathbf{g}' : \cup\mathbb{V}'_j \to \mathbb{U}'$ *over* D' *is also proper and has winding number 1.*

Thus, the "parapuzzle piece" D admits a tiling into parapuzzle pieces D'_i according to the route of return of the critical point back to the central domain V_λ. The residual set $D \cap \mathbb{M} \setminus \cup D_i$ is a Cantor set of Misiurewicz parameters λ for which the critical point never returns back to V_λ.

2.7. Through the central cascades. We will now describe the cascade renormalization of a generalized quadratic-like family, which will be then treated as a single step in the procedure of parameter subdivisions. Let us consider a holomorphic family $(\mathbf{g} : \cup\mathbb{V}_i \to \mathbb{U}, \mathbf{h})$ of generalized quadratic-like maps over $(\Delta, *)$. We will subdivide Δ according to the combinatorics of the central cascades of maps g_λ. To this end let us first stratify the parameter values according to the length of their central cascade. This yields a nest of parapuzzle pieces

$$\Delta \equiv D \supset D^{'} \supset \dots \supset D^{(N)} \supset \dots$$

check N For $\lambda \in D^{(N)}$, the map g_λ has a central cascade

$$V_\lambda^{(0)} \equiv U_\lambda \supset V_\lambda \equiv V_\lambda^{(1)} \supset \dots \supset V_\lambda^{(N)} \tag{5.8}$$

of length N, so that $g_\lambda 0 \in V_\lambda^{(N-1)} \setminus V_\lambda^{(N)}$. Note that the puzzle pieces $V_\lambda^{(k)}$ are organized into the tubes $\mathbb{V}^{(k)}$ over $D^{(k-1)}$ (with the convention that $D^{(-1)} \equiv D$).

The intersection of these puzzle pieces, $\cap D^{(N)}$, is the little Mandelbrot set $M(\mathbf{g})$ centered at the superattracting parameter value $c = c(\mathbf{g})$ such that $g_c(0) = 0$. Let us call c the *center* of D.

Let $* \in D^{(N-1)} \setminus D^{(N)}$. Let us consider the Bernoulli map

$$\mathbf{G} : \cup \mathbb{W}_j \to \mathbb{U} \tag{5.9}$$

associated with the cascade (5.8) (see §1.5). Here the tubes $\mathbb{W}_j$ over $D^{(N-1)}$ are the pull-backs of the tubes $\mathbb{V}_i | D^{(N-1)}$, $i \neq 0$, by the covering maps

$$\mathbf{g}^k : (\mathbb{V}^{(k)} \setminus \mathbb{V}^{(k+1)}) | D^{(N-1)} \to (\mathbb{U} \setminus \mathbb{V}) | D^{(N-1)}, \quad k = 0, 1 \dots, N-1. \tag{5.10}$$

In the same way as in §2.6, to any string $\bar{j} = (j_0, \dots, j_{t-1})$ corresponds the tube over $D^{(N-1)}$,

$$\mathbb{W}_{\bar{j}} = \{p \in \mathbb{U} | D^{(N-1)} : \mathbf{G}^n p \in \mathbb{W}_{j_n},\ n = 0, \dots, t-1\}.$$

Note that $\mathbf{G}^t$ univalently maps each $\mathbb{W}_{\bar{j}}$ onto $\mathbb{U}|D^{(N-1)}$. Thus $\mathbb{W}_{\bar{j}}$ contains a tube $\mathbb{L}_{\bar{j}}$ which is univalently mapped by $\mathbf{G}^t$ onto the central tube $\mathbb{V}^{(N)}$. These maps altogether form the first landing map to $\mathbb{V}^{(N)}$,

$$\mathbf{T} : \cup \mathbb{L}_{\bar{j}} \to \mathbb{V}^{(N)}. \tag{5.11}$$

Remark. Note that

$$\operatorname{mod}(W_{\bar{j},\lambda} \setminus L_{\bar{j},\lambda}) = \operatorname{mod}(U_\lambda \setminus V_\lambda^{(N)}) \geq \operatorname{mod}(U_\lambda \setminus V_\lambda), \tag{5.12}$$

since G_λ^t univalently maps the annulus $W_{\bar{j},\lambda} \setminus L_{\bar{j},\lambda}$ onto $U_\lambda \setminus V_\lambda^{(N)}$.

Let us now consider the itinerary $\bar{j}_*$ of the critical value $\phi(*) \equiv g_*(0)$ through the tubes W_j until its first return to $V^{(N)}$, so that $\Phi(*) \in \mathbb{L}_{\bar{j}_*} \equiv \mathbb{L}_*$. Let $\mathbb{W}_* \equiv \mathbb{W}_{\bar{j}_*}$ and

$$\Delta^\diamond(*) = \Phi^{-1}\mathbb{L}_*, \quad \Lambda^\diamond(*) = \Phi^{-1}\mathbb{W}_*. \tag{5.13}$$

Thus, the annuli $D^{(N-1)} \setminus D^{(N)}$ are tiled by the parapuzzle pieces $\Delta^\diamond(\lambda)$ according as the itinerary of the critical point through the Bernoulli scheme (5.9) until the first return to $V_\lambda^{(N)}$. Altogether these tilings form the desired new subdivision of Δ. (Again, the new tiles do not cover the whole domain Δ: the residual set consists of the Mandelbrot set $M(\mathbf{g})$ and of the parameter values $\lambda \in D^{(N-1)} \setminus D^{(N)}$ for which the critical orbit never returns back to $V_\lambda^{(N)}$.)

The affiliated quadratic-like family over $\Delta^\diamond(*)$ is defined as the first return map to $V_\lambda^{(N)} \equiv U_\lambda^\diamond$. Its domain $\cup \mathbb{V}_i^\diamond$ is obtained by pulling back the tubes $\mathbb{L}_{\bar{j}}$ from (5.11) by the double branched covering $\mathbf{g} : \mathbb{V}^{(N)} \to \mathbb{V}^{(N-1)}|\Delta^\diamond(*)$, and the return map itself is just $\mathbf{T} \circ \mathbf{g}$.

The affiliated holomorphic motion is also constructed naturally, as in §2.6. Let us first lift the holomorphic motion $\mathbf{h}$ from the condensator $\mathbb{U} \smallsetminus \mathbb{V}$ to the condensators $(\mathbb{V}^{(k)} \smallsetminus \mathbb{V}^{(k+1)})|D^{(N-1)}$ via the coverings (5.10). This provides us with a holomorphic motion of $(\mathbb{U} \smallsetminus \mathbb{V}^{(N)}, \cup \mathbb{W}_j)$ over $D^{(N-1)}$. Extend it through $\mathbb{V}^{(N)}$ by the λ-lemma, lift it to the tubes $(\mathbb{W}_{\bar{j}}, \mathbb{L}_{\bar{j}})$ and then extended again by the λ-lemma to the whole domain $\mathbb{U}$ over D^{N-1}. Let us denote it by $\mathbf{H}$.

Lifting this motion via the fiberwise analytic double covering over $\Delta^\diamond(*)$,

$$\mathbf{g} : (\mathbb{U}^\diamond \smallsetminus \mathbb{V}^\diamond, \bigcup_{i \neq 0} \mathbb{V}_i^\diamond) \to (\mathbb{V}^{(N-1)} \smallsetminus \mathbb{L}_*, \bigcup_{\bar{j} \neq \bar{j}_*} \mathbb{L}_{\bar{j}}),$$

we obtain the desired motion of $(\mathbb{U}^\diamond \smallsetminus \mathbb{V}^\diamond, \bigcup_{i \neq 0} \mathbb{V}_i^\diamond)$ over $\Delta^\diamond(*)$. By the λ-lemma it extends through $\mathbb{V}_0^\diamond$.

2.8. Principal parapuzzle nest. Let us now summarize the above discussion. Given a quadratic-like family $(\mathbf{f}, \mathbf{h})$ over $D \equiv \Delta^0$, we consider the first tiling $\mathcal{D}^1$ of a Misiurewicz wake Λ as described in §2.4. Each tile $\Delta \in \mathcal{D}^1$ comes together with a generalized quadratic-like family $(\mathbf{g}_\Delta, \mathbf{h}_\Delta)$ over Δ.

Now assume inductively that we have constructed the tiling $\mathcal{D}^l$ of level l. Then the tiling of the next level, $\mathcal{D}^{l+1}$ is obtained by decomposing each tile $\Delta \in \mathcal{D}^l$ by means of the cascade renormalization as described in §2.7.

Let $\Delta^l(\lambda)$ stand for the tile of $\mathcal{D}^l$ containing λ, while $\Delta^l(\lambda) \subset \Lambda^l(\lambda) \subset \Delta^{l-1}(\lambda)$ stand for the other tile defined in (5.13). Each tile $\Delta = \Delta^l(\lambda)$ contains a *central sub-tile* $\Pi^l(\lambda) = \Phi_\Delta^{-1}\mathbb{V}_0$ corresponding to the central return of the critical point (here $\Phi_\Delta(\lambda) = (\lambda, \mathbf{g}_\Delta(\lambda))$). Note that $\Pi^l(\lambda)$ may or may not contain λ itself.

Let us then consider the sequence of renormalized families $(\mathbf{g}_{l,\lambda}, \mathbf{h}_{l,\lambda})$ over topological discs $\Delta^l(\lambda)$. We call the nest of topological discs $\Delta^0 \supset \Delta^1(\lambda) \supset \Delta^2(\lambda) \supset \dots$ (supplied with the corresponding families) *the principal parapuzzle nest* of λ. If λ is not Misiurewicz and not renormalizable, then this nest is infinite.

Let $c_{l,\lambda} \in \Delta^l(\lambda)$ be the centers of the corresponding parapuzzle pieces. Let us call them the *principal superattracting approximations* to λ. If λ is not renormalizable, then $c_{l,\lambda} \to \lambda$ as $l \to \infty$, since $\operatorname{diam} \Delta^l(\lambda) \to 0$ (see the next section).

The $\mathrm{mod}(\Delta^l(\lambda) \smallsetminus \Delta^{l+1}(\lambda))$ are called the *principal parameter moduli* of $\lambda \in D$.

When we fix a base point $*$, we will usually skip label $*$ in the above notations, so that $\Delta^l \equiv \Delta^l(*)$, $\mathbf{g}_l \equiv \mathbf{g}_{l,*}$, $\mathbf{h}_l \equiv \mathbf{h}_{l,*}$ etc.

3. Parapuzzle geometry

We are now ready to formulate the main geometric property of the parapuzzle:

THEOREM 5.19. *Let us consider a proper unfolded quadratic-like family* $(\mathbf{f}, \mathbf{h})$ *over* D, *and a Misiurewicz wake* $\Lambda \subset D$. *Then for any* $\lambda \in M(\mathbf{f}) \cap \Lambda$,

$$\mathrm{mod}(\Delta^l(\lambda) \smallsetminus \Delta^{l+1}(\lambda)) \geq Bl, \quad \textit{and} \quad \mathrm{mod}(\Delta^l(\lambda) \smallsetminus \Pi^l(\lambda)) \geq Bl,$$

where the constant $B > 0$ *depends only on* Λ *and* $\mathrm{mod}(\mathbf{f})$.

In the rest of this section we will outline the proof of this theorem.

3.1. Initial parameter geometry. Let us start with a bound on the geometry of the first level parapuzzle. Fix a quadratic-like family $(\mathbf{f}, \mathbf{h})$ and its Misiurewicz wake $\Lambda = \Lambda^{\sigma,i}_{q/p}$, $|\sigma| = t$, as in §2.3.

LEMMA 5.20. *All parapuzzle pieces of the first level are well inside the corresponding wake:* $\mathrm{mod}(\Lambda \smallsetminus \Delta^1) \geq \nu > 0$. *Moreover, the holomorphic motion* $\mathbf{h}_1$ *of the condensator* $\mathbb{U}^1 \smallsetminus \mathbb{V}^1$ *over* Δ^1 *is* K*-qc. The constants* ν *and* K *depend only on the geometry of* $(\mathbf{f}, \mathbf{h})$ *and the choice of* Λ.

3.2. Inductive estimate of the parameter geometry. The following lemma shows that the geometry of the parapuzzle changes gradually under the cascade renormalization.

LEMMA 5.21. *Let us consider a generalized quadratic-like family* $(\mathbf{g} : \cup \mathbb{V}_i \to \mathbb{U}, \mathbf{h})$ *over* Δ. *Assume that the dilatation of* $\mathbf{h}$ *on* $\mathbb{U} \smallsetminus \mathbb{V}_0$ *is bounded by* K *and* $\mathrm{mod}(U_* \smallsetminus V_{0,*}) \geq \mu > 0$, $\lambda \in D$. *Then the dilatation of the cascade renormalized motion* $\mathbf{h}^\diamond$ *on* $\mathbb{U}^\diamond \smallsetminus \mathbb{V}^\diamond_0$ *over* $D^\diamond$ *(as described in §2.7) is bounded by* $K^\diamond = K^\diamond(\mu, K)$.

3.3. Inscribing rounds condensators. In this section we will show that the parameter annuli have definite moduli. Given a holomorphic motion h_λ and a holomorphic family of affine maps $g_\lambda : z \mapsto a_\lambda z + b_\lambda$, we can consider an "affinely equivalent" motion $g_\lambda \circ h_\lambda$. In this way the motion can be normalized such that any two points $z, \zeta \in U_*$ don't move (that is, $h_\lambda(z) \equiv z$ and $h_\lambda(\zeta) \equiv \zeta$ for $\lambda \in D$). Let us start with a technical lemma:

LEMMA 5.22. *Let us consider a holomorphic motion* $h : (U_*, V_*, 0) \to (U_\lambda, V_\lambda, 0)$ *of a pair of nested topological discs over a domain* D. *Assume that the maps* $h_\lambda : (\partial U_*, \partial V_*) \to (\partial U_\lambda, \partial V_\lambda)$ *admit* K*-qc extensions* $H_\lambda : (\mathbb{C}, U_*) \to (\mathbb{C}, V_\lambda)$ *(not necessarily holomorphic in* λ *but with uniform dilatation* K*). Then there exists an* $M = M(K)$ *such that if* $\operatorname{mod}(U_* \smallsetminus V_*) > M$ *then after appropriate normalization of the motion, there exists a round condensator* $D \times \mathbb{A}(q, 2q)$ *embedded into* $\mathbb{U} \smallsetminus \mathbb{V}$.

PROOF. Let z_* be a point on ∂U_* closest to 0. Normalize the motion in such a way that $z_* = 1$, and this point does not move. With this normalization, $V_* \subset \mathbb{D}_\varepsilon(0)$ where $\varepsilon = \varepsilon(m) \to 0$ as $m \equiv \operatorname{mod}(U_* \smallsetminus V_*) \to \infty$.

Since the space of normalized K-qc maps is compact, $|H_\lambda(\varepsilon e^{i\theta}|) < \delta$, where $\delta = \delta(\varepsilon, K) \to 0$ as $\varepsilon \to 0$, K being fixed, and $|H_\lambda(e^{i\theta})| > r$ where $r = r(K) > 0$. It follows that the domain $\mathbb{U}$ contains the round cylinder $D \times \mathbb{A}(\delta, r)$, and we are done. □

COROLLARY 5.23. *Under the circumstances of Lemma 5.22, let* $\Phi : D \to \mathbb{U}$ *be a proper analytic map with winding number 1. Let* $D' = \Phi^{-1}\mathbb{V}$. *If* $\operatorname{mod}(U_* \smallsetminus V_*) > M = M(K)$ *then* $\operatorname{mod}(D \smallsetminus D') \geq \log 2$.

PROOF. By Lemma 5.22, $\mathbb{U} \smallsetminus \mathbb{V} \supset D \times A$ where $A = \mathbb{A}(q, 2q)$. Let $Q = \Phi^{-1}(D \times A)$. By the Argument Principle, $\phi = \pi_2 \circ \Phi$ univalently maps Q onto A, so that $\operatorname{mod}(D \smallsetminus D') \geq \operatorname{mod} Q = \operatorname{mod} A = \log 2$. □

3.4. Pseudo-conjugacies revisited. Let us consider a quadratic-like family $(\mathbf{f}, \mathbf{h})$ and its parameter tilings. Let $\tilde{\lambda} \in \Delta^l(\lambda)$. Let us consider the corresponding l-fold generalized renormalizations of these two maps $g_l : \cup V_i \to U$ and $\tilde{g}_l : \cup \tilde{V}_i \to \tilde{U}$. Then the holomorphic motion transforms the domains of g_l to the corresponding domains of $\tilde{g}_l$ respecting the boundary marking. In this sense f_λ and $f_{\tilde{\lambda}}$ have "the same combinatorics up to level" l.

Proofs of Theorems 4.26 and 4.28 show:

THEOREM 5.24. *Assume that* $\tilde{\lambda} \in \Lambda^{l+1}(\lambda)$, *where the tile* $\Lambda^{l+1}(\lambda)$ *is defined by (5.13). Then the corresponding generalized renormalizations* g *and* $\tilde{g}$ *are* K*-qc pseudo-conjugate, with* K *depending only on the Misiurewicz wake* $\Lambda(\lambda)$ *and geometry of* $(\mathbf{f}, \mathbf{h})$.

3.5. Uniform bound of dilatation.

LEMMA 5.25. *Let* $*$ *belong to a Misiurewicz wake* Λ. *For any principal parapuzzle piece* $\Delta = \Delta^{l+1}(*)$, *the corresponding holomorphic motion* $\mathbf{h}_\Delta$ *of* $\mathbb{U}^{l+1} \smallsetminus \mathbb{V}_0^{l+1}$ *over* Δ *has a uniformly bounded dilatation, depending only on the choice of the Misiurewicz wake* Λ *and the geometry of* $(\mathbf{f}, \mathbf{h})$.

Proof. Let K be a dilatation bound given by Theorem 5.24. Find an $M = M(K)$ by Corollary 5.23. By Theorem 3.6 and (5.12), there exists an l_0 such that $\operatorname{mod}(W_*^l \setminus L_*^l) \geq M$ for $l \geq l_0$.

For $l \leq l_0$, the desired dilatation bound is guaranteed by Lemmas 5.20 and 5.21.

Fix an $l \geq l_0$. Let us consider the generalized quadratic-like family $(\mathbf{g} : \cup \mathbb{V}_i \to \mathbb{U}, \mathbf{h})$ over $D \equiv \Delta^l(*)$. In what follows we will use the notations of §2.7. Let $* \in D^{(N-1)} \setminus D^{(N)}$.

By Theorem 5.24, for $\lambda \in \Lambda^{l+1}(*)$, there is a K-qc pseudo conjugacy $\psi_\lambda : (U_*, \cup V_{i,*}) \to (U_\lambda, \cup V_{i,\lambda})$, with K depending only on the choice of wake Λ and geometry of $(\mathbf{f}, \mathbf{h})$. As $\operatorname{mod}(W_*^l \setminus L_*^l) \geq M$, Corollary 5.23 can be applied. We conclude that

$$\operatorname{mod}(\Lambda^{l+1}(*) \setminus \Delta^{l+1}(*)) \geq \log 2 \tag{5.14}$$

for l sufficiently big (depending on Λ and geometry of $(\mathbf{f}, \mathbf{h})$).

In §2.7 we have constructed a holomorphic motion $\mathbf{H}$ of $(\mathbb{U}, \mathbb{W}_{\bar{j}}, \mathbb{L}_{\bar{j}})$ over $D^{(N-1)}$. By the λ-lemma and (5.14), $\mathbf{H}$ is L-qc over $\Delta = \Delta^{l+1}(*)$, with an absolute L provided l is big enough. But the holomorphic motion $\mathbf{h}_\Delta$ on $\mathbb{U}^{l+1} \setminus \mathbb{V}^{l+1}$ is the lift of $\mathbf{H}$ on $\mathbb{V}^{(N-1)} \setminus \mathbb{L}_{\bar{j}_*}$ over Δ by means of the fiberwise analytic double covering

$$\mathbf{g} : \mathbb{U}^{l+1} \setminus \mathbb{V}^{l+1} \to \mathbb{V}^{(N-1)} \setminus \mathbb{L}_*.$$

Hence $\mathbf{h}_\Delta$ on $\mathbb{U}^{l+1} \setminus \mathbb{V}^{l+1}$ is also L-qc. □

3.6. Proof of Theorem 5.19. We are now prepared to complete the proof:

$$\operatorname{mod}(\Delta^l \setminus \Delta^{l+1}) \geq K^{-1} \operatorname{mod}(W_{\bar{i}_*} \setminus L_{\bar{i}_*}) \geq Bl.$$

The first estimate in the above row follows from Lemma 5.25 and transverse quasiconformality of holomorphic motions (§II.4.3). The last estimate is due to Theorem 3.6.

For the same reason,

$$\operatorname{mod}(\Delta^l \setminus \Pi^l) \asymp \operatorname{mod}(U_*^l \setminus V_{0,*}^l) \geq Bl.$$

4. Proof of Theorem A

With Theorem 5.19 in hands, we can turn the heuristic argument of §1.4.3 into a rigorous proof. By renormalizing the quadratic family over some Mandelbrot copy, we reduce the statement to the set $\mathcal{N}$ of non-renormalizable parameters (of some full unfolded quadratic-like family). The above discussion provides us with complete combinatorial understanding and strong geometric control of this set.

Given measurable sets $X, Y \subset \mathbb{R}$, with $|Y|) > 0$, let $\text{dens}(X|Y)$ stand for the $|X \cap Y|/|Y|$.

We will restrict all tilings $\mathcal{D}^l$ constructed above to the real line, without change of notations. We will use the same notation, $\mathcal{D}^l$, for the union of all pieces of $\mathcal{D}^l$. For every $\Delta = \Delta^l(\lambda) \in \mathcal{D}^l$, let us consider the central piece $\Pi \subset \Delta$ corresponding to the central return of the critical point. By Theorem A, $\text{dens}(\Pi|\Delta) \leq Cq^l$ for absolute $C > 0$ and $q < 1$. Let Γ^l be the union of these central pieces. Summing up over all $\Delta \in \mathcal{D}^l$, we conclude that

$$|\Gamma^l| \leq \text{dens}(\Gamma^l|\mathcal{D}^l) \leq Cq^l \tag{5.15}$$

(the whole interval is normalized so that its length is equal to 1).

It follows that for l sufficiently big,

$$\text{dens}(\bigcup_{k\geq 0} \Gamma^{l+k}|\mathcal{D}^l) \leq C_1 q^l < 1,$$

which means that with positive probability central returns will never occur again. This proves that $\text{meas}(\mathcal{Y}) > 0$.

To prove that almost all points $c \in \mathcal{Y}$ are stochastic, just notice that (5.15) together with the Borel-Cantelli Lemma yield that infinite number of central returns occurs with zero probability.

5. Shapes of the Mandelbrot copies

The above geometric results also provide us with control of the shape of M-copies:

THEOREM 5.26. *Let O be a Misiurewicz wake in a full unfolded proper quadratic-like family* $\mathbf{f}$. *Then all maximal M-copies in O have a K-quasistandard shape, with K depending only on the geometry of* $\mathbf{f}$ *and the choice of O.*

PROOF. By Theorem 5.19 and Lemma 5.25, the quadratic-like families creating the M-copies in question have a bounded geometry. By Lemma 2.15, these M-copies have a quasistandard shape. □

Since all maximal real M-copies, except the doubling one, are contained in a single Misiurewicz wake, we conclude:

COROLLARY 5.27. *All maximal real M-copies in the quadratic family, except the doubling one, have a K-quasistandard shape with an absolute K.*

6. Bibliographical notes

The central result of this lecture is the proof of Theorem A [**L5**]. §2 and §3 on the parapuzzle combinatorics and geometry are based upon [**L5**].

The Collet-Eckmann criterion for existence of a.c.i.m. (Theorem 5.10) appeared in [**CE**]; the Nowicki-van Strien criterion appeared in [**NS**]. The Martens-Nowicki criterion appeared in [**MN**].

The result that the set of stochastic parameters has positive Lebesgue measure was first proved by Jacobson [**J**] and Benedicks - Carleson [**BC**]. The proof given in these notes follows [**L5**].

Theorem 5.3 was proven in [**L3**]. The results of 1.3 on measurable dynamics of S-unimodal maps are mostly taken from [**BL2**]. However, Theorem 5.8 is due to Ledrappier [**Le**].

For an introduction to the basic ergodic theory see, e.g., [**KFS**].

For a further deep exploration of stochastic properties of typical non-renormalizable quadratics, see [**AM2**]. In particular, it is proven over there that for almost any non-renormalizable $c \in [-2, 1/4]$, the polynomial f_c satisfies the Collet-Eckmann property.

LECTURE 6

Universality

1. Set-up

1.1. Discovery. In mid 1970's a truly remarkable discovery was made by Feigenbaum and independently by Coullet & Tresser. Consider the real quadratic family $x \mapsto x^2 + c$, and let c decrease from $1/4$ to -2. In the beginning we see the cascade of doubling bifurcations c_n converging to the Feigenbaum point c_F (see §I.5.1 and Figure 1.5).

With the help of a calculator Feigenbaum observed that this convergence is exponential: $c_n - c_\infty \sim C\lambda^{-n}$, where $\lambda = 4.669....$ It was curious but what was really surprising is that if you take a similar family of unimodal maps, say $x \mapsto b \sin x$ on $[0, \pi]$, then you will observe a similar sequence of doubling bifurcations b_n exponentially converging to a limit point b_* with the *same rate*: $b_* - b_n \sim C'\lambda^{-n}$, where $\lambda = 4.669....$ In other words, the rate of convergence is *universal*, independent of the particular family of unimodal maps under consideration.

1.2. Renormalization Conjecture. Motivated by the renorm-group method in statistical mechanics, Feigenbaum and Coullet & Tresser formulated a beautiful conjecture which would completely explain the above universality. Imagine an infinite dimensional space $\mathcal{U}$ of unimodal maps, and consider the doubling renormalization operator R in this space defined on the set of maps renormalizable with period two (see §II.6.1). The conjecture asserted that:

(i) R has a unique fixed point f_*, i.e., a unique solution of the *Feigenbaum-Cvitanović equation* $f(z) = \mu^{-1} f \circ f(\mu z)$ with an appropriate scaling factor μ.

(ii) R is hyperbolic at this fixed point, that is, there exist two transverse R-invariant manifolds $\mathcal{W}^s$ and $\mathcal{W}^u$ through f_* such that the orbits $\{R^n f\}$, $f \in \mathcal{W}^s$, exponentially converge to f_*, while the orbits $\{R^n f\}$, $f \in \mathcal{W}^u$, are exponentially repelled from f_*.

(iii) $\dim \mathcal{W}^u = 1$.

(iv) $\mathcal{W}^u$ transversally intersects the doubling bifurcation locus $\mathcal{B}_1$, where an attracting fixed point bifurcates into an attracting cycle of period 2.

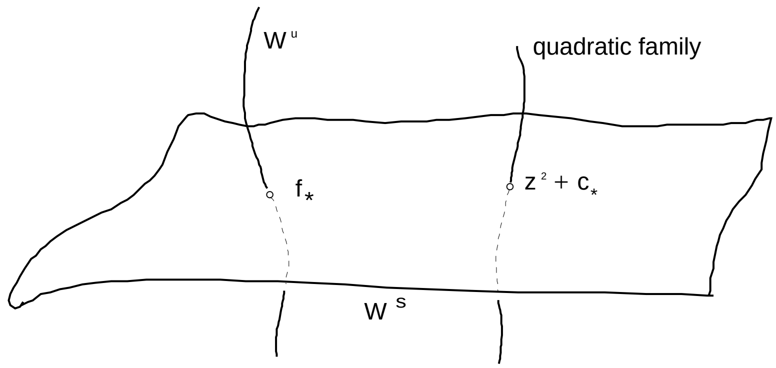

FIGURE 6.1. Hyperbolic fixed point of the renormalization operator.

Since the doubling bifurcations loci $\mathcal{B}_n$ of higher periods (from 2^n to 2^{n+1}) are obtained by taking preimages of $\mathcal{B}_1$ under R^n, one can readily see that any one parameter family of unimodal maps (i.e., a curve in $\mathcal{U}$) which is transverse to W^s intersects the $\mathcal{B}_n$ at the points b_n exponentially converging to a limit point $b_* \in W^s$, where the rate of convergence, λ, is just the *unstable eigenvalue* of $DR(f_*)$. Thus, it is independent of the particular family under consideration.

2. Renormalization Theory with stationary combinatorics

In this section we will discuss the Renormalization theory in a slightly bigger generality than stated before, namely for an arbitrary real stationary combinatorics. In the next section we will give a radical generalization of the theory.

2.1. Set-up. Let us fix some finite kneading sequence $\boldsymbol{\kappa}$ of length $p \geq 2$ (see §I.5.3). Then we have the following associated objects:

- $R : \mathcal{T} \to \mathcal{Q}$ is the complex renormalization operator with combinatorics $\boldsymbol{\kappa}$ (see §II.6.2), where $\mathcal{Q}$ is the space of quadratic-like germs (see §II.4) and $\mathcal{T}$ is the subspace of quadratic-like germs renormalizable with combinatorics $\boldsymbol{\kappa}$;
- $\mathcal{Q}_{\mathbb{R}}$ is the space of real quadratic like maps and $\mathcal{T}_R = \mathcal{T} \cap \mathcal{Q}_{\mathbb{R}}$; thus, $R : \mathcal{T}_{\mathbb{R}} \to \mathcal{Q}_{\mathcal{R}}$;
- M is the corresponding little M-copy, and $\sigma : M \to \mathbb{M}$ is the stretching homeomorphism (2.12);
- $J = M \cap \mathbb{R}$ is the corresponding renormalization window; thus,

$$\sigma : J \to [-2, 1/4].$$

2.2. Stable-manifold-to-be. By the Rigidity Theorem 4.24, the stretching $\sigma : J \to [-2, 1/4]$ has a unique fixed point $c \in J$, which is the unique infinitely renormalizable real parameter with stationary combinatorics $(\boldsymbol{\kappa}, \boldsymbol{\kappa}, \dots)$. Consider the hybrid class $\mathcal{H}_c \subset \mathcal{C}$ through this point (see §II.4.6). Since $\sigma = \chi \circ R$, $\mathcal{H}_c$ is invariant under the renormalization R. This is going to be the stable manifold of the renormalization fixed point. It is big luck that the stable-manifold-to-be can be recognized before the fixed point f_* is found! In fact, it gives us the first approximation to f_*.

By the complex *a priori* bounds (Theorem 3.21) and Proposition 2.5, the orbits of the operator $R : \mathcal{H}_* \to \mathcal{H}_*$ are non-escaping (that is, pre-compact) in $\mathcal{H}_*$.

Remark. Note that the maps in $\mathcal{H}_c$ are not necessarily real. However, they are qc conjugate to a real map f_c. Since f_c has *a priori bounds*, all the maps in $\mathcal{H}_c$ also do.

Pick now some $g \in \mathcal{H}_c$ and consider the ω-limit set $\omega(g) \subset \mathcal{H}_c$ of the orbit $\{R^n g\}_{n=0}^{\infty}$. It is an R-invariant compact set. It gives us the next approximation to the fixed point. Note that the map $R : \omega(g) \to \omega(g)$ is surjective. Indeed, if $g = \lim R^{n_k} f$ then $g = Rg_{-1}$, where $g_{-1} \in \omega(g)$ is a limit point of the subsequence $\{R^{n_k - 1} g\}$. Hence for any $f \equiv f_0 \in \omega(g)$ we can consider a two-sided orbit

$$\cdots \mapsto f_{-1} \underset{R}{\mapsto} f_0 \underset{R}{\mapsto} f_1 \underset{R}{\mapsto} \dots \tag{6.1}$$

of quadratic-like germs in $\omega(g)$. It was McMullen's insight that the corresponding sequence of quadratic-like maps related by the renormalization should be viewed as a single dynamical system called a "tower".

2.3. McMullen towers.

2.3.1. *Definitions.* A *McMullen tower* $\mathbf{f}$ (with stationary combinatorics $\boldsymbol{\kappa}$) is a sequence of quadratic-like maps $f_n : U_n \to U'_n$ representing a renormalization orbit (6.1), $l \le n \le r$. Thus, $f_{n+1} = f_n^p | U_n$. The dimensions l and r can be finite or infinite. If $r = \infty$, $l = -\infty$, the tower is called *bi-infinite*.

The towers are considered up to affine equivalence, i.e., the simultaneous conjugacy of the maps f_n by $z \mapsto \lambda z$. The Julia set of the tower, $J(\mathbf{f})$, is the union of the Julia sets $J(f_n)$. It is not necessarily closed.

Two towers $\mathbf{f}$ and $\tilde{\mathbf{f}}$ are called *topologically conjugate* if there exists a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ conjugating f_n to $\tilde{f}_n$ on some neighborhoods of their Julia sets, $n \in \mathbb{Z}$. If h can be selected to be qc then the towers are called *qc conjugate.* If additionally $\bar{\partial} h = 0$ a.e. on $J(\mathbf{f})$ then

$\mathbf{f}$ and $\tilde{\mathbf{f}}$ are called *hybrid* equivalent. (One can also define naturally *affine, smooth, etc.* conjugacies between towers.)

A tower has *a priori* bounds if $\mathrm{mod}(U'_n \setminus U_n) \geq \mu > 0$, $n \in \mathbb{Z}$. It represents a non-escaping renormalization orbit in $\mathcal{C}$.

LEMMA 6.1. *Let $\{f_n\}$ and $\{\tilde{f}_n\}$ be two towers with a priori bounds. If f_n is hybrid equivalent to $\tilde{f}_n$ for all $n \in \mathbb{Z}$ then the towers are qc equivalent.*

PROOF. By Proposition 2.3(ii), there exists a K such that the maps f_n and $\tilde{f}_n$ are K-qc conjugate, $n \in \mathbb{Z}$. A limit qc map provides a conjugacy between the towers. □

2.3.2. *Compactness.* The space of towers is naturally endowed with coordinatewise topology: A sequence of towers $\mathbf{f}_m = \{f_{m,n}\}_n$ converges to a bi-infinite tower $\mathbf{g} = \{g_n\}$ as $m \to \infty$ if for any $n \in \mathbb{Z}$, $f_{m,n} \to g_n$ as $m \to \infty$. Proposition 2.5 implies: (Note that the size of the towers $\mathbf{f}_m$ can vary but eventually the maps $f_{m,n}$ should be well defined for any $n \in \mathbb{Z}$.)

LEMMA 6.2. *The space of towers with stationary combinatorics $\boldsymbol{\kappa}$ and common* a priori *bounds is compact.*

2.3.3. *Expanding property.* Given a tower $\mathbf{f} = \{f_n : U_n \to U'_n\}$, let $O_{\mathbf{f}} = \cup O_{f_n}$ stand for its *postcritical set.* Endow the domain $\mathbb{C} \setminus O_{\mathbf{f}}$ with the hyperbolic metric.

THEOREM 6.3. *Let $\mathbf{f}$ be a tower with combinatorics $\boldsymbol{\kappa}$ and a priori bounds. Let $z \in U_0 \setminus J(\mathbf{f})$. Then there exists a $\lambda > 1$ $s \in \mathbb{N}$, and a sequence $z_n \to \infty$ as $n \to +\infty$ such that $z_{n+1} = f_{-sn}^{l_n} z_n$ for some moments $l_n \in \mathbb{N}$ and*

$$\|Df_{-sn}^{l_n}(z_n)\|_{\mathrm{hyp}} \geq \lambda > 1.$$

PROOF. It is not hard to see that there is an $s \in \mathbb{N}$ such that after some adjustment of the quadratic-like maps f_n (compare Proposition 2.3), we obtain: $f_n^{-1} U_n \supset U'_{n+s}$. Moreover, one can make the annuli $U'_n \setminus f_n^{-2} U_n$ disjoint from the postcritical set $O_{\mathbf{f}}$.

To make notations easier, let us assume that $s = 1$.

Note that for any quadratic-like map $f : U \to U'$, the (multi-valued) inverse map $f^{-1} : U' \setminus O_f \to U \setminus O_f$ can be decomposed as the inverse of the covering $f : U \setminus f^{-1} O_f \to U' \setminus O_f$ and the natural embedding

$$i : U \setminus f^{-1}(O_f) \to U' \setminus O_f.$$

The former map preserves the respective hyperbolic metrics while the latter is contracting by the Schwarz Lemma. Moreover, the amount of contraction at $z \in U \setminus f^{-1}(O_f)$ depends only on the hyperbolic distance from z to $f^{-1}(O_f)$ in $U' \setminus O_f$.

Hence the map $f : U \setminus O_f \to U' \setminus O_f$ is *expanding* with respect to the corresponding hyperbolic metrics. Moreover, amount of expansion at point $z \in U \setminus O_f$ depends only on the hyperbolic distance from z to $f^{-1}(O_f)$ in $U \setminus O_f$. This amount is uniform if z belongs to the fundamental annulus $A = U \setminus f^{-1}U$.

Now, as $z \in U_0 \setminus J(f)$, there is a moment l_0 such that $z_1 = f^{l_0} z_0$ escapes through the fundamental annulus $A_0 = U_0 \setminus f_0^{-1} U_0$ of f_0. For the same reason, there is a moment l_1 when $z_2 = f_{-1}^{l_1} z_1$ escapes through the fundamental annulus A_{-1} of f_{-1}, etc. At every escaping moment, the corresponding map enjoys some definite expansion with respect to the corresponding hyperbolic metrics.

One can show (using disjointness of the fundamental annuli and $O_{\mathbf{f}}$) that these metrics are in fact comparable with the hyperbolic metric in $\mathbb{C} \setminus O_{\mathbf{f}}$, which yields the desired assertion. □

2.3.4. *Tower Rigidity.*

THEOREM 6.4. *Let* $\mathbf{f}$ *and* $\tilde{\mathbf{f}}$ *be two towers with a priori bounds. If* $\mathbf{f}$ *and* $\tilde{\mathbf{f}}$ *are qc equivalent then they are affinely equivalent.*

PROOF. Let $h : \mathbb{C} \to \mathbb{C}$ be a qc equivalence between the towers. If this map is not affine, then the Beltrami differential $\bar{\partial} h / \partial h$ induces an $\mathbf{f}$-invariant line field μ on some set of positive measure (i.e., a line field which is invariant under the maps f_n on their domains).

However, by Theorem 4.27, there are no invariant line fields on the Julia set $J(\mathbf{f})$. Hence μ is supported on $\mathbb{C} \setminus J(\mathbf{f})$. Let us take a density point z for μ. Then the line field is almost parallel on a set of almost full measure near z. Consider the orbit $\{z_n\}$ of this point constructed in Theorem 6.3, and let g_n be the appropriate iterates of the tower maps that carry z to z_n. Then $\|Dg_n\|_{\text{hyp}} \to \infty$.

Consider a disk D_n of hyperbolic radius 1 centered at z_n, and let $\Delta_n \ni z$ be the univalent pullback of D_n under the branch of g_n. Then $\operatorname{diam} \Delta_n \to 0$ and (by the Koebe Distortion Theorem) the maps $g_n : \Delta_n \to D_n$ have uniformly bounded distortion. It follows that after rescaling the disks D_n and Δ_n to the unit (Euclidean) size, we can pass to a limit ϕ of the rescaled maps g_n and the corresponding line fields. In the limit, we obtain a *real analytic* line field, the image of the horizontal line field under ϕ.

Let us now consider a sequence of towers $\mathbf{f}_n$ obtained from $\mathbf{f}$ by shifting it by ns (so that the map f_{-ns} serves as the zero coordinate for $\mathbf{f}_n$) and rescaling (so that the disk D_n is scaled to the unit Euclidean size). By Lemma 6.2, we can pass to a limit tower $\mathbf{g}$. This tower has an invariant line field which is real analytic on some domain. It is easy to see that such a line field cannot exist. $\square$

2.4. Renormalization fixed point. Now we will show how the tower rigidity yields existence of the renormalization fixed point which is the global attractor in $\mathcal{H}_c$.

THEOREM 6.5. *There exists a unique renormalization fixed point f_* in the hybrid class $\mathcal{H}_c$ (hence f_* if the unique real renormalization fixed point). Moreover, $R^n g \to f_*$ for all $g \in \mathcal{H}_c$.*

PROOF. Let us show that the limit set $\omega(g)$ consists of a single point f_*. Indeed, pick two points f and $\tilde{f}$ in ω and include them into towers representing two-sided orbits in $\omega(g)$. By Lemma 6.1, these towers are qc equivalent. By the Tower Rigidity Theorem, they are affinely equivalent. Hence $f = \tilde{f}$ (recall that quadratic-like maps are considered up to affine conjugacy) as was asserted.

Since $\omega(g) = \{f_*\}$ is R-invariant, f_* is a renormalization fixed point.

If there were another fixed point $\tilde{f}_* \in \mathcal{H}_c$, then for the same reason two stationary towers

$$(\dots \mapsto f_* \mapsto f_* \mapsto f_* \mapsto \dots) \quad \text{and} \quad (\dots \mapsto \tilde{f}_* \mapsto \tilde{f}_* \mapsto \tilde{f}_* \mapsto \dots)$$

would be affinely equivalent – contradiction.

Hence $\omega(g) = \{f_*\}$ for any $g \in \mathcal{H}_c$, so that f_* is the global attractor in $\mathcal{H}_c$.

Finally, f_* is the only *real* renormalization fixed point since by the Straightening Theorem 2.2 and the Rigidity Theorem 4.24, any such a point belongs to $\mathcal{H}_c$. $\square$

2.5. Exponential convergence. Exploiting more carefully the Tower Rigidity Theorem, one can strengthen Theorem 6.5 to make convergence *uniform*. The following lemma gives a precise meaning to this statement. It makes use of the conceptual background developed in §§4.1–4.6 of the 2nd lecture.

LEMMA 6.6. *Let $g_* = \pi(f_*) \in \mathcal{E}$ be the circle map corresponding to the quadratic-like map f_*. Let $T = \pi \circ R \circ i_c : \mathcal{E} \to \mathcal{E}$. Then there exists a Banach ball $\mathcal{E}_V(g_*, r)$ such that:*

- *$T^l(\mathcal{E}_V(g_*, r)) \subset \mathcal{E}_V(g_*, r/2)$ for some $l \in \mathbb{N}$;*
- *for any $g \in \mathcal{E}$ there exists an N depending only on the above Banach ball and on $\operatorname{mod}(g)$ such that $T^n f \in \mathcal{E}_V(g_*, r)$.*

To proceed further to the exponential contracting property of the renormalization, we need to recall some basics of the analytic function theory in Banach spaces (*see also* §I.4.2). Given a complex Banach space $\mathcal{B}$, let $\mathcal{B}_r$ stand for the ball in $\mathcal{B}$ of radius r centered at 0 (if the space is called differently, say $\mathcal{D}$, then the corresponding notation, $\mathcal{D}_r$, will be used).

Cauchy Inequality. *Let $f : (\mathcal{B}_1, 0) \to (\mathcal{D}_1, 0)$ be a holomorphic map between two unit Banach balls. Then $\|Df(0)\| \leq 1$. Moreover, for $x \in \mathcal{B}_1$,*

$$\|Df(x)\| \leq \frac{1}{1 - \|x\|}.$$

This immediately yields:

Schwarz Lemma. *Let $r < 1/2$ and let $f : (\mathcal{B}_1, 0) \to (\mathcal{D}_r, 0)$ be a holomorphic map between two Banach balls. Then the restriction of f onto the ball $\mathcal{B}_r$ is contracting: $\|f(x) - f(y)\| \leq q\|x - y\|$, where $q = r/(1 - r) < 1$.*

Applying the Schwarz lemma to the operator

$$T^l : \mathcal{E}_V(g_*, r) ra \mathcal{E}_V(g_*, r/2)$$

from Lemma 6.6, we conclude that it is contracting. Passing back to the renormalization operator $R|\mathcal{H}_c$, we see that it is *exponentially contracting* in the following sense:

THEOREM 6.7. *There exist $C > 0$, $\lambda > 1$, and a Banach ball $\mathcal{B}_U(f_*, r)$ with the following property. For any $f \in \mathcal{H}_c$ there exists an N depending only on* mod(f) *such that $R^N f \in \mathcal{B}_U(f_*, r)$ and*

$$\|R^{n+N} f - f_*\|_U \leq C\lambda^n, \quad n = 0, 1, \dots$$

It is a remarkable virtue of holomorphic dynamics that the exponential contraction comes for free from some purely topological qualities of the map.

2.6. Small Orbits Theorem. To complete the proof of the hyperbolicity of the renormalization operator at f_* we will need one more remarkable property of holomorphic operators in Banach spaces which will allow us to detect the unstable eigenvalue for a purely topological reason.

2.6.1. *One-dimensional case.* Consider a local holomorphic map

$$R : (\mathcal{B}, 0) \to (\mathcal{B}, 0)$$

in a Banach space $\mathcal{B}$ fixing the origin. We say that it has *small orbits* if for any $\delta > 0$ there exists a point $x \in \mathcal{B}$ such that $\|R^n x\| < \delta$, $n = 0, 1, 2, \dots$.

In-one dimensional situation, the Small Orbit Theorem says that any analytic map $R : z \mapsto e^{2\pi i\theta} z + bz^2 + \dots$ near the origin has small orbits. This situation is well understood. There are three possible cases (cf. §I2.1):

• *Parabolic case* when $\theta = q/p$ is rational. In this case R is either of finite order, that is $R^p = \mathrm{id}$, or there exist orbits converging to 0 (within the attracting petals).

• *Siegel case* when R is conformally equivalent to the rotation $z \mapsto e^{2\pi i\theta} z$. In this case all orbits which start sufficiently close to 0 don't escape a small neighborhood of 0.

• *Cremer case* (none of the above). In this case, for all sufficiently small $\varepsilon > 0$, the connected component K_ε of the set

$$\{z : |R^n z| \le \varepsilon,\ n = 0, 1, \dots\}$$

is a continuum intersecting the boundary circle $\{z : \ |z| = \varepsilon\}$.

Thus, in all three cases small orbits exist.

2.6.2. *Basin of attraction.* Consider now a Banach space $\mathcal{B}$ decomposed into two subspaces: $\mathcal{B} = E^s \oplus E^c$. Let $D^s = D^s(\delta)$ and $D^c = D^c(\delta)$ stand for the open disks of radius δ centered at 0 in E^s and E^c respectively. Let us consider the bidisk $D = D(\delta) = D^s(\delta) \times D^c(\delta)$. Let $\partial^c D$ stand for $D^s \times \partial D^c$, and let $\partial^s D$ have the similar meaning.

For $h \in \mathcal{B}$, let h^s and h^c denote the horizontal and vertical components of h, i.e, the projections of h onto E^s and E^c respectively. Define the angle $\theta(h) \in [0, \pi/2]$ (between h and E^s) by the condition:

$$\operatorname{tg} \theta(h) = \frac{\|h^c\|}{\|h^s\|}.$$

Let $C_f^\theta = \{h \in \mathcal{B} : \theta(h) \ge \theta\}$ stand for the cone with angle $\pi/2 - \theta$ about its axis E^c.

Lemma 6.8 (Basin of attraction). *Let $\mathcal{B}$ be a Banach space as above and let $\mathcal{B}'$ be another Banach space compactly included into $\mathcal{B}$.*[1] *Let $R : (\mathcal{B}, 0) \to (\mathcal{B}', 0)$ be a local holomorphic map fixing 0.*

Assume that the decomposition $\mathcal{B} = E^s \oplus E^c$ is invariant with respect to the differential $DR(0)$, and moreover, the following properties are satisfied.

H0. *The origin is attracting:* $\operatorname{spec} DR^p(0) \subset \mathbb{D}$*;*

H1. *Horizontal contraction: There exists a $q \in (0,1)$ such that for any $h \in E^s_f$, $\|(DR(f)h)^s\| \le q\|h\|$, provided $f \in D$, $Rf \in D$;*

[1] i.e., there is a linear injection $i : \mathcal{B}' \to \mathcal{B}$ such that the image of the unit ball of $\mathcal{B}'$ is relatively compact in $\mathcal{B}$.

H2. *Invariant cone field: There exists a* $\theta \in (0, \pi/2)$ *such that the tangent cone field* C_f^θ *over* D *is* R*-invariant:*

$$(DR_f)\, C_f^\theta \subset C_{Rf}^\theta \quad , provided \quad f \in D,\, Rf \in D.$$

Then there is a point $f \in \partial^c D$ *such that* $\mathrm{orb}(f) \subset D$ *and* $\|R^m f\| \to 0$.

Note that there are no assumptions relating the spectrum of $DR^p(0)$ and the size of the bidisk D.

2.6.3.

THEOREM 6.9. *Let* $\mathcal{B}$ *and* $\mathcal{B}'$ *be two complex Banach spaces, and let* $i : \mathcal{B}' \to \mathcal{B}$ *be a compact inclusion. Let* $T : (\mathcal{B}, 0) \to (\mathcal{B}', 0)$ *be a local holomorphic map and let*

$$R = i \circ T : (\mathcal{B}, 0) \to (\mathcal{B}, 0).$$

Assume that the spectrum of $DR(0)$ *belongs to the closed unit disk and is not empty on the unit circle. Then* R *has "slow small orbits", that is, for any neighborhood* $\mathcal{V} \ni 0$*, there is an* $\mathrm{orb}(f) \subset \mathcal{V}$ *such that*

$$\lim \frac{1}{m} \log \|R^m f\| = 0.$$

PROOF. We will deduce this result from Lemma 6.8.

Let E^s stand for the spectral subspace of R corresponding to the part of $\mathrm{spec}\, R$ inside the unit disk $\mathbb{D}$, and let E^c correspond to the part on the unit circle $\mathbb{T}$. After replacing R by its iterate, R becomes horizontally contracting and a cone field preserving on a sufficiently small bidisk $D = D(\delta)$.

For $\lambda \in (0, 1)$, let as consider the perturbation $R_\lambda = \lambda R$ which makes the origin attracting. This operator is even stronger horizontally contracting than R and preserves the same cone field. Thus, it satisfies assumptions H0-H2 of Lemma 6.8. Hence there is a point

$$f_\lambda \in \partial^c D \cap A_\lambda,$$

where A_λ is the attracting basin of 0 for R_λ.

Since the set $\{Rf_\lambda\}$ is pre-compact in $\mathcal{B}$, there is a convergent subsequence $Rf_{\lambda_n} \to g$ as $\lambda_n \to 1$. It is obvious that $\mathrm{orb}(f) \subset \bar{D}$ and it is easy to see that this orbit is slow. □

2.7. Unstable direction. We are now ready to complete the proof that the renormalization R is hyperbolic at the fixed point f_*. It is easy to see that for any quadratic-like representative $f_* : V \to V'$ and any domain $U \Subset V$ containing $J(f_*)$, there exists an N and a domain $W \subset U$ such that the restriction $f_*^{p^N} | W$ is a pre-renormalization $PR^N f_*$ is affinely conjugate to $f_* : V \to V'$.

Since $U \Subset V$, the natural inclusion $i : \mathcal{B}_V \to \mathcal{B}_U$ is compact (by Montel's Theorem). Moreover, by continuity argument, there exists an $\varepsilon > 0$ such that if $f \in \mathcal{B}(f_*, \varepsilon)$ then $PR^N f$ has a quadratic-like representative on W. Hence the renormalization $R^N f$ has a quadratic-like representative defined on V, so that $R^N : B_U(f_*, \varepsilon) \to B_V$.

If V and ε are selected sufficiently small then by Lemma 2.12 the Banach slice $\mathcal{H}_c \cap \mathcal{B}_V(f_*, \varepsilon)$ is a codimension-one submanifold in $\mathcal{B}_U(f_*, \varepsilon)$. Let E^s be the tangent space to this manifold at f_*. Since R is contracting in $\mathcal{H}_c$, $\operatorname{spec} DR(f_*)$ belongs to the unit disk.

This brings us to the setting of the Small Orbits Theorem (6.9). Consider the eigenvalue λ of the quotient operator $DR^N(f_*)$ on $\mathcal{B}_U/E^s$. If $|\lambda| \leq 1$ then R^N would have a small orbit $\{R^{nN} f\} \not\subset \mathcal{H}_c$. But then f would be an infinitely renormalizable quadratic-like map with *a priori* bounds. By the Rigidity Theorem (4.28), then $f \in \mathcal{H}_c$ – contradiction.

3. Full Renormalization Horseshoe

In this section we will develop the Renormalization Theory for all real combinatorial types simultaneously.

3.1. Renormalization Theorem. Recall that $\mathcal{M}_{\mathbb{R}}$ stands for the family of maximal real M-copies (see §II.6.4). For any $M \in \mathcal{M}_{\mathbb{R}}$, let

$$\mathcal{T}_M = \bigcup_{c \in M} \mathcal{H}_c \in \mathcal{C}$$

stand for the corresponding *renormalization strip.* It consists of quadratic-like maps which are renormalizable with combinatorics M. So, for every strip we have a well-defined renormalization operator $R : \mathcal{T}_M \to \mathcal{C}$. Let us put these operators together to obtain a single renormalization operator

$$R : \bigcup_{M \in \mathcal{M}_{\mathbb{R}}} \mathcal{T}_M \to \mathcal{C}$$

defined simultaneously for all real combinatorial types (see Figure 6.2).

Theorem 6.10. *There is a set $\mathcal{A} \subset \cup_{M \in \mathcal{M}} \mathcal{T}_M \cap \mathcal{Q}_{\mathbb{R}}$ (called the full renormalization horseshoe), a constant $\rho \in (0, 1)$, and a neighborhood $V \subset \mathbb{C}$ of the origin such that:*

(i) *$\mathcal{A}$ is precompact in $\mathcal{Q}$, R-invariant, and $R|\mathcal{A}$ is topologically conjugate to the two-sided shift $\omega : \Sigma \to \Sigma$ in countably many symbols (whose points represent all possible two-sided strings of real combinatorial types).*

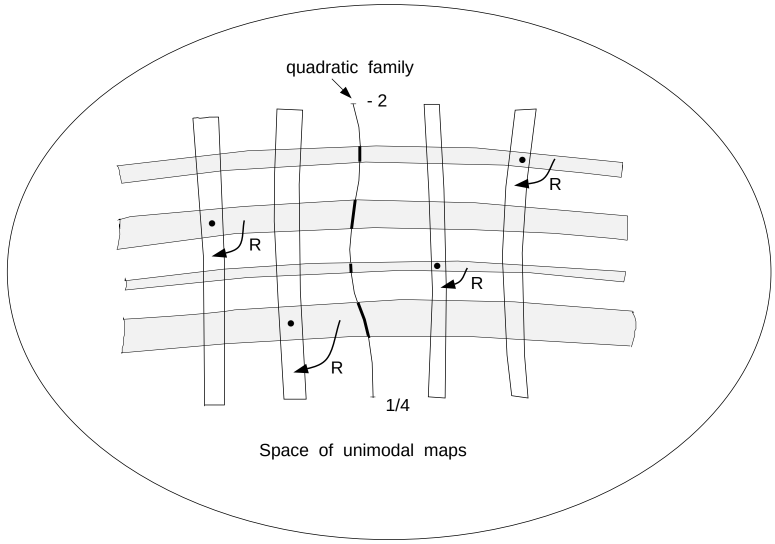

FIGURE 6.2. Full renormalization horseshoe.

(ii) *The hybrid classes* $\mathcal{H}(f)$*,* $f \in \mathcal{A}$*, are codimension-one complex analytic submanifolds in* $\mathcal{Q}$ *("stable leaves") which form an* R*-invariant lamination in* $\mathcal{Q}$*. Moreover, if* $g \in \mathcal{H}(f)$ *and* $\mathrm{mod}(g) \geq \nu$*, then* $R^n g \in \mathcal{B}_V$ *and* $\|R^n f - R^n g\|_V \leq C\rho^n$ *for* $n \geq N(\nu)$*.*

(iii) *There exists an* R^{-1}*-invariant family of holomorphic curves* $W^u_{\mathbb{R}}(f)$*,* $f \in \mathcal{A}$*, ("unstable leaves") which transversally pass through all hybrid classes* $\mathcal{H}_c$*,* $c \in [-2, 1/4 - \epsilon]$*, and such that*

$$\|R^{-n} f - R^{-n} g\|_V \leq C\rho^n, \quad n \geq 0,$$

provided $g \in W^u(f)$*.*

(iv) *The renormalization operator has uniformly bounded distortion on the unstable leaves.*

(v) *The stable lamination is transversally quasisymmetric.*

This Renormalization Theorem encodes diverse universality properties of the bifurcation sets in one-parameter families of unimodal maps. In this way *uniform hyperbolicity* of an *infinite dimensional* operator shed light on the dynamics of just *one-dimensional* but badly *non-hyperbolic* maps.

This theorem is the culmination of the theory developed in this lecture notes: its proof is based upon all the above machinery (puzzle and parapuzzle geometry, rigidity theorems, analysis and geometry in manifolds modeled on sheaves of Banach Spaces,...) plus several extra ingredients, like rigidity of towers with essentially bounded combinatorics and a Shadowing Theorem generalizing the Small Orbits Theorem to arbitrary combinatorics.

In the rest of the section we will outline the proof of the Renormalization Theorem emphasizing new ingredients as compared with the stationary case.

3.2. Parabolic towers. Parabolic towers are geometric limits of McMullen towers with essentially bounded combinatorics. The following result extends the Rigidity Theorem 6.4 to this class of towers:

THEOREM 6.11. *If two parabolic towers $\bar{\Phi}$ and $\bar{\Psi}$ in $\hat{\mathcal{T}}_p(K)$ are combinatorially equivalent then they are affinely equivalent.*

3.3. Exponential Contraction. To prove exponential contraction of the renormalization operator on the hybrid lamination we split the analysis into two combinatorial cases: essentially bounded and "high". The essentially bounded is treated similarly to the stationary case by means of the above Rigidity Theorem for parabolic towers (6.11). The treatment of the high combinatorics case will be based upon Theorem 3.6 on the growth of the principal moduli. Altogether, this yields the macroscopic contraction property for the renormalization. The Schwarz Lemma in Banach spaces rounds up the argument.

Let us now detail the argument. Let $\mathcal{S} \subset \mathcal{C}$ stand for the union of (complex) quadratic-like germs f with the real straightening, i.e., such that $\chi(f) \in [-2, 1/4]$. Let $\mathcal{S}(\mu) = \mathcal{S} \cap \mathcal{C}(\mu)$ and let $\mathcal{S}_n(\mu)$ be the set of n times renormalizable germs of $\mathcal{S}(\mu)$.

We will make use of the notion of Montel metric dist_{Mon} introduced in §4.1.3 of the 2nd lecture.

LEMMA 6.12. *The renormalization is macroscopically contracting in the following sense: For any $\varepsilon > 0$ there is an $N = N(\mu)$ such that*

$$\text{dist}_{Mon}(R^m f, R^m g) < \varepsilon, \; m = N, N+1, \dots, n,$$

provided f and g are hybrid equivalent and belong to $\mathcal{S}_n(\mu)$.

Remark. We call this property "macroscopic" since it provides contraction only in "big" scales but allows expansion in "small" scales ($< \varepsilon$).

PROOF. The contracting property of the renormalization in Sullivan's Teichmüller metric (see §II.6.2) implies that it is Lyapunov stable in the Montel metric (see §II.4), i.e., there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\text{dist}_{Mon}(f, g) < \delta \Rightarrow \text{dist}_{Mon}(R^m f, R^m g) < \varepsilon, \; m = 1, \dots, n,$$

provided f and g belong to $\mathcal{S}_n(\mu)$ (where $\delta = \delta(\varepsilon)$ is independent of n and the combinatorics of f and g).

Take a renormalizable map $f : V \to V'$ with $\operatorname{mod}(V \smallsetminus V') \geq \mu$. By Theorem 3.6, if $p_e(f) \geq \bar{p}_e(\delta)$, then the renormalization Rf is $\delta/2$-close to a quadratic map P_c, where $c = \chi(Rf)$. Thus for two hybrid equivalent maps like that we have: $\operatorname{dist}_{Mon}(Rf, Rg) < \delta$.

Furthermore, let us show that there is an $N = N(\mu, \bar{p}_e)$ with the following property: If for $2N$ consecutive renormalizations of maps f and g in $\mathcal{S}_{2N}(\mu)$, their essential periods stay bounded by $\bar{p}_e$, then $\operatorname{dist}_{Mon}(R^N f, R^N g) < \delta$. Otherwise there would be a sequence of maps f_N and g_N as above with $\operatorname{dist}_{Mon}(f_N, g_N) \geq \delta$. Let $F_{N,m} = R^{N+m} f_N$ and $G_{N,m} = R^{N+m} g_N$, $m = -N, \dots, N$. Consider a sequence of hybrid equivalent finite towers $\mathbf{F}_N = \{F_{N,m}\}_{m=-N}^{N}$ and $\mathbf{G_N} = \{G_{N,m}\}_{m=-N}^{N}$. By compactness (Lemma 6.2), these towers converge along a subsequence to bi-infinite parabolic towers $\mathbf{F} = \{F_m\}$ and $\mathbf{G} = \{G_m\}$ with essentially bounded combinatorics and *a priori* bounds. By the Rigidity Theorem for parabolic towers (6.11), $\mathbf{F} = \mathbf{G}$ up to rescaling. On the other hand, $\operatorname{dist}_{Mon}(F_0, G_0) \geq \delta$ - contradiction.

Thus, in any case (with no restrictions on the combinatorics), there is an $l \leq N(\mu, \bar{p}_e)$ such that $\operatorname{dist}_{Mon}(R^l f, R^l g) < \delta$. By the choice of δ, $\operatorname{dist}_{Mon}(R^m f, R^m g) < \varepsilon$ for all further moments $m = l+1, \dots, n$. $\square$

As in the stationary case, by means of the Schwarz Lemma in Banach spaces, we can pass from the macroscopic to the exponential contraction:

THEOREM 6.13. *Let us consider two hybrid equivalent quadratic-like maps $f \in \mathcal{S}_{n+1}(\nu)$ and $g \in \mathcal{S}_{n+1}(\nu)$. Then*

$$\operatorname{dist}_{Mon}(R^m f, R^m g) \leq C\rho^m, \ m = 0, 1, \dots, n,$$

where $\rho \in (0,1)$ depends only on the choice of $\operatorname{dist}_{Mon}$*, while $C > 0$ depends also on ν.*

3.4. Realization and rigidity of general towers. In this section we will construct the horseshoe $\mathcal{A}$.

3.4.1. *Contraction in the middle of the tower.* Let us consider an orbit $\{R^m f\}_{m=-l}^{n}$ (assuming that f is n times renormalizable and l times anti-renormalizable and using notation $R^m f$ with negative m for some anti-renormalization of f). Its (l,n)-*itinerary* is a sequence of M-copies $\{M_m\}_{m=-l}^{n}$ such that $R^m f \in \mathcal{T}_{M_m}$.

LEMMA 6.14. *Consider two maps f and g in $\mathcal{S}$ with the same (l,n)-itinerary and such that*

$$\operatorname{mod}(R^k f) \geq \mu > 0 \ \textit{and} \ \operatorname{mod}(R^k g) \geq \mu > 0, \ -l \leq k \leq n.$$

Then $\mathrm{dist}_{Mon}(f,g) < \varepsilon = \varepsilon(\mu, l, n)$*, where* $\varepsilon \to 0$ *as* $l, n \to \infty$ *(*μ *being fixed).*

Proof. Let $\chi(f) = P_c$ and $\chi(g) = P_b$, where b and c are real by the assumption. Theorem 4.30 implies that the renormalization windows of order n in the parameter interval $[-2, 1/4]$ (i.e., the connected components of the set of real n times renormalizable maps) uniformly shrink as the order grows. Thus, $|b - c| < \delta(n) \to 0$ as $n \to \infty$, so that f and g lie on the nearby leaves of the foliation $\mathcal{F}$. The same is applicable to $f_k \equiv R^k f$ and $g_k \equiv R^k g$, $k = -l, \dots, N$, for any given N.

For any integer $k \in [-l, 0]$, let us consider a map $h_k \in \mathcal{H}(f_k)$ belonging to the vertical fiber via g_k, i.e., $\pi(h_k) = \pi(g_k)$. Then $\mathrm{mod}(h_k) = \mathrm{mod}(g_k)$. By Theorem 6.13, there exist ρ and N depending only on μ such that

$$\mathrm{dist}_{Mon}(R^N f_k, R^N h_k) \le \rho\, \mathrm{dist}_{Mon}(f_k, h_k).$$

The results of §II.5.4 imply that the vertical fibers through g_k near the connectedness locus can be equipped so that they become quadratic-like families of some class $\mathcal{G}_{L,\lambda}$, with geometry (i.e., the constants L, λ) depending only on μ. Hence by Theorem 5.19 and Lemma 5.25, $R^N h_k$ and $R^N g_k$ belong to the same quadratic-like family with bounded geometry. By Lemma 2.8, the holonomy $\mathcal{Q} \to \mathcal{S}$, $\mathcal{S} \in \mathcal{G}^{\epsilon}_{C,\nu}$, is equicontinuous. Hence

$$\mathrm{dist}_{Mon}(R^N h_k, R^N g_k) < \delta_1(n) \to 0 \text{ as } n \to \infty.$$

It follows that R^N uniformly contracts the distance between the f_k and g_k, as long as it stays greater than ε. Hence in a bounded number of steps (depending on ε) this distance must become smaller than ε. ☐

3.4.2. *Realization and rigidity.* We will now prove that any real combinatorics $\tau = \{M_k\}_{k=-\infty}^{\infty}$, $M_k \in \mathcal{M}_{\mathbb{R}}$, can be be realized by a unique real tower. Let $\bar{\mathcal{S}}$ stand for the space of towers $\bar{f}$ with $f_k \in \mathcal{S}$. make it script

Theorem 6.15. *For any two-sided real combinatorics* τ *there is a unique tower* $\bar{f} \in \bar{\mathcal{S}}$ *with this combinatorics and a priori bounds. Moreover, this tower is real and* $\mathrm{mod}(\bar{f}) \ge \nu$ *with an absolute* $\nu > 0$*.*

Proof. By Theorem III.3.21, there is an absolute $\nu > 0$ such that for any infinitely renormalizable quadratic polynomial $f = P_c \in \mathcal{I}$, $R^n f \in \mathcal{Q}(\nu)$, $n = 0, 1, \dots$.

Let us take a combinatorial sequence $\tau = \{M_k\}$. For any $l \ge 0$, there is a real infinitely renormalizable quadratic polynomial $P_l \equiv P_{c_l}$ with combinatorics $\tau(P_l) = \{M_{-l}, \dots, M_0, \dots\}$. Let $f_{0,l} = R^l P_l$. These are infinitely renormalizable real quadratic-like maps with common combinatorics $\{M_0, M_1, \dots\}$ and $\text{mod}(f_{0,l}) \geq \nu$. Since the set of such maps is compact, we can pass to a quadratic-like limit $f_0 = \lim_{l\to\infty} f_{0,l}$ (along a subsequence) with the same properties.

Let us now do the same thing for every $i \leq 0$. Let $f_{i,l} = R^{l+i}P_l$, and let $f_i = \lim_{l\to\infty} f_{i,l}$ be a limit point. The map f_i is real and has combinatorics $\tau_i = \{M_i, M_{i+1}, \dots\}$.

Selecting the above converging subsequences diagonally, we construct a sequence of real infinitely renormalizable quadratic-like maps $\{f_i\}_{i=-\infty}^{\infty}$ such that $Rf_i = f_{i+1}$, $\chi(f_i) \in M_i$ and $\text{mod}(f_i) \geq \nu$. This sequence represents a real tower $\bar{f}$ with combinatorics $\bar{\tau}$ and a moduli bound ν.

Thus, any real combinatorics τ is represented by a tower $\bar{f} \in \bar{\mathcal{S}}$ with *a priori* bounds. Moreover, this tower is unique. Indeed, if $\bar{f}$ and $\bar{g}$ are two such towers, then by Lemma 6.14 $\text{dist}_{Mon}(f_0, g_0)$ is arbitrary small, so that $f_0 = g_0$. For the same reason $f_i = g_i$ for any i. □

Let us say that an infinitely renormalizable map $f \in \mathcal{C}$ is *completely non-escaping* under the renormalization if some full renormalization orbit $\{R^n f\}_{n=-\infty}^{\infty}$ is well-defined, $R^n f \in \mathcal{C}$, and

$$\text{mod}(f_n) \geq \mu = \mu(f) > 0, \quad n \in \mathbb{Z}.$$

Note that we do not ask $R^n f$ to be uniquely determined for negative n but by Lemma II.2.26 this is the case for real maps.

Let $\mathcal{A} \subset \mathcal{C}$ stand for the set of completely non-escaping maps with real combinatorics. This is the full renormalization horseshoe promised in Theorem 6.10. It follows from Theorem 6.15 that $R|\mathcal{A}$ is topologically conjugate to the shift ω on the space Σ of all possible real combinatorial types $\tau = \{M_k\}_{k=-\infty}^{\infty}$, $M_k \in \mathcal{M}_{\mathbb{R}}$. This is part (i) of Theorem 6.10. Part (ii) of Theorem 6.10 now follows from Theorem 6.13. Let us pass to part (iii).

4. Unstable foliation

4.1. Family of special bidisks. To capture hyperbolicity, we construct a family of "special Banach bidisks" Q_f, $f \in \mathcal{A}$, nicely transformed by the renormalization. These bidisks are contained in certain Banach slices $\mathcal{B}_f$ which are in turn locally contained in some vertical tubes $\mathcal{P}_f$ (see II.§5.4), and have the following structure. There is a neighborhood $\mathcal{W}_f \subset E^h_f \cap \mathcal{B}_f$ and a neighborhood $\mathcal{S}_f \subset E^v_f$ (recall the horizontal-vertical decomposition (2.10)) such that Q_f is obtained from $\mathcal{S}_f$ by the holomorphic motion along the foliation $\mathcal{F}$ over $\mathcal{W}_f$.

Let $\mathcal{V}_f$ denote the leaf of the above motion through f (the "base" of Q_f), and let $\mathcal{S}_f(g)$ denote the vertical cross-section of Q_f through a point $g \in Q_f$. (For special disks $\tilde{Q}_f \subset \mathcal{B}_f$, the corresponding objects will be marked with tilde.)

Let $\|\cdot\|_f$ denote the Banach norm in $\mathcal{B}_f$.

LEMMA 6.16. *There exists an $N \in \mathbb{N}$ and a family of special bidisks*

$$Q_f \subset \tilde{Q}_f \subset \mathcal{B}_f, \quad f \in \mathcal{A},$$

based on the $\mathcal{V}_f$, satisfying the following properties:

(i) *The renormalization $R^N \mid \mathcal{H}_f$ admits the analytic continuation to Q_f, and $R^N Q_f \subset \mathcal{B}_{R^N f}$;*
(ii) *Horizontal contraction: If $g \in Q_f$ and $v \in T_g\mathcal{B}_f$ is tangent to the leaf of $\mathcal{F}$ through g, then $\|DR^N v\|_{R^N f} \le \rho\|v\|_f$, where $\rho \in (0,1)$;*
(iii) *Invariance of the cone fields: If $g \in Q_f$ and $R^n g \in \tilde{Q}_{R^n f}$, then*

$$DR^N(\Lambda_f(g)) \subset \Lambda_{R^N f}(R^N g),$$

where $\Lambda_f(g)$ stands for the vertical $\pi/4$-cone in $\mathcal{B}_f$ based at g;
(iv) *Overflowing property for high periods: There exists p such that if the renormalization period of f is greater than p, then for $g \in \mathcal{V}_f$, $R^N(\mathcal{S}_f(g)) \cap \tilde{Q}_{R^N f}$ is a manifold with vertical slope bounded by 1 properly embedded into $\tilde{Q}_f$;*
(v) *The leaves of $\mathcal{F}$ in Q_f have horizontal slopes bounded by $1/2$;*
(vi) *The bidisks Q_f and $\tilde{Q}_f$ have a definite horizontal size in $\mathcal{B}_f$;*
(vii) *The cross-sections $\mathcal{S}_f(g)$, $g \in \mathcal{V}_f$, have a bounded shape;*
(viii) *The cross-sections $\tilde{\mathcal{S}}_f(g)$, $g \in \mathcal{V}_f$, have a bounded shape and an absolute size;*

4.2. Slow shadowing orbits. We can now use the above family of special bidisks to prove the following generalization of the Small Orbits Theorem:

THEOREM 6.17. *If the renormalization R is not hyperbolic, then there exist quadratic-like maps $f \in \mathcal{A}$ and $g \notin \mathcal{H}_f$ such that $R^n g \in Q_{R^n f}$.*

We say that g "slowly shadows f". It is easy to see that the shadowing map g is infinitely renormalizable with the same combinatorics as f. By the Rigidity Theorem (4.28), it must be hybrid equivalent to f, i.e., $g \in \mathcal{H}_f$ – contradiction. This proves hyperbolicity of the horseshoe $\mathcal{A}$.

The last assertion of Theorem 6.10(iii) can be derived from *a priori* bounds.

Point (iv) of the theorem follows from the Koebe Distortion Theorem, while the last point, (v), follows from the λ-lemma.

5. Proof of Theorem B

We are now ready to prove Theorem B from the Introduction (and thus, to complete the proof of the Regular or Stochastic Theorem).

THEOREM 6.18. *The set $\mathcal{I}$ of real infinitely renormalizable parameters has zero Lebesgue measure.*

PROOF. By the Lebesgue Density Points Theorem, it is sufficient to show that the set $\mathcal{I}$ is "porous", that is, it has definite gaps in arbitrary small scales near any point $c \in \mathcal{I}$. Since this property is quasisymmetrically invariant, by Theorem 6.10(v) it is enough to check that the sets $\mathcal{A} \cap W^u(f)$ are uniformly porous on the (real) unstable manifolds $W^u(f)$.

By Theorem 6.10(iii), for any N, there is an interval $I_N \subset W^u(f)$ which is stretched by R^N to the whole manifold $W^u(R^N f)$. Since this manifold transversally intersects all hybrid classes $\mathcal{H}_c$ with $c \in [-2, 0]$, it contains an interval J_N corresponding under the straightening χ to the hyperbolic window $(-3/4, 0)$. Since χ is uniformly quasisymmetric, this window occupies a definite proportion of $W^u(R^N f)$.

But Theorem 6.10(iv) implies (by standard distortion estimates) that the map $R^N : I_N \to W^u(R^N f)$ has a bounded distortion (independent of N). Hence the interval $R^{-N} J_N$ occupies a definite size in I_N. Since the maps $g \in R^{-N} J_N$ are only N times renormalizable, this interval is a gap in $\mathcal{I}$. Since $\operatorname{diam} I_N \to 0$, we are done. □

6. Bibliographical notes

The Renormalization Conjecture was formulated in [**F1, F2**] and [**CT, TC**]. The first, computer-assisted, proof of the Renormalization Conjecture for the period doublings was given by Lanford in 1982 [**La**]. The idea was to find numerically an approximation to the solution of the Feigenbaum-Cvitanović equation and then to prove rigorously that there exists a true hyperbolic solution nearby. In this way the original conjecture was formally checked, at least locally, near the fixed point f_*.(This was perhaps the first experience with rigorous computer-assisted proofs, which nowadays have become quite widespread.)

Still, the nature of the universality phenomenon remained mysterious. Also, computer-assisted proofs can conceivably handle only a few small renormalization periods, while the renormalization operator is well defined for arbitrary periods (triplings, quadruplings etc.), not to mention arbitrary infinite strings of periods. So, people kept looking for a "conceptual" proof of the Renormalization Conjecture: see [**VSK, E, Mar**] for advances in this direction.

A complete proof of the conjecture in the stationary case (and in fact, in the case of bounded combinatorics) was given in the works of Sullivan [**S2**], McMullen [**McM2**] and the author [**L6**] as outlined in §2 of this lecture. Namely, the renormalization fixed point and it stable manifold were constructed in [**S2, McM2**] (see also [**MS**]), while hyperbolicity of this fixed point was proven in [**L6**].

The Full Renormalization Horseshoe was treated in [**L7**].

For the one-dimensional Small Orbits Theorem (going back to Birkhoff) see Perez-Marco [**PM**]. Theorem 6.11 was proven in the thesis of Hinkle (Stony Brook, 1997), see [**Hi**].

For further advances in the Renormalization Theory, see [**FMP, Y2**]. For further advances in the theory of regular and stochastic dynamics, see [**ALM, AM2**].

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