ALMOST EVERY REAL QUADRATIC MAP IS EITHER REGULAR OR STOCHASTIC

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ABSTRACT. In this paper we complete a program of study measurable dynamics in the real quadratic family whose goal was to prove that almost any real quadratic map $P_c : z \mapsto x^2 + c$, $c \in [-2, 1/4]$, has either an attracting cycle or an absolutely continuous invariant measure. The final step filled in here is to prove that the set of infinitely renormalizable parameter values $c \in [-2, 1/4]$ has zero Lebesgue measure. We derive it from a Renormalization Theorem which asserts uniform hyperbolicity of the full renormalization operator. This theorem gives the most general real version of the Feigenbaum-Coullet-Tresser Universality, simultanuously for all combinatorial types.

1. INTRODUCTION

1.1. **The goal of dynamics.** The main goal of dynamical systems theory is to describe the typical behavior of orbits for a typical dynamical system. There can be different points of view on the meaning of "typical" but it is generally accepted today that the probabilistic notion based on Lebesgue measure makes the best physical sense. This viewpoint going back to Boltzman, Poincaré, Birkhoff and Kolmogorov, was precisely shaped in the sixties and seventies by Arnol'd, Moser, Sinai and Ruelle, and has been a guiding principle since then. Roughly speaking, the goal is the following: Given a finite dimensional manifold M and a "representative" finite parameter family $f_t : M \to M$ of dynamical systems on M, describe the asymptotic behavior of almost all orbits of f_t for almost all parameters t. (By "almost all", we always mean "all outside of a set of Lebesgue measure zero".)

It was soon realised that this problem is transcendentally hard in general, so that one has to start with the simplest models. The model that shortly attracted a great deal of attention was the one-dimensional real quadratic family $P_c : x \mapsto x^2 + c$, with $c \in [-2, 1/4]$. (For these parameter values, the map P_c has an invariant interval I_c.) It turned out that even this elementary formula hides very rich dynamics, which

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depends sensitively on the parameter c. In this paper, we will give a complete measure-theoretic picture for this family.

1.2. **The main theorem on measurable dynamics in the real quadratic family.** There are two opposite types of dynamics observed in the quadratic family. A quadratic map P_c is called *regular* if it has an attracting cycle (i.e., a cycle whose multiplier has absolute value less than 1). In this case, the attracting cycle is unique and attracts (Lebesgue) almost all orbits of the invariant interval I_c ([Fa], [Ju], [Si], [G1]).

A quadratic map is called *stochastic* if it has an absolutely continuous invariant probability measure μ. Such a measure is unique, has a positive characteristic exponent $\chi_\mu(f) = \int \log |Df| \, d\mu$, and Lebesgue almost all orbits on I_c are asymptotically equidistributed with respect to it ([BL2]). Moreover it has support

$$\operatorname{supp} \mu = \bigcup_{k=0}^{p-1} L_k,$$

where the L_k are intervals with disjoint interiors which are cyclically permuted under f, and $f^p|L_0$ is weakly Bernoulli [Le].

So, in both the regular and stochastic cases, the asymptotic behavior of almost all orbits is well understood.

Theorem 1.1 (regular or stochastic)**.** *Almost every real quadratic polynomial* $P_c(z) = z^2 + c$, $c \in [-2, 1/4]$, *is either regular or stochastic.*

Regular quadratic maps are said to be (*uniformly*) *hyperbolic*, since they are uniformly expanding outside the basin of the attracting cycle [Fa], [Ju], [G1]. On the other hand, stochastic maps are always (*nonuniformly*) *hyperbolic* in the sense of Oseledets-Pesin theory, since $\chi_\mu(f) > 0$. Thus, one can say that *almost any real quadratic map is hyperbolic*, in either the uniform or the nonuniform sense.

Previously it was known that stochastic maps are observable with positive probability ([J], [BC1]), but are nowhere dense. In fact Yoccoz showed that any interval of nonregular maps would have to be infinitely renormalizable (see [H]); but it was known that no infinitely renormalizable map can be stochastic [G2],[BL1], [S2]. More recently it was proven that the open set consisting of all regular maps is actually dense. (See [L3] for the proof of this result and further references.) So, neither regular nor stochastic phenomena can be neglected in the real quadratic family.

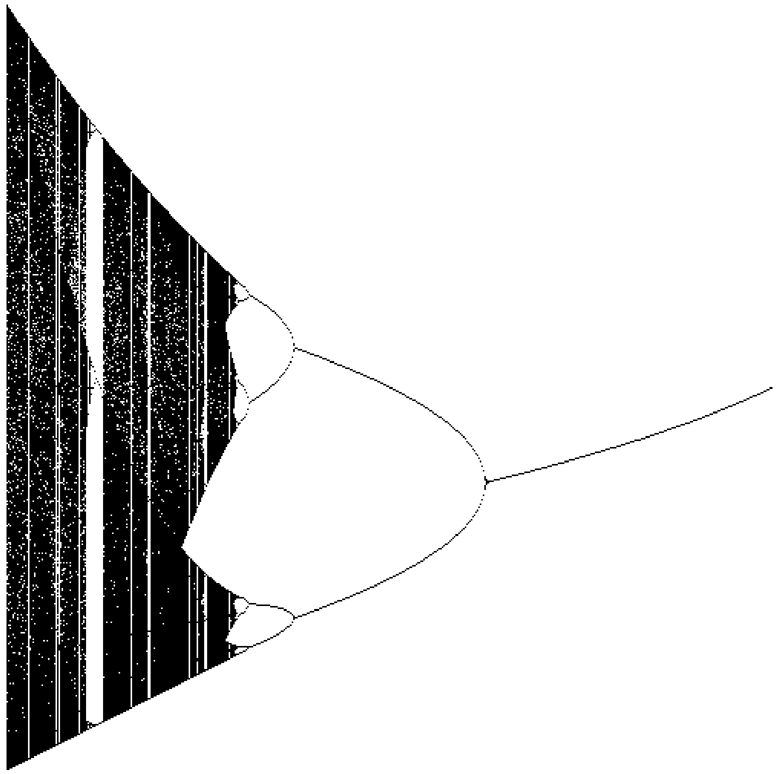

FIGURE 1. Real quadratic family. This picture presents how the limit set of the orbit $\{f_c^n(0)\}_{n=0}^{\infty}$ bifurcates as the parameter c changes from $1/4$ on the right to -2 on the left. Two types of regimes are intertwined in an intricate way. The gaps correspond to the regular regimes. The black regions correspond to the stochastic regimes (though of course there are infinitely many narrow invisible gaps therein).

1.3. **Two parts of the main theorem.** To put the main theorem into context, let us recall the topological classification of unimodal maps (see [MS] for a detailed discussion). Take a closed interval $I \ni 0$. A smooth map $f : I \to I$ is called *unimodal* if it has only one critical point in $\operatorname{int} I$, and this point is an extremum. In what follows we assume that the critical point is located at 0.

A unimodal map $f = P_c$ is called *renormalizable* if there exists a $p > 1$ and an interval $L \ni 0$ such that $f^p(L, \partial L) \subset (L, \partial L)$, while $\operatorname{int}(f^k L) \cap \operatorname{int} L = \emptyset$ for $1 < k < p$. The smallest $p = p(f)$ with this property is called the *renormalization period.* The restriction $f^p|L$ is again a unimodal map. If L is the maximal interval as above, then

$f^p|L$, considered up to rescaling, is called the *renormalization* of f and is denoted by Rf. If Rf is also renormalizable, f is called "twice renormalizable", and its second renormalization is denoted by R^2f. Continuing this process, we naturally define "n times renormalizable" maps, "infinitely renormalizable" maps, and "at most finitely renormalizable" maps.

Evidently we have the following decomposition of the parameter interval of the quadratic family: $[-2, 1/4] = \mathcal{R} \cup \mathcal{N} \cup \mathcal{I}$, where $\mathcal{R}$ stands for the set of regular parameter values, $\mathcal{N}$ stands for the set of nonregular at most finitely renormalizable parameter values, and $\mathcal{I}$ stands for the set of infinitely renormalizable parameter values.

The set $\mathcal{S}$ of stochastic parameter values is contained in $\mathcal{N}$,[1] and it is known that the difference $\mathcal{N} \smallsetminus \mathcal{S}$ is non-empty ([Jo], [HK], [Bru]). In [MN], Martens and Nowicki gave an efficient geometric condition for a map P_c, $c \in \mathcal{N}$, to have an absolutely continuous invariant measure.

Thus, Theorem 1.1 follows from the following two results:

Theorem 1.2 ([L4]). *Almost every nonregular real quadratic polynomial which is at most finitely renormalizable satisfies the Martens-Nowicki condition. Hence,* $\text{meas}(\mathcal{N} \smallsetminus \mathcal{S}) = 0$.

Theorem 1.3. *The set of infinitely renormalizable real quadratics has zero Lebesgue measure:* $\text{meas}(\mathcal{I}) = 0$.

We will derive the last result from a renormalization theorem for all possible real combinatorial types. Roughly speaking, this theorem says that the renormalization operator R is hyperbolic in an appropriate space of unimodal maps. A proof of this theorem will be the main subject of this paper.

Remark. It is worthy to compare the Density Theorem of [L3] with Theorem 1.3. The former asserts that the set $\mathcal{I}$ is nowhere dense; the measure-theoretic assertion of the latter is much stronger.

1.4. **Renormalization theory.** The renormalization conjecture was stated by Feigenbaum [F1, F2] and independently by Coullet and Tresser [CT, TC] in 1976-78, for the particular case of doubling combinatorics. It suggested a renorm-group explanation (motivated by statistical physics) of numerically observed universal properties of unimodal maps. Its importance both for mathematics and physics was soon realized (see [C]), and there has been a sustained effort since then to explore this phenomenon and to give its mathematical justification.

[1] This follows from a theorem that for $f \in \mathcal{I}$, almost all orbits converge to an attractor of measure 0, see [G2, BL1, S2].

In the work of Lanford, Epstein, Eckmann, Sinai, Sullivan, de Melo, McMullen (see [La1], [E], [EE], [VSK], [S1], [S2], [MS], [McM2]), among others, spectacular progress in this problem has been achieved (see [L5] for more historical comments and references). However, until recently even the doubling case was not completely resolved. In [L5] the conjecture was proved for bounded combinatorics. In the present paper, we extend the conjecture and prove it simultaneously for all real combinatorial types. (Compare Lanford's conjecture [La2] for circle maps.)

Following Sullivan's program [S1], we approach renormalization theory from the point of view of holomorphic dynamics. This approach is based on the notion of quadratic-like map introduced by Douady and Hubbard [DH2], which is a complex analogue of the notion of unimodal map. A *quadratic-like map* is a holomorphic double branched covering $f : U \to U'$, where $U \Subset U'$ are topological disks. Let $\mathcal{Q}_\mathbb{R}$ stand for the space of real analytic unimodal maps $f : I \to I$ which admit quadratic-like extensions $f : U \to U'$ to the complex plane, $U \supset I$ (compare §2.1).

Renormalizable unimodal maps f differ combinatorially not only by the renormalization period p but also by the order of the intervals L, $fL, \dots, f^{p-1}L$ on the real line (up to change of orientation), where L is the domain of the renormalization $f^p|L$. Thus the set of renormalizable maps $f \in \mathcal{Q}_\mathbb{R}$ is decomposed into the union of *renormalization strips* $\mathcal{T}_{k,\mathbb{R}}$ consisting of maps renormalizable with the same combinatorial type.

The intersection of the strip $\mathcal{T}_{k,\mathbb{R}}$ with the real quadratic family $\{P_c : \ -2 \le c \le 1/4\}$ is a closed interval $J_k \subset [-2, 1/4]$ called the *renormalization window*. In fact, any map $f \in \mathcal{T}_{k,\mathbb{R}}$ is topologically conjugate to some quadratic map P_c in the corresponding window J_k. The renormalization period $p_k = p(J_k) = p(\mathcal{T}_{k,\mathbb{R}})$ is constant throughout the renormalization strip.

On each renormalization strip $\mathcal{T}_{k,\mathbb{R}}$ one can consider the corresponding renormalization operator $R_k : \mathcal{T}_{k,\mathbb{R}} \to \mathcal{Q}_\mathbb{R}$. These operators are real analytic with respect to the natural real analytic structure on $\mathcal{Q}_\mathbb{R}$, and altogether, they form a single piecewise analytic operator $R : \cup\mathcal{T}_{k,\mathbb{R}} \to \mathcal{Q}_\mathbb{R}$ (see Figure 2).

Let us now define *real hybrid classes* $\mathcal{H}_\mathbb{R}(f)$, $f \in \mathcal{Q}_\mathbb{R}$ (the motivation for this notion will become clear in the complex setting). If f is not hyperbolic then let $\mathcal{H}_\mathbb{R}(f)$ be the topological class of f (i.e., the set of maps $g \in \mathcal{Q}_\mathbb{R}$ topologically conjugate to f). If f is hyperbolic then it has an attracting cycle with multiplier $\lambda(f)$. Then let $\mathcal{H}_\mathbb{R}(f)$ be the set of $g \in \mathcal{Q}_\mathbb{R}$ in the topological class of f such that $\lambda(g) = \lambda(f)$.

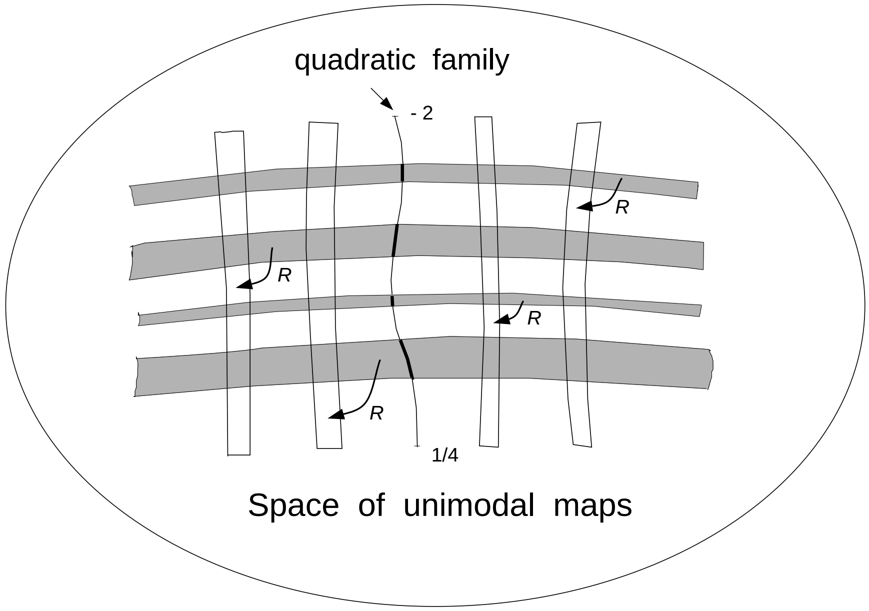

FIGURE 2. Full renormalization operator

Let Σ stand for the space of two-sided sequences of natural numbers, and ω stand for the shift on this symbolic space.

Now we will state a somewhat simplified version of the Renormalization Theorem for all real combinatorial types.

Theorem 1.4 (Renormalization Theorem)**.** *There is a set $\mathcal{A} \subset \cup \mathcal{T}_{k,\mathbb{R}}$ ("the full renormalization horseshoe") such that:*

- *$\mathcal{A}$ is R-invariant and $R|\mathcal{A}$ is topologically conjugate to the two-sided shift ω.*
- *The topological classes $\mathcal{H}_\mathbb{R}(f)$, $f \in \mathcal{A}$, are codimension-one real analytic submanifolds in $\mathcal{Q}_\mathbb{R}$ which form an R-invariant lamination in $\mathcal{Q}_\mathbb{R}$. Moreover, if $g \in \mathcal{H}_\mathbb{R}(f)$, then the forward orbits of f and g under R are exponentially asymptotic.*
- *Through any point $f \in \mathcal{A}$ there is a real analytic curve $W^u_\mathbb{R}(f)$ transverse to $\mathcal{H}_\mathbb{R}(f)$. This family of curves is R^{-1}-invariant. If $g \in W^u_\mathbb{R}(f)$ then the orbits of f and g under R^{-1} are exponentially asymptotic.*

This theorem expresses the fact that the renormalization operator is *hyperbolic* in the space $\mathcal{Q}_\mathbb{R}$, with the topological classes $\mathcal{H}_\mathbb{R}(f) \equiv W^s_\mathbb{R}(f)$ as stable manifolds and the curves $W^u_\mathbb{R}(f)$ as unstable ones. In fact, there are several technical issues to be addressed to make this statement precise:

- What is the real analytic structure on the space $\mathcal{Q}_{\mathbb{R}}$?

- With respect to what metric are the expansion and contraction measured?

Also, for the sake of applications, finer properties of the above hyperbolicity picture are needed (uniform bounds on the expansion/contraction rates and on the unstable non-linearity of R, uniform size of the unstable leaves, and a certain transverse regularity of the stable foliation).

The complete technical formulation of the Renormalization Theorem will be given in Section 3.

1.5. **More consequences.** Let us state a few more consequences of the renormalization theorem. For any renormalization window J_i, there is a canonical map $\sigma : J_i \to [-2, 1/4]$ defined as the renormalization postcomposed with the straightening (where the straightening associates to a unimodal map $f \in \mathcal{Q}_{\mathbb{R}}$ a unique quadratic polynomial $P_{\chi(f)}$ in the hybrid class $\mathcal{H}_{\mathbb{R}}(f)$). Let $\{J_i^n\}$ stand for the collection of domains of definition of σ^n, that is, the windows for the n-fold renormalization, and let $J_i^n(\epsilon) = \sigma^{-n}[-2, 1/4 - \epsilon] \cap J_i^n$.

Theorem 1.5. *The maps $\sigma^n : J_i^n(\epsilon) \to [-2, 1/4 - \epsilon]$ are uniformly quasi-symmetric (with dilatation independent of n and i).*

Given a renormalizable map f, let $p(f)$ stand for the period of the first renormalization. The following result improves Theorem VIII of [L3]:

Theorem 1.6. *There is a number $\bar{p}$ with the following property. If $f = P_c$ is an infinitely renormalizable real quadratic map with $p(R^{n_k} f) \geq \bar{p}$ for a subsequence $n_k \to \infty$, then the Mandelbrot set is locally connected at c. Moreover, the corresponding little Mandelbrot sets M^{n_k} shrinking to c are uniformly quasi-conformally equivalent to the standard Mandelbrot set M^0.*

1.6. **Structure of the proof of the Renormalization Theorem.** There are three types of combinatorics to take care of: bounded, "essentially bounded", and "high". For any bounded combinatorics, Sullivan [S2] and McMullen [McM2] constructed the corresponding renormalization horseshoe and its strong stable foliation. It was proven in [L5] that the renormalization horseshoe is hyperbolic. The idea of the proof is that, in the complex analytic context, lack of hyperbolicity yields existence of "slowly shadowing orbits". On the other hand, such orbits are ruled out by the rigidity theorem [L3].

Note that an important part of [L5] is the construction of a complex analytic structure for the space of quadratic-like germs (modulo affine conjugacy), and a proof that the Douady-Hubbard hybrid classes [DH2] form a foliation of the connectedness locus by analytic leaves of complex codimension one. By means of the λ-lemma, this allows us to relate parameter geometry of the quadratic family and other quadratic-like families.

The unbounded combinatorics can be split into two types: essentially bounded and high. In the former case, the unboundedness is produced by the saddle-node behavior of the critical point (see [L3], [LY]). This phenomenon can be analyzed by means of parabolic bifurcation theory (see [D2]). Extending the work of A. Epstein [Ep] and McMullen [McM2], Ben Hinkle has proved a rigidity theorem for "parabolic towers" [Hi], that is, geometric limits of dynamical systems generated by infinitely renormalizable maps with essentially bounded combinatorics. Using this result, we prove hyperbolicity of the renormalization operator with essentially bounded combinatorics. Note that McMullen's argument for exponential contraction does not seem to work in this case, and instead we make use of the Schwarz Lemma in Banach spaces.

To treat the remaining case of high combinatorics we need the extensive analytic preparation on the geometry of the puzzle and parapuzzle which was carried out in [L3] and [L4]. The main geometric result of these papers is the linear growth of the conformal moduli of the "principal nest" of dynamical and parameter annuli. This implies that the image of a renormalization "horizontal" strip of high type is a narrow "vertical strip" close to the quadratic family. Using the Schwarz lemma and the Koebe distortion theorem, this yields strong hyperbolicity of the high type renormalization, with large contraction and expansion factors. Note that it is crucial for our argument that the results of [L3, L4] are proved for complex parameter values, even though in the present paper we are ultimately interested in real quadratics.

Moreover, the parameter results of [L4] (accompanied by the Koebe Distortion Theorem and the λ-lemma) provide us with transverse control of the renormalization which allows us to relate global expansion to the multipliers of periodic points.

Finally, we construct a special family of Banach bidisks which capture hyperbolic properties of the renormalization. These bidisks allow us to generalize the argument of [L5] (slowly shadowing orbits versus rigidity) which yields Theorem 3.1.

This paper completes a program of studying the real quadratic family by complex methods, carried in the series of papers [LM], [L2], [L3], [LY], [L4], [MN], [Hi], [L5]. The results were announced in the Proceedings of the National Academy of Sciences [L6] (1998) and in the

Notices of the AMS (October 2000). Their preprint version appeared in the IMS at Stony Brook preprint series (1997, # 8).

1.7. **Why the quadratic family?** One can ask what is special about the quadratic family? Besides the attractive simple formula and the advantage of being globally holomorphic, there are several good reasons why the quadratic family plays a distinguished role.

First, it is a global transversal to the lamination of the space of quadratic-like maps into real hybrid classes. For this reason, the dynamical picture in the quadratic family is also valid in generic quadratic-like families.

Second, there are several tight links between (sufficiently) smooth and holomorphic dynamics. One of the links is that the renormalizations of smooth unimodal maps (with a nondegenerate critical point) are asymptotically quadratic-like [S2]. This should make it possible to reduce many questions about generic smooth families to the quadratic situation. In particular, the renormalization horseshoe and its unstable foliation as described above will surely yield universal properties of generic families of smooth unimodal maps. Another link is that smooth maps admit an asymptotically conformal extension to the complex plane and can be studied there by the methods of holomorphic dynamics [L7], [L3, §12.2].

For these reasons, holomorphic dynamics is not an exotic branch but rather an intrinsic part of smooth dynamics, and provides a powerful approach to the latter. Work in this direction is already on the way (see [LM, K, MP, FMP, ALM]).

1.8. **Further perspective.** Theorem 1.2 was recently strengthened by Avila and Moreira [AM] who have proven that almost all quadratic maps which are at most finitely renormalizable satisfy the Collet-Eckmann condition for existence of an absolutely continuous invariant measure. One more natural question is whether the complement to the set of regular or stochastic maps has Hausdorff dimension strictly less than 1.

In one-dimensional theory, it will be natural to proceed to higher degree polynomials and then to polymodal C^2-smooth maps. We expect an analogous "regular or stochastic" result to be valid in generic one parameter families. There is still a lot of interesting work to be done in this direction, but we believe that the "quadratic theory" has prepared basic tools to deal with this more general situation.

One can also formulate an analogous conjecture for the complex quadratic family $z \mapsto z^2 + c$. Here absolute continuity of an invariant measure can be understood with respect to Sullivan's conformal measure on the Julia set. "Almost all" in the parameter plane can be understood in the sense of Hausdorff dimension as "outside a set of strictly smaller dimension". Of course, such a conjecture cannot be proved prior to the MLC conjecture (though it could be disproved).

There is a parallel development in the closely related branch of one-dimensional dynamics, the theory of critical circle maps. In this field, some pieces go easier but others are more involved. Work in this direction is also close to completion (see Yampolsky [Ya] and further references therein).

A general program in higher-dimensional dynamics was explicitly formulated by Palis [Pa]. Roughly speaking, it asserts that in a generic finite parameter family of smooth dynamical systems, there are typically only finitely many attractors each of which carries an SRB measure and such that almost any orbit is equidistributed with respect to one of them. This program initiated by the work of Benedicks & Carleson [BC2] on the dynamics in the Hénon family has been intensively carried on (see Viana [V], Young [Y]).

There are many issues in the higher-dimensional situation which make it significantly different from the one-dimensional case. Among them is the necessity to deal with infinitely many "critical points" (which are not even precisely defined) and the related necessity to deal with infinitely many sinks (the Newhouse phenomenon). Another problem is to find a transparent combinatorial model for higher-dimensional maps (compare [CH]). And last but not least is the lack of the complex analytic machinery (quasi-conformal maps, conformal invariants), which is so powerful in the one-dimensional theory. Note, however, that the work in complex two-dimensional dynamics which has been intensively carried out since the mid eighties, by Hubbard, Sibony, Bedford and Smillie, among others, may eventually provide useful tools for real dynamics.

1.9. **Basic notations and definitions.** $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote as usual the complex plane, the real line, and the sets of integer and natural numbers respectively;

$\bar{\mathbb{C}}$ is the Riemann sphere;

$\mathbb{D}(a,r) = \{z : |z-a| < r\}$ is the open disk of radius r;

$\mathbb{D}_r \equiv \mathbb{D}(0,r)$, $\mathbb{D} \equiv \mathbb{D}_1$;

$\mathbb{T}_r = \partial\mathbb{D}_r$ is the circle of radius r, $\mathbb{T} \equiv \mathbb{T}_1$.

The closure of a set X will be denoted either by $\bar{X}$ or by $\mathrm{cl}(X)$;

$U \Subset V$ means that U is *compactly contained* in V, that is, the closure $\bar{U}$ is compact and is contained in V;

Given two subsets X and Y in a metric space Z, we let

$$\operatorname{dist}(X,Y) = \inf_{x\in X, y\in Y} \operatorname{dist}(x,y).$$

(To specify other meanings of the distance we will use subscripts, e.g., dist_T, $\operatorname{dist}_{\text{hyp}}$.)

The *Hausdorff distance* between two compact subsets X and Y of Z is defined as the infimum of $\varepsilon > 0$ such that X is contained in an ε-neighborhood of Y and the other way around. It determines the *Hausdorff topology* on the space of compact subsets of M.

A *pointed space* (X, a) is a space X with a *base point* $a \in X$.

A sequence of pointed open sets (V_n, a_n) in $\mathbb{R}^l$ *Carathéodory converges* to a pointed open set (V, a) if:

(i) $a_n \to a$;

(ii) Any compact subset $K \subset V$ is contained in all but finitely many sets V_n;

(iii) Any open connected set $K \ni a$ contained in infinitely many sets V_n is contained in V.

If $A \subset \mathbb{C}$ is a topological annulus (perhaps with boundary) , then $\operatorname{mod}(A)$ stands for $\log(R/r)$, provided $\operatorname{int} A$ is conformally equivalent to $\{z : r < |z| < R\}$.

Let

$$\partial \equiv \frac{\partial}{\partial z}, \quad \bar\partial \equiv \frac{\partial}{\partial \bar z}.$$

We refer to the book of Ahlfors [A] for the basic theory of quasi-conformal maps. The quasi-conformality property will be often abbreviated as “qc”. Similarly, “qs” will stand for “quasi-symmetric”. Let

$$\operatorname{Dil}(h) = \left\| \frac{\partial h + \bar\partial h}{\partial h - \bar\partial h} \right\|_\infty$$

stand for the dilatation of a qc map h.

A Jordan curve $\gamma \subset \mathbb{C}$ is called a κ-*quasi-circle* if for any two points $x, y \in \gamma$ there is an arc $\delta \subset \gamma$ bounded by these points such that

$$\operatorname{diam} \delta \le \kappa |x - y|.$$

A curve is called a quasi-circle if it is a κ-quasi-circle for some κ. The best possible κ in the above definition is called the *dilatation* $\operatorname{Dil}(\gamma)$ of the quasi-circle. A Jordan disk is called (κ-)*quasi-disk* if it is bounded by a (κ-)quasi-circle.

Let $P_c(z) = z^2 + c$;

M^0 is the Mandelbrot set;

Given two maps f and $\tilde{f}$ partially defined on sets X and $\tilde{X}$ respectively, we say that a map $h : V \to \tilde{V}$ is $(f, \tilde{f})$-*equivariant* if $h(fz) = \tilde{f}(hz)$, whenever both z and fz belong to X. (Thus, "equivariance" means "semi-conjugacy for partially defined maps".) We will skip reference to $(f, \tilde{f})$ unless it may lead to a confusion.

The notation $\alpha \asymp \beta$ means as usual that the ratio α/β is bounded away from 0 and ∞.

1.10. **Acknowledgement.** This work was done in fall 1996 during author's visit to IHES and was partially written during a brief visit to ETH in Zürich (January 1997). I thank both Institutes for their hospitality. The main results were first announced at the analysis seminar at KTH in Stockholm (November 1996), and I wish to thank Lennart Carleson and Michael Benedicks for their inspiring interest. It is my pleasure to thank Ben Hinkle, Marco Martens, Welington de Melo, John Milnor, and Michael Yampolsky for many useful comments, and Christian Henriksen for making Figure 3. Finally, I thank Stony Brook University for generous sabbatical support in the academic year 1996 – 1997 and the NSF for its continuing support.

2. Quadratic-like germs, puzzle, renormalization, and towers

2.1. **Quadratic-like germs.** In this section we give a refined summary of [L5, §§3,4].

2.1.1. *Notion of quadratic-like map and germ.* A map $f : U \to U'$ is called *quadratic-like* if it is a double branched covering between topological disks U, U' such that $U \Subset U'$. It has a single critical point which is assumed to be located at the origin 0, unless otherwise is stated. For expositional reasons, we will also make the following assumptions:

- The boundaries ∂U and $\partial U'$ are quasi-circles. Hence f is continuous on $\bar{U}$ and maps ∂U onto $\partial U'$ as a double covering.
- U is symmetric with respect to the origin and f is even, i.e., $f(-z) = f(z)$.

The filled Julia set is defined as the set of non-escaping points: $K(f) = \{z : f^n z \in U, n = 0, 1 \dots\}$. Its boundary is called the *Julia set*, $J(f) = \partial K(f)$. The sets $K(f)$ and $J(f)$ are connected if and only if the critical point itself is non-escaping: $0 \in K(f)$. Otherwise these sets are Cantor.

The *fundamental annulus* of a quadratic-like map $f : U \to U'$ is the annulus between the domain and the range of f, $A = U' \setminus \bar{U}$. We let

$\mathrm{mod}(f) = \mathrm{mod}(A)$. For a quadratic polynomial $P_c : \mathbb{C} \to \mathbb{C}$, we let $\mathrm{mod}(P_c) = \infty$.

Any quadratic-like map has two fixed points counted with multiplicity. In the case of connected Julia set these two points can be dynamically distinguished. One of them, usually denoted by α, is either non-repelling or *dividing*, i.e., removing it makes the Julia set disconnected. Another one, denoted by β, is always *non-dividing.*

A quadratic-like map $f : U \to U'$ is called *real* if the domains U and U' are $\mathbb{R}$-symmetric and f commutes with the conjugacy $z \mapsto \bar{z}$.

We allow to change the domains (U, U') of a quadratic-like map without changing "its germ" near the Julia set. More precisely, let us say that a quadratic-like map $g : V \to V'$ is an *adjustment* of another quadratic-like map $f : U \to U'$ if $V \subset U$, $g = f|V$, and $\partial V' \subset \bar{U}' \smallsetminus U$. (In particular, we can restrict f to $V = f^{-1}U$, provided $f(0) \in U$.) Let us say that two quadratic-like maps f and $\tilde{f}$ represent the same *quadratic-like germ* if there is a sequence of quadratic-like maps $f = f_0, f_1, \dots, f_n = \tilde{f}$, such that f_{i+1} is obtained by an adjustment of f_i or the other way around. Clearly a quadratic-like germ has a well-defined Julia set.

Remark. Note that this notion of a quadratic-like germ is slightly different from the one given in [L5]. We found the modified notion more convenient to work with.

We will consider quadratic-like maps/germs up to *affine* conjugacy (rescaling), so that near the origin they can be normalized as follows:

$$f(z) = c + z^2 + \sum_{k=2}^{\infty} a_k z^{2k}$$

(quadratic-like germs modulo affine conjugacy will still be called briefly "quadratic-like maps/germs"). We will usually not make notational difference between quadratic-like germs and quadratic-like maps representing them but in some cases we will use notation $f_U : U \to f(U)$ for the quadratic-like representative of a germ f on a disk U.

A quadratic-like germ is called *real* if it has a real representative.

Note also that any quadratic polynomial $P_c : z \mapsto z^2 + c$ determines a quadratic-like germ by restricting it to the preimage $P_c^{-1}(\mathbb{D}_r)$ of a sufficiently big round disk $\mathbb{D}_r$. These germs will still be called "quadratic polynomials".

2.1.2. *Space of quadratic-like germs.* Let $\mathcal{QM}$ be the union of the space of normalized quadratic-like maps and the quadratic family $\{P_c : \mathbb{C} \to \mathbb{C}\}_{c \in \mathbb{C}}$. The space $\mathcal{QM}$ is endowed with the following *convergence* structure [McM1, §5.1]. A sequence of maps $f_n : V_n \to V_n'$ converges to a map $f : V \to V'$ if the pointed domains $(V_n', V_n, 0)$ Carathéodory converge to $(V', V, 0)$, and the maps f_n converge to f uniformly on compact subsets of V.

For $\mu > 0$, let $\mathcal{QM}(\mu, \rho)$ stand for the union of the disk in the quadratic family $\{P_c\}_{|c|\leq\rho}$ and the space of normalized quadratic-like maps $f : V \to V'$ such that the curves ∂V and $\partial V'$ are ρ-quasi-circles,

$$\operatorname{mod}(V' \setminus V) \geq \mu, \ |f(0)| \leq \rho, \text{ and } \operatorname{dist}_{\text{hyp}}(0, f(0)) \leq \rho, \tag{2.1}$$

where the hyperbolic distance is measured in V'.

The following two compactness lemmas are slight variations of [McM1, Theorems 5.6, 5.8] and [L5, Lemma 4.1].

Lemma 2.1. *The space $\mathcal{QM}(\mu, \rho)$ is compact. Moreover, if $f_n \in \mathcal{QM}(\mu_n, \rho)$ with $\mu_n \to \infty$, then all limits of the sequence $\{f_n\}$ are quadratic polynomials.*

Similarly, let $\mathcal{Q}$ stand for the space of quadratic-like germs, and let $\mathcal{C}$ be its connectedness locus, that is, the subset of germs with connected Julia set. We endow $\mathcal{Q}$ with topology and complex analytic structure described in Appendix 2. Recall the main notations from there:

- $\mathbf{V}$ is the set of topological disks $V \ni 0$ with piecewise smooth boundary symmetric with respect to the origin;
- $\mathcal{B}_V$ is the space of normalized even analytic functions $f(z) = c + z^2 + \dots$ on $V \in \mathbf{V}$ continuous up to the boundary supplied with sup-norm $\|\cdot\|_V$;
- $\mathcal{B}_V(f, \varepsilon)$ is the ε-ball in this space centered at f;
- For $\mathcal{X} \subset \mathcal{Q}$, $\mathcal{X}_V = \mathcal{X} \cap \mathcal{B}_V(f, \varepsilon)$ is a *Banach slice* of $\mathcal{X}$ (f and ε are implicit in this notation);
- For a germ $f \in \mathcal{Q}$, $\mathbf{V}_f$ is the set of topological disks $V \in \mathbf{V}$ such that f has a quadratic-like representative $f_V : V \to f(V)$ in the space $\mathcal{B}_V$.

By Lemma 6.2, compactness in $\mathcal{Q}$ is equivalent to sequential compactness. Moreover, any compact set $\mathcal{K} \subset \mathcal{Q}$ sits in a finite union of Banach slices $\mathcal{B}_V$ and possesses a *Montel metric* $\operatorname{dist}_{Mon}$ well-defined up to Hölder equivalence.

Let $\mathcal{QP} = \{P_c\}_{c\in\mathbb{C}}$ stand for the quadratic family. It is a complex one-dimensional submanifold of $\mathcal{Q}$. By definition, the Mandelbrot set $M^0 \subset \mathcal{QP}$ is equal to $\mathcal{QP} \cap \mathcal{C}$.

Given a quadratic-like germ f, let $\operatorname{mod}(f) = \sup \operatorname{mod}(A)$ where A runs over the fundamental annuli of quadratic-like representatives of

f. For $\mu > 0$, let $\mathcal{Q}(\mu, \rho)$ stand for the set of normalized quadratic-like germs which have representatives $f : V \to V'$ satisfying (2.1). Furthermore, let

$$\mathcal{Q}(\mu) = \{f \in \mathcal{Q} : \mathrm{mod}(f) \geq \mu\}.$$

Given a set $\mathcal{X} \subset \mathcal{Q}$, let $\mathcal{X}(\mu) = \mathcal{X} \cap \mathcal{Q}(\mu)$.

Lemma 2.1 and the remark afterwards yield:

Lemma 2.2. *For any $\mu > 0$ and $\rho > 0$, the sets $\mathcal{Q}(\mu, \rho)$ and $\mathcal{C}(\mu)$ are compact. Moreover, if $f_n \in \mathcal{Q}(\mu_n, \rho)$ with $\mu_n \to \infty$ then the limit points of the f_n are quadratic polynomials.*

Let $\mathcal{Q}_{\mathbb{R}}$ stand for the space of real quadratic-like germs.

2.1.3. *Hybrid lamination.* Two quadratic-like maps/germs f and g are called *hybrid equivalent* if they are conjugate by a qc map h with $\bar{\partial} h = 0$ a.e. on $K(f)$. By the Douady-Hubbard Straightening Theorem [DH2], any quadratic-like map $f : V \to V'$ is hybrid equivalent to a quadratic polynomial

$$P_c : \Omega_c(\sqrt{r}) \to \Omega_c(r),$$

where $\Omega_c(r) \subset \mathbb{C}$ is the topological disk bounded by the equipotential of P_c of some radius $r > 1$. Moreover, the polynomial P_c and the qc map h conjugating it to f are uniquely determined by the choice of an equivariant qc map

$$H : \mathbb{C} \setminus V \to \mathbb{C} \setminus \mathbb{D}_r, \quad H(fz) = P_0(Hz) \text{ for } z \in \partial V. \tag{2.2}$$

Such a map H is called a *tubing* of the fundamental annulus $V' \setminus V$, and the quadratic polynomial P_c (as well as the corresponding parameter value $c = \chi_H(f)$) is called the *straightening of* f. We will also say that f is *equipped* with the tubing H.

In the case of connected Julia set the straightening P_c is, in fact, independent of the choice of H. Thus, every hybrid class $\mathcal{H}(f)$ in $\mathcal{C}$ intersects the quadratic family $\mathcal{QP}$ at a single point $c = \chi(f)$ of the Mandelbrot set M^0. Such hybrid classes can be also labeled as $\mathcal{H}_c$, $c \in M^0$.

The hybrid classes in the connectedness locus can be endowed with the Teichmüller-Sullivan metric (see [S1]). Below we modify its definition so that it takes into account that the maps are considered up to affine (rather than conformal) conjugacy. Let

$$\mathrm{dist}_T(f, g) = \inf_h \log \mathrm{Dil}(h),$$

where $h : \mathbb{C} \to \mathbb{C}$ runs over all qc maps which are hybrid conjugacies between f and g near their filled Julia sets. If $f : V \to V'$ is a quadratic-like representative of f, then let us also define $\mathrm{dist}_{T,V}$ as a similar infimum where $h : \mathbb{C} \to \mathbb{C}$ provides a hybrid conjugacy on V (warning: unlike dist_T, $\mathrm{dist}_{T,V}$ is not a metric).

Lemma 2.3. *Let $W \Subset V \Subset V' \subset \mathbb{C}$ be three topological disks. Let us consider a normalized quadratic-like map $f : V \to V'$ with connected Julia set. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any normalized quadratic-like map $g \in \mathcal{H}(f)$ we have:*

- *If $\mathrm{dist}_{T,V}(f, g) < \delta$ then g belongs to $\mathcal{B}_W$ and $\|f - g\|_W < \varepsilon$.*
- *Vice versa, if $g \in \mathcal{B}_V(f, \delta)$ then $\mathrm{dist}_{T,W}(f, g) < \varepsilon$.*

Proof. Let $h : \mathbb{C} \to \mathbb{C}$ be a qc map with dilatation $K < e^{2\delta} < 2$ which provides on V a hybrid conjugacy of f to g. Since h transfers $V' \setminus V$ to a fundamental annulis of g, $\mathrm{mod}(g) \geq \mathrm{mod}(V' \setminus V)/2$. By Lemma 2.2, g stays within a compact family of maps. It follows that its β-fixed point β_g stays away from 0 and ∞.

As $h(\beta_f) = \beta_g$, h belongs to a compact family of maps in the uniform topology (by compactness of the space of normalized at two points qc maps $\mathbb{C} \to \mathbb{C}$ with bounded dilatation). Hence h with sufficiently small dilatation, $\mathrm{Dil}(h) < \delta$, is uniformly close to an affine map $z \mapsto az$. Since g is normalized at the origin, $a = 1$, so that h is uniformly close to id. This proves the first statement.

To prove the second one, observe that if $g \in \mathcal{B}_V(f, \delta)$, then f and g have $(1+\varepsilon)$-qc equivalent fundamental annuli A_f and A_g such that the corresponding $(1+\varepsilon)$-qc map $h : (\mathbb{C}, A_f) \to (\mathbb{C}, A_g)$ is equivariant on ∂A_f, i.e., $h(fz) = g(hz)$ for z on the inner boundary of A_f (see [L5, Lemma 4.2]). By a standard pullback procedure, such an h can be turned into a hybrid conjugacy between f and g, with the same dilatation $(1 + \varepsilon)$. □

It is proven in [L5] that the hybrid classes $\mathcal{H}_c$, $c \in M^0$, are connected codimension-one holomorphic submanifolds of $\mathcal{Q}$ in the sense defined in Appendix 2. They form a foliation (or rather, a lamination) $\mathcal{F}$ called *horizontal.* This foliation is transversally quasi-conformal everywhere, and holomorphic on $\mathrm{int}\,\mathcal{C}$.

Let us state the last result more precisely. Take two hybrid equivalent germs $f_i \in \mathcal{C}$, and two holomorphic transversals $\mathcal{S}_i$ to the leaf $\mathcal{H} \equiv \mathcal{H}(f_i)$ through f_i. The holonomy $\gamma : \mathcal{C} \cap \mathcal{S}_1 \to \mathcal{C} \cap \mathcal{S}_2$ along $\mathcal{F}$ is called *locally quasi-conformal* at f_1 if it admits a local qc extension $\tilde{\gamma} : \Omega_1 \to \Omega_2$, where $\Omega_i \subset S_i$ are neighborhoods of the f_i in the transversals S_i.

The local dilatation of γ at f_1 is defined as

$$\inf_{\tilde{\gamma}} \mathrm{Dil}(\tilde{\gamma}),$$

where the infimum is taken over all local qc extensions $\tilde{\gamma}$ of γ.

Theorem 2.4 ([L5], Theorem 4.19)**.** *Given two quadratic-like germs and two transversals as above, the holonomy $\gamma : \mathcal{C} \cap S_1 \to \mathcal{C} \cap S_2$ is locally quasi-conformal. If the transversals S_i are represented by holomorphic one-parameter families $\{f_{i,\lambda}\}$ of quadratic-like maps such that* $\mathrm{mod}(f_{i,\lambda}) \geq \mu > 0$*, then the local dilatation of γ at f_1 is bounded by $K(\mu)$.*

2.1.4. *External maps.* Let $\mathcal{E}$ denote the space of real analytic *expanding circle endomorphisms* $g : \mathbb{T} \to \mathbb{T}$ of degree 2 considered up to conjugacy by rotation and such that $g(z) = g(-z)$. Any $g \in \mathcal{E}$ admits a holomorphic extension to a double covering $g : V \to V'$, where V and V' are two symmetric annuli neighborhoods of the circle $\mathbb{T}$ with piecewise smooth boundary such that $V \Subset V'$ (here "symmetric" means being invariant under the circle involution $z \mapsto 1/\bar{z}$ and the central involution $z \mapsto -z$). Note that such a map can be normalized so that $g(1) = 1$.

There is a natural projection

$$\pi : \mathcal{Q} \to \mathcal{E}, \tag{2.3}$$

which associates to $f \in \mathcal{Q}$ its *external map* $g = \pi(f) \in \mathcal{E}$ (see [DH2], [L5, §3.2]). For readers' convenience we will outline the construction of the external map. Take a quadratic-like representative $f : U \to U'$ and consider the fundamental annulus $A = U' \smallsetminus U$. Let $\mu = \mathrm{mod}(A)$. Using the map $f : I \to O$ from the inner to outer boundary of A, attach to the inner boundary of A an abstract annulus A_1 of modulus $\mu/2$. It comes together with a double covering $A_1 \to A$ which extends $f : I \to O$. Using this covering, attach in a similar way an annulus A_2 of modulus $\mu/4$ to the inner boundary of A_1, etc. Taking the infinite union of these annuli together with $\mathbb{C} \smallsetminus U'$, we obtain a conformal punctured disk S (with the puncture corresponding to ∞) and a double covering F between annuli neighborhoods of its ideal boundary. Let us uniformize it, $\phi_f : S \to \mathbb{C} \smallsetminus \mathbb{D}$, and consider a map $\phi \circ F \circ \phi^{-1}$. It is a double covering between outer annuli neighborhoods of $\mathbb{T}$. Reflecting it about the circle, we obtain the desired external map $g : V \to V'$, where V and V' are symmetric annuli neighborhoods of $\mathbb{T}$, and $V' \Supset V$.

Since the uniformization ϕ_f is uniquely defined up to post-composition with rotation, the external map g is well-defined up to conjugacy by rotation. We normalize it so that $g(1) = 1$. By [L5, §3.2], the external map depends only on the germ of f.

Note further that the uniformization ϕ_f provides an (f, g)-equivariant conformal isomorphism between the fundamental annuli $U' \smallsetminus U$ and $V' \smallsetminus (V \cup \mathbb{D})$. Let n be the smallest natural number such that $f(0) \notin$

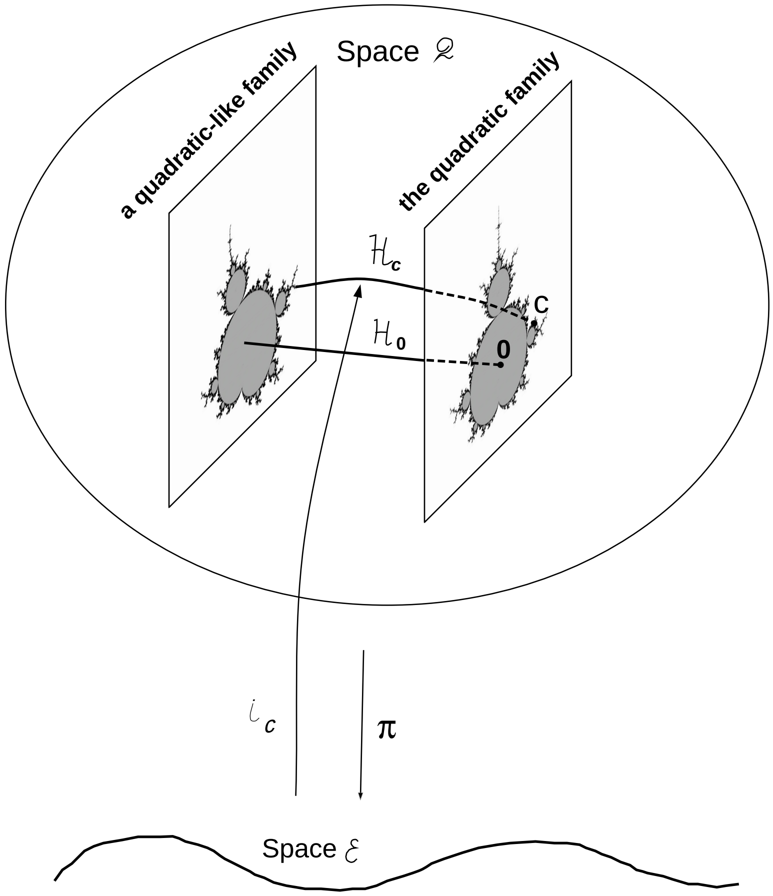

FIGURE 3. Space of quadratic-like germs

$\mathrm{cl}(f^{-n}U)$. Using the equivariance equation $\phi_f \circ f = g \circ \phi_f$, the uniformization can be analytically extended to the domain

$$\Omega_f = \mathbb{C} \smallsetminus \mathrm{cl}(f^{-n}U) \tag{2.4}$$

containing the critical value $f(0)$. (Note that Ω_f depends on the *map* f rather than its germ.) Thus we can consider the image of the critical point under this map:

$$\xi(f) = \phi_f(f(0)) \tag{2.5}$$

(it is well-defined once g is normalized).

Lemma 2.5. *The point $\xi(f)$ depends only on the germ of f.*

Proof. We need to verify that $\xi(f) = \xi(\tilde{f})$, provided $\tilde{f} : \tilde{U} \to \tilde{U}'$ is an adjustment of $f : U \to U'$. It is easy to see that $\phi_f | (\mathbb{C} \smallsetminus \mathrm{cl}(U))$ is the restriction of $\phi_{\tilde{f}} | (\mathbb{C} \smallsetminus \mathrm{cl}(\tilde{U}))$.

Let $U^n = f^{-n}U$, $\tilde{U}^n = f^{-n}\tilde{U}$. Take n as in (2.4). Note that $U^n \subset \tilde{U}^n \subset U^{n-1}$. If $c \notin \mathrm{cl}(\tilde{U}^n)$ then $\Omega_{\tilde{f}} = \mathbb{C} \setminus \mathrm{cl}(\tilde{U}_n) \subset \Omega_f$ and $\phi_{\tilde{f}}$ if the restriction of ϕ_f to $\Omega_{\tilde{f}}$. Otherwise $\Omega_{\tilde{f}} = \mathbb{C} \setminus \mathrm{cl}(\tilde{U}_{n+1}) \supset \Omega_f$, and ϕ_f is the restriction of $\phi_{\tilde{f}}$ to $\Omega_{\tilde{f}}$. □

The "Green function" $G = \log|\xi| : \mathcal{Q} \setminus \mathcal{C} \to \mathbb{R}_+$ provides us with a dynamically natural way to measure the "distance" from an $f \in \mathcal{Q} \setminus \mathcal{C}$ to the connectedness locus.

The inverse map $\psi_f = \phi_f^{-1}$ will be called the *uniformization of f at ∞*.

Restricted to any hybrid class $\mathcal{H}_c$, $c \in M^0$, the projection π becomes a homeomorphism onto $\mathcal{E}$. The inverse map $i_c : \mathcal{E} \to \mathcal{H}_c$ is provided by the "mating" of a circle map $g \in \mathcal{E}$ with the quadratic polynomial P_c (see [DH2, L5]).

The homeomorphism $\pi : \mathcal{H}_0 \to \mathcal{E}$ allows us to transfer the complex analytic structure from the hybrid class $\mathcal{H}_0$ of $z \mapsto z^2$ to the space $\mathcal{E}$. This complex structure makes the projection $\pi : \mathcal{Q} \to \mathcal{E}$ and all the parametrizations $i_c : \mathcal{E} \to \mathcal{H}_c$, $c \in M^0$, holomorphic (see [L5, §4.3]). The fibers of π turn out to be holomorphic curves in $\mathcal{Q}$ [L5, Theorem 4.23]. They are called *vertical fibers*. The vertical fiber through a point $f \in \mathcal{Q}$ will be denoted $\mathcal{Z}(f)$.

In what follows we will often use the hybrid class $\mathcal{H}_0$ (in place of $\mathcal{E}$) to parametrize holomorphically all other hybrid classes. Let us introduce the corresponding notations:

$$\Pi = i_0 \circ \pi : \mathcal{Q} \to \mathcal{H}_0, \quad I_c = i_c \circ \pi : \mathcal{H}_0 \to \mathcal{H}_c,\ c \in M^0. \tag{2.6}$$

Note that for $G \in \mathcal{H}_0$, $\mathcal{Z}(G) = \Pi^{-1}(G)$.

2.1.5. *Dependence of the uniformization on f.* The following statement on the continuous dependence of the uniformization ψ_f on f is a slight variation of Lemma 4.15 of [L5].

Lemma 2.6. *Consider a quadratic-like map $f : U \to U'$ and let $W_f = \phi_f(\Omega_f)$. Let a sequence of quadratic-like maps $f_n \in \mathcal{B}_U$ converges to f in $\mathcal{B}_U$. Then the uniformizations ψ_{f_n} converge to ψ_f uniformly on compact subsets of W_f.*

Let us consider a conformal representation $\Psi_f = \psi_f \circ \psi_G^{-1}$, where $G = \Pi(f)$. In the case of connected Julia set $J(f)$, Ψ_f is the conformal mapping $\mathbb{C} \setminus K(G) \to \mathbb{C} \setminus K(f)$. In general, Ψ_f conformally maps some domain Δ_G onto Ω_f, where Ω_f is defined in (2.4).

Lemma 2.7. *Consider a quadratic-like map $f_0 : U \to U'$ and let $\tilde{\Delta} \subset \Delta_{f_0}$. If $\varepsilon > 0$ is sufficiently small, then for $f \in \mathcal{B}_U(f_0, \varepsilon)$, the conformal representation Ψ_f is well-defined on $\tilde{\Delta}$ and depends holomorphically on f.*

Proof. If $\varepsilon > 0$ is sufficiently small, then by Lemma 2.6, Ψ_f is well-defined on $\tilde{\Delta}$ and depends continuously on f.

To prove holomorphic dependence on f, let us consider a one-parameter holomorphic family of quadratic-like maps $f_\lambda \in \mathcal{B}_U(f_0.\varepsilon)$, $\lambda \in \Lambda \subset \mathbb{C}$, where $f_0 = f_{\lambda_0}$. Objects corresponding to f_0 will be labeled with 0.

Select a fundamental annulus A_0 for f_0 with smooth boundary, and let W_0 denote the outer component of $\mathbb{C} \setminus A_0$. For λ near λ_0, we can select a fundamental annulus A_λ holomorphically moving with λ (so that its outer boundary is not moving at all) in such a way that the corresponding holomorphic motion $h_\lambda : A_0 \to A_\lambda$ is equivariant and is equal to the identity on W_0 (see [DH2, Prop. 9] or [L5, Lemma 4.2]). Let us consider the corresponding holomorphic family of conformal structures $\mu_\lambda = (h_\lambda \circ \psi_0)^*(\sigma)$, where σ is the standard structure on $A_\lambda \cup W_0$. Pulling them back by the external map g_0, we obtain a holomorphic family of conformal structures on $\mathbb{C} \setminus \bar{\mathbb{D}}$. Extend these structures to $\bar{\mathbb{D}}$ as the standard ones. We obtain a holomorphic family of complex structures on $\mathbb{C}$ which will still be denoted as μ_λ.

By the Measurable Riemann Mapping Theorem, there is a family of qc maps ω_λ holomorphically depending on λ which solves the Beltrami equations $(\omega_\lambda)_* \mu_\lambda = \sigma$. It maps $\mathbb{C} \setminus \bar{\mathbb{D}}$ onto $\mathbb{C} \setminus K(G)$, where $G = \Pi(f) \in \mathcal{H}_0$. Then $\Psi_\lambda = h_\lambda \circ \psi_0 \circ \omega_\lambda^{-1}$, and we conclude that it depends holomorphically on λ (see *Remark* on p. 345 of [L5]). ☐

2.2. **Quadratic-like families.**

2.2.1. *Basic definitions.* The reader is referred to [DH2, L4] for the background in the theory of quadratic-like families. Let us consider a domain $\Lambda \Subset \mathbb{C}$. A domain $\mathbb{V} \subset \Lambda \times \mathbb{C}$ is called a *topological bidisk* over Λ if it is homeomorphic over Λ to a straight bidisk $\Lambda \times \mathbb{D}$. Let $V_\lambda = \pi_1^{-1}\{\lambda\}$ stand for the vertical fibers of a bidisk $\mathbb{V}$, where $\pi_1 : V \to \Lambda$ is the natural projection. We will assume that they are quasi-disks containing 0. Denote by $\partial^h \mathbb{V} = \cup_{\lambda \in \Lambda} \partial V_\lambda$ the *horizontal boundary* of $\mathbb{V}$.

By definition, a map $\mathbf{f} : \mathbb{V} \to \mathbb{V}'$ between two bidisks $\mathbb{V} \subset \mathbb{V}'$ over Λ is called a *quadratic-like family* over $\Lambda = \Lambda_\mathbf{f}$ if $\mathbf{f}$ is a holomorphic endomorphism preserving the fibers such that every fiber restriction $f_\lambda : V_\lambda \to V'_\lambda$, $z \mapsto z^2 + c(\lambda) + \dots$, is a normalized quadratic-like map with a critical point at 0. Clearly any quadratic-like family $\mathbf{f}$ represents a holomorphic curve in $\mathcal{Q}$. We will use the same notation, $\mathbf{f}$, for this curve.

Let us say that a quadratic-like family $\mathbf{f} : \mathbb{V} \to \mathbb{V}'$ over $(\Lambda, *)$ is *equipped* if the base map f_* is equipped with a tubing H_* (see (2.2)) and there is an equivariant holomorphic motion $\mathbf{h}$,

$$h_\lambda : (\mathbb{C},\ \mathrm{cl}(V'_* \smallsetminus V_*)) \to (\mathbb{C},\ \mathrm{cl}(V'_\lambda \smallsetminus V_\lambda)), \quad \lambda \in \Lambda,$$

where equivariance means that $h_\lambda(f_* z) = f_\lambda(h_\lambda z) \quad$ for $z \in \partial V_*$.

For instance, the Böttcher coordinate naturally equips the restricted quadratic family $P_c : \Omega_c(\sqrt{r}) \to \Omega_c(r)$ over the domain $\Delta_r \subset \mathbb{C}$ bounded by the parameter equipotential of radius r.

In what follows, all quadratic-like families are assumed to be equipped. In particular, H_* and $\mathbf{h}$ will often be implicit in the notations.

Remark. Any quadratic-like family can be equipped (Kahn & Lyubich, 1998).

Let $M^0_{\mathbf{f}} = \{\lambda \in \Lambda :\ 0 \in K(f_\lambda)\} = \{\lambda \in \Lambda :\ f_\lambda \in \mathcal{C}\}$ stand for the Mandelbrot set of $\mathbf{f}$. The family $\mathbf{f}$ is called *full* if $M^0_{\mathbf{f}}$ is compact.

Let $\phi(\lambda) = f_\lambda(0)$ denote the critical value of f_λ, and let $\Phi(\lambda) = (\lambda, \phi(\lambda))$. Let

$$\Lambda^1 \equiv \Lambda^1_{\mathbf{f}} = \{\lambda \in \Lambda :\ \phi(\lambda) \in V_\lambda.\}$$

Consider a natural map

$$\eta \equiv \eta_{\mathbf{f}} : \Lambda \smallsetminus \Lambda^1 \to V'_* \smallsetminus V_*, \quad \eta(\lambda) = h_\lambda^{-1}(\phi(\lambda)) \tag{2.7}$$

from the parameter region $\Lambda \smallsetminus \Lambda^1$ to the dynamical annulus $V'_* \smallsetminus V_*$. A family $\mathbf{f}$ is called *proper* if the map η is proper, i.e., $\eta(\lambda) \to \partial V'_*$ as $\lambda \to \partial\Lambda$. Any proper family is full.

For a full family, one defines the *winding number* $w(\mathbf{f})$ as the winding number of the curve $\lambda \mapsto \phi(\lambda)$ about the origin (which is the critical point), as λ goes once anti-clockwise around a Jordan curve $\gamma \subset \Lambda \smallsetminus M^0_{\mathbf{f}}$ surrounding $M^0_{\mathbf{f}}$. A full family is called *unfolded* if $w(\mathbf{f}) = 1$. In this case there is a single superattracting parameter value $*$ (the root of ϕ) called the *center* of Λ. It is a natural base point in Λ.

The straightening (see §2.1.3) provides a continuous map

$$\chi = \chi_{\mathbf{f}} : (\Lambda, M^0_{\mathbf{f}}) \to (\Delta_r, M^0) \tag{2.8}$$

(which depends on the choice of the equipment but is canonical on $M^0_{\mathbf{f}}$). If $\mathbf{f}$ is full and unfolded, then $\chi : M^0_{\mathbf{f}} \to M^0$ is a homeomorphism. If $\mathbf{f}$ is proper than χ is a homeomorphism on the whole domain Λ (Douady & Hubbard [DH2]).

There is a special situation of doubling renormalization when we need to truncate the Mandelbrot set near its cusp. For $\epsilon \in (0, 1/4]$, let

$$M^\epsilon = \{c \in M^0 : \ |c - 1/4| \geq \epsilon\}, \quad M^\epsilon_{\mathbf{f}} = \chi_{\mathbf{f}}^{-1}(M^\epsilon), \quad \mathcal{C}^\epsilon = \chi^{-1}(M^\epsilon).$$

We say that a family $\mathbf{f}$ is *almost full* if $M^\epsilon_{\mathbf{f}} \Subset \Lambda$. For such families the winding number, and hence the notion of being unfolded, are well-defined. We say that a family $\mathbf{f}$ is *almost proper* if the straightening $\chi_{\mathbf{f}}$ homeomophically maps Λ onto some neighborhood of M^ϵ.

From now on we will fix the truncation parameter ϵ (e.g., $\epsilon = 1/4$) and will not mention dependence of different parameters on this choice.

2.2.2. *Compactness and shapes of Mandelbrot sets.* Let $\mathcal{G}$ stand for the class of proper unfolded equipped quadratic-like families up to affine change of variable in λ. We will normalize such a family so that the superattracting parameter value $*$ sits at the origin and $\operatorname{diam} M^0_{\mathbf{f}} = 1$. We will impose the following convergence structure on $\mathcal{G}$: A sequence of normalized families $(\mathbf{f}_n : \mathbb{V}_n \to \mathbb{V}'_n, \mathbf{h}_n)$ over $(\Lambda_n, *)$ is declared to converge to a family $(\mathbf{f} : \mathbb{V} \to \mathbb{V}', \mathbf{h})$ over $(\Lambda, *)$ if:

(i) Parameter domains $(\Lambda_n, *)$ Carathéodory converge to $(\Lambda, *)$;
(ii) Holomorphic motions $\mathbf{h}_n$ converge to $\mathbf{h}$ uniformly over any domain $\Omega \Subset \Lambda$; i.e., $h_{n,\lambda}(z) \to h_\lambda(z)$ uniformly for $(\lambda, z) \in \Omega \times \bar{\mathbb{C}}$ (where $\bar{\mathbb{C}}$ is endowed with the spherical metric);
(iii) The maps $\mathbf{f}_n$ converge to $\mathbf{f}$ uniformly on compact subsets of $\mathbb{V}$.

Note that the convergence of quadratic-like families yields uniform on compact sets convergence of the corresponding holomorphic curves in $\mathcal{Q}$.

Given an equipped quadratic-like family $(\mathbf{f}, \mathbf{h})$ over $(\Lambda, *)$, let

$$\operatorname{mod}(\mathbf{f}) = \inf_{\lambda \in \Lambda} \operatorname{mod}(V'_\lambda \setminus V_\lambda), \quad \operatorname{Dil}(\mathbf{h}) = \sup_{\lambda \in \Lambda} \operatorname{Dil}(h_\lambda).$$

For $C, \mu > 0$, let

$$\mathcal{G}_{C,\mu} = \{(\mathbf{f}, \mathbf{h}) \in \mathcal{G} : \operatorname{diam} \mathbb{V}' \leq C, \ f_* \in \mathcal{QM}(\mu, C), \\ \operatorname{Dil}(H_*) \leq C, \ \operatorname{Dil}(\mathbf{h}) \leq C\}, \tag{2.9}$$

where H_* is the tubing of the fundamental annulus of f_*.

Similarly, let $\mathcal{G}^\epsilon$ stand for the class of almost proper quadratic-like families. Let $\mathcal{G}^\epsilon_{C,\mu} \subset \mathcal{G}^\epsilon$ be its subclass consisting of families $(\mathbf{f}, \mathbf{h})$, satisfying (2.9) and such that $\operatorname{mod}(\Lambda \setminus M^\epsilon_{\mathbf{f}}) \geq \mu$.

We will say that a quadratic-like family (taken from some collection under consideration) has a "bounded geometry" if it belongs to a certain class $\mathcal{G}_{C,\mu}$ or $\mathcal{G}^\epsilon_{C,\mu}$ (depending on whether we consider full or truncated families) with C and μ being uniform over the collection.

A statement that certain bound "depends only on the geometry" of a quadratic-like family means that this bound is uniform over any class $\mathcal{G}_{C,\mu}$ (resp. $\mathcal{G}^{\epsilon}_{C,\mu}$).

Theorem 2.4 provides a control of the shape of the Mandelbrot sets in quadratic-like famillies:

Lemma 2.8 ([L4], Lemma 3.2)**.** *Let us consider a quadratic-like family* $(\mathbf{f}, \mathbf{h})$ *over* $(\Lambda, *)$ *of class* $\mathcal{G}_{C,\mu}$. *Then the straightening* $\chi_{\mathbf{f}} : (\Lambda, M^0_{\mathbf{f}}) \to (\Delta, M^0)$ *is a* $K(C, \mu)$*-qc map onto an appropriate neighborhood* Δ *of the Mandelbrot set* M^0. *For a family of class* $\mathcal{G}^{\epsilon}_{C,\mu}$*, the similar statement holds for the straightening* $\chi_{\mathbf{f}} : (\Lambda, M^{\epsilon}_{\mathbf{f}}) \to (\Delta, M^{\epsilon})$.

We will briefly say that the Mandelbrot sets $M^0_{\mathbf{f}}$ (respectively, $M^{\epsilon}_{\mathbf{f}}$) have a $K(C, \mu)$*-standard shape.* If we do not want to specify dilatation K, we say that the sets have *quasi-standard shape.*

Lemma 2.9. *For any* $C, \mu > 0$*, the spaces* $\mathcal{G}_{C,\mu}$ *and* $\mathcal{G}^{\epsilon}_{C,\mu}$ *are compact.*

Proof. Let us consider a sequence of normalized quadratic-like families $(\mathbf{f}_n, \mathbf{h}_n) \in \mathcal{G}_{C,\mu}$ over $(\Lambda_n, *)$. The maps of these families will be naturally denoted as $f_{n,\lambda} : V_{n,\lambda} \to V'_{n,\lambda}$ and $h_{n,\lambda}$. We will consecutively select several subsequences of this sequence without using double indices.

By Lemma 2.1, we can select a convergent subsequence

$$f_{n,*} \to (f_* : V_* \to V'_*) \in \mathcal{QM}(\mu, C).$$

Since $\operatorname{mod}(\Lambda_n \smallsetminus M^0_{\mathbf{f}_n}) \geq \mu$ and $\operatorname{diam} M^0_{\mathbf{f}_n} = 1$, the family of pointed domains $(\Lambda_n, *) \equiv (\Lambda_{\mathbf{f}_n}, *)$ is Carathéodory precompact (see [McM1, Thm 5.2]). Select a converging subsequence: $(\Lambda_n, *) \to (\Lambda, *)$.

Let $\Omega \Subset \Lambda$. It is easy to see that the family of holomorphic functions $\lambda \mapsto h_{n,\lambda}(z)$, $z \in \bar{\mathbf{C}}$, is normal, and hence equicontinuous over Ω. On the other hand, since $\operatorname{Dil}(\mathbf{h}_n) \leq C$, the family of maps $h_{n,\lambda}$, $\lambda \in \Lambda$, is equicontinuous on $\bar{\mathbf{C}}$. Putting these two remarks together, we conclude that the family of maps $(\lambda, z) \mapsto h_{n,\lambda}(z)$ is equicontinuous on $\Omega \times \bar{\mathbf{C}}$. Hence, an appropriate subsequence of these maps converges, uniformly on compact subsets of $\Lambda \times \bar{\mathbf{C}}$, to a continuous map $\mathbf{h} : (\lambda, z) \mapsto h_\lambda(z)$.

Since uniform limits of holomorphic functions are holomorphic, the functions $\lambda \mapsto h_\lambda(z)$, $z \in \bar{\mathbf{C}}$, are holomorphic. Further, the graphs of these functions do not intersect, for otherwise the graphs of the corresponding approximating functions $\lambda \mapsto h_{\lambda,n}(z)$ would intersect at nearby points for all sufficiently big n. Hence $\mathbf{h}$ is a holomorphic motion. Moreover, $\operatorname{Dil}(\mathbf{h}) \leq C$, since uniform limits of C-qc maps are C-qc.

The limit holomorphic motion $\mathbf{h}$ provides us with topological bidisks $\mathbb{V}' \supset \mathbb{V}$ such that $(V'_\lambda, V_\lambda) = h_\lambda(V'_*, V_*)$. Clearly the bidisks $(\mathbb{V}'_n, \mathbb{V}_n)$

converge to $(\mathbb{V}', \mathbb{V})$ in the Hausdorff topology on compact subsets of $\Lambda \times \bar{\mathbf{C}}$. Hence, the maps $\mathbf{f}_n$ are eventually defined on any compact subset of $\Lambda \times \bar{\mathbf{C}}$. Since this sequence of maps is uniformly bounded, it admits a further subsequence converging, uniformly on compact subsets of $\mathbb{V}$, to some holomorphic map $\mathbf{f} : \mathbb{V} \to \mathbb{V}'$. It represents a quadratic-like family $\{f_\lambda\}$ over Λ.

Let us show that the family $(\mathbf{f}, \mathbf{h})$ is proper. Consider a compact subset $Q \subset V'_*$. Then there exists an annulus $A \Subset V'_* \smallsetminus Q$ separating Q from $\partial V'_*$. Let $\eta_n \equiv \eta_{\mathbf{f}_n}$ (see (2.7)). Then the annuli $\eta_n^{-1} A$ separate $\eta_n^{-1} Q$ from $\partial \Lambda_n$. Since the holomorphic motions $\mathbf{h}_n$ are C-qc, $\operatorname{mod} \eta_n^{-1} A \geq \nu \equiv C^{-1} \operatorname{mod} A$. It follows that $\eta^{-1} Q$ is also separated from $\partial \Lambda$ by an annulus of modulus ν. Hence $\eta^{-1} Q$ is compact.

The family $(\mathbf{f}, \mathbf{h})$ is unfolded by continuity of the winding number.

As the bounds of (2.9) are clearly carried to the limit, $(\mathbf{f}, \mathbf{h})$ is an equipped quadratic-like family of class $\mathcal{G}_{C,\mu}$.

The argument for the space $\mathcal{G}^\epsilon_{C,\mu}$ is similar. If we are given a sequence of quadratic-like families $(\mathbf{f}_n, \mathbf{h}_n)$ in the space $\mathcal{G}^\epsilon_{C,\mu}$, then in the same way we construct a limit quadratic-like family $(\mathbf{f}, \mathbf{h})$ over Λ. By continuous dependence of the straightening on the (equipped) quadratic-like map, $\chi_{\mathbf{f}_n}(\lambda) \to \chi_{\mathbf{f}}(\lambda)$ for any $\lambda \in \Lambda$. Since the straightenings $\chi_{\mathbf{f}_n}$ are uniformly qc (Lemma 2.8), this convergence is uniform. Hence $\operatorname{mod}(\Lambda \smallsetminus M^\epsilon_{\mathbf{f}}) \geq \mu$ and thus the family $\mathbf{f}$ is almost proper. All other required properties of $\mathbf{f}$ are obvious. ☐

2.2.3. *Vertical tubes.* Recall from §2.1.4 that $\mathcal{Z}(G) = \Pi^{-1}(G)$ is the vertical fiber though $G \in \mathcal{H}_0$. For a set $\mathcal{P} \subset \mathcal{Q}$, let $\mathcal{Z}_{\mathcal{P}}(G) = \mathcal{P} \cap \mathcal{Z}(G)$ stand for the vertical fibers in $\mathcal{P}$.

Let us say that $\mathcal{P}$ is a *vertical tube* over a Banach neighborhood $\mathcal{V} \subset \mathcal{H}_{0,V}$ if its vertical fibers are topological disks and it has a topological product structure over $\mathcal{V}$ (i.e., there exists a topological disk $W \subset \mathbb{C}$ such that $\mathcal{P}$ and $\mathcal{V} \times W$ are homeomorphic over $\mathcal{V}$). Let us say that a vertical tube $\mathcal{P}$ is *equipped* if

- There is a base map $G_* \in \mathcal{V}$ equipped with a tubing H_* (2.2);
- There is an equivariant holomorphic motion of the fundamental annulus A_f,

$$h_f : (\mathbb{C}, A_*) \to (\mathbb{C}, A_f), \ f \in \mathcal{P};$$

- The vertical fibers $\mathcal{Z}_{\mathcal{P}}(G)$ equipped with the above motion are proper unfolded quadratic-like families.

By §2.1.3, for any equipped tube $\mathcal{P}$, there is a well defined straightening

$$\chi_{\mathcal{P}} : \mathcal{P} \to \mathbb{C}. \tag{2.10}$$

Lemma 2.10. *Any* $G_* \in \mathcal{H}_0$ *belongs to an equipped vertical tube* $\mathcal{P}$ *over a Banach neighborhood* $\mathcal{V} \subset \mathcal{H}_0$*. The straightening* $\chi_{\mathcal{P}}$ *is a trivial fibration over some domain* $\Delta \supset M^0$ *whose fibers are holomorphic leaves* $\mathcal{L}_{\mathcal{P}}(f)$*,* $f \in \mathcal{P}$*, parametrized by* $\mathcal{V}$*. If* $\mathcal{B} \equiv \mathcal{B}_V(f, \delta)$ *is a Banach ball in* $\mathcal{P}$ *with sufficiently small* $V \supset K(f)$ *and* $\delta > 0$*, then* $\mathcal{B}$ *it foliated by codimension-one holomorphic submanifolds, the slices of the leaves* $\mathcal{L}_{\mathcal{P}}(g)$*.*

Proof. By [L5, Theorem 4.23], external fibers $\mathcal{Z}(G)$ are holomorphic curves in $\mathcal{Q}$. In order to equip them, we will make use of the conformal representations $\Psi_f : \Delta_G \to \Omega_f$, $f \in \mathcal{Z}(G)$ (see §2.1.5).

Consider a quadratic-like representative $G_* : W_* \to W'_*$ with the filled Julia set $K_* \equiv K(G_*)$. By [L5, Lemma 4.2], there is a Banach neighborhood $\mathcal{V} = \mathcal{H}_0 \cap \mathcal{B}_{W_*}(G_*, \varepsilon)$ that can be equipped with an equivariant holomorphic motion

$$h_G : \mathbb{C} \smallsetminus W_* \to \mathbb{C} \smallsetminus W_G, \quad G \in \mathcal{V},$$

(i.e., $G : W_G \to G(W_G) \equiv W'_G$ is a quadratic-like map and h_G conjugates $G_* : \partial W_* \to \partial W'_*$ to $G : \partial W_G \to \partial W'_G$). This motion admits a natural dynamical extension to an equivariant holomorphic motion

$$h_G : (\mathbb{C}, W'_*, W_*) \to (\mathbb{C}, W'_G, W_G)$$

which is conformal on K_*.

By [L5, Theorem 3.4], for any $G \in \mathcal{V}$ and $a \in \mathrm{cl}(W'_*) \smallsetminus K_*$, there exists a unique germ $f = m(G, a) \in \mathcal{Z}(G)$ such that $h_G(a) = \Psi_f^{-1}(f(0))$ ("mating" of G and a). Define $\mathcal{P}$ as the union of $\mathcal{C}_{\mathcal{P}} \equiv \Pi^{-1}\mathcal{V} \cap \mathcal{C}$ (the connectedness locus of the tube) and the set of all matings of $G \in \mathcal{V}$ with $a \in W'_*$.

For $G = \Pi(f)$, the conformal representation Ψ_f is well defined on $\mathbb{C} \smallsetminus W_G$. By Lemma 2.7, $\Psi_f(A_G)$, $G = \Pi(f)$, is a holomorphically moving fundamental annulus of $f \in \mathcal{P}$. Moreover, $\mathcal{Z}_{\mathcal{P}}(G) \smallsetminus \mathcal{C} \approx W'_* \smallsetminus K(G_*)$ is a topological annulus, and hence $\mathcal{Z}_{\mathcal{P}}(G)$ is a topological disk representing a full unfolded quadratic-like family. If $(f : U_f \to U'_f) \in \partial \mathcal{Z}_{\mathcal{P}}(G)$ then by definition, $a \in \partial W'_*$ and hence $f(0) \in \partial U'_f$. Thus, this family is proper.

The mating m gives a homeomorphism $\mathcal{P} \smallsetminus \mathcal{C} \to \mathcal{V} \times (W'_* \smallsetminus K_*)$ fibered over $\mathcal{V}$. Moreover, any map $f = m(G, a) \in \mathcal{P} \smallsetminus \mathcal{C}$ is equipped with a tubing $H_f = H_* \circ h_G^{-1} \circ \Psi_f^{-1}$, where H_* is a tubing of the fundamental annulus of the base map G_*. For this tubing, $H_f(f(0)) = H_*(a)$, which implies that the sets $\mathcal{L}_a = \{m(a, G) : \ G \in \mathcal{V}\}$ coincide with the fibers of the straightening $\chi_{\mathcal{P}}$.

On the other hand, we have the mating $m : \mathcal{V} \times M^0 \to \mathcal{C}_{\mathcal{P}}$ inverse to the map $f \mapsto (\Pi(f), \chi(f))$ on the connectedness locus (see

§2.1.4). Putting these ingredients together, we see that the map $f \mapsto (\Pi(f), \chi_{\mathcal{P}}(f))$ is a homeomorphism $\mathcal{P} \to \mathcal{V} \times \Delta$ fibered over $\mathcal{V}$, where $\Delta = \chi_{\mathcal{P}}(\mathcal{Z}_{\mathcal{P}}(G_*))$. Thus, $\mathcal{P}$ is a tube and $\chi_{\mathcal{P}}$ is a trivial fibration.

The property that its fibers (and their Banach slices) are holomorphic manifolds is a slight variation of Theorem 4.18 and Lemma 4.17 of [L5]. □

Denote the above foliation by $\mathcal{F}_{\mathcal{P}}$. It will be naturally called the horizontal foliation in $\mathcal{P}$.

For $f \in \mathcal{C}$, let E^h_f stand for the tangent space to the hybrid class $\mathcal{H}(f)$ at f (the *horizontal space*), and let E^v_f stand for the complementary *vertical line* tangent to the vertical fiber $\mathcal{Z}(f)$. For a tangent vector $u \in \mathrm{T}_f\mathcal{Q}$, let u^h and u^v denote its "horizontal" and "vertical" projections onto the spaces E^h_f and E^v_f respectively.

Consider a vertical tube $\mathcal{P}$. For $f \in \mathcal{C} \cap \mathcal{P}$, we have the horizontal-vertical decomposition

$$\mathrm{T}_f\mathcal{P} = E^h_{f,\mathcal{P}} \oplus E^v_f,$$

where $E^h_{f,\mathcal{P}} = E^h_f \cap \mathrm{T}\mathcal{P}$. These two distributions admit an extension to the whole tube $\mathcal{P}$ as the tangent distributions to the foliation $\mathcal{F}_{\mathcal{P}}$ and to the vertical foliation respectively. To simplify notations, we will often make the label "$\mathcal{P}$" implicit in the notation for the horizontal spaces in $\mathcal{P}$.

2.2.4. *Transverse overflowing.* Let $r > 0$. We say that a quadratic-like family $\mathbf{f}$ over Λ *transversally r-overflows* the connectedness locus $\mathcal{C}$ (or the truncated connectedness locus $\mathcal{C}^\epsilon$) if there exists a finite collection of vertical tubes $\mathcal{P}_i$ and a domain $\Omega \Subset \Lambda$ containing $M_{\mathbf{f}}$ (resp. $M^\epsilon_{\mathbf{f}}$) such that

(i) $f_\lambda \in \cup\mathcal{P}_i$ for $\lambda \in \bar{\Omega}$;
(ii) If $f_\lambda \in \mathcal{P}_i$, $\lambda \in \partial\Omega$, then $\mathrm{dist}(\chi_{\mathcal{P}_i}(f_\lambda), M) \geq r$ (resp. $\mathrm{dist}(\chi_{\mathcal{P}_i}(f_\lambda), M^\epsilon) \geq r$);
(iii) The curve $\lambda \mapsto f_\lambda$, $\lambda \in \Omega$, is transverse to the leaves of $\mathcal{P}_i$.

Lemma 2.11. *Given $C > 0$ and $\mu > 0$, there is an $r > 0$ such that any quadratic-like family $\mathbf{f} \in \mathcal{G}_{C,\mu}$ (resp. $\mathbf{f} \in \mathcal{G}^\epsilon_{C,\mu}$) transversally r-overflows $\mathcal{C}$ (resp. $\mathcal{C}^\epsilon$).*

Proof. The arguments in the full and truncated cases are the same, and we will restrict ourselves to the former case.

There exists a $\rho > 0$ such that any curve $\mathbf{f} \in \mathcal{G}_{C,\mu}$ belongs to the compact set $\mathcal{Q}(\mu, \rho)$. Since $\Pi(\mathcal{Q}(\mu, \rho)) \subset \mathcal{H}_0$ is also compact, it can be covered with finitely many Banach neighborhoods $\mathcal{U}_i \Subset \mathcal{H}_0$ satisfying the property of Lemma 2.10, i.e., such that there exist equipped vertical tubes $\mathcal{P}_i$ based on the $\mathcal{U}_i$. Then $\mathbf{f} \in \cup \mathcal{P}_i$ for any $\mathbf{f} \in \mathcal{G}_{C,\mu}$.

Let

$$r(\lambda) = \min_{i:\, f_\lambda \in \mathcal{P}_i} \operatorname{dist}(\chi_{\mathcal{P}_i}(f_\lambda),\, M), \quad \lambda \in \Lambda.$$

Compactness of the class $\mathcal{G}_{C,\mu}$ (Lemma 2.9) implies that $r(\lambda) > r(C,\mu)$ for all $\lambda \in \Lambda$ sufficiently close to $\partial\Lambda$. Hence for $r < r(C,\mu)$, the domains $\Omega_r = \{\lambda \in \Lambda : r(\lambda) < r\}$ satisfy property (ii)

Let us show that (iii) is also satisfied for r sufficiently close to 1. Otherwise, by compactness of $\mathcal{G}_{C,\mu}$, some curve $\mathbf{f} \in \mathcal{G}_{C,\mu}$ would be tangent to some leaf $\mathcal{L} \subset \mathcal{C}$ of $\mathcal{F}_{\mathcal{P}}$. Let us apply to it (or rather, to its Banach slice $\mathcal{L}_V$) the results of Appendix 1. By the Hurwitz Theorem, $\mathbf{f}$ would have the same number of intersection points (counted with multiplicity) with the slices of all nearby leaves of $\mathcal{F}_{\mathcal{P}}$. But by the Intersection Lemma, the intersection points with the slices of nearby leaves in $\operatorname{int} \mathcal{C}$ are simple, so that there would be more than one such a point. On the other hand, any unfolded family $\mathbf{f}$ has a single point of intersection with any hybrid class – contradiction. □

2.2.5. *Uniform transversality and Montel metric on transverse curves.* We will now reformulate the above result in terms of Banach slices. Let us cover $\mathcal{Q}(\mu,\rho)$ with finitely many Banach balls $\mathcal{B}_j$ each of which is contained in some tube $\mathcal{P}_k$ and foliated by the slices of the leaves $\mathcal{L}_{\mathcal{P}_k}$ (see Lemma 2.10). Denote the corresponding foliations by $\mathcal{F}_j$ (if the ball is contained in several tubes, make an arbitrary choice). Let $\mathcal{L}_j(f)$ denote the leaf of this foliation through f.

For $f \in \mathcal{B}_j$, the tangent space $\mathrm{T}_f \mathcal{B}_j$ splits into the direct sum of the horizontal and vertical subspaces, $E^h_{f,j} = \mathrm{T}_f \mathcal{L}_j(f)$ and E^v_f. A *Banach bidisk* T centered at f is the product of a horizontal and a vertical Banach balls, $T^h \subset E^h_{f,j}$ and $T^v \subset E^v_f$.

Consider a collection $\mathcal{W}$ of holomorphic curves Γ in $\mathcal{Q}$ and a subset Y of their union. We say that the curves $\Gamma \in \mathcal{W}$ are *uniformly transverse to* $\mathcal{F}$ *over* Y (or, that the collection $\mathcal{W}$ is *normal* over Y) if there exists a $\kappa \in (0, 1/2)$, a finite collection of vertical tubes $\mathcal{P}_k$ covering Y, and a finite collection of Banach bidisks $T_i \subset \cup \mathcal{P}_k$ centered at some points $f_i \in Y$, each of which contains a family $\mathcal{W}_i$ of graphs of holomorphic functions $T^v_i \to T^h_i$, satisfying the following properties:

- Bidisks T'_i obtained from T_i by two-fold shrinking in the vertical direction, cover Y.
- If $f \in T'_i \subset \mathcal{B}_j$ then the slice of the leaf $\mathcal{L}_j(f)$ by T_i is a graph over T^h_i with horizontal slope bounded by some $\kappa/2$.
- The curves $\gamma \in \mathcal{W}_i$ have vertical slopes bounded by $1/\kappa$.

• Any curve $\gamma \in \mathcal{W}_i$ is a slice of some curve $\Gamma \in \mathcal{W}$ by T_i.

• For any curve $\Gamma \in \mathcal{W}$, the intersection $\Gamma \cap Y$ is covered by the union of some curves $\gamma_i' \in \mathcal{W}_i'$, where $\mathcal{W}_i'$ stands for the family of slices of curves $\gamma_i \in \mathcal{W}_i$ by T_i'.

If we do not specify Y, then we let

$$Y = \cup_{\Gamma \in \mathcal{W}} \Gamma \cap \mathcal{C} \quad \text{or} \quad Y = \cup_{\Gamma \in \mathcal{W}} \Gamma \cap \mathcal{C}^\epsilon$$

in the above definition, depending on whether we consider full or truncated families.

Lemmas 2.11 and 2.9 imply:

Corollary 2.12. *Given $C > 0$ and $\mu > 0$, the holomorphic curves $\mathbf{f} \in \mathcal{G}_{C,\mu}$ (resp. $\mathbf{f} \in \mathcal{G}^\epsilon_{C,\mu}$) are uniformly transverse to the foliation $\mathcal{F}$.*

Hence for any $\mathbf{f} \in \mathcal{G}_{C,\mu}$, the projections of $\mathbf{f} \cap T_i$ onto the vertical disks T_i^v are uniformly Lipschitz. On the other hand, by [L5, Lemma 4.10], the norms on the vertical disks induced from different Banach slices are comparable (with constants depending on the slices but independent of the particular choice of the disk). Hence the metrics on different slices of $\mathbf{f}$ by bidisks T_i are comparable as well. Gluing these metrics together by means of some partition of unity we obtain a "Montel metric" on $\mathbf{f}$ (in a neighborhood of $M_{\mathbf{f}}$) whose Lipschitz class is independent of the particular choices made. In this sense we have the *Lipschitz structure* on curves $\mathbf{f} \in \mathcal{G}_{C,\mu}$ near the connectedness locus. Moreover, the Koebe Distortion Theorem implies that the hyperbolic metric on $\mathbf{f}$ induced from the disk $\Lambda_{\mathbf{f}}$ has a bounded distortion (independent of the particular curve $\mathbf{f} \in \mathcal{G}_{C,\mu}$) with respect to the Montel metric.

Similarly, the curves of class $\mathcal{G}^\epsilon_{C,\mu}$ possess a Montel metric near the truncated connectedness locus.

2.2.6. *Perturbations of the quadratic family.* Let us show that if $\text{mod}(\mathbf{f})$ is big then the family $\mathbf{f}$ is close to the quadratic family $\mathcal{QP}$:

Lemma 2.13. *For any positive ε, L, and r, there is a μ and a Banach space $\mathcal{B}_W$ containing disk $\mathcal{QP}_r = \{P_c : \ c \in \mathbb{D}_r\}$ in the quadratic family $\mathcal{QP} \approx \mathbb{C}$ with the following property. If $(\mathbf{f}, \mathbf{h})$ is a proper unfolded quadratic-like family over Λ with $\text{mod}(\mathbf{f}) > \mu$ and $\text{Dil}(\mathbf{h}) \le L$, then there is topological disk $\Delta \subset \Lambda$ such that the family $\{f_\lambda\}_{\lambda \in \Delta}$ belongs to $\mathcal{B}_W$ and is represented in that space as a graph of a holomorphic function $\phi : \mathcal{QP}_r \to E$ with $\|\phi\| < \varepsilon$ (where E is a complement of $\mathcal{QP}$ in $\mathcal{B}_W$).*

Proof. In what follows, μ is assumed to be greater than 1. Let $\nu \in (0, \mu/2)$. By restricting dynamical and parameter domains, the family

$(\mathbf{f} : \mathbb{V} \to \mathbb{V}', \mathbf{h})$ can be adjusted to a family $(\mathbf{f} : \tilde{\mathbb{V}} \to \tilde{\mathbb{V}}', \tilde{\mathbf{h}})$ over $\tilde{\Lambda}$ which belongs to class $\mathcal{G}_{C,\nu}$ with some $C = C(\nu, L)$ such that

$$\operatorname{mod}(\tilde{V}'_* \smallsetminus \tilde{V}_*) \in [\nu/2, \nu], \quad \operatorname{Dil}(\tilde{\mathbf{h}}) = \operatorname{Dil}(\mathbf{h}) \le L.$$

If ν is fixed and μ is sufficiently big (depending on ν), then there is a symmetric (with respect to 0) topological disk $W \subset \mathbb{C}$ such $f_\lambda \in \mathcal{B}_W$ and $K(f_\lambda) \subset W$, $\lambda \in \tilde{\Lambda}$.

Select a complement E to $\mathcal{QP}$ in $\mathcal{B}_W$, and let $p : \mathcal{B}_W \to \mathcal{QP}$ be the projection of $\mathcal{B}_W$ onto $\mathcal{QP}$ parallel to E. By Lemma 2.1,

$$\|P_{c(\lambda)} - f_\lambda\|_V = \varepsilon < \varepsilon(\mu, \nu), \tag{2.11}$$

where $P_{c(\lambda)} = p(f_\lambda)$ and $\varepsilon(\mu, \nu) \to 0$ as $\mu \to \infty$, ν being fixed.

Consider a curve $\Gamma = \{P_{c(\lambda)}(0)\}_{\lambda \in \partial\tilde{\Lambda}}$ in $\mathbb{C}$. Let us show that if $\nu > \nu(r)$ and $\mu > \mu(\nu)$ then Γ encloses the disk $\mathbb{D}_r$. Indeed, since $\tilde{\mathbf{f}}$ is proper, $f_\lambda(0) \in \partial\tilde{V}'_\lambda$ for $\lambda \in \partial\tilde{\Lambda}$. Hence the critical value $f_\lambda(0)$ is separated from the Julia set $J(f_\lambda)$ by a fundamental annulus of modulus at least $\nu/(2L)$, $\lambda \in \partial\tilde{\Lambda}$. By (2.11), the critical value $c(\lambda)$ is separated from the Julia set $J(P_{c(\lambda)})$ by a fundamental annulus of modulus at least $\nu/(2L) - \delta(\varepsilon)$ where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. It follows that for $\lambda \in \partial\tilde{\Lambda}$ we have: $|c(\lambda)| \ge r$, provided $\nu > \nu(r)$ and $\mu > \mu(\nu)$.

Since $\mathbf{f}$ is unfolded, it has the winding number 1 about the origin. Hence the winding number of Γ about the disk $\mathbb{D}_r$ is equal to 1. Hence, the projection of $\mathbf{f}$ onto $\mathcal{QP}$ univalently covers disk $\mathcal{QP}_r$. Together with (2.11) this yields the assertion (with $\Delta \subset \tilde{\Lambda}$ being the pullback of $\mathcal{QP}_r$ under the projection). ☐

2.3. **Puzzle, parapuzzle and renormalization.**

2.3.1. *Complex renormalization.* The notion of *complex renormalization* was introduced by Douady and Hubbard [DH2, D1] in order to explain computer observable little Mandelbrot copies inside the Mandelbrot set (see [L3, M, McM1] for an extensive discussion of this notion).

Let f be a quadratic-like map. Assume that we can find topological disks $U' \Subset U$ around 0 and an integer p such that $g = f^p : U \to U'$ is a quadratic -like map with connected Julia set. Assume also that the "little Julia sets" $f^k J(g)$, $k = 0, \dots, p-1$, are pairwise disjoint except, perhaps, touching at their non-dividing β-fixed points. Then the map f is called renormalizable (with period p) and the map g is called its *pre-renormalization.* The quadratic-like germ of g considered up to rescaling is called a *renormalization* Rf of f.

Take a quadratic-like representative $f : V \to V'$. If the pre-renormalization $g : U \to U'$ above is selected in such a way that $f^k U \Subset V$, $k =$

$0, 1 \dots, p-1$, then we say that g is *subordinate* to V. The germ of subordinate pre-renormalizations considered up to rescaling will be called a *subordinate renormalization* Rf_V of f. Moreover, $\operatorname{mod}(Rf_V) = \sup \operatorname{mod} g$, where g runs over all subordinate pre-renormalizations of $f : V \to V'$.

The map f can be renormalizable with different periods, finitely or infinitely many. Accordingly it is called *at most finitely* or *infinitely renormalizable.*

Lemma 2.14. *No quadratic polynomial P_c can be realized as the renormalization Rf of a quadratic-like map.*

Proof. Indeed, the renormalization Rf admits the analytic continuation to the domain of f^p as a branched covering of degree $2^p > 2$. It is certainly not compatible with the quadratic extension to the whole complex plane. □

We will now describe a canonical way to produce the *first* renormalization Rf of f, with the smallest period.

2.3.2. *Principal nest of the Yoccoz puzzle.* The reader can consult [L3], §3, for a detailed discussion of the combinatorics of the Yoccoz puzzle. That discussion is based on the notion of *principal nest* of puzzle pieces $V^0 \supset V^1 \supset \dots$. The first puzzle piece V^0 is the domain bounded by two external rays landing at the dividing fixed point α, two rays landing at the symmetric point α', and two arcs of some equipotential. Then V^{n+1} is inductively defined as the pull-back of V^n corresponding to the first return of the critical point back to V^n. The corresponding return map $g_n : V^n \to V^{n-1}$ is a branched double covering. The return to level $n-1$ is called *central* if $g_n(0) \in V^n$. Let n_k count the non-central levels. If this sequence is infinite then the map f is non-renormalizable. Otherwise the principal nest ends up with an infinite central cascade $V^{n-1} \supset V^n \supset \dots$, and the map $g_n : V^n \to V^{n-1}$ (after perhaps little thickening of the domain and the range, in the doubling case) is a quadratic-like map with connected Julia set. The germ of this map (up to rescaling) is called the first renormalization Rf of f.

The number of the non-central levels in the principal nest is called the *height* of f.

The map $g_n : V^n \to V^{n-1}$ is a restriction of the full first return map $g_n : \cup V_i^n \to V^{n-1}$ (denoted by the same letter). Here $V_i^n \subset V^{n-1}$ are puzzle pieces with disjoint interiors, $V_0^n \equiv V^n$, and the restrictions $g_n : V_i^n \to V^{n-1}$ are univalent for $i \neq 0$. If f is renormalizable, then only finitely many puzzle pieces V_i^n meet the ω-limit set $\omega(0)$ of the critical point. Restriction of g_n to the union of these puzzle pieces is called the *generalized renormalization* of f on V^{n-1}. It will be denoted by g_n as well.

2.3.3. *Parapuzzle and Mandelbrot copies.* Let us consider the quadratic family $P_c : z \mapsto z^2 + c$. For any parameter value $c_0 \in M^0$ outside the main cardioid, there is a nest of *parapuzzle pieces*

$$\Delta^1(c_0) \supset \Delta^2(c_0) \supset \cdots \ni c_0$$

corresponding to the dynamical principal nest. For parameter values $c \in \Delta^n(c_0)$, the combinatorics of the first return maps to the puzzle piece V^{n-1} stay the same (see [L4] for the precise definition which, however, does not matter for the following discussion).

If P_c is non-renormalizable then the parapuzzle pieces $\Delta^n(c)$ shrink to c (Yoccoz, see [H] or [L4]). Otherwise the return maps

$$g_{n,c} = P_c^p : V^n \to V^{n-1}, \quad c \in \Delta^n,$$

on some level (called a "renormalization level") form a quadratic-like family $\mathbf{g}$ naturally equipped with a holomorphic motion $\mathbf{j}$.

In the primitive case (when the little Mandelbrot set $\chi(M^0_{\mathbf{g}})$ is not attached to the main cardioid), $\mathbf{g}$ is a proper unfolded family. In the satellite case, $\mathbf{g}$ is unfolded and almost proper, which means that the straightening χ homeomorphically maps $M^0_{\mathbf{g}}$ onto "unrooted" Mandelbrot set $M^0 \setminus \{1/4\}$ (see [D1]).

We allow to restrict the parameter domain of the family $(\mathbf{g}, \mathbf{j})$ keeping these properties. The quadratic-like family $(\mathbf{g}, \mathbf{j})$ up to such restrictions will be called the renormalization of $(\mathbf{f}, \mathbf{h})$. By saying that the renormalization belongs to some class $\mathcal{G}_{L,\nu}$ or $\mathcal{G}^{\epsilon}_{L,\nu}$ we mean that there is restricted family $(\mathbf{g}, \mathbf{j})$ in this class.

The little copies $M = \chi(M^0_{\mathbf{g}})$ produced by the renormalized families are *maximal* in the sense that they are not contained in any other copy except for the whole set M^0. Each maximal copy encodes the combinatorial data of the renormalization: all maps P_c with $c \in M$ are "renormalizable with the same combinatorics". The period of this renormalization is certainly constant throughout the copy, $p = p(M)$.

A Mandelbrot copy M is called *real* if it is centered on the real line. The real slice $J = M \cap \mathbb{R} \subset (-2, 1/4)$ of a real Mandelbrot copy is an interval called the *renormalization window.* Denote by $\mathcal{M}$ the family of maximal real Mandelbrot copies. The set of maximal renormalization windows (formally coinciding with $\mathcal{M}$) will be denoted by $\mathcal{J}$.

For any $M \in \mathcal{M}$, there is a canonical *stretching* $\sigma : M \to M^0$ defined as the composition of the renormalization and the straightening, $\sigma = \chi \circ R$ ([DH2, M]). (Note that though R is not defined at the root of the doubling Mandelbrot copy, the stretching σ admits a continuous extension to it.) If a map $f \in \mathcal{Q}$ is renormalizable a few times, then its combinatorics is encoded by a sequence (finite or infinite) $\tau(f) = \{M_0, M_1, \dots\}$ such that $\sigma^n f \in M_n$, $n = 0, 1 \dots$. One says that an infinitely renormalizable map f has a *bounded type* if the periods $p(M_n)$ are bounded.

2.3.4. *Renormalization strips.* For the doubling Mandelbrot set $M \in \mathcal{M}$, let $\hat{M} = M \setminus \{-3/4\}$ be the corresponding unrooted set; for all others $M \in \mathcal{M}$, let $\hat{M} = M$. Let $\mathcal{T}_M \subset \mathcal{Q}$ stand for the set of quadratic-like germs which are hybrid equivalent to the quadratic maps P_c with $c \in \hat{M}$ (that is, $\mathcal{T}_M$ is the union of the hybrid classes passing through $\hat{M}$). We call it a *renormalization strip.* We say that the maps $f \in \mathcal{T}_M$ are renormalizable with *real* combinatorics encoded by the little Mandelbrot set M. Thus the renormalization operator R is canonically defined on the union of all renormalization strips. We let $R_M \equiv R|\mathcal{T}_M$.

Recall that $\mathcal{Q}_{\mathbb{R}}$ denotes the space of real quadratic-like germs. The real slice of the renormalization strip $\mathcal{T}_M$ will be denoted as $\mathcal{T}_J$, where $J = M \cap \mathbb{R}$ is the corresponding renormalization window.

2.3.5. *Injectivity.*

Lemma 2.15 (de Melo & van Strien [MS]). *The renormalization operator*

$$R : \cup_{J \in \mathcal{J}} \mathcal{T}_J \to \mathcal{Q}_{\mathbb{R}}$$

is injective.

2.3.6. *Contracting property.* Moreover, the renormalization is non-expanding with respect the Teichmüller-Sullivan metric on the hybrid classes:

$$\mathrm{dist}_T(Rf, Rg) \leq \mathrm{dist}_T(f, g). \tag{2.12}$$

This immediately follows from the fact that a hybrid conjugacy h between f and g provides a hybrid conjugacy between the renormalizations Rf and Rg. This observation was a starting point for Sullivan's renormalization theory [S1].

2.3.7. *Analytic extension.* Any R_M admits a complex analytic extension to Banach neighborhoods of maps $f \in \mathcal{T}_M$. Namely, if $R_M f_V = f^p : U \to U'$ is a subordinated quadratic-like pre-renormalization of $f \in \mathcal{C}_V$, then any nearby map $g \in \mathcal{Q}_V$ admits a quadratic-like return map $g^p : U_g \to U'$ with the same range. Since g^p analytically depends on g, this provides us with the desired extension (see [L5, §5.3] for a more detailed discussion).

Let us say that a map f is *non-escaping* under the renormalization of type $\tau = \{M_0, M_1, \dots\}$ if all the maps

$$f_n = R_{M_n} \circ \cdots \circ R_{M_0} f \qquad (2.13)$$

are well-defined (where the R_{M_k} stand for the analytic extensions of the renormalizations) and $\mathrm{mod}(f_n) \geq \varepsilon > 0$, $n = 0, 1, \dots$.

Lemma 2.16 ([L5], Lemma 5.7)**.** *If a quadratic-like map f is non-escaping under the renormalization of type $\tau = \{M_0, M_1, \dots\}$ then it is infinitely renormalizable with type τ.*

Any equipped quadratic-like family $(\mathbf{f} = \{f_\lambda\}_{\lambda\in\Lambda}, \mathbf{h}) \in \mathcal{G}$ can be tiled into the parapuzzle pieces, the pullbacks of the parapuzzle pieces in $\mathcal{QP}$ under the straightening $\chi_{\mathbf{f}}$ (see [L4]). They depend on the equipment but canonical on the Mandelbrot set. Let us take a renormalizable map $f_0 \equiv f_{\lambda_0} \in \mathbf{f}$ of type M. Then as in the quadratic case, the renormalization $g_0 \equiv R_M f_0$ is included into a full or almost full unfolded quadratic-like family $\mathbf{g} \equiv R_M \mathbf{f} = \{g_\lambda : V_\lambda \to V'_\lambda\}$ over an appropriate parapuzzle piece $\Delta \ni \lambda_0$. This provides us with the analytic continuation of R_M to Δ. Moreover, this quadratic-like family is equipped with a holomorphic motion $\mathbf{j}$, so that we can write that $(\mathbf{g}, \mathbf{j}) = R_M(\mathbf{f}, \mathbf{h})$.

In particular, by Lemma 2.10 this discussion is applied to the vertical fibers. Moreover, if we consider an equipped tube $\mathcal{P}$, then the parapuzzle pieces Δ^n_G in the vertical fibers $\mathcal{Z}_{\mathcal{P}}(G)$ will holomorphically move with $G \in \mathcal{V}$. This motion is obtained by the holonomy along the foliation $\mathcal{F}_{\mathcal{P}}$. Indeed, the parapuzzle pieces are specified by the external coordinates of the critical value, which determine the leaves of the foliation $\mathcal{F}_{\mathcal{P}}$ (see the discussion in §2.2.3). Thus, we obtain an equipped tube $\cup_{G\in\mathcal{V}} \Delta_G$ of the parapuzzle pieces to which the renormalization R_M analytically extends along the vertical fibers.

2.3.8. *The iterate R^N.* In what follows, it will be sometimes handy to work with an iterate R^N of the renormalization operator instead of R itself. Let us give a brief description of the structure of this operator which is quite similar to the structure of R. Let us say that a little Mandelbrot copy M has rank N if it is included in the nest of distinct Mandelbrot copies of length $N + 1$,

$$M \equiv M_0 \subset M_1 \subset \cdots \subset M_N \equiv M^0,$$

and cannot be included in a longer nest like this. (In particular, maximal Mandelbrot copies have rank 1.) Let $\mathcal{M}^N$ stand for the family of real Mandelbrot copies of rank N.

If M is obtained by the doubling bifurcation from some hyperbolic component, then let $\hat{M}$ stand for the unrooted M. Otherwise let $\hat{M} = M$. Then $\cup_{M\in\mathcal{M}^N}\hat{M}$ is exactly the set of N times renormalizable quadratics.

The renormalization strip $\mathcal{T}_M$ is the union of the hybrid classes through $c \in \hat{M}$. The domain of the renormalization operator R^N is the union of the renormalization strips $\mathcal{T}_M$, $M \in \mathcal{M}^N$. For $f \in \mathcal{T}_M$, $M \in \mathcal{M}^N$, the renormalization $R^N f$ is equal to a restricted iterate f^p considered up to rescaling. Here $p = p_N(f) \equiv p_N(M) \equiv p(M)$ is naturally called the period of M (or the renormalization period of f under R^N).

2.4. **Essentially bounded combinatorics.**

2.4.1. *Central cascades.* There is a special type of combinatorics related to the parabolic bifurcation which usually requires a special treatment. In this section we will describe this phenomenon.

Let f be a renormalizable map of period p with real combinatorics $M \in \mathcal{M}$. Let us consider a *central cascade*

$$V^m \supset V^{m+1} \supset \cdots \supset V^{m+N} \tag{2.14}$$

meaning that the levels $m-1$ and $m+N-1$ are non-central, while all the levels $m, m+1, \dots, m+N-2$, are central: $g_{m+1}0 \in V^{m+N-1} \smallsetminus V^{m+N}$. Then the quadratic-like map $g_{m+1} : V^{m+1} \to V^m$ is combinatorially close to either the Ulam-Neumann map $z \mapsto z^2 - 2$, or to the parabolic map $z \mapsto z^2 - 1/4$ (depending on whether $g_{m+1}(V^{m+1}) \ni 0$ or otherwise, see [L3], §§8, 12). In the former case the cascade is called *Ulam-Neumann*, while in the latter it is called *saddle-node.*

Remark. If $N = 1$ then the "cascade" (2.14) does not have central levels at all. Still, we will consider it as a (degenerate) cascade of length 1.

2.4.2. *Essentially bounded combinatorics.* Recall that

$$g_{m+l} : \cup V_i^{m+l} \to V^{m+l-1}$$

is a generalized renormalization of f on V^{m+l-1}, i.e., the first return map to V^{m+l-1} restricted to (finitely many) puzzle pieces V_i^{m+l} meeting the postcritical set $\omega(0)$. To make the notations simpler, let us temporarily suppress m from all labels so that $V^{m+l} \equiv V^l$, $g_{m+l} \equiv g_l$, and $g_1 \equiv g$. Let $A^l = V^{l-1} \smallsetminus V^l$, $l = 1, \dots, N$.

Let us say that the combinatorics of the cascade is *essentially bounded* by p if

- The g-orbit of any point $x \in \omega(0) \cap V^N$ passes through the top annulus A^1 at most p times before it returns back to V^N;

• If $x \in \omega(0) \cap A^1$ then $gx \notin A^l$ with $p \le l \le N-p$.

The levels $l = p, p+1, \dots, N-p$ are "neglectable" for the cascade like that: there is a simple transit machinery acting on these levels. All other levels are "essential".

Note that the length N is not incorporated in the essential bound on the combinatorics of the cascade. This will allow us to consider geometric limits of the cascades as $N \to \infty$.

2.4.3. *Markov machinery.* Let us describe the Markov machinery associated with a saddle-node cascade (2.14) whose combinatorics is essentially bounded by p (compare [L3, §3.6]). First, let $U_i^1 \equiv V_i^1 \subset A^1$. Then define puzzle pieces U_i^l, $l = 2, \dots, N+1$, inductively in l as components of $(g|V^l)^{-1}(U_j^{l-1})$ which meet $\omega(0)$. Note that $U_i^l \subset A^l$ for $l = 1, \dots, N$, while $U_i^{N+1} \subset V^N$. Among the latter puzzle pieces there is one, U_0^{N+1}, which contains the critical point 0. It is called *critical.* (Off-critical puzzle pieces U_i^l are labeled by $i > 0$ in an arbitrary way.)

Let $d(l) = \min(l, N-l)$ stand for the "depth" of level $l \in [1, N]$ in the cascade. Consider the transit map

$$h = g^{N-2p} : \bigcup_i U_i^{N-p} \to \bigcup_j U_j^p \tag{2.15}$$

and the Markov map

$$\phi : \bigcup_{d(l) \le p} \bigcup_i U_i^l \to V^0 \tag{2.16}$$

defined as follows: $\phi|U_i^l = g$ if $l \ne N-p$ and $\phi|U_i^{N-p} = h$. It univalently maps every puzzle piece U_i^1 onto V^0, and every off-critical puzzle piece U_i^l, $l > 1$, onto some puzzle piece $U_{j(i)}^{k(l)}$ with $k(l) < l$. The critical piece U_0^{N+1} is mapped onto its image $U_{j(0)}^N$ as a double covering branched at 0.

Let us define the *renormalization* of a saddle-node cascade (2.14) as the generalized renormalization of g on the critical puzzle piece $P \equiv U_0^{N+1}$, i.e., the first return map $\cup V_i' \to P$ restricted to the puzzle pieces V_i' meeting $\omega(0)$.

2.4.4. *Modified nest.* Let us modify the principal nest as follows. Once we see a saddle-node cascade (2.14) of length ≥ 100, we renormalize it as just described. Then we continue the nest in the usual way (as the nest of consecutive first return maps) starting with this renormalization. If in this process we observe another saddle-node cascade of length ≥ 100, we again modify the nest by renormalizing the cascade, and so on.

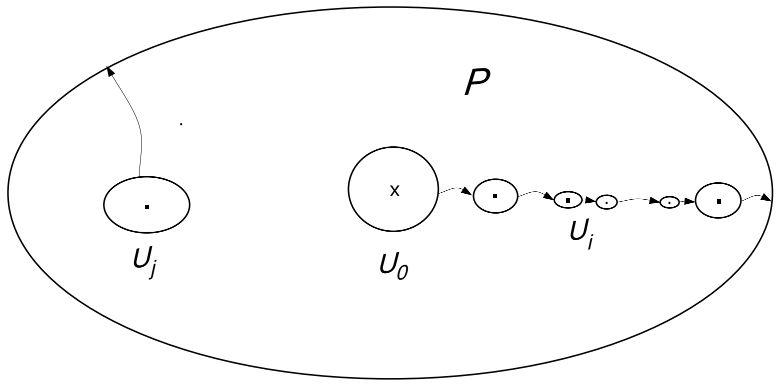

FIGURE 4. A map of class $\mathcal{L}$

The modified nest is naturally decomposed into χ central cascades (some of which may be degenerate), where $\chi = \chi(f)$ is called the "height" of f.

Let us say that the combinatorics of f is *essentially bounded by* p if

- $\chi \leq p$;
- the length of all Ulam-Neumann cascades is bounded by p;
- all cascades have essentially p-bounded combinatorics.

The best essential bound on the combinatorics will be called the *essential period* $p_e(f)$ of f.

Let $\mathcal{Q}_p$ stand for the space of quadratic-like maps with real combinatorics essentially bounded by p.

2.4.5. *Class* $\mathcal{L}$. Let us consider the space $\mathcal{L}_s$ of Markov maps $\phi : \cup_{0\leq i \leq s} U_i \to P$ (up to rescaling) defined on the disjoint union of $s+1$ topological disks U_i compactly contained in a topological disk $P \ni 0$. The disk U_0 is assumed to be symmetric with respect to 0 and is called *critical*. The map $\phi|U_0$ is a double covering onto a disk $U_{j(0)}$ branched at 0. Moreover, we assume that this map is even: $\phi(z) = \phi(-z)$ for $z \in U_0$. Each off-critical disk U_i is univalently mapped onto either P or another disk $U_{j(i)}$.

Moreover, we assume that this Markov map does not have cycles, i.e., there are no i and m such that $\phi^m U_i = U_i$. It follows that any U_i, $i \neq 0$, is eventually mapped onto P under some iterate $\phi^{q(i)}$. The corresponding map

$$G : \cup U_i \to P, \quad G|U_i = \phi^{q(i)} \tag{2.17}$$

will be called the *Bernoulli map* associated with ϕ.

Let $\mathcal{L} = \cup \mathcal{L}_s$.

Let us mark the point $x_i = (\phi|U_i)^{-q(i)}(0)$ in every off-critical disk U_i. In the critical disk U_0 mark the origin $x_0 = 0$. Impose a Carathéodory topology on the spaces $\mathcal{L}_s$. In this topology, a sequence of maps $\phi_n : \cup U_i^n \to P^n$ converges to a map $\phi : \cup U_i \to P$ if $(U_i^n, x_i^n) \to (U_i, x_i)$, $(P^n, 0) \to (P, 0)$ in the sense of Carathéodory, and $\phi_n \to \phi$ uniformly on compact subsets of $\cup U_i$ (where x_i^n stand for the marked points in U_i^n).

Let us say that a map ϕ of class $\mathcal{L}$ has a K-*bounded geometry* if all the distances $\operatorname{dist}(\partial U_i, x_i)$ and $\operatorname{dist}(U_i, U_j)$, $i \neq j$, are at least $K^{-1} \operatorname{diam} P$. Let $\mathcal{L}_s(K)$ denote the space of maps of class $\mathcal{L}_s$ with K-bounded geometry.

Lemma 2.17. *The space $\mathcal{L}_s(K)$ is compact.*

Proof. Consider a sequence of maps $\phi_n : \cup U_i^n \to P^n$ in $\mathcal{L}_s(K)$ normalized so that $\operatorname{diam} P^n = 1$. Since the space of pointed disks (W, x) with

$$C^{-1} \leq \operatorname{dist}(\partial W, x) \leq \operatorname{diam} W \leq C$$

is Carathéodory compact, we can select convergent subsequences

$$(P^n, 0) \to (P, 0), \quad (U_i^n, x_i^n) \to (U_i, x_i).$$

The limit configuration $\{(P, 0), (U_i, x_i)\}$ has obviously a K-bounded geometry as well. By normality argument, we can select a further subsequence of the maps ϕ_n uniformly convergent on compact subsets of $\cup U_i$ to some Markov map ϕ. □

2.4.6. *Markov nests.* Let us define a *Markov nest* Φ *of height* χ as a sequence of maps $\phi_n : \cup U_i^n \to P^n$, $n = 0, 1, \ldots, \chi$, of class $\mathcal{L}$ such that

(i) $\phi_0 : U^0 \to P^0$ and $\phi_\chi : U^\chi \to P^\chi$ are quadratic-like maps with connected Julia set;

(ii) $P^1 \equiv V^0$ is the first puzzle piece of the principal nest of ϕ_0;

(iii) $U_0^n = P^{n+1}$;

(iv) If $\phi_{n+1}(U_i^{n+1}) = P^{n+1}$ then $\phi_{n+1}|U_i^{n+1}$ is the first return map of the U_i^{n+1} to P^{n+1} under iterates of ϕ_n.

(A single quadratic-like map $\phi : U \to P$ is considered to be a nest of height 0.)

A nest is called *real* if all the domains P^n, U_i^n and the corresponding maps ϕ_n are symmetric with respect to the real line. A nest is called *Jordan* if all its domains are Jordan domains.

Let us consider a domain U_i^n of the nest such that $\phi_n(U_i^n) = P^n$, $n \geq 1$. Then according to property (iv) , $\phi_n|U_i^n = \phi_{n-1}^p|U_i^{n+1}$, where

$p = p(n,i)$ is the first return time of U_i^n to P^n under the iterates of ϕ_{n-1}. Let

$$U_{i,j}^n = \phi_n^j(U_i^n), \quad j = 0, \dots, p(n,i) - 1,$$

stand for the orbit of U_i^n under iterates of ϕ_{n-1} until its first return to P^n; $U_{i,0}^n \equiv U_i^n$.

Two Jordan Markov nests $\Phi = \{\phi_n\}$ and $\tilde{\Phi} = \{\tilde{\phi}_n\}$ have *the same combinatorics* if there is a homeomorphism

$$H : (\mathbb{C},\ \cup P^n,\ \cup U_{i,j}^n) \to (\mathbb{C},\ \cup \tilde{P}^n,\ \cup \tilde{U}_{i,j}^n)$$

which is equivariant on the boundary of the pieces, i.e.,

$$H(\phi_{n-1} z) = \tilde{\phi}_{n-1}(Hz) \quad \text{for} \quad z \in \bigcup_{n,i,j} \partial U_{i,j}^n$$

If the Markov nests in question are not Jordan, to define "the same combinatorics" one should shrink the domains a little to make them Jordan.

We say that the combinatorics of the nest is bounded by p if all the numbers:

- the height;
- the number of the domains U_i^n on every level;
- the return times of the domains U_i^{n+1} to P^{n+1} under iterates of ϕ_n

are bounded by p. Let $\mathcal{N}_p$ stand for the space of Markov nests with combinatorics bounded by p.

Let us say that the geometry of the Markov nest is K-bounded if all the maps ϕ_n have K-bounded geometry. Let $\mathcal{N}_p(K)$ stand for the space of Markov nests (up to rescaling) with p-bounded combinatorics and K-bounded geometry. Endow it with the Carathéodory topology: convergence of a sequence of nests means the Carthéodory convergence of the corresponding maps on every level. Lemma 2.17 immediately yields:

Lemma 2.18. *The space $\mathcal{N}_p(K)$ is compact.*

2.4.7. *Compactification.* To any map $f \in \mathcal{Q}_p$ we have associated in §2.4.4 a modified principal nest. The Markov maps associated to the cascades of this nest (§2.4.3) form a Markov nest. This provides us with an embedding $i : \mathcal{Q}_p \to \mathcal{N}_p$. (The map i is obviously continuous, and it is injective by Lemma 2.15 since the last map ϕ_χ in this nest is the renormalization Rf).

Recall that $\mathcal{Q}_p(\mu) = \{f \in \mathcal{Q}_p : \mathrm{mod}(f) \geq \mu\}$. The following statement is a variation of [L3, Lemma 8.8].

Lemma 2.19. *The Markov nest $\Phi = i(f)$ corresponding to $f \in \mathcal{Q}_p(\mu)$ has a K-bounded geometry, with $K = K(p, \mu)$.*

Proof. The map f is combinatorially equivalent to a real map g. By Main Lemma of [L3, §11], the Markov cascades $i(f)$ and $i(g)$ are qc equivalent, with dilatation depending only on μ. Hence without loss of generality we can assume that f is real.

Let $\Phi = \{\phi_n\}$, where $\phi_n : \cup_i U_i^n \to P^n$. Let $G_n : \cup U_i^n \to P^n$ be the associated Bernoulli map (2.17). Then by [L3, Theorem II] and [L3, Lemma 8.8] (for the last property) we have:

- Every U_i^n has a collar $R_i^n \subset P^n \smallsetminus \cup_i U_i^n$ with a definite modulus;
- The maps $G_n | U_i^n$, $i \neq 0$, have bounded distortion;
- The maps $G_n | U_0^n$ are compositions of the quadratic map $z \mapsto z^2$ and maps with bounded distortion;
- The real slices $U_i^n \cap \mathbb{R}$ are commensurable with $\operatorname{diam} P^n$.

(All the above bounds depend only on μ and p.)

These properties easily imply the assertion. □

Thus, we can embed the space $\mathcal{Q}_p(\mu)$ into a compact space $\mathcal{N}_p(K)$. Let $\bar{\mathcal{Q}}_p(\mu)$ denote the closure of $\mathcal{Q}_p(\mu)$ in $\mathcal{N}_p(K)$, and $\partial\mathcal{Q}_p(\mu) = \bar{\mathcal{Q}}_p(\mu) \smallsetminus \mathcal{Q}_p(\mu)$ be the "boundary at infinity" of $\mathcal{Q}_p(\mu)$. Boundary Markov nests $\Phi \in \partial\mathcal{Q}_p(\mu)$ will be called *parabolic* (since the base map of Φ, ϕ_0, is parabolic). Parabolic nests are *geometric limits* of quadratic-like maps with essentially bounded combinatorics and a definite modulus, as the renormalization period goes to ∞.

2.4.8. *Mandelbrot copies with essentially bounded combinatorics.* Let us now describe the structure of the family $\mathcal{M}_p$ of Mandelbrot copies with essential combinatorics bounded by p. These copies are in one-to-one correspondence with the different real combinatorial types essentially bounded by p. So let us fix some essentially p-bounded combinatorial type τ of height χ (by selecting some nest $\Phi \in \mathcal{N}_p$). Let us consider the family of Mandelbrot copies $\mathcal{M}_\tau^k$ of height $k \leq \chi$ whose essential combinatorics coincides with τ on the first k levels. These copies can be distinguished by specifying the lengths

$$\bar{N} = (N(1), N(2), \dots, N(k))$$

of the cascades in the principal nest. We will describe the hierarchical structure of these families by induction in k.

Let $k = 1$. Then we have a sequence $\mathcal{M}^1$ of Mandelbrot sets M_N^1 with the cascade of length N. As $N \to \infty$, these cascades converge to a parabolic combinatorics representing a cusp of some maximal Mandelbrot set M^0. Thus, the copies M^1_N converge to this cusp.

Remark. The convergence of Mandelbrot sets can be understood either combinatorially, or as convergence of their centers (since combinatorics uniquely determines the cusp). However, we will see below (Corollary 2.23) that the maximal real Mandelbrot sets shrink, so that they converge to the cusp uniformly.

For $k = 2$ we have a double sequence $\mathcal{M}^2$ of Mandelbrot sets $M^2_{N(1),N(2)}$. If $N(1) \to \infty$, then these Mandelbrot sets converge to M^0. If $N(2) \to \infty$ while $N(0)$ is eventually constant, then the corresponding Mandelbrot sets accumulate on the cusp of $M^1_{N(0)}$.

Proceeding in this way, we construct a hierarchical sequence of families $\mathcal{M}^k$ of Mandelbrot set $M^k_{N(1),\dots,N(k)}$ accumulating on the cusps of the previous families. Namely if $N(l+1) \to \infty$ while $N(s)$ are eventually constant for $s \le l$, then the corresponding Mandelbrot sets converge to the cusp of $M^l_{N(1),\dots,N(l)}$ (independently of the behavior of $N(s)$ for $s > l+1$).

We refer to Hinkle [Hi] for a further discussion of the essentially bounded combinatorics.

2.5. **Geometric bounds.** An infinitely renormalizable map $f : V \to V'$ is said to have *a priori* bounds if $\operatorname{mod}(R^n f_V) \ge \nu > 0$, $n = 0, 1, \dots$, where the $R^n f_V$ stand for the subordinate renormalizations of f_V (see §2.3.1). We say that a map $f \in \mathcal{C}$ is *close to the cusp* if $|\chi(f) - 1/4| < \epsilon$. Note that renormalizable maps are not close to the cusp.

Theorem 2.20 (A priori bounds [LS, LY])**.** *Let $f : V \to V'$ be n times renormalizable real quadratic-like map with* $\operatorname{mod}(V' \smallsetminus V) \ge \mu > 0$. *Then*

$$\operatorname{mod}(R^n f_V) \ge \nu_n(\mu) \ge \nu(\mu) > 0,$$

unless the last renormalization is of doubling type and $R^n f$ is close to the cusp. Moreover, $\liminf \nu_n(\mu) \ge \nu > 0$, where ν is an absolute constant. Thus all real infinitely renormalizable maps have a priori bounds.

The following two geometric results are crucial for our study.

Theorem 2.21 (Big dynamical moduli [L3], Theorem V)**.** *Let* $\operatorname{mod}(f) \ge \mu > 0$. *Then for any $M \in \mathcal{M}$,*

$$\operatorname{mod}(R_M f) \ge \nu(\mu, M) \ge \nu(\mu) > 0, \quad f \in M,$$

unless $p(M) = 2$ and f is close to the cusp. Moreover, $\nu(\mu, M) \to \infty$ as $p_e(M) \to \infty$ (μ being fixed).

Remark. A related result on moduli growth for real quadratics was independently proven in [GS]. Note in this respect that in this paper we need in a crucial way the above Theorem 2.21 for complex parameter values (even though in this paper we are ultimately interested in the real case).

The corresponding parapuzzle result is:

Theorem 2.22 (Parameter moduli [L4]). *Consider an equipped quadratic-like family* $(\mathbf{f}, \mathbf{h}) \in \mathcal{G}_{C,\mu}$ *and its renormalization* $(\mathbf{g}, \mathbf{j}) = R_M(\mathbf{f}, \mathbf{h})$, $M \in \mathcal{M}$. *If* $p(M) > 2$, *then*

$$\operatorname{mod}(\mathbf{g}) \geq \lambda(M, C, \mu) \geq \lambda(C, \mu) > 0, \quad \operatorname{Dil}(\mathbf{j}) \leq K(C, \mu),$$

where $\lambda(M, C, \mu) \to \infty$ *as* $p_e(M) \to \infty$ *($C, \mu > 0$ being fixed). Moreover, the family* $(\mathbf{g}, \mathbf{j})$ *can be restricted so that it belongs to some class* $\mathcal{G}_{L,\nu}$ *with* L, ν *depending only on* C, μ. *If* $p(M) = 2$ *then* $(\mathbf{g}, \mathbf{j}) \in \mathcal{G}^{\epsilon}_{L,\nu}$.

This result will give us a transverse control of the full renormalization operator.

Corollary 2.23. *Let us consider an equipped quadratic-like family* $(\mathbf{f}, \mathbf{h}) \in \mathcal{G}_{C,\mu}$, *and let* $M_i = M_{i,\mathbf{f}} \subset \mathbb{D}$ *be the corresponding family of maximal real Mandelbrot copies except the doubling copy. Then the sets* M_i *have* $K(C, \mu)$*-standard shape and* $\operatorname{diam}(M_i) \to 0$ *as* $p(M_i) \to 0$ *at rate depending only on* C *and* μ.

Proof. By Lemma 2.8, the Mandelbrot set $M^0_{\mathbf{f}}$ has $K(C, \mu)$-standard shape. Hence it is enough to check shrinking of the M_i in the case of the quadratic family $\mathcal{QP}$. By the same lemma and Theorem 2.22, all the sets M_i have quasi-standard shape. Hence it is enough to have shrinking of their real traces $M_i \cap \mathbb{R}$. But these traces are pairwise disjoint as the copies M_i are maximal. □

2.6. Combinatorial rigidity.

Theorem 2.24 ([L3]). *Let* $f \in \mathcal{C}(\mu)$ *and* $g \in \mathcal{C}(\mu)$ *be two infinitely renormalizable quadratic-like germs with the same real combinatorial type*

$$\tau = \{M_0, M_1, \dots\}, \ M_k \in \mathcal{M},$$

(but not necessarily real), and with a priori bounds. Then f *and* g *are hybrid equivalent. Moreover, the dilatation of the hybrid conjugacy depends only on* μ.

Together with *a priori* bounds (Theorem 2.20) this Rigidity Theorem yields:

Corollary 2.25. *For any real combinatorial type* $\tau = \{M_0, M_1, \dots\}$, *there is a single real quadratic* P_c *with this combinatorics.*

2.7. **McMullen towers.** A *McMullen tower* $\bar{f}$ is a sequence $\{f_k\}_{k=l}^n$ of quadratic-like maps with connected Julia sets such that $f_{k+1} = Rf_k$. The numbers l and t are called the *dimensions* of the tower. A tower is called *two-sided* if $t = -l = \infty$.

Combinatorial type $\tau(\bar{f})$ of a tower $\bar{f}$ is a sequence of maximal Mandelbrot copies M_k such that $f_k \in M_k$. We will consider towers with real combinatorics only, so that $M_k \in \mathcal{M}$.

Let $p(\bar{f}) = \sup p(f_k)$ and $p_e(\bar{f}) = \sup p_e(f_k)$ stand respectively for the "period" and the "essential period" of the tower. One says that the tower has a p-bounded (or essentially p-bounded) combinatorics if $p(\bar{f})$ (respectively $p_e(\bar{f})$) is bounded by p.

The space of towers is supplied with the weak topology: $\bar{g}_m = \{g_{m,k}\}_k \to \bar{f}$ as $m \to \infty$ if for each index k, $g_{m,k} \to f_k$ (where the dimensions of the towers in a converging sequence can vary but should converge to the dimensions of the limit tower). Let $\mathcal{T}$ stand for the spaces of towers.

The modulus of the tower, $\operatorname{mod}(\bar{f})$, is defined as $\inf \operatorname{mod}(f_k)$. One says that a tower has *a priori* bounds if $\operatorname{mod}(\bar{f}) > 0$. Compactness of $\mathcal{C}(\mu)$ yields:

Lemma 2.26. *The space of towers with uniformly bounded combinatorics and common a priori bounds is compact.*

Theorem 2.27 (Towers rigidity)**.** *Two bi-infinite towers with the same bounded combinatorics and a priori bounds are affinely equivalent.*

Proof. By the Rigidity Theorem 2.24, two bi-infinite towers with the same combinatorics are quasi-conformally equivalent. By McMullen's Rigidity Theorem [McM2], qc equivalent towers are affinely equivalent. □

Later on we will prove a similar rigidity theorem for towers with arbitrary real combinatorics (see Theorem 3.7).

2.8. **Parabolic towers.**

2.8.1. *Rigidity.* Let us say that a Markov nest $\Psi = \{\psi_m\}$ is a *renormalization* of a Markov nest $\Phi = \{\phi_n\}_{n=0}^{\chi}$, $\Psi = R\Phi$, if $\phi_\chi = \psi_0$ (this kind of renormalization is also called "parabolic renormalization"). A *Markov tower* $\bar{\Phi}$ is a sequence of Markov nests $\{\Phi_k\}_{k=l}^t$ such that $\Phi_{k+1} = R\Phi_k$.

A tower has a *p-bounded combinatorics* if all the nests Φ_k belong to $\mathcal{N}_p$. It has a K-bounded geometry if all the nests Φ_k belong to $\mathcal{N}_p(K)$. We will use the following notations:

- $\hat{\mathcal{T}}$ is the space Markov towers endowed with the topology of coordinatewise convergence;

• $\hat{\mathcal{T}}_p$ is the space of Markov towers $\bar{\Phi} \in \mathcal{T}$ with p-bounded combinatorics;

• $\hat{\mathcal{T}}_p(K)$ is the subspace of towers $\bar{\Phi} \in \mathcal{T}_p$ with essentially K-bounded geometry.

Lemma 2.18 implies that the latter space is compact.

Two towers are *combinatorially equivalent* if the corresponding Markov nests are. Two towers are *qc equivalent* if there is a qc map $h : \mathbb{C} \to \mathbb{C}$ conjugating the corresponding nests.

Let $\mathcal{T}_p$ be the space of McMullen towers with *essentially* p-bounded combinatorics. Then we have a natural coordinatewise embedding

$$i : (\mathcal{T}, \mathcal{T}_p) \to (\hat{\mathcal{T}}, \hat{\mathcal{T}}_p).$$

Markov towers in the closure of $\mathcal{T}_p$ in $\hat{\mathcal{T}}_p$ will be called *parabolic towers*. They represent geometric limits of McMullen towers $\bar{f}_m$ with essentially bounded period $p_e(f_m)$ as the period $p(\bar{f}_m)$ goes to ∞.

Theorem 2.28 (Hinkle [Hi]). *If two parabolic towers $\bar{\Phi}$ and $\bar{\Psi}$ in $\hat{\mathcal{T}}_p(K)$ are combinatorially equivalent then they are affinely equivalent.*

2.8.2. *Injectivity.* Let us finish with an extension of Lemma 2.15 to the case of parabolic renormalization.

Lemma 2.29. *Let f be a renormalizable real quadratic-like map, and let $\Phi = \{\phi_n\}_{n=0}^{\chi}$ be a real parabolic nest. Then $Rf \neq R\Phi \equiv \phi_\chi$.*

Proof. Let $A = K(f) \cap \mathbb{R}$; this is the maximal f-invariant interval. Let $g : J \to J$ be the real pre-renormalization of f, where $J = K(g) \cap \mathbb{R} \subset A$ is the maximal g-invariant interval.

Recall that the nest Φ consists of maps

$$\phi_n : \cup U_i^n \to P^n, \quad n = 0, 1, \dots, \chi,$$

of class $\mathcal{L}$, where the last map $\psi \equiv \phi_\chi : U^\chi \to P^\chi$ is a quadratic-like map with connected Julia set. Normalize it so that $\psi | J = g | J$.

Note that the relation $g = f^p | J$ gives the analytic extension of g to A. Hence, it provides the analytic extension of ψ to A as well.

Let $I^n = P^n \cap \mathbb{R}$. We will prove by induction that A contains all the intervals I^n, $n = \chi, \chi - 1, \dots, 0$.

Since

$$\bigcap_{k \in \mathbb{N}} \psi^{-k} I^\chi = K(\psi) \cap \mathbb{R} = J,$$

the map ψ does not have any invariant intervals in I^χ disjoint from J. Since g must have such an invariant interval in A (and since both intervals A and I^χ are 0-symmetric), we conclude that $A \supset I^\chi$.

Assume by induction that $A \supset I^n$. Consider the Bernoulli map G (2.17) associated with $\phi \equiv \phi_{n-1} : \cup U_i^{n-1} \to P^{n-1}$. Let r be the first return moment of the orbit $\{G^k(0)\}$ back to J. By definition, $\psi \mid J = G^r \mid J$.

Let us push I^n forward under "cut-off iterates" of G until the first return of the critical point back to J. More precisely, let us consider the itinerary of the critical point through the intervals $T_i = U_i^{n-1} \cap \mathbb{R}$;

$$G^k(0) \in T_{i(k)}, \quad k = 0, 1, \dots, r.$$

Let $H_0 = I^n$, and define H_k inductively as $G(H_{k-1} \cap T_{i(k-1)})$. By the Bernoulli property of G, one of the endpoints of H_k belongs to ∂I^{n-1}. Hence the interval H_r contains one of the components of $I^{n-1} \setminus J$. Call it S.

But $H_r = G^r \mid L$, where $0 \in L \subset A$ and G^r is continuous on L. By analytic continuation, $G^r \mid L = \psi \mid L$. Since A is ψ-invariant, $A \supset H_r \supset S$. Since the interval A is 0-symmetric, it contains the convex hull of S and $-S$, which is equal to I^{n-1}.

Thus, ψ admits analytic extension to the whole interval I^0 and coincides with g over there.

Furthermore, the base map of the nest, $\phi \equiv \phi_0 : T^0 \to I^0$, is parabolic, with the interval J contained in its parabolic basin B (in fact, already the interval I^1 is contained in B). Hence the orbit $\{\phi^n(J)\}$ converges to the parabolic fixed point of ϕ.

But one of the endpoints of J, say β, is fixed under ψ. Since ψ commutes with ϕ, the whole orbit $\{\phi^n \beta\}$ consists of fixed points of $\psi = g$. But g has only finitely many fixed points – contradiction. $\square$

3. Hyperbolicity of the renormalization operator

3.1. **Renormalization Theorem.** Let us start with a complete technical formulation of the Renormalization Theorem whose simplified version was stated in the Introduction. Recall from §2.3.4 that $\mathcal{T}_J$, $J \in \mathcal{J}$, stand for the real renormalization strips, whose union form the domain of definition of the renormalization R in the space $\mathcal{Q}_\mathbb{R}$ of real quadratic-like maps.

Theorem 3.1. *There is a set $\mathcal{A} \subset \cup \mathcal{T}_J$ (called the full renormalization horseshoe), a constant $\rho \in (0,1)$, and a neighborhood V of the origin in $\mathbb{C}$ such that:*

(i) *$\mathcal{A}$ is precompact in $\mathcal{Q}_\mathbb{R}$, R-invariant, and $R|\mathcal{A}$ is topologically conjugate to the two-sided shift $\omega : \Sigma \to \Sigma$ in countably many symbols.*

(ii) *The topological classes* $\mathcal{H}_{\mathbb{R}}(f)$*,* $f \in \mathcal{A}$*, are codimension-one real analytic submanifolds in* $\mathcal{Q}_{\mathbb{R}}$ *("stable leaves") which form an* R*-invariant lamination in* $\mathcal{Q}_{\mathbb{R}}$*. Moreover, if* $g \in \mathcal{H}_{\mathbb{R}}(f)$ *and* $\operatorname{mod}(g) \geq \nu$*, then* $R^n g \in \mathcal{B}_V$ *and* $\|R^n f, R^n g\|_V \leq C\rho^n$ *for* $n \geq N(\nu)$*.*

(iii) *There exists an* R^{-1}*-invariant family of real analytic curves* $W^u_{\mathbb{R}}(f)$*,* $f \in \mathcal{A}$*, ("unstable leaves") which transversally pass through all real hybrid classes* $c \in [-2, 1/4 - \epsilon]$*, and such that*

$$\|R^{-n} f - R^{-n} g\|_V \leq C\rho^n, \quad n \geq 0,$$

provided $g \in W^u_{\mathbb{R}}(f)$*.*

(iv) *The renormalization operator has uniformly bounded distortion with respect to the Montel metric on the unstable leaves (see §3.5.1).*

(v) *The stable lamination is transversally quasi-symmetric.*

3.2. **Exponential contraction.**

3.2.1. *Macroscopic contraction.* Let

$$\mathcal{C}_n(\mu) = \{f : f \text{ is } n \text{ times renormalizable and } \operatorname{mod} R^m(f) \geq \mu,\ m = 0, 1, \dots, n.\}$$

Lemma 3.2. *The renormalization is macroscopicly contracting in the following sense: For any* $\varepsilon > 0$ *there is an* $N = N(\mu)$ *such that*

$$\operatorname{dist}_{Mon}(R^m f, R^m g) < \varepsilon,\ m = N, N+1, \dots, n,$$

provided f *and* g *are hybrid equivalent and belong to* $\mathcal{C}_n(\mu)$*.*

Remark. We call this property "macroscopic" since it provides contraction only in big scales but allows expansion in small scales ($< \varepsilon$).

Proof. By the contracting property with respect to the Teichmüller-Sullivan metric (see §2.3.6) and the relation between this metric and Banach metrics (Lemma 2.3), the renormalization is Lyapunov stable: There exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\operatorname{dist}_{Mon}(f, g) < \delta \Rightarrow \operatorname{dist}_{Mon}(R^m f, R^m g) < \varepsilon,\ m = 1, \dots, n,$$

provided f and g belong to $\mathcal{C}_n(\mu)$ (where $\delta = \delta(\varepsilon)$ is independent of n and the combinatorics of f and g).

Take a renormalizable map $f : V \to V'$ with $\operatorname{mod}(V \setminus V') \geq \mu$. By Theorem 2.21, if $p_e(f) \geq \bar{p}_e(\delta)$, then the renormalization Rf is $\delta/2$-close to a quadratic map P_c, where $c = \chi(Rf)$. Thus for two hybrid equivalent maps like that we have: $\operatorname{dist}_{Mon}(Rf, Rg) < \delta$.

Furthermore, let us show that there is an $N = N(\mu, \bar{p}_e)$ with the following property: If for $2N$ consecutive renormalizations of maps f and g in $\mathcal{C}_{2N}(\mu)$, their essential periods stay bounded by $\bar{p}_e$, then $\operatorname{dist}_{Mon}(R^N f, R^N g) < \delta$. Otherwise there would be a sequence of maps

f_N and g_N as above with $\mathrm{dist}_{Mon}(f_N, g_N) \geq \delta$. Let $F_{N,m} = R^{N+m} f_N$ and $G_{N,m} = R^{N+m} g_N$, $m = -N, \dots, N$. Consider a sequence of hybrid equivalent finite towers $\mathbf{F}_N = \{F_{N,m}\}_{m=-N}^{N}$ and $\mathbf{G_N} = \{G_{N,m}\}_{m=-N}^{N}$. By compactness (see §2.4.7), these towers converge along a subsequence to bi-infinite parabolic towers $\mathbf{F} = \{F_m\}$ and $\mathbf{G} = \{G_m\}$ with essentially bounded combinatorics and *a priori* bounds. By the Rigidity Theorem 2.28 for parabolic towers, $\mathbf{F} = \mathbf{G}$ up to rescaling. On the other hand, $\mathrm{dist}_{Mon}(F_0, G_0) \geq 1/2$ - contradiction.

Thus in any case (with no restrictions on the combinatorics), there is an $l \leq N(\mu, \bar{p}_e)$ such that $\mathrm{dist}_{Mon}(R^l f, R^l g) < \delta$. By the choice of δ, $\mathrm{dist}_{Mon}(R^m f, R^m g) < \varepsilon$ for all further moments $m = l+1, \dots, n$. ☐

3.2.2. *Absolute a priori bounds.* Let $\mathcal{S} \subset \mathcal{C}$ stand for the union of (complex) quadratic-like germs f with the real straightening, i.e., such that $\chi(f) \in [-2, 1/4]$. Let $\mathcal{S}(\mu) = \mathcal{S} \cap \mathcal{C}(\mu)$ and let $\mathcal{S}_n(\mu)$ be the set of n times renormalizable germs of $\mathcal{S}(\mu)$. The following lemma provides us with absolute *a priori* bounds in $\mathcal{S}$:

Lemma 3.3. *There is an absolute $\mu > 0$ such that if the germ of $f : V \to V'$ belongs to $\mathcal{S}_{n+1}(\nu)$, then $\mathrm{mod}(R^m(f_V)) \geq \mu$ for $m = N(\nu), \dots, n$.*

Proof. It is true for real maps by Theorem 2.20. In particular, it is true for the straightening $g = P_c$ of f, $c = \chi(f) \in [-2, 1/4]$.

Let us take quadratic-like representatives $g : V \to V'$ and $f : U \to U'$ with $\mathrm{mod}(V' \smallsetminus V) \geq 1$, $\mathrm{mod}(U' \smallsetminus U) \geq \nu/2$ conjugated by a qc map $h : V' \to U'$ with a dilatation K depending on ν only. Consider representatives $g_m : V_m \to V'_m$ and $f_m : U_m \to U'_m$ of the renormalizations subordinate to the maps g and f and qc conjugate by the same map h. By Theorem 2.20, there is a choice of these representatives such that

$$\mathrm{mod}(V'_m \smallsetminus V_m) > \mu > 0, \quad m = 0, 1, \dots, n,$$

with an absolute μ. Hence

$$\mathrm{mod}(U'_m \smallsetminus U_m) \geq \mu/K, \quad m = 0, 1, \dots, n.$$

Thus, we can apply Lemma 3.2 to these two maps and conclude that for some $l = l(\nu)$, there exist 2-qc conjugate representatives $g_l : \tilde{V}_l \to \tilde{V}'_l$ and $f_l : \tilde{U}_l \to \tilde{U}'_l$ with a definite modulus (depending only on ν). But this conjugacy provides a 2-qc conjugacy between the further renormalizations of $g_m : \tilde{V}_m \to \tilde{V}'_m$ and $f_m : \tilde{U}_m \to \tilde{U}'_m$, $m \geq N(\nu) \geq l$, subordinate to the above representatives. Since $\mathrm{mod}(\tilde{V}'_m \smallsetminus \tilde{V}_m) \geq \mu$, we conclude that $\mathrm{mod}(\tilde{U}'_m \smallsetminus \tilde{U}_m) \geq \mu/2$ for $m \geq N(\nu)$. ☐

3.2.3. *Invariant family of Banach slices.* The above result allows us to select a family of Banach slices invariant with respect to some iterate of the renormalization:

Lemma 3.4. *Let μ be an absolute bound from Lemma 3.3 and $0 < \nu \le \mu$. There exist $N = N(\nu)$, $\delta > 0$, and a family of quadratic-like representatives $f : V(f) \to V'(f)$ of germs $f \in \mathcal{C}(\nu)$, with the following properties:*

- $\operatorname{mod}(V'(f) \setminus V(f)) > \gamma(\nu) > 0$*;*
- *If $f \in \mathcal{S}_{N+1}(\nu)$ and $g \in \mathcal{B}_f(\delta) \cap \mathcal{H}(f)$, then $R^N g \in \mathcal{B}'_{R^N f}(\rho)$, where $\rho = \rho(\nu, \delta) \to 0$ as $\delta \to 0$ (ν being fixed), $\mathcal{B}_f \equiv \mathcal{B}_{V(f)}$, $\mathcal{B}_f(\delta) \equiv \mathcal{B}_{V(f)}(f, \delta)$, and the notations with "prime" have a similar meaning.*

Proof. Since $\operatorname{diam} K(f)$ depends continuously on $f \in \mathcal{C}$, it is bounded away from 0 for $f \in \mathcal{C}(\nu)$. Hence there exists an $\varepsilon = \varepsilon(\nu) > 0$ such that $\operatorname{dist}(\partial U, K(f)) > \varepsilon$ for any $f \in \mathcal{C}(\nu)$ and any topological disk U containing $K(f)$ with $\operatorname{mod}(U \setminus K(f)) \ge \mu/2$.

Recall that $\Omega_c(r)$ is the domain bounded by the equipotential of radius $r > 1$ of a quadratic map P_c.

For any $f \in \mathcal{C}(\nu)$, there exists a quadratic-like representative $f : U(f) \to U'(f)$ which is $K(\nu)$-qc conjugate to the quadratic polynomial $P_c : \Omega_c(2) \to \Omega_c(4)$, where $c = \chi(f)$. Since for given $r > 1$ $\operatorname{dist}(K(P_c), \partial\Omega_c(r))$ depends continuously on $c \in M$, the domains $\Omega_c(r)$ uniformly shrink to $K(P_c)$ as $r \to 1$, $c \in M$. Hence the domains $U^l(f) = f^{-l}(U(f))$ uniformly shrink to $K(f)$ as $l \to \infty$, $f \in \mathcal{C}(\nu)$. Thus, there exists an $l = l(\nu)$ such that $\operatorname{dist}(\partial U^{l-2}(f), K(f)) < \varepsilon$ for any $f \in \mathcal{C}(\nu)$.

Let $V(f) = U^l(f)$, $V'(f) = U^{l-1}(f)$. Then

$$\operatorname{mod}(V'(f) \setminus V(f)) \ge \nu/2^l \equiv \gamma(\nu) \text{ for any } f \in \mathcal{C}(\nu).$$

By Lemma 3.3, there exists an $N = N(\nu)$ and a renormalization $g \equiv R^N f : W(g) \to W'(g)$ subordinate to $f : V(f) \to V'(f)$ such that $\operatorname{mod}(W'(g) \setminus W(g)) \ge \mu$. By the choice of ε,

$$W(g) \supset U^{l-2}(g) \supset U^{l-1}(g) = V'(g) \text{ and } \operatorname{mod}(W(g) \setminus V'(g)) \ge \nu/2^{l-1}.$$

Furthermore, by Lemma 2.3, any map $\tilde f \in \mathcal{H}(f) \cap \mathcal{B}_f(\delta)$ is conjugate (on an appropriate domain) to $f : V(f) \to V'(f)$ by a globally defined K-qc map $h : \mathbb{C} \to \mathbb{C}$, where $K = K(\nu, \delta) \to 1$ as $\delta \to 0$. The appropriately rescaled h conjugates the renormalizations $R^N \tilde f$ and $R^N f$ on the corresponding W-domains. Hence $\tilde g \equiv R^N \tilde f$ is defined and ρ-close to g on $V'(g)$, where $\rho = \rho(\nu, \delta) \to 0$ as $\delta \to 0$ (ν being fixed). $\square$

Remark. Using Lemma 6.2(iv), one can make the choice of the domains $V(f)$ in such a way that there are only finitely many distinct domains among them.

3.2.4. *Exponential contraction.* We will now apply the Schwarz Lemma in Banach spaces (see Appendix 1) in order to pass from the macroscopic contraction to the exponential contraction.

Theorem 3.5. *Let us consider two hybrid equivalent quadratic-like maps $f \in \mathcal{S}_{n+1}(\nu)$ and $g \in \mathcal{S}_{n+1}(\nu)$. Then*

$$\mathrm{dist}_{Mon}(R^m f, R^m g) \le C\rho^m, \ m = 0, 1, \dots, n,$$

where dist_{Mon} is a Montel distance on $\mathcal{C}(\nu)$, $\rho \in (0,1)$ depends only on the choice of dist_{Mon}, while $C > 0$ depends also on ν.

Remark. It is *a priori* clear that this statement is qualitatively independent of the particular choice of Montel metric since all of them are Hölder equivalent (see Lemma 6.2(iii)). In similar statements to follow dependence of the constants on the Montel metric will be implicit.

Proof. By Lemma 3.3, there exists an $N = N(\nu)$ such that $R^m f$ and $R^m g$ belong to $\mathcal{S}(\mu)$ with an absolute μ. It follows that it is enough to consider the case $\nu = \mu$.

Let us consider the projection $\Pi : \mathcal{C} \to \mathcal{H}_0$ and the parametrizations $I_c : \mathcal{H}_0 \to \mathcal{H}_c$ (see (2.6)). We know that both of them are continuous (and, in fact, analytic). Moreover, the family of parametrizations I_c, $c \in M$, is equicontinuous on $\mathcal{H}_0(\mu)$. It follows from the observation that $I_c(\mathcal{H}_0(\mu)) \subset \mathcal{C}(\mu)$ and compactness of $\mathcal{C}(\mu)$.

Let us now consider a family of operators

$$T_c^n = \Pi \circ R^n \circ I_c : \mathcal{H}_0 \to \mathcal{H}_0$$

defined for n times renormalizable parameter values $c \in [-2, 1/4]$. Lemma 3.2 together with the above equicontinuity property imply that these operators are macroscopicly contracting: For any $\varepsilon > 0$ there exists an l such that

$$\mathrm{dist}_{Mon}(T_c^m f, T_c^m g) < \varepsilon, \quad m = l, \dots, n,$$

provided that $c \in [-2, 1/4]$ is $n+1$ times renormalizable, and f & g belong $\mathcal{H}_0(\mu)$.

Furthermore, since $\mathrm{mod}(\Pi(f)) = \mathrm{mod}(f)$, the absolute bounds of Lemma 3.3 are carried to the operators T_c^n: If $f \in \mathcal{H}_0$ and $\mathrm{mod}(f) \ge \nu$, then $\mathrm{mod}(T_c^N) > \mu$ (where the relations between μ, ν and N are the same as in the lemma). It implies by repeating the argument of Lemma 3.4 that there is a family of quadratic-like representatives $f : V(f) \to V'(f)$, $f \in \mathcal{H}_0(\nu)$, and a $\delta > 0$ such that

- $\mathrm{mod}(V'(f) \smallsetminus V(f)) > \gamma(\nu) > 0$;
- If $g \in \mathcal{B}_{0,f}(\delta)$ then $T_c^N g \in \mathcal{B}'_{0,T_c^N f}(\rho)$, where $\rho = \rho(\nu, \delta)$ and subscript "0" indicates slicing by $\mathcal{H}_0$, e.g., $\mathcal{B}_{0,f} \equiv \mathcal{B}_f \cap \mathcal{H}_0$.

Since the Banach distance in the $\mathcal{B}'_{0,T_c^N f}(\rho)$ induced from $\mathcal{B}_{0,T_c^N f}$ is uniformly Hölder equivalent to the Montel distance, we conclude that the operators $T_c^N : \mathcal{B}_{0,f}(\delta) \to \mathcal{B}_{0,R^N f}$ are macroscopicly contracting with respect to the Banach norms in the corresponding spaces. By the Schwarz Lemma in Banach spaces this contraction is actually microscopic:

$$\|T_c^N f - T_c^N g\|_{T_c^n f} < \frac{1}{2}\|f - g\|_f, \quad g \in \mathcal{B}_f(\delta),$$

where $\|\cdot\|_f$ stands for the norm in $\mathcal{B}_f$. This implies that the operators T_c^n are exponentially contracting with respect to the Banach metrics involved. Since these metrics dominate over the Montel metric, the operators T_c^n are exponentially contracting with respect to the latter metric too. By uniform continuity of the I_c, the iterates of the renormalization R are exponentially contracting as well. ☐

3.3. **Realization and rigidity of general towers.**

3.3.1. *Contraction in the middle of the tower.* Let us consider an orbit $\{R^m f\}_{m=-l}^n$ (assuming that f is n times renormalizable and l times anti-renormalizable and using notation $R^m f$ with negative m for some anti-renormalization of f). Its (l,n)-*itinerary* is a sequence of Mandelbrot copies $\{M_m\}_{m=-l}^n$ such that $R^m f \in \mathcal{T}_{M_m}$.

Lemma 3.6. *Consider two maps f and g in $\mathcal{S}$ with the same (l,n)-itinerary and such that*

$$\mathrm{mod}(R^k f) \geq \mu > 0 \textit{ and } \mathrm{mod}(R^k g) \geq \mu > 0,\ -l \leq k \leq n.$$

Then $\mathrm{dist}_{Mon}(f,g) < \varepsilon = \varepsilon(\mu, l, n)$, *where $\varepsilon \to 0$ as $l, n \to \infty$ (μ being fixed).*

Proof. Let $\chi(f) = P_c$ and $\chi(g) = P_b$, where b and c are real by the assumption. Corollary 2.25 implies that the renormalization windows of order n in the parameter interval $[-2, 1/4]$ (i.e., the connected components of the set of real n times renormalizable maps) uniformly shrink as the order grows. Thus, $|b - c| < \delta(n) \to 0$ as $n \to \infty$, so that f and g lie on the nearby leaves of the foliation $\mathcal{F}$. The same is applicable to $f_k \equiv R^k f$ and $g_k \equiv R^k g$, $k = -l, \dots, N$, for any given N.

For any integer $k \in [-l, 0]$, let us consider a map $h_k \in \mathcal{H}(f_k)$ belonging to the vertical fiber via g_k, i.e., $\pi(h_k) = \pi(g_k)$. Then $\mathrm{mod}(h_k) =$

$\mathrm{mod}(g_k)$. By Theorem 3.5, there exist ρ and N depending only on μ such that

$$\mathrm{dist}_{Mon}(R^N f_k, R^N h_k) \le \rho\,\mathrm{dist}_{Mon}(f_k, h_k).$$

The results of §2.2.3 imply that the vertical fibers through g_k near the connectedness locus can be equipped so that they become quadratic-like families of some class $\mathcal{G}_{L,\lambda}$, with geometry (i.e., the constants L, λ) depending only on μ. Hence by Theorem 2.22, $R^N h_k$ and $R^N g_k$ belong to the same quadratic-like family of class $\mathcal{G}^{\epsilon}_{C,\nu}$ with C and ν depending only on μ. Since by Lemma 2.9 $\mathcal{G}^{\epsilon}_{C,\nu}$ is a compact class of families, the holonomy $\mathcal{QP} \to \mathcal{S}$, $\mathcal{S} \in \mathcal{G}^{\epsilon}_{C,\nu}$, is equicontinuous. Hence

$$\mathrm{dist}_{Mon}(R^N h_k, R^N g_k) < \delta_1(n) \to 0 \text{ as } n \to \infty.$$

Take some $\varepsilon > 0$ and $\rho' \in (\rho, 1)$, and find an n such that $\delta_1 = \delta_1(n) < (\rho' - \rho)\varepsilon/(\rho + 1)$. If $\mathrm{dist}_{Mon}(f_k, g_k) \ge \varepsilon > 0$ then

$$\mathrm{dist}_{Mon}(R^N f_k, R^N g_k) \le \mathrm{dist}_{Mon}(R^N f_k, R^N h_k) + \mathrm{dist}_{Mon}(R^N h_k, R^N g_k) \le$$

$$\rho\,\mathrm{dist}_{Mon}(f_k, h_k) + \delta_1 \le \rho((\mathrm{dist}_{Mon}(f_k, g_k) + \delta_1) + \delta_1 < \rho'\,\mathrm{dist}_{Mon}(f_k, g_k).$$

Thus, R^N uniformly contracts the distance between the f_k and g_k, as long as it stays greater than ε. Hence in a bounded number of steps (depending on ε) this distance must become smaller than ε. □

3.3.2. *Realization and rigidity.* Let us now prove that any real combinatorics $\tau = \{M_k\}_{k=-\infty}^{\infty}$, $M_k \in \mathcal{M}$, can be be realized by a unique real tower. Let $\bar{\mathcal{S}}$ stand for the space of towers $\bar{f}$ with $f_k \in \mathcal{S}$.

Theorem 3.7. *For any two-sided real combinatorics τ there is a unique tower $\bar{f} \in \bar{\mathcal{S}}$ with this combinatorics and a priori bounds. Moreover, this tower is real and* $\mathrm{mod}(\bar{f}) \ge \nu$ *with an absolute $\nu > 0$.*

Proof. By Theorem 2.20, there is an absolute $\nu > 0$ such that for any infinitely renormalizable quadratic polynomial $f = P_c \in \mathcal{I}$, $R^n f \in \mathcal{Q}(\nu)$, $n = 0, 1, \dots$.

Let us take a combinatorial sequence $\tau = \{M_k\}$. For any $l \ge 0$, there is a real infinitely renormalizable quadratic polynomial $P_l \equiv P_{c_l}$ with combinatorics $\tau(P_l) = \{M_{-l}, \dots, M_0, \dots\}$. Let $f_{0,l} = R^l P_l$. These are infinitely renormalizable real quadratic-like maps with common combinatorics $\{M_0, M_1, \dots\}$ and $\mathrm{mod}(f_{0,l}) \ge \nu$. Since the set of such maps is compact, we can pass to a quadratic-like limit $f_0 = \lim_{l\to\infty} f_{0,l}$ (along a subsequence) with the same properties.

Let us now do the same thing for every $i \le 0$. Let $f_{i,l} = R^{l+i} P_l$, and let $f_i = \lim_{l\to\infty} f_{i,l}$ be a limit point. The map f_i is real and has combinatorics $\tau_i = \{M_i, M_{i+1}, \dots\}$.

Selecting the above converging subsequences diagonally, we construct a sequence of real infinitely renormalizable quadratic-like maps $\{f_i\}_{i=-\infty}^{\infty}$ such that $Rf_i = f_{i+1}$, $\chi(f_i) \in M_i$ and $\operatorname{mod}(f_i) \geq \nu$. This sequence represents a real tower $\bar{f}$ with combinatorics $\bar{\tau}$ and a moduli bound ν.

Thus, any real combinatorics τ is represented by a tower $\bar{f} \in \bar{\mathcal{S}}$ with *a priori* bounds. Moreover, this tower is unique. Indeed, if $\bar{f}$ and $\bar{g}$ are two such towers, then by Lemma 3.6 $\operatorname{dist}_{Mon}(f_0, g_0)$ is arbitrary small, so that $f_0 = g_0$. For the same reason $f_i = g_i$ for any i. □

Let us now state a more general realization and rigidity theorem for one-sided towers.

Theorem 3.8. *For any real combinatorial past $\tau = \{M_k\}_{k=-1}^{-\infty}$, $M_k \in \mathcal{M}$, and any $c \in [-2, 1/4)$, there is a unique tower $\bar{f} = \{f_k\}_{k=0}^{-\infty}$ in $\bar{\mathcal{S}}$ with a priori bounds such that $\chi(f_0) = c$ and $\chi(f_k) \in M_k$ for $k < 0$. Moreover, this tower is real and $\operatorname{mod}(\bar{f}) \geq \nu(\epsilon) > 0$, provided $c < 1/4 - \epsilon$.*

Proof. A priori bounds for real finitely renormalizable quadratic-like maps (Theorem 2.20) imply the existence part of the theorem in the same way as for two-sided towers.

Since the stretching $\sigma : J \to [-2, 1/4]$ is homeomorphic on every renormalization window $J \in \mathcal{J}$ (see §2.3.3), all the parameter values $\chi(f_k) \in J_k \equiv M_k \cap \mathbb{R}$ are uniquely determined by the combinatorics τ and the parameter $c = \chi(f_0)$. Hence, for any other tower $\bar{g} = \{g_k\}_{k=0}^{-\infty}$ with the same data, the maps g_k are hybrid equivalent to the f_k. If both towers have *a priori* bounds, then by Lemma 3.2,

$$\operatorname{dist}_{Mon}(f_k, g_k) = \operatorname{dist}_{Mon}(R^N f_{k-N}, R^N g_{k-N}) < \varepsilon(N) \to \text{ as } N \to \infty,$$

and hence $f_k = g_k$, $k = -1, -2, \ldots$. □

3.4. **Full renormalization horseshoe.** Let us now consider the space Σ of all possible real combinatorial types $\tau = \{M_k\}_{k=-\infty}^{\infty}$, where the $M_k \in \mathcal{M}$ are selected arbitrarily from the family of real maximal Mandelbrot copies. Supply Σ with the weak topology. Let $\omega : \Sigma \to \Sigma$ stand for the left shift on this space.

Let us say that an infinitely renormalizable map $f \in \mathcal{Q}$ is *completely non-escaping* under the renormalization if some full renormalization orbit $\{R^n f\}_{n=-\infty}^{\infty}$ is well-defined, $R^n f \in \mathcal{C}$, and

$$\operatorname{mod}(f_n) \geq \mu = \mu(f) > 0, \quad n \in \mathbb{Z}.$$

Note that we do not ask $R^n f$ to be uniquely determined for negative n but by Lemma 2.15 this is the case for real maps.

Let $\mathcal{A} \subset \mathcal{Q}$ stand for the set of completely non-escaping maps with real combinatorics. We call this set the *(full) renormalization horseshoe.*

Theorem 3.9. *There exist absolute $\mu > 0$ and $\rho \in (0,1)$ with the following properties. The set $\mathcal{A}$ belongs to $\mathcal{Q}_{\mathbb{R}}(\nu)$ and $R : \mathcal{A} \to \mathcal{A}$ is a homeomorphism. There exists a homeomorphism $\eta : \Sigma \to \mathcal{A}$ conjugating ω and $R|\mathcal{A}$. Moreover, for any infinitely renormalizable map $f \in \mathcal{S}$ there exists a $g \in \mathcal{A}$ such that*

$$\operatorname{dist}_{Mon}(R^n f, R^n g) \le C\rho^n, \tag{3.1}$$

where C depends only on $\operatorname{mod} f$*.*

Proof. By Theorem 3.7, any combinatorics $\tau \in \Sigma$ can be realized by a unique real tower $\bar{f} = \{f_i\}$ with absolute *a priori* bounds. Thus, we can define a map $\eta : \Sigma \to \mathcal{C}(\mu)$ by associating to a combinatorics $\tau \in \Sigma$ the zero coordinate f_0 of the corresponding tower $\bar{f} = \{f_i\}$. This map is continuous by Lemma 3.6. Let $\mathcal{A}$ be its image. Clearly, $\mathcal{A}$ is R-invariant, consists of completely non-escaping maps, and η conjugates the shift ω and $R|\mathcal{A}$. Moreover, by Lemma 2.15 this map is injective and thus bijective.

Since ω is a homeomorphism, $R : \mathcal{A} \to \mathcal{A}$ is bijective as well. Let us show that it is a homeomorphism.

Recall that $\mathcal{T}_J$ stand for the strips of real quadratic-like maps corresponding to renormalization windows $J \in \mathcal{J}$ (see §2.3.4). Let $\mathcal{A}_J = \mathcal{A} \cap \mathcal{T}_J \subset \mathcal{A} \cap \mathcal{T}_J(\mu)$. As the boundary points of J are at most once renormalizable, $\mathcal{A} \cap \partial\mathcal{T}_J = \emptyset$ for any $J \in \mathcal{J}$. Hence any map $f \subset \mathcal{A}_J$ belongs to $\mathcal{T}_J$ together with some neighborhood $\mathcal{U}$. Since every branch R_M of the renormalization is continuous, $R \,|\, \mathcal{A}$ is continuous at f.

Let us show that $(R|\mathcal{A})^{-1}$ is also continuous. Let $f \in \mathcal{A}$ and $g = R^{-1}f \in \mathcal{A}_J$. Let I be any other interval of family $\mathcal{J}$. Then by Lemma 2.15 $R(\mathcal{T}_I(\mu)) \not\ni f$. Since the strip $\mathcal{T}_I(\mu)$ is compact, its image $R(\mathcal{T}_I(\mu))$ misses some neighborhood of f.

Let us show that these images cannot accumulate on f. Indeed, assume that $Rh_k \to f$, where h_k belong to distinct strips $\mathcal{T}_{I_k}(\mu)$. If the essential period $p_e(h_k)$ is unbounded then by Theorem 2.21 f must be a quadratic polynomial. But it is impossible since by Lemma 2.14 quadratic polynomials are not anti-renormalizable. On the other hand, if the h_k have uniformly essentially bounded combinatorics (i.e, $p_e(h_k) \le p$) then by §2.4.7 we can pass to a geometric limit, a Markov nest $\Phi \in \mathcal{N}_p(K)$, $K = K(\mu)$. Then f is equal to a parabolic renormalization of the Markov nest Φ, so that $R\Phi = Rg$ contrary to Lemma 2.29.

Thus, there is a neighborhood $\mathcal{U}$ of f which misses all the images $R(\mathcal{T}_I(\mu))$ with $I \neq J$. Hence on this neighborhood $(R|\mathcal{A})^{-1}$ is the restriction of $(R|\mathcal{T}_J(\mu))^{-1}$. But the latter map is continuous since $\mathcal{T}_J(\mu)$ is compact.

Let us now show that $\eta : \Sigma \to \mathcal{A}$ is also a homeomorphism. The only thing to check is that the inverse map is continuous. Let $f \in \mathcal{A}$ be a map with itinerary $\{J_k\}_{k=-\infty}^{\infty}$. Let $n \geq 0$. Since $R^k f \in \operatorname{int} \mathcal{A}_{J_k}$ for all k and $R|\mathcal{A}$ is a homeomorphism, all the maps $g \in \mathcal{A}$ near f have the same itinerary $(J_{-n}, \dots, J_n)$. But this exactly translates into continuity of η^{-1}.

Finally, for any infinitely renormalizable quadratic-like map $f \in \mathcal{S}$, there is a map $g \in \mathcal{A}$ with the same combinatorics (by the Realization Theorem 3.7). By Corollary 2.25, f and g are hybrid equivalent, and Theorem 3.5 yields (3.1). □

3.5. **Transverse control of the renormalization.**

3.5.1. *Transverse distortion.* Let us consider a conformal map $f : \mathcal{S} \to \mathcal{S}'$ between two Riemann surfaces endowed with conformal metrics. The *distortion* (or *non-linearity*) of f is defined as follows:

$$n(f) = \sup_{z,\zeta \in \mathcal{S}} \log \frac{\|Df(z)\|}{\|Df(\zeta)\|}.$$

The following statement shows that the renormalization has transversally bounded distortion with respect to the Montel metric on quadratic-like families:

Lemma 3.10. *Consider a quadratic-like family* $\mathbf{f}$ *of class* $\mathcal{G}_{C,\mu}$. *Take a little Mandelbrot set* $M \in \mathcal{M}$ *and let* $M_{\mathbf{f}}$ *be the corresponding set in the family* $\mathbf{f}$. *If* $p(M) > 2$, *then there is* $\lambda = \lambda(C, \mu) > 0$ *and a domain* $\Omega_{\mathbf{f}} \subset \mathbf{f}$ *of the renormalization* R_M *with* $\operatorname{mod}(\Omega_{\mathbf{f}} \smallsetminus M_{\mathbf{f}}) \geq \lambda$ *such that the curve* $R_M(\Omega_{\mathbf{f}})$ *is uniformly (in terms of* C *and* μ*) transverse to the foliation* $\mathcal{F}$, *and* R_M *on* $\Omega_{\mathbf{f}}$ *has a* $K(C,\mu)$*-bounded distortion (independent of* M*) with respect to the Montel metric. If* $p(M) = 2$, *then the analogous statement holds for the truncated set* M^ϵ.

Proof. Assume for definiteness that $P(M) > 2$ (the argument in the doubling case is analogous).

Let $\Delta'_{\mathbf{f}} \equiv \Delta^n_{\mathbf{f}} \subset \mathbf{f}$ be the parapuzzle piece around $M_{\mathbf{f}}$ on which the renormalization R_M is defined (see §2.3.3), and let $\Delta_{\mathbf{f}} \equiv \Delta^{n+1}_{\mathbf{f}}$. By [L4], $\operatorname{mod}(\Delta'_{\mathbf{f}} \smallsetminus \Delta_{\mathbf{f}}) \geq \varepsilon(C, \mu) > 0$. Moreover, the holomorphic curve $R_M : \Delta_{\mathbf{f}} \to \mathcal{Q}$ belongs (after appropriate restriction of the domain $\Delta_{\mathbf{f}}$) to some class $\in \mathcal{G}_{L,\nu}$, where L, ν depend only on C, μ (Theorem 2.22). By Corollary 2.12, it is uniformly transverse to the foliation $\mathcal{F}$.

Hence on some domain $\Omega_{\mathbf{f}} \subset \Delta_{\mathbf{f}}$ with $\operatorname{mod}(\Omega_{\mathbf{f}} \setminus M_{\mathbf{f}}) \geq \lambda$ it is endowed with a Montel metric which has a bounded distortion with respect to the hyperbolic metric induced from $\Delta'_{\mathbf{f}}$ (see §2.2.5). Now the assertion follows from the Koebe Distortion Theorem. □

3.5.2. *An estimate of the transverse derivative.* It is proven in [L5] that any real infinitely renormalizable map f with periodic combinatorics is a hyperbolic periodic point for R. For what follows we will need the following weaker statement:

Lemma 3.11. *Real periodic points of R are not attracting.*

Proof. Let $R^p f_0 = f_0$, $P_{c_0} = \chi(f_0)$. If f_0 is attracting than there is a neighborhood of the hybrid class $\mathcal{H}(f_0)$ attracted to the cycle of f_0. By Lemma 2.16, all the maps in this neighborhood are infinitely renormalizable with the same combinatorics as f_0. In particular, if c is nearby to c_0 then P_c is an infinitely renormalizable map with the same combinatorics as P_{c_0} contradicting Corollary 2.25. □

Consider the one-dimensional quotient bundle $\mathrm{T}\mathcal{Q}/\mathrm{T}\mathcal{F}$ over $\mathcal{C}$ and supply it with the transverse metric $\|\cdot\|_{\mathrm{tr}}$ induced by the Montel metric on the line bundle $\{E^v_f\}$ (see §2.2.4). Let DR_{tr} stand for the tangent action of DR in the quotient bundle. If $R^p f = f$ then the value $\lambda(f) = \|DR^p_{\mathrm{tr}}(f)\|^{1/p}$ will be called *mean transverse multiplier* at f (note that it is independent of the choice of the norm on the one-dimensional space $\mathrm{T}_f\mathcal{Q}/\mathrm{T}_f\mathcal{H}(f)$). Let

$$\bar{\lambda} = \inf_p \inf_{f: R^p f = f} \lambda(f) \tag{3.2}$$

stand for the "smallest" mean transverse multiplier of the periodic points of R. By Lemma 3.11, $\bar{\lambda} \geq 1$. If $\bar{\lambda} > 1$ then we say that *the periodic points of R are uniformly hyperbolic* (this term is justified as by Theorem 3.5 R is uniformly contracting on the foliation $\mathcal{F}$).

Lemma 3.12. *For any $q \in (0,1)$, there exist $\delta > 0$ and $c > 0$ with the following property. For any $f \in \mathcal{A}$,*

$$\|DR^p_{tr}(f)\| \geq c(q\bar{\lambda}^{\delta})^p.$$

Proof. Fix some $\nu > 0$. Let us consider two hybrid equivalent germs $f, g \in \mathcal{C}$ with the modulus at least $\nu > 0$. Let $\mathrm{dist}_{Mon}(f,g) \leq \varepsilon$. Then

$$\|DR_{\mathrm{tr}}(f)\| \geq q\, \|DR_{\mathrm{tr}}(g)\|^{\delta}, \tag{3.3}$$

where $\delta = \delta(\varepsilon) > 0$, $q = q(\varepsilon) \in (0,1)$, and $q \to 1$ as $\varepsilon \to 0$. Moreover, there is a $\bar{p}$ such that if $p(M) \leq \bar{p}$ then we can let $\delta = 1$, and otherwise we can let $q = 1$.

This follows from the fact that the holonomy from g to f is transversally qc (Theorem 2.4). Indeed, (3.3) is obviously true for any particular renormalization R_M with $\delta = 1$ and $q = q(M)$, since R_M is smooth. So let us take a Mandelbrot copy M with a big period $p(M)$. Then by Corollary 2.23, $\operatorname{diam} M$ is small.

Let us now take the vertical fibers $\mathcal{S}, \mathcal{X}$ through g and f respectively. By Theorem 2.22, there is a disk $D \subset \mathcal{S}$ around g of small size $\xi > 0$ whose image under the renormalization has size of order 1 representing a quadratic-like family of class $\mathcal{G}_{C,\mu}$ with absolute C and μ. By Lemma 3.10, R has a bounded distortion on D. Hence $\|DR_{\mathrm{tr}}(g)\| \asymp \xi^{-1}$.

Furthermore, since the holonomy $\gamma : \mathcal{S} \to \mathcal{X}$ is quasi-conformal, it is Hölder continuous with some exponent $\delta = \delta(\mu) > 0$ and an absolute constant. Hence $\operatorname{diam}(\gamma D) = O(\xi^\delta)$, so that

$$\|DR_{\mathrm{tr}}(f)\| \asymp (\operatorname{diam}(\gamma D))^{-1} \geq q\|DR_{\mathrm{tr}}(g)\|^\delta.$$

Finally, since $\|DR_{\mathrm{tr}}(g)\|$ is big, we can kill the constant q by a small decreasing of the exponent. This yields (3.3).

Since by Lemma 3.10 the transverse distortion of R is bounded, we conclude that the same estimate holds under the assumption that f and g belong to the same renormalization strip and $|\chi(Rf) - \chi(Rg)| < \varepsilon$ (with the constants independent of the strip).

Given an $f \in \mathcal{A}$, let us consider the periodic point $g \in \mathcal{A}$ of period p which has the same itinerary $(M_0, \dots M_{p-1})$ as f. Then by Lemma 3.6 the orbit $\{R^k g\}_{k=N}^{p-N}$ ε-shadowes the corresponding orbit of f, where $N = N(\varepsilon)$, and the desired estimate follows from (3.3) by the chain rule (with the constant c coming from the first and the last N points of the trajectory). □

3.6. **Invariant cone field and line bundle.** Let us consider the family $\mathcal{B}_f \equiv \mathcal{B}_{V(f)}$, $f \in \mathcal{A}$, of Banach slices constructed in Lemma 3.4. Let $\|\cdot\|_f$ denote the norm in $\mathcal{B}_f$ and let $\mathcal{H}_f = \mathcal{B}_f \cap \mathcal{H}(f)$, $E^h_f = \mathrm{T}_f\mathcal{H}_f$. For $f \in \mathcal{A}$, let us consider the vertical θ*-cone*

$$C^\theta_f = \{u \in \mathcal{B}_f : \ \|u^v\|_f \geq \operatorname{tg}\theta\, \|u^h\|_f\}. \tag{3.4}$$

Lemma 3.13. *There exist $\theta > 0$ and N such that $DR^N(C^\theta_f) \subset C^{2\theta}_{R^N f}$ for any $f \in \mathcal{A}$.*

Proof. By Theorem 3.5, R is uniformly exponential contracting in the $\mathcal{F}$-direction. On the other hand, by Lemmas 3.11 and 3.12, R can only slowly contract in the transverse direction: $\|DR^n_{\mathrm{tr}}(f)\| \geq cq^n$, $f \in \mathcal{A}$, with q arbitrary close to 1 (and $c = c(q)$).

Recall that given a tangent vector $u \in \mathcal{B}_f$, u^h and u^v stand for its horizontal and vertical components, i.e., the projections to E^h_f and E^v_f

respectively. Then for N big enough, there exist $\rho' > \rho > 0$ with arbitrary small ratio ρ/ρ' such that

$$\|(R^N u)^h\|_{R^N f} \leq \rho \|u^h\|_f, \quad \|(R^N u)^v\|_{R^N f} \geq \rho' \|u^v\|_f. \tag{3.5}$$

Let $\rho < r < \rho'/2$. Then for any $u \in \partial C_f^\theta$,

$$\begin{aligned} \|(R^N u)^h\|_{R^N f} &= \|R^N u^h + (R^N u^v)^h\|_{R^N f} \leq \\ &\leq \rho \|u^h\|_f + O(\theta \|u^h\|_f) \leq r\|u^h\|_f, \end{aligned} \tag{3.6}$$

provided θ is sufficiently small. By the second inequality of (3.5) and (3.6), $R^N u \in C^{2\theta}_{R^N f}$.

Thus, $R^N(\partial C_f^\theta) \subset C^{2\theta}_{R^N f}$. Since the cones under consideration consist of two convex parts with bounded base ("above" and "below" the horizontal hyperplanes), this implies the assertion. □

We are ready to construct the unstable-to-be line bundle $\mathcal{O}^u$ over $\mathcal{A}$.

Lemma 3.14. *The renormalization operator has a continuous invariant tangent line field $\mathcal{O}^u = \{E^u_f \subset \mathcal{B}_f\}$, $f \in \mathcal{A}$, transverse to $\mathcal{F}$.*

Proof. This is a standard construction by going backwards and pushing the cones forward. Take θ and N from Lemma 3.13. Then for any $f \in \mathcal{A}$, the cones $C_f^{\theta,n} = R^{Nn} C^\theta_{R^{-Nn} f}$ are nested. Let $E^u_f = \bigcap_{n\geq 0} C_f^{\theta,n}$.

Let us consider the projective cone $\hat{C}_f^\theta$, i.e., the space of lines in C_f^θ. It can be realized as the cross-section of C_f^θ by the hyperplane $\{u : u^v = \text{const}\}$.

Supply the projective cones with the projective distance as follows: For $\hat{u}, \hat{v} \in \hat{C}_f$, consider the line interval $I(\hat{u}, \hat{v}) = \{w = \hat{u} + t\hat{v} \in \hat{C}_f^\theta\}$, and view it as the one-dimensional hyperbolic line $\mathbb{H}^1$ Then the projective distance between $\hat{u}$ and $\hat{v}$ is defined as the hyperbolic distance between $\hat{u}$ and $\hat{v}$ in $I(\hat{u}, \hat{v})$.

The embedding $C_f^{2\theta} \to C_f^\theta$ uniformly contracts the projective distance on these cones, while the differential $DR^N : C^\theta_{R^{-N} f} \to C^{2\theta}_f$ is at least simply contracting. Hence $DR^N : C^\theta_{R^{-N} f} \to C^\theta_f$ is uniformly contracting.

It follows that the projective cones $C_f^{\theta,n}$ uniformly exponentially shrink to some projective points. These points represent the tangent lines E^u_f transverse to $\mathcal{F}$. This line field, $\mathcal{O}^u$, is clearly R^N-invariant. Moreover, these properties uniquely determine the line field $\mathcal{O}^u$ (since R^N is projectively contracting). But $R(\mathcal{O}^u)$ is also a transverse one-dimensional R^N-invariant line field. Hence $R(\mathcal{O}^u) = \mathcal{O}^u$.

Finally, the line field $\mathcal{O}^u$ is continuous, since the cone field $\{C_f^{\theta,n}\}$ is continuous for any given n and well localizes the line field for n big enough. □

3.7. **Slow shadowing and hyperbolicity.**

3.7.1. We will now prove that $R|\mathcal{A}$ is uniformly hyperbolic. The idea is to construct (assuming the contrary) an $\mathrm{orb}(g)$, $g \notin \mathcal{H}(f)$, which slowly shadows some $\mathrm{orb}(f)$ on $\mathcal{A}$, which would contradict the Rigidity Theorem 2.24.

Let $\mathcal{O}^s$ stand for the field of tangent subspaces to $\mathcal{F}$ over $\mathcal{A}$ (the "horizontal field") and let $\mathcal{O}^u$ denote as above the transverse line field given by Lemma 3.14.

Theorem 3.15. *The renormalization operator $R : \mathcal{A} \to \mathcal{A}$ is uniformly hyperbolic with $\mathcal{O}^s$ and $\mathcal{O}^u$ serving as the stable and unstable fields.*

3.7.2. *Special bidisks.* We will begin the proof with a special choice of Banach slices and bidisks. Let $\mu > 0$ be an absolute bound from Lemma 3.3. In particular, $\mathcal{A} \subset \mathcal{C}(\mu)$.

Consider a number $N = N(\mu)$, a bound $\gamma = \gamma(\mu)$, and and a family of Banach slices $\mathcal{B}_f$, $f \in \mathcal{C}(\mu)$, from Lemma 3.4. In what follows we will allow ourselves to increase N (replacing it with some multiple of it) without changing the notations.

Select a $\delta_0 > 0$ in such a way that

$$\Pi\left(\bigcup_{f\in\mathcal{C}(\mu)} \mathcal{B}_f(2\delta_0)\right) \subset \mathcal{H}_0(\gamma/2).$$

Since $\mathcal{H}_0(\gamma/2)$ is compact, it can be covered with finitely many Banach neighborhoods $\mathcal{U}_i \Subset \mathcal{H}_0$ satisfying the property of Lemma 2.10, i.e., such that there exist equipped vertical tubes $\mathcal{P}_i \subset \Pi^{-1}\mathcal{U}_i$.

There exists a $\delta \in (0, \delta_0)$ such that for any $f \in \mathcal{C}(\mu)$, the ball $\mathcal{B}_f(2\delta)$ belongs to some tube $\mathcal{P}_f$ of the above finite family, and this property is persistent (i.e., it is satisfied for all $\tilde{f} \in \mathcal{C}(\mu)$ near f).

Let

$$\Pi_f = i_{\chi(f)} \circ \pi : \mathcal{Q} \to \mathcal{H}(f); \quad \mathcal{U}_f = \Pi_f(\mathcal{P}_f).$$

Take a small $\rho \in (0, 1/2)$. Then by Lemma 3.4 and Theorem 3.5, $N = N(\delta, \rho)$ can be increased so that

$$R^N(\mathcal{U}_f) \subset \mathcal{B}_{R^N f}(\rho\delta/8). \tag{3.7}$$

Furthermore, by Lemma 2.10, the tube $\mathcal{P}_f$ is foliated by holomorphic leaves $\hat{\mathcal{L}}_f(g)$, $g \in \mathcal{P}_f$, parametrized by the neighborhood $\mathcal{U}_f$. Denote this foliation by $\hat{\mathcal{F}}_f$. The slices of these leaves,

$$\mathcal{L}_f(g) = \hat{\mathcal{L}}_f(g) \cap \mathcal{B}_f(2\delta),$$

are codimension-one complex analytic submanifolds in $\mathcal{B}_f(2\delta)$. We denote by $\mathcal{F}_f$ the foliation of $\mathcal{B}_f(2\delta)$ by these slices.

Let $E^s_f(g) \subset \mathcal{B}_f$ stand for the tangent plane to the leaf $\mathcal{L}_f(g)$ at g, and $E^u_f(g) \subset \mathcal{B}_f$ stand for the complementary line through g parallel to the unstable line $E^u_f(f) \equiv E^u_f$ from Lemma 3.14. The δ-balls in these spaces centered at g and f will be denoted as $E^{s/u}_f(g,\delta)$ and $E^{s/u}_f(\delta) \equiv E^{s/u}_f(f,\delta)$ respectively.

Since the leaves $\hat{\mathcal{L}}_f(g)$ are almost parallel to $\hat{\mathcal{L}}_f(f)$ for g near f, their slices $\mathcal{L}_f(g)$ are locally graphs of holomorphic functions

$$\psi_{f,g} : E^s_f(1.5\,\delta) \to E^u_f \text{ with a small slope } \sigma_{f,g} < 1/2. \tag{3.8}$$

Let

$$\mathcal{V}_f = \psi_f(E^s_f(\delta)), \tag{3.9}$$

where $\psi_f \equiv \psi_{f,f}$.

Let us now define a *special bidisk* $Q_f \subset \mathcal{B}_f$ centered at f. Take a small topological disk $\mathcal{S}_f \subset E^u_f$ containing f, and consider its motion $\mathcal{S}_f \to \mathcal{S}_f(g)$ under the holonomy along $\mathcal{F}_f$, as g runs over $\mathcal{V}_f$ (in Lemma 3.16 below we will make a specific choice of $\mathcal{S}_f$). Then

$$Q_f = \bigcup_{g \in \mathcal{V}_f} \mathcal{S}_f(g). \tag{3.10}$$

We call the disks $\mathcal{S}_f(g)$ the *vertical cross-sections* of Q_f and we call the domain $\mathcal{V}_f$ the *base* of Q_f. Let

$$\partial^s Q_f = \bigcup_{g \in \mathcal{V}_f} \partial \mathcal{S}_f(g) \quad \text{and} \quad \partial^u Q_f = \bigcup_{g \in \partial \mathcal{V}_f} \mathcal{S}_f(g) \tag{3.11}$$

stand respectively for the horizontal and vertical boundaries of the bidisk Q_f.

Given a tangent vector $v \in \mathrm{T}_g \mathcal{B}_f$, its projections to $E^s_f(g)$ and $E^u_f(g)$ will be respectively denoted as v^s and v^u. Let

$$\Lambda_f(g) = \{v \in \mathrm{T}_g \mathcal{B}_f : \ \|v^u\| > \|v^h\|\}$$

stand for the tangent vertical "$\pi/4$-cone" at g.

Let Υ_f stand for the family of holomorphic curves Γ *properly embedded* into Q_f in the sense that $\partial\Gamma \subset \partial^s Q_f$, whose tangent lines $\mathrm{T}_g\Gamma$ belong to the cones $\Lambda_f(g)$.

For another special bidisk $\tilde{Q}_f$, we will naturally mark the corresponding objects with "tilde": $\tilde{S}_f(g)$, $\tilde{\Upsilon}_f$ etc. However, we let

$$\tilde{\Lambda}_f(g) = \{v \in \mathrm{T}_g\mathcal{B}_f : \ \|v^u\| > 8\|v^h\|\} \subset \Lambda_f(g)$$

For an N times renormalizable map $f \in \mathcal{C}(\mu)$, let $\mathcal{T}_f$ denote the renormalization strip $\mathcal{T}_M$, $M \in \mathcal{M}^N$, containing f (see §2.3.8).

Lemma 3.16. *There exists a family of special bidisks*

$$Q_f \subset \tilde{Q}_f \subset \mathcal{B}_f(\delta), \quad f \in \mathcal{A},$$

based on the $\mathcal{V}_f$, satisfying the following properties:

(i) *The renormalization $R^N \,|\, \mathcal{T}_f$ admits the analytic continuation to Q_f, and $R^N Q_f \subset \mathcal{B}_{R^N f}$.*

(ii) *Horizontal contraction: If $g \in Q_f$ and $v \in E^s_f(g)$, then*

$$\|DR^N v\|_{R^N f} \le \rho \|v\|_f,$$

where $\rho \in (0, 1/8)$ is as above.

(iii) *Invariance of the cone fields: If $g \in Q_f$ and $R^n g \in \tilde{Q}_{R^n f}$, then*

$$R^N(\Lambda_f(g)) \subset \tilde{\Lambda}_{R^N f}(R^N g).$$

(iv) *Overflowing property for high periods: There exists a $\bar{p}$ such that if $p(f) \ge \bar{p}$, $g \in \mathcal{V}_f$, then*

$$R^N(\partial\mathcal{S}_f(g)) \cap \tilde{Q}_{R^N f} = \emptyset \quad \textit{and} \quad R^N(\mathcal{S}^0_f(g)) \in \tilde{\Upsilon}_{R^N f},$$

where $\mathcal{S}^0_f(g)$ is the connected component of $\mathcal{S}_f(g) \cap R^{-N}(\tilde{Q}_{R^N f})$ containing g.

(v) *The slopes of the leaves $\mathcal{L}_f(g)$ are bounded by $1/2$.*

(vi) *The bidisks Q_f and $\tilde{Q}_f$ have a definite horizontal size in $\mathcal{B}_f$:*

$$\mathrm{dist}(f, \partial\mathcal{V}_f) > \xi > 0.$$

(vii) *The cross-sections $\mathcal{S}_f(g)$, $g \in \mathcal{V}_f$, have a bounded shape: There exists a $K > 0$ such that for any $M \in \mathcal{M}^N$, there exist $\varepsilon'(M) > \varepsilon(M) > 0$ with $\varepsilon' < K\varepsilon$ such that*

$$E^u_f(g, \varepsilon) \subset \mathcal{S}_f(g) \subset E^u_f(g, \varepsilon').$$

(Thus, the cross-sections $\mathcal{S}_f(g)$ have a definite size on every strip.)

(viii) *The cross-sections $\tilde{\mathcal{S}}_f(g)$, $g \in \mathcal{V}_f$, have a bounded shape and an absolute size:*

$$E^u_f(g, \tilde{\varepsilon}) \subset \tilde{\mathcal{S}}_f(g) \subset E^u_f(g, \tilde{\varepsilon}'),$$

where $0 < \tilde{\varepsilon} \le \tilde{\varepsilon}' \le K\tilde{\varepsilon}$ are absolute constants.
Moreover, $\tilde{\varepsilon}'$ can be assumed to be arbitrary small compared with ξ: $\tilde{\varepsilon}'/\xi < \varepsilon$ for any a priori chosen ε
(thus, the bidisks Q_f are "stretched" in the horizontal direction).

(ix) *Bounded vertical distortion: the renormalization $R^N|\mathcal{S}^0_f(g)$ composed with the projection $\mathcal{B}_{R^N f} \to E^u_{R^N f}$ has bounded distortion.*

Remark. One can add to this list one more nice property: the image $R^N(Q_f)$ is precompact in $\mathcal{B}_{R^N f}$.

Proof. By the discussion in §§2.3.3 and 2.2.3, the renormalization R^N analytically extends from $\mathcal{T}_f$ to the (truncated) parameter puzzle pieces $\hat{\Delta}_f(g)$ on the vertical fibers $\mathcal{P}_f(g)$ of the tube $\mathcal{P}_f$, $g \in \mathcal{U}_f$. By Theorem 2.22, $R^N(\hat{\Delta}_f(g))$ is a quadratic-like family of some class $\mathcal{G}^{\epsilon}_{C,\nu}$, where C and ν are independent on the particular f and g in question. Moreover, these puzzle pieces are related by holonomy along the foliation $\hat{\mathcal{F}}_f$, and thus form a tube $\hat{\mathcal{Y}}_f$ over $\mathcal{U}_f$. We let $\hat{\Delta}_f \equiv \hat{\Delta}_f(f)$.

Take a small $r \in (0, 1/2)$ and consider the hyperbolic disk Δ_f of radius r centered at f in the puzzle piece $\hat{\Delta}_f$ (where $\hat{\Delta}_f$ is supplied with the hyperbolic metric). For $g \in \mathcal{U}_f$, let $\Delta_f(g) \subset \hat{\Delta}_f(g)$ stand for the image of Δ_f under the holonomy $\hat{\Delta}_f \to \hat{\Delta}_f(g)$ along the foliation $\hat{\mathcal{F}}_f$. Since this holonomy is uniformly quasi-conformal, the hyperbolic diameter of $\Delta_f(g)$ in $\hat{\Delta}_f(g)$ is bounded by some $\kappa = \kappa(r)$ such that $\kappa(r) \to 0$ as $r \to 0$.

By the λ-lemma, the holomorphic motion which equips the family $R^N(\hat{\Delta}_f(g))$ has qc dilatation $O(\kappa)$ on $R^N(\Delta_f(g))$. It follows (compare Lemma 2.3) that for a sufficiently small r,

$$R^N(\Delta_f(g)) \subset \mathcal{B}_{R^N f}, \qquad \operatorname{diam}(R^N(\Delta_f(g))) < \rho\delta/8, \tag{3.12}$$

where the diameter is measured in $B_{R^N f}$.

For $g \in \mathcal{V}_f$, let us define $\mathcal{S}_f(g) \subset E^u_f(g)$ as the image of Δ_f under the holonomy to $E^u_f(g)$ along the foliation $\hat{\mathcal{F}}_f$. If r is sufficiently small, then

$$\mathcal{S}_f(g) \subset \mathcal{B}_f(g, \delta/2). \tag{3.13}$$

Taking the union of these cross-sections over $g \in \mathcal{V}_f$, we obtain a special bidisk Q_f of type (3.10), see Figure 5.

By the definition of the bases $\mathcal{V}_f$ (3.9), the 1/2-bound on their slope (3.8), and bound (3.13), the bidisks Q_f are contained in $\mathcal{B}_f(2\delta)$ and satisfies property (vi).

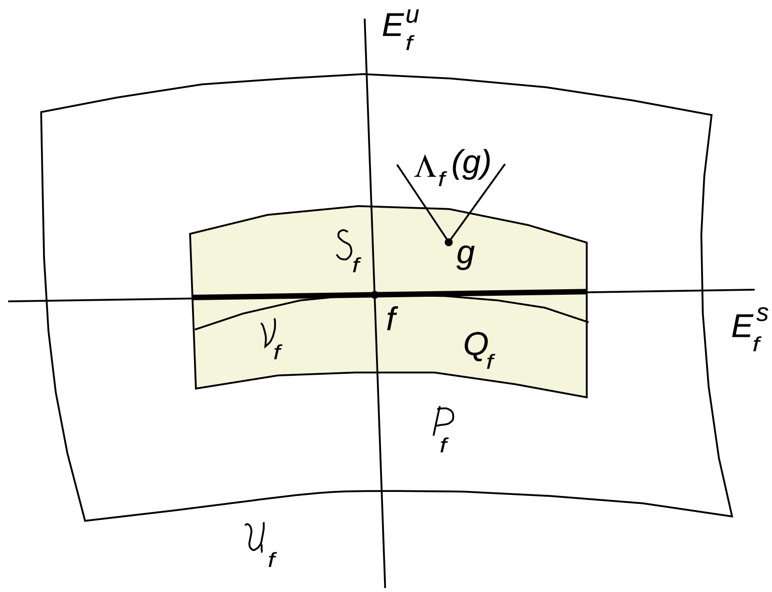

FIGURE 5. Special bidisk

Note that Q_f can also be defined as the intersection of the ball $\mathcal{B}_f(2\delta) \subset \mathcal{P}_f$ with the tube

$$\mathcal{Y}_f = \bigcup_{g \in \mathcal{U}_f} \Delta_f(g).$$

Thus, the analytic continuation of R^N to the tube $\hat{\mathcal{Y}}_f \supset \mathcal{Y}_f$ provides us with the analytic continuation of R^N to the bidisk Q_f. Together with (3.12), this implies (i).

Property (v) is ensured by (3.8).

Each vertical tube $\mathcal{P}_i$ is endowed with the qc straightening $\chi_i : \mathcal{P}_i \to \mathbb{C}$ (2.10). These straightenings restrict to straightenings $\chi_f : Q_f \to \mathbb{C}$ on the bidisks. By construction of the bidisks, χ_f maps every vertical cross-section of Q_f onto some topological disk D_f with bounded shape (independent of $p(f)$ and definite size (depending on $p(f)$):

$$\mathrm{dist}(\chi_f(f),\, \chi_f(\partial^s Q_f)) \geq \delta_0(p(f)) > 0. \tag{3.14}$$

These properties yield (vii).

Let us verify property (ii). By (3.7) and (3.12),

$$R^N(\mathcal{Y}_f) < \rho\delta/4.$$

Let $g_0 \in E^s_f(\delta)$ denote the projection of $g \in Q_f$ to E^s_f. By the Schwarz Lemma, the derivative of the map

$$R^N \circ \psi_{f,g} : E^s_f(g_0, \delta/2) \to \mathcal{B}_{R^N f}(\rho\delta/4)$$

at g_0 is bounded by $(\rho\delta/4) : (\delta/2) = \rho/2$. Hence the derivative of $R^N|\mathcal{L}_f(g)$ at g is bounded by $(\rho/2) : (1 - \sigma_{f,g}) < \rho$, as asserted by (ii).

Let us now construct bidisks $\tilde{Q}_f$. Take a small $\tilde{r} \in (0, 1/2)$, to be specified below. For $f \in \mathcal{C}(\mu)$, let $\mathcal{X}_f$ denote the hyperbolic disk of radius $\tilde{r}$ in the vertical fiber of the tube $\mathcal{P}_f$ through f. Similarly to (3.10), let us define $\tilde{\mathcal{S}}_f(g) \subset E^u_f(g)$ as the image of $\mathcal{X}_f$ under the holonomy to $E^u_f(g)$ along the foliation $\hat{\mathcal{F}}_f$. If $\tilde{r}$ is sufficiently small, then $\tilde{\mathcal{S}}_f(g) \subset \mathcal{B}_f(g, \delta/2)$. Let

$$\tilde{Q}_f = \bigcup_{g \in \mathcal{V}_f} \tilde{\mathcal{S}}^u_f(g).$$

This is a family of special bidisks in $\mathcal{B}_f(2\delta)$ satisfying condition (viii).

Furthermore, Corollary 2.12 implies that the curves $R^N(S_f(g))$ are uniformly transverse to the foliations $\tilde{\mathcal{F}}_{R^N f}$. By the Koebe Distortion Theorem, this implies the bounded distortion property (ix) (after some shrinking of the radius r). By the almost repelling of the renormalization in the transverse direction (Lemma 3.12), there exists a $\rho' \in (8\rho, 1)$ such that for any $f \in \mathcal{A}$ and $g \in Q_f$,

$$\|DR^N(v)\|_{R^N f} \geq \rho' \|v\|_f, \; v \in E^u_g \tag{3.15}$$

(after increasing N if necessary). Together with (ii) this implies (iii).

We will now adjust the parameters to ensure the overflowing property (iv) as well as the inclusions

$$Q_f \subset \tilde{Q}_f. \tag{3.16}$$

Let $f \in \mathcal{A}$, $g \in \mathcal{V}_f$, and $X = \mathcal{S}^0_f(g)$. By (iii), $R^N(X) \subset \mathcal{B}_{R^N f}(2\delta)$ is a graph over $E^u_{R^N f}$ with small vertical slope (bounded by 1/8). By (ii), $R^N f$ and $R^N g$ are very close:

$$\|R^N g - R^N f\| < \rho\delta/4 < \delta/4.$$

Hence the horizontal projection of $R^N(X)$ onto $E^s_{R^N f}$ belongs to $E^s_{R^N f}(\delta/2)$. On the other hand, the vertical boundary $\partial^u \tilde{Q}_{R^N f}$ is projected to $\partial E^s_{R^N f}(\delta)$. Hence $R^N(X) \cap \partial^u \tilde{Q}_{R^N f} = \emptyset$.

Assume now that there exists an $h \in \partial X$ such that $R^N h \in \tilde{Q}_{R^N f}$. Then $h \in \partial \mathcal{S}_f(g)$ and hence $h \in \partial \Delta_f(g)$. It follows that the hyperbolic distance between $R^N h$ and $R^N g$ in $R^N(\hat{\Delta}_f(g))$ is bounded from below by some $\kappa' = \kappa'(r) > 0$. But since $R^N(\hat{\Delta}_f(g))$ belongs to a compact class $\mathcal{G}^\epsilon_{C,\nu}$ of quadratic-like families (Theorem 2.22), $\mathrm{dist}_{\mathrm{hyp}}(R^N h, a) \geq \varepsilon = \varepsilon(\kappa') > 0$, where $a = \Pi_f(R^N g) \in \mathcal{U}_{R^N f}$ and the hyperbolic distance is measured in the vertical fiber of the tube $\mathcal{P}_{R^N f}$. But it is impossible, provided $\tilde{r} < \varepsilon$.

So, if $\tilde{r}$ satisfies the last estimate, then the R^N-image of each box Q_f overflows $\tilde{Q}_{R^N f}$. However, with this choice we can violate inclusions (3.16). But by Corollary 2.23, the vertical diameter of the tubes $\mathcal{Y}_f$ goes to 0 as the renormalization period $p(f)$ goes to ∞. Hence there exists a period $\bar{p}$ such that (3.16) is satisfied provided $p(f) \geq \bar{p}$. Let us redefine the box Q_f with $p(f) \leq \bar{p}$ to coincide with $\tilde{Q}_f$. Now all the inclusions (3.16) are satisfied, while the overflowing property is still satisfied for sufficiently high periods as required by (iv). □

Remark. It might be possible to work directly with the vertical tubes $\mathcal{Y}_f$ instead of the bidisks Q_f.

3.7.3. *Completion of the proof of Theorem 3.15.* Let $\mathcal{X}$ be the disjoint union of the Banach spaces $\mathcal{B}_f$, $f \in \mathcal{A}$, and $\mathcal{X}^0 \subset \mathcal{X}$ be the disjoint union of the corresponding bidisks Q_f. The spaces $\mathcal{B}_f$ will be called the *Banach fibers* of $\mathcal{X}$ over $\mathcal{A}$. Due to property (i) of Lemma 3.16, the renormalization $R^N : \mathcal{A} \to \mathcal{A}$ naturally lifts to an operator $\bar{R}^N : \mathcal{X}^0 \to \mathcal{X}$ fibered over $\mathcal{A}$.

By Lemma 3.12, hyperbolicity of R on $\mathcal{A}$ would follow from the uniform hyperbolicity of the periodic points of R. Assume the contrary: $\bar{\lambda} = 1$.

For $\tau \in (0, 1)$ near 1, let us consider the fiberwise linear contraction

$$T_\tau : \mathcal{X} \to \mathcal{X}, \quad f + \phi \mapsto f + \tau\phi, \quad f \in \mathcal{A},\ \phi \in \mathcal{B}_f.$$

Consider the perturbation $L_\tau : \mathcal{X}^0 \to \mathcal{X}$ of $\bar{R}^N$ by postcomposing $\bar{R}^N$ with this contraction: $L_\tau = T_\tau \circ \bar{R}^N$. Since $\bar{\lambda} = 1$, some periodic point f_τ of period p becomes attracting under this perturbation. Consider its basin of attraction:

$$A = \{h \in \mathcal{X}^0 : \ L_\tau^k(h) \in Q_{R^{(k+l)} f_\tau},\ k = 0, 1, \dots,$$
$$\|L_\tau^k(h) - R^{kN}(f_\tau)\|_{R^{kN} f} \to 0 \text{ as } k \to \infty\}.$$

If the vertical size of the bidisks Q_{f_τ} is sufficiently small compared with their horizontal size (property (viii) of Lemma 3.16), then by Lemma 7.1 from Appendix 3, there is an l and a map

$$g_\tau \in \partial^s Q_{R^l f_\tau} \cap A. \tag{3.17}$$

The overflowing property (iv) implies that

$$L_\tau(\partial^s Q_{f_\tau}) \cap Q_{R^N f_\tau} = \emptyset \text{ if } p(f_\tau) \geq \bar{p}.$$

Hence f_τ shadowed by g_τ satisfying (3.17) may belong only to finitely many renormalization strips $\mathcal{T}_M$, with $p(M) < \bar{p}$. By (3.14),

$$|\chi_{f_\tau}(f_\tau) - \chi_{f_\tau}(g_\tau)| \geq \delta_0 > 0. \tag{3.18}$$

Since the curves $R^N(\hat{\Delta}_f)$, $f \in \mathcal{A}$, belong to some class $\mathcal{G}^\epsilon_{C,\nu}$, a similar estimate is valid for them:

$$|\chi_{R^N f_\tau}(R^N f_\tau) - \chi_{R^N f_\tau}(L_\tau(g_\tau))| \geq \delta_1 > 0,$$

and hence $R^N f_\tau$ also may belong only to finitely many renormalization strips. Repeating this argument for the further iterates, we conclude that all the renormalizations $R^{Nk} f_\tau$ may belong only to finitely many renormalization strips (depending on k).

By Lemma 2.2, we can pass to limits $f = \lim f_{\tau_k}$ and $g = \lim g_{\tau_k}$ along some sequence $\tau_k \to 1$. Since the renormalizations of the maps f_τ may belong only to finitely many renormalization strips, the limit map f is infinitely renormalizable. By Lemma 2.16, the map g is also infinitely renormalizable with the same combinatorial type as f. By the Rigidity Theorem 2.24, g must be hybrid equivalent to f. But on the other hand, by (3.18) $\chi_f(g) \neq \chi_f(f)$ - contradiction. □

3.8. **Unstable foliation.**

3.8.1. *Statement.* We will now construct the global unstable foliation of the horseshoe $\mathcal{A}$. We will show, in particular, that the global unstable leaves transversally pass through all real hybrid classes except the cusp one.

Theorem 3.17. *There is a family $\mathcal{W}^u$ of holomorphic curves, "unstable leaves", $W^u(f)$, $f \in \mathcal{A}$, satisfying the following properties:*

a) *There is a well-defined branch $R^{-1}|W^u(f)$ such that*

$$R^{-1}W^u(f) \subset W^u(R^{-1}f);$$

b) *If $g \in W^u(f)$ then* $\mathrm{dist}_{Mon}(R^{-n}g, R^{-n}f) \leq C\rho^n$ *with $C = C(\varepsilon) > 0$ and $\rho = \rho(\varepsilon) \in (0,1)$;*

c) *Each unstable leaf $W^u(f)$ transversally intersects every hybrid class $\mathcal{H}_c$ with $c \in [-2, 1/4 - \epsilon]$ at a single point $g \in \mathcal{Q}_\mathbb{R}$;*

d) *The family $\mathcal{W}^u$ of the unstable leaves is normal*[2] *over*

$$Y = \bigcup_{f \in \mathcal{A}} W^u(f) \cap \chi^{-1}[-2, 1/4 - \epsilon];$$

e) *The renormalization R has uniformly bounded distortion on all the unstable leaves;*

[2]See the definition of normality in §2.2.5

f) *The straightening* $\chi : W^u(f) \to \mathcal{QP}$ *is uniformly quasi-conformal;*
g) *The real traces* $W^u(f) \cap \mathcal{Q}_{\mathbb{R}}$ *of the leaves are pairwise disjoint.*

The proof will be based on the discussion in §3.7, and the notations will be adopted from there without further comments. We will split the proof into several steps.

3.8.2. *Vertical expansion.* We can now add to the properties (i)-(ix) from Lemma 3.16 the property of *uniform vertical expansion*:

For any $\rho \in (0, 1)$, there exists an N such that in the corresponding family of special bidisks Q_f the following estimate holds:

$$\|DR^N(g)\, v\|_{R^N f} \geq \rho^{-1}\|v\|_f, \quad f \in \mathcal{A},\ g \in Q_f,\ v \in \Lambda_f(g). \tag{3.19}$$

Indeed, hyperbolicity of the horseshoe (Theorem 3.15) implies that there exists an N such that estimate (3.19) is valid for $g = f$ and $v \in \Lambda_f$. By the Hölder estimate for the transverse derivative (3.3), it is also valid for $g \in \mathcal{V}_f$ and $v \in \Lambda_f(g)$. By the bounded vertical distortion (ix), it is valid for any $g \in Q_f$.

3.8.3. *Local unstable manifolds.* Take a small number $q \in (0, 1)$ and scale down all the vertical cross-sections $\mathcal{S}_f$, $f \in \mathcal{A}$, by this factor. We obtain a family of special bidisks $Q'_f \subset Q_f$. Let us consider the family $\mathcal{X}_f$ of holomorphic curves $\gamma \subset Q_f$ via f whose tangent lines stay within the corresponding family of cones $\Lambda_f(g)$, $g \in Q_f$. Consider also a similar family $\mathcal{X}'_f$ in Q'_f but with additional assumption that these curves are properly embedded into Q'_f, i.e., $\partial\gamma \in \partial^s Q'_f$.

If $\gamma \in \mathcal{X}'_f$ and $R^k\gamma \subset Q_{R^k f}$, $k = 0, \dots, l$, then $R^k\gamma$ is a curve of $\mathcal{X}_{R^k f}$. Let D_k be the projection of $R^k\gamma$ onto $\mathcal{S}_{R^k f}$ via the holonomy along the foliation $\mathcal{F}_{R^k f}$. The vertical expanding property (3.19), bounded distortion property (ix), and the uniform transverse quasi-conformality of the foliations $\mathcal{F}_f$ imply that the domains D_k are quasi-disks with bounded shape around f_k (i.e., these disks have comparable outer and inner radii around f_k). By the vertical expanding property, the size of these disks grows exponentially in k.

It follows that there exists an l such that the curves $R^k\gamma$ intersect $\partial^s Q_{R^k f}$ for $k \geq l$. Since the disks D_k have bounded shape, the curves $R^k\gamma$ are properly embedded into $Q'_{R^k f}$:

$$R^k\gamma \cap Q'_{R^k f} \in \mathcal{X}'_{R^k f}. \tag{3.20}$$

Now we can construct the local unstable leaves W^u_{loc} in the usual way by letting

$$W^u_{\text{loc}}(f) = \lim_{k\to\infty} R^k \gamma_{-k},$$

where γ_{-k} is an arbitrary curve of $\mathcal{X}_{R^{-k}f}$ and $R\gamma$ is understood as the "cut-off" iterate: $R\gamma \cap Q_{Rh}$ for $\gamma \in \mathcal{X}_h$. Moreover,

$$W^u_{\mathrm{loc}}(f) = \{g \in Q_f : \exists\, g_{-n} \in Q_{R^{-n}f},\ Rg_{-n-1} = g_{-n},\ n = 0, 1, \dots\}, \tag{3.21}$$

where $R^{-n}f \in \mathcal{A}$ is well defined by Lemma 2.15. By the overflowing property (3.20), these leaves intersect Q'_f properly

Let now $\tilde{W}^u_{\mathrm{loc}}(f) = R^N(W^u_{\mathrm{loc}}(R^{-N}f))$. Property (iv) and (viii) of Lemma 3.16 imply that for some $\delta > 0$ and $\varepsilon > 0$, the leaves $\tilde{W}^u_{\mathrm{loc}}(f)$ intersect bidisks $E^h_f(\delta) \times E^v_f(\varepsilon) \subset \tilde{Q}_f$ properly. This implies that this family of leaves is normal.

3.8.4. *Global unstable leaves.* We will now globalize the local leaves $\tilde{W}^u_{\mathrm{loc}}(f)$ by iterating them forward. For $f \in \mathcal{A}$, let J^n_f stand for the set of real quadratic-like maps $g \in W^u_{\mathrm{loc}}(f)$ which are n times renormalizable with the same combinatorics as f and such that $\chi(R^n g) \in [-2, 1/4 - \epsilon]$. In other other words, this is the truncated renormalization window of R^n in $W^u_{\mathrm{loc}}(f)$ around f (see §1.5 and §2.3.2). Let $\tilde{J}^n_f = R^N(J^{n+N}_{R^{-N}f})$.

Lemma 3.18. *For all sufficiently big n, the leaves $R^n(\tilde{J}^n_f)$, $f \in \mathcal{A}$, intersect all hybrid classes $\mathcal{H}_c$ with $c \in [-2,\ 1/4 - \epsilon]$.*

Proof. By Theorem 3.8, any combinatorial past

$$\tau_- = \{\dots, J_{-2}, J_{-1}, c\}, \quad J_i \in \mathcal{J},\ c \in [-2,\ 1/4 - \epsilon]$$

can be realized by a one-sided tower $\{\dots, g_{-1}, g_0\}$ with absolute bounds: $\mathrm{mod}(g_{-k}) > \nu > 0$. On the other hand, take a two-sided tower $\{f_k\}_{k=-\infty}^{\infty} \subset \mathcal{A}$ with combinatorics $\{J_k\}_{k=-\infty}^{\infty}$ which has the same combinatorial past as τ_-. By Lemma 3.6, $\mathrm{dist}_{Mon}(f_{-k}, g_{-k}) < \varepsilon$ for all $k \ge l_0(\varepsilon)$. Hence for sufficiently big k's, $g_{-k} \subset \mathcal{B}_{f_{-k}}$ (if the slices $\mathcal{B}_f$ were chosen appropriately), and

$$\|g_{-k} - f_{-k}\|_{f_{-k}} \to 0 \textit{ as } k \to \infty. \tag{3.22}$$

Remark. The "appropriate" choice of the slices $\mathcal{B}_f \equiv \mathcal{B}_{V(f)}$ means under the circumstances that $\mathrm{dist}_H(\partial V(f), J(f)) < \eta(\nu)$, (where dist_H stands for the Hausdorff distance between sets). This $\eta > 0$ should be selected so small that if $g : U \to U'$ is a quadratic-like map with $\mathrm{mod}(U' \setminus U) > \nu$ and $\mathrm{dist}_H(K(g), K(f)) < \eta$, then $U \supset V(f)$.

Properties (iv) and (viii) of Lemma 3.16 imply that the straightening $\chi(R^N(Q_f \cap \mathcal{C}))$ covers an ε-neighborhood of $\chi(f)$ in M^0, with an absolute $\varepsilon > 0$. By (3.22), $\chi(g_{-(n+N)})$ eventually (for $n \ge l = l(\varepsilon)$) belongs to the ε-neighborhood of $f_{-(n+N)}$ in $[-2, 1/4 - \epsilon]$. Hence

$\chi(g_{-(n+N)}) \in \chi(Q_{f_{-(n+N)}})$. Since the bidisks Q_f have a definite horizontal size (property (vi)), $g_{-(n+N)} \in Q_{f_{-(n+N)}}$ for all $n \geq l$. By (3.21), $g_{-(n+N)} \in W^u_{\text{loc}}(f_{-(n+N)})$, Since $g_{-(n+N)}$ is clearly $n+N$ times renormalizable, it belongs to $J^{n+N}_{R^{-(n+N)}f}$. Thus, $g_{-n} \in \tilde{J}^n_{R^{-n}f}$ for $n \geq l(\varepsilon)$, and the conclusion follows. □

Normality of the family of local unstable manifolds $\tilde{W}^u_{\text{loc}}(f)$ and transverse control of the renormalization (Lemma §3.10) imply (similarly to the proof of the overflowing property (iv) from Lemma 3.16) that for any n there are simply connected domains $\tilde{\Omega}^n_f \subset \tilde{W}^u_{\text{loc}}(f)$ containing $\tilde{J}^n_f$ such that $R^n|\,\tilde{J}^n_f$ admits an analytic continuation to $\tilde{\Omega}^n_f$ and the family of curves $R^n(\tilde{\Omega}^n_f)$, $f \in \mathcal{A}$, is normal. Let us define the global unstable leaves as the images of these domains:

$$W^u(f) = R^n(\tilde{\Omega}^n_f),\ \ f \in \mathcal{A}.$$

These global leaves $W^u(f)$ satisfy condition a) of the theorem since the local leaves do. They satisfy b) by (3.19).

By Lemma 3.18, any leaf $W^u(f)$ intersects any hybrid class $\mathcal{H}_c$, $c \in [-2, 1/4 - \epsilon]$. By Theorem 3.8, the intersection point is unique. Transversality of the intersection follows from the corresponding property of the local unstable leaves and transverse non-singularity of R.

The normality condition d) is satisfied by construction. It implies e) by the Koebe Distortion Theorem and f) by Theorem 2.4. The last condition g) follows from Lemma 2.15.

Theorem 3.17 (and thus the Renormalization Theorem) is proven. □

4. Consequences

4.1. **Proof of Theorem 1.3.** Let us take any infinitely renormalizable parameter value $c \in \mathcal{I}$. By Theorem 3.7, there is a point $f \in \mathcal{A}$ with $\chi(f) = c$. Let

$$W^u_{\mathbb{R}}(f) = W^u(f) \cap \chi^{-1}[-2, 1/4 - \epsilon].$$

Then $J^n_f = R^{-n}(W^u_{\mathbb{R}}(R^n f)) \subset W^u_{\mathbb{R}}(f)$ (see §3.8.4 for the definition of J^n_f). By Theorem 3.17,

$$\operatorname{diam} J^n_f \to 0 \text{ as } n \to \infty.$$

Moreover, the same theorem implies (by means of the standard hyperbolic estimate of the distortion) that the map

$$R^n : J^n_f \to W^u_{\mathbb{R}}(R^n f) \tag{4.1}$$

has a uniformly bounded distortion.

For $g \in \mathcal{A}$, let us consider the interval

$$L(g) = W^u_{\mathbb{R}}(g) \cap \chi^{-1}(-3/4,\ 1/4-\epsilon)$$

consisting of maps $h \in W^u_{\mathbb{R}}(g)$ with attracting fixed point. Since the straightening $\chi : W^u_{\mathbb{R}}(g) \to [-2,\ 1/4-\epsilon]$ is uniformly quasi-symmetric (by Theorem 3.17),

$$\operatorname{diam} L(g)/\operatorname{diam} W^u_{\mathbb{R}}(g) \geq \delta > 0$$

for all $g \in \mathcal{A}$.

Let now $S_n(f) = R^{-n}(L(R^n f)) \subset J^n_f$. Since the distortion of (4.1) is bounded, $\operatorname{diam} S_n(f)/\operatorname{diam} J^n_f \geq \delta_1 >$ for all f and n. But the maps in $S_n(f)$ are only n times renormalizable. Hence the set of infinitely renormalizable maps has definite gaps in arbitrary small scales on $W^u_{\mathbb{R}}(f)$ near f. Using once more that the straightening is uniformly quasi-symmetric we conclude that the same property holds in the parameter interval $[-2,\ 1/4-\epsilon]$ near c. Thus, c is not a density point of $\mathcal{I}$, and the Lebesgue Density Points Theorem completes the proof. □

4.2. **Proof of Theorem 1.5.** Let $J = J^n_i(\varepsilon)$. As in the above proof, let us consider the interval $I = J^n_f \subset W^u(f)$, $f \in \mathcal{A}$, such that $\chi(I) = J$. Then

$$\sigma^n|J = \chi \circ R^n \circ \chi^{-1}|J.$$

As $R^n|I$ has bounded distortion and χ is uniformly quasi-symmetric, the conclusion follows. □

4.3. **Proof of Theorem 1.6.** Since by Theorem 3.17, the family $\mathcal{W}^u$ of unstable leaves is normal, there is a neighborhood $\Omega \subset M^0$ of $[-2,\ 1/4-\epsilon]$ in the Mandelbrot set covered by the straightening $\chi(W^u(f))$ of any leaf. On the other hand, by Lemma 2.23, the maximal Mandelbrot copies $M \in \mathcal{M}$ shrink as $p(M) \to \infty$. Hence there is a $\bar{p}$ such that $\chi(W^u(f)) \supset M$ for any $f \in \mathcal{A}$ and any $M \in \mathcal{M}$ with $p(M) \geq \bar{p}$.

Take a map $f \in \mathcal{A}$ with $\chi(f) = c$. Let

$$M^0(f) \supset M(f) \equiv M^1(f) \supset M^2(f) \supset \cdots \ni f$$

stand for the nest of the Mandelbrot copies in the unstable leaf $W^u(f)$ containing f. We have shown that if $p(R^n f) \geq \bar{p}$ then $M(R^n f) \Subset W^u(R^n f)$. But $M^n(f) = R^{-n} M(R^n f)$, and the map R^{-n} is contracting on the unstable foliation. It follows that $\operatorname{diam} M^n(f) \to 0$, provided there is a subsequence $n_k \to \infty$ such that $p(R^{n_k} f) \geq \bar{p}$.

The Mandelbrot sets M^{n_k} have a quasi-standard shape because on the unstable foliation the renormalization iterates R^{-n} have bounded non-linearity and the straightening χ has bounded dilatation. □

5. Appendix 1: Elements of analytic function theory in Banach spaces

Here we will state several basic facts of the analytic function theory in Banach spaces referring to [L5, §11.1] for proofs or references.

Given a complex Banach space $\mathcal{B}$, let $\mathcal{B}_r(x)$ stand for the ball of radius r centered at x in $\mathcal{B}$, and let $\mathcal{B}_r \equiv \mathcal{B}_r(0)$. Given another Banach space $\mathcal{D}$, $\mathcal{D}_r(x)$ and $\mathcal{D}_r$ will have the similar meaning.

Cauchy Inequality. *Let $f : (\mathcal{B}_1, 0) \to (\mathcal{D}_1, 0)$ be a holomorphic map between two Banach balls. Then $\|Df(0)\| \leq 1$. Moreover, for $x \in \mathcal{B}_1$,*

$$\|Df(x)\| \leq \frac{1}{1 - \|x\|}.$$

The Cauchy Inequality yields:

Schwarz Lemma. *Let $r < 1/2$ and $f : (\mathcal{B}_1, 0) \to (\mathcal{D}_r, 0)$ be a complex analytic map between two Banach balls. Then the restriction of f onto the ball $\mathcal{B}_r$ is contracting: $\|f(x) - f(y)\| \leq q\|x - y\|$, where $q = r/(1-r) < 1$.*

Let us state a couple of facts on the intersection properties between analytic submanifolds which provide a tool to the transversality results.

Let $\mathcal{X}$ and $\mathcal{S}$ be two connected submanifolds in the Banach space $\mathcal{B}$ intersecting at point x. Assume that $\operatorname{codim} \mathcal{X} = \dim \mathcal{S} = 1$ and $\mathcal{S} \subset \mathcal{X}$. Let us define the *intersection multiplicity* σ between $\mathcal{X}$ and $\mathcal{S}$ at x as follows. Select a local coordinate system (w, z) near x in such a way that $x = 0$ and $\mathcal{X} = \{z = 0\}$. Let us analytically parametrize $\mathcal{S}$ near 0: $z = z(t), w = w(t)$, $z(0) = 0, w(0) = 0$. Then by definition, σ is the multiplicity of the root of $z(t)$ at $t = 0$.

Hurwitz Theorem. *Under the above circumstances, let us consider a submanifold $\mathcal{Y}$ of codimension one obtained by a small perturbation of $\mathcal{X}$. Then $\mathcal{S}$ has σ intersection points with $\mathcal{Y}$ near x counted with multiplicity.*

As usual, a foliation of some analytic Banach manifold is called holomorphic/smooth if it can be locally straightened by a holomorphic/smooth change of variable.

Intersection Lemma. *Let $\mathcal{F}$ be a codimension-one holomorphic foliation in a domain of a Banach space. Let $\mathcal{S}$ be a one-dimensional complex analytic submanifold intersecting a leaf $\mathcal{L}_0$ of the foliation at a point x with multiplicity σ. Then $\mathcal{S}$ has σ simple intersection points with any nearby leaf.*

Corollary 5.1. *Under the circumstances of the above lemma, $\mathcal{S}$ is transverse to $\mathcal{L}_0$ at x if and only if it has a single intersection point near x with all nearby leaves.*

Let $X \subset \mathbb{C}$ be a subset of the complex plane. A *holomorphic motion* of X over a Banach ball $(\mathcal{B}_1, 0)$ (or, more generally, over a pointed complex analytic Banach manifold) is a a family of injections $h_\lambda : X \to \mathbb{C}$, $\lambda \in \mathcal{B}_1$, with $h_0 = \mathrm{id}$, holomorphically depending on $\lambda \in \mathcal{B}_1$ (for any given $z \in X$). The graphs of the functions $\lambda \mapsto h_\lambda(z)$, $z \in X$, form a foliation $\mathcal{F}$ (or rather a lamination as it is partially defined) in $\mathcal{B}_1 \times \mathbb{C}$ with codimension-one complex analytic leaves. This is a ""dual" viewpoint on holomorphic motions.

Given two complex one-dimensional transversals $\mathcal{S}$ and $\mathcal{T}$ to the lamination $\mathcal{F}$ in $\mathcal{B}_1 \times \mathbb{C}$, we have a partially defined holonomy $\mathcal{S} \to \mathcal{T}$. We say that this map is locally quasi-conformal if it admits local quasi-conformal extensions near any $(\lambda, z) \in \mathcal{S}$.

Given two points $\lambda, \mu \in \mathcal{B}_1$, let us define the hyperbolic distance $\rho(\lambda, \mu)$ in $\mathcal{B}_1$ as the hyperbolic distance between λ and μ in the one-dimensional complex slice $\lambda + t(\mu - \lambda)$ passing through these points in $\mathcal{B}_1$.

λ-Lemma. *Holomorphic motion h_λ of a set X over a Banach ball $\mathcal{B}_1$ is transversally locally quasi-conformal. The local dilatation K of the holonomy from $(\lambda, z) \in \mathcal{S}$ to $(\mu, \zeta) \in \mathcal{T}$ depends only on the hyperbolic distance ρ between the points λ and μ in $\mathcal{B}_1$. Moreover, $K = 1 + O(\rho)$ as $\rho \to 0$.*

6. Appendix 2: Complex structure on the space of quadratic-like germs

Let us consider the space $\mathcal{Q}$ of quadratic-like germs.

Let $\mathbf{V}$ be the set of topological discs $V \ni 0$ with piecewise smooth boundary symmetric with respect to the origin. Let $\mathcal{B}_V$ denote the affine space of normalized even analytic functions $f(z) = c + z^2 + \sum_{k>1} a_k z^{2k}$ on $V \in \mathbf{V}$ continuous up to the boundary supplied with sup-norm $\|\cdot\|_V$. Let $\mathcal{B}_V(f, \varepsilon)$ stand for the ε-ball in this space centered at f. We will identify the affine space $\mathcal{B}_V$ with its tangent linear space by putting the origin at the point $f(z) = z^2$. For $U \subset V$, let $j_{U,V} : \mathcal{B}_V \to \mathcal{B}_U$ stand for the restriction operator.

If a map $f : V \to V'$ is quadratic-like then all nearby maps $g \in \mathcal{B}_V$ are also quadratic-like on a slightly smaller domain. Thus, we have a natural inclusion j_V of some Banach ball $\mathcal{B}_V(f, \varepsilon)$ into $\mathcal{Q}$. We will call it a *Banach ball* or a *Banach slice* of $\mathcal{Q}$. Somewhat loosely, we will also use notation $\mathcal{Q}_V$ for such a Banach slice (without specifying f and ε). The inclusions $j_V : \mathcal{Q}_V \to \mathcal{Q}$ play a role of *Banach charts* on $\mathcal{Q}$ (though $\mathcal{Q}$ is not going to be a Banach manifold).

Lemma 6.1. *The family of local charts j_V satisfies the following properties:*

P1: *countable base and compactness. There exists a countable family of Banach slices $\mathcal{Q}_n \equiv \mathcal{Q}_{V_n}$ with the following property. For any $f \in \mathcal{Q}_V$, there is a $\delta > 0$ and a Banach slice $\mathcal{Q}_n$ such that $V_n \Subset V$, and the Banach ball $\mathcal{B}_V(f, \delta) \subset \mathcal{Q}$ is compactly embedded into $\mathcal{Q}_n$.*

P2: *lifting of analyticity. For $W \subset V$, the inclusion $j_{W,V} : \mathcal{Q}_V \to \mathcal{B}_W$ is complex analytic. Moreover, let $U \Subset V$. Let us consider a locally bounded map $\phi : \mathcal{V} \to \mathcal{B}_V$ defined on a domain $\mathcal{V}$ in some Banach space. Assume that the map $j_{W,V} \circ \phi : \mathcal{V} \to \mathcal{B}_W$ is analytic. Then the map $j_{U,V} \circ \phi : \mathcal{V} \to \mathcal{B}_U$ is analytic as well.*

P3: *density. If $W \subset V$, then the space $\mathcal{B}_V$ is dense in $\mathcal{B}_W$.*

Proof. P1. Consider a countable family $\mathbf{V}_0$ of topological disks $V_n \in \mathbf{V}$ with polygonal boundary and rational vertices. Then any disk $V \in \mathbf{V}$ can be approximated by some $V_n \Subset V$ from this family. Hence for any quadratic-like map $f \in \mathcal{Q}_V$, there exists a domain $V_n \Subset V$ such that the restriction of f to V_n is still quadratic-like, i.e., $f \in \mathcal{Q}_n \equiv \mathcal{Q}_{V_n}$. Then the same is true for all nearby $g \in \mathcal{B}_V(f, \delta)$.

Thus we have embedding $\mathcal{B}_V(f, \delta) \to \mathcal{Q}_n$. Since $V_n \Subset V$, $\mathcal{B}_V(f, \delta)$ is compact in $\mathcal{Q}_n$ by Montel's theorem.

P2. The first statement is obvious. Let us prove the lifting property. Without loss of generality ϕ can be assumed to be bounded. Let us first assume that V and $\mathcal{V}$ are round disks in $\mathbb{C}$ centered at 0, $V = \mathbb{D}_R$, $\mathcal{V} = \mathbb{D}_\varepsilon$. Let us use the notation $\lambda \mapsto \phi_\lambda(z)$, $\lambda \in \mathcal{V}$, $z \in V$. Since for any $\lambda \in \mathcal{V}$, ϕ_λ is a holomorphic in the round disk V, it admits an expansion

$$\phi_\lambda(z) = \sum_{k=0}^{\infty} a_k(\lambda) z^k, \quad \lambda \in \mathcal{V},\ z \in V.$$

Let us show that this series converges uniformly on compact subsets of $\mathcal{V} \times V$.

Since the image of ϕ bounded in $\mathcal{B}_V$, $|\phi_\lambda(z)| \le M$ for all $\lambda \in \mathcal{V}$, $z \in V$. By the Cauchy inequality,

$$|a_k(\lambda)| \le \frac{M}{R^k}, \quad \lambda \in \mathcal{V}. \tag{6.1}$$

Since $j_{W,V} \circ \phi : \mathcal{V} \to \mathcal{B}_W$ is holomorphic, $\phi_\lambda(z)$ is holomorphic in both variables on $\mathcal{V} \times W$. Hence the Taylor coefficients $a_k(\lambda)$ are holomorphic in $\lambda \in \mathcal{V} = \mathbb{D}_\varepsilon$:

$$a_k(\lambda) = \sum_{m=0}^{\infty} a_{k,m}\lambda^m.$$

Then by (6.1) and the Cauchy inequality,

$$|a_k(\lambda)| \leq \frac{M}{R^k \cdot \varepsilon^m}.$$

Hence for $|\lambda| < \delta < \varepsilon$, $|z| < \rho < R$ we obtain:

$$\sum_{m=0}^{\infty} |a_{k,m}||\lambda|^m |z|^k \leq \sum_{m=0}^{\infty} \frac{M}{R^k \varepsilon^m}\, \delta^m \rho^k = M \sum_{m=0}^{\infty} \left(\frac{\delta}{\varepsilon}\right)^m \sum_{k=0}^{\infty} \left(\frac{\rho}{R}\right)^k < \infty.$$

Thus this series converges unformly in the bidisk $\mathbb{D}_\delta \times \mathbb{D}_\rho$, so that $\phi_\lambda(z)$ is holomorphic in $\mathcal{V} \times V$. Hence $\phi : \mathcal{V} \to \mathcal{B}_U$ is holomorphic for any $U \Subset V$.

Let now $\mathcal{V}$ be an arbitrary Banach domain. Then we have proven that $\phi : \mathcal{V} \to \mathcal{B}_U$ is holomorphic on analytic curves in $\mathcal{V}$. Since ϕ is bounded, it follows that ϕ is holomorphic (see [D1]).

Finally, if V is an arbitrary domain in $\mathbb{C}$, then considering analytic continuation of ϕ_λ along chains of round disks in V, we conclude that ϕ is holomorphic on $\mathcal{V} \times V$, which yields the assertion.

P3. This property follows from a classical theorem saying that any function $f \in \mathcal{B}_W$ can be uniformly approximated by a polynomial. ☐

Given a set $\mathcal{X} \subset \mathcal{Q}$, the intersections $\mathcal{X}_V = \mathcal{X} \cap \mathcal{Q}_V = j_V^{-1}\mathcal{X}$ well be called a Banach slice of $\mathcal{X}$. By the intrinsic (or Banach) topology/metric on the slice $\mathcal{Q}_V$ we mean the topology/metric induced from the Banach space $\mathcal{B}_V$. We endow $\mathcal{Q}$ with the finest topology which makes all the local charts j_V continuous. In other words, a set $\mathcal{V} \subset \mathcal{Q}$ is declared to be open if and only if all its Banach slices $\mathcal{V}_V$ are intrinsically open.

Lemma 6.2. *The topological space $\mathcal{Q}$ satisfies the following properties:*

(i) *A sequence $\bar{f} = \{f_n\} \subset \mathcal{Q}$ converges to $f \in \mathcal{Q}$ if and only if there exists a finite family of Banach slices $\mathcal{Q}_i = \mathcal{Q}_{V_i}$ such that $f \in \cap \mathcal{Q}_i$, $\bar{f} \subset \cup \mathcal{Q}_i$, and the corresponding subsequences $\bar{f}^i = \bar{f} \cap \mathcal{Q}_i$ converge to f in the intrinsic topology of $\mathcal{Q}_i$.*

(ii) *A set $\mathcal{K} \subset \mathcal{Q}$ is compact (or sequentially compact) if and only if there exists a finite family of Banach slices $\mathcal{Q}_i$ and intrinsically compact subsets $\mathcal{K}_i \subset \mathcal{Q}_i$ such that $\mathcal{K} = \cup \mathcal{K}_i$. Thus, compactness and sequential compactness in $\mathcal{Q}$ are equivalent.*

(iii) *A compact set $\mathcal{K} \subset \mathcal{Q}$ is metrizable with a "Montel metric" dist_{Mon} induced from some $\mathcal{B}_V$ containing $\mathcal{K}$. The Montel metrics induced from different domains V are Hölder equivalent.*

(iv) *For a compact set $\mathcal{K} \subset \mathcal{Q}$, there exists $d = d(\mathcal{K}) > 0$ with the following property. Any covering of $\mathcal{K}$ with a family of Banach balls $\mathcal{B}_{V(f)}(f, \varepsilon(f))$ such that $\mathrm{dist}(J(f), \partial V(f)) < d$ admits a finite subcovering.*

Proof. (i) Since the inclusions $\mathcal{Q}_V \to \mathcal{Q}$ are continuous, any sequence $\bar{f} = \{f_n\} \subset \mathcal{Q}_V$ converging to f in $\mathcal{Q}_V$ converges to f in $\mathcal{Q}$ as well. Hence if $\bar{f}$ splits in finitely many subsequences converging to f in slices $\mathcal{Q}_i$, then the whole sequence converges to f in $\mathcal{Q}$.

Vice versa, let us consider the family $\mathcal{F}$ of Banach balls $B_V(f, \varepsilon)$ compactly contained in some $\mathcal{Q}_W$, $W \in \mathbf{V}$. Let us assume that a sequence $\bar{f}$ converges to f in $\mathcal{Q}$ but is not covered by finitely many Banach balls of $\mathcal{F}$. By countability property P1, we can select a subsequence which hits each Banach ball of $\mathcal{F}$ at most finitely many times (and never hits f itself). By definition of the $\mathcal{Q}$ topology, the complement of this sequence is a neighborhood of f - contradiction.

So let is consider finitely many Banach balls $\mathcal{V}_s \in \mathcal{F}$ compactly contained in $\mathcal{Q}_{k(s)}$ whose union covers $\bar{f}$. Any limit point $g \in \mathcal{Q}_{k(s)}$ of the subsequence $\bar{f}^s = \bar{f} \cap \mathcal{V}_s$ in the intrinsic topology of $\mathcal{Q}_{k(s)}$ is also a limit point of this subsequence in $\mathcal{Q}$. Since $\bar{f}$ converges to f in $\mathcal{Q}$, g must coincide with f. Hence $\bar{f}^s$ converges to f in the intrinsic topology of $\mathcal{Q}_{k(s)}$.

(ii) The argument is similar to the previous one.

The "if" direction is obvious. So assume $\mathcal{K}$ is compact (or sequentially compact). Let us consider the family $\mathcal{F}$ of Banach balls as above. If $\mathcal{K}$ is not covered with finitely many of these balls, then there is a sequence $\{f_n\} \subset \mathcal{K}$ which hits every V_s only finitely many times. By definition of the $\mathcal{Q}$ topology, this sequence does not have accumulation points - contradiction.

Thus $\mathcal{K}$ is covered with finitely many Banach balls $\mathcal{V}_s$. As each $\mathcal{V}_s$ is compactly contained in some slice $\mathcal{Q}_{i(s)}$, the statement follows.

(iii) By the previous result, $\mathcal{K}$ belongs to a finite union of Banach slices $\mathcal{Q}_{V_i}$. Hence there is a neighborhood V of 0 (not necessarily of family $\mathbf{V}$) such that $V \Subset \cap V_i$, so that $\mathcal{K} \subset \mathcal{B}_V$. Thus we can endow $\mathcal{K}$ with the Banach metric induced from $\mathcal{B}_V$. Since the inclusion $j_V : \mathcal{K} \to \mathcal{B}_V$ is continuous, it is a homeomorphism onto the image. Thus the above Banach metric is compatible with the topology of $\mathcal{K}$.

All these metrics are Hölder equivalent by the Hadamard Three Circles Theorem (see [L5, Lemma 11.5]).

(iv) Let us consider compact sets $\mathcal{K}_i \subset \mathcal{Q}_{V_i} \equiv \mathcal{Q}_i$ from (ii). Select a $d = d(\mathcal{K}) > 0$ in such a way that for $f \in \mathcal{K}_i$, $\operatorname{dist}(J(f), \partial V_i) > d > 0$. Then for $f \in \mathcal{K}_i$, $V(f) \subset V_i$ and hence $\mathcal{Q}_i \subset \mathcal{B}_{V(f)}$. It follows that the slice $\mathcal{B}_{V(f)}(f, \varepsilon(f)) \cap \mathcal{Q}_i$ is open in the intrinsic topology of $\mathcal{Q}_i$. Since $\mathcal{K}_i$ is compact in this topology, it admits a finite subcovering by these slices. □

We say that the family of local charts j_V endows $\mathcal{Q}$ with *complex analytic structure* modeled on the family of Banach spaces $\mathcal{B}_V$. More generally, if we have a set $\mathcal{S}$ and a family of inclusions $j_V : \mathcal{S}_V \to \mathcal{S}$, where $\mathcal{S}_V$ is an open set in $\mathcal{B}_V$, satisfying properties P1-P1 we say that $\mathcal{S}$ is endowed with complex analytic structure modeled on the family of Banach spaces $\mathcal{B}_V$. In what follows we will say briefly that $\mathcal{S}$ is a *complex space*. For instance, the hybrid class of z^2,

$$\mathcal{H}_0 = \{f \in \mathcal{Q} : \ f(0) = 0\},$$

is clearly a complex space.

Remark. Since the transit maps $j_{U,V}$ in $\mathcal{Q}$ are affine, $\mathcal{Q}$ is actually endowed with a *complex affine* structure. Then the hybrid class $\mathcal{H}_0$ becomes a codimension-one affine subspace in $\mathcal{Q}$.

Let us consider two complex spaces $\mathcal{S}^1$ and $\mathcal{S}^2$. A map $\phi : \mathcal{S}^1 \to \mathcal{S}^2$ is called *holomorphic* if for any $f \in \mathcal{S}^1$ and any Banach slice $\mathcal{S}^1_U \ni f$, there is an $\varepsilon > 0$ and a Banach slice $\mathcal{S}^2_V$ such that

$$\phi(\mathcal{B}_U(f, \varepsilon)) \subset \mathcal{S}^2_V, \tag{6.2}$$

and the restriction $\phi : \mathcal{B}_U(f, \varepsilon) \to \mathcal{S}^2_V$ is analytic in the Banach sense. Note that by P2, this property is independent of the choice of slice $\mathcal{S}^2_V$ satisfying (6.2) if to allow a little shrinking of V. In the case when Ω is a domain in $\mathbb{C}$, a holomorphic map $\gamma : \Omega \to \mathcal{S}$ is called a *holomorphic curve* in $\mathcal{S}$.

A subset $\mathcal{Q}^\#$ will be called a *slice* of $\mathcal{Q}$ if it is a union of some family of Banach balls $\mathcal{B}_V(f, \varepsilon) \subset \mathcal{Q}_V$. It naturally inherits from $\mathcal{Q}$ complex analytic structure.

Let us consider a complex space $\mathcal{S}$ and a point $f \in \mathcal{S}$. Let $\mathbf{V}_f = \{V \in \mathbf{V} : \ f \in \mathcal{Q}_V\}$. Let us call a point f of a complex space $\mathcal{S}$ *regular* if $\mathbf{V}_f$ is a directed set, i.e., for any U and V in $\mathbf{V}_f$, there exists a $W \in \mathbf{V}_f$ contained in $U \cap V$. At such a point we can define the tangent space $\mathrm{T}_f\mathcal{S}$ as the inductive limit of the Banach spaces $\mathcal{B}_V$, $V \in \mathbf{V}_f$ (see [L5, Appendix 2] for a discussion of inductive limits). If all points of a space $\mathcal{S}$ are regular, we call it a complex *manifold* modeled on the family of Banach spaces.

In the case of $\mathcal{S} = \mathcal{Q}$, $\mathbf{V}_f$ is the set of topological disks V on which f is quadratic-like. All points of the connectedness locus $\mathcal{C}$ are regular (in particular, all points of the space $\mathcal{H}_0$ are regular). The tangent space $\mathrm{T}_f\mathcal{Q}$, $f \in \mathcal{C}$, is identified with the space of germs of holomorphic vector fields $v(z)$ near the filled Julia set $K(f)$.

If $\phi : \mathcal{S}^1 \to \mathcal{S}^2$ is a holomorphic map between complex spaces and f, $\phi(f)$ are regular points in the corresponding spaces, then we can naturally define the differential $D\phi(f) : \mathrm{T}_f\mathcal{S}^1 \to \mathrm{T}_{\phi(f)}\mathcal{S}^2$ by restricting ϕ to the Banach slices. The differential continuously depends on f in the following sense. If $f_n \to f$ and $V \in \mathbf{V}_f$, then for sufficiently big n, the differentials $D\phi(f_n)$ are well-defined on the space $\mathcal{B}_V$, map it into some $\mathcal{B}_U$ (independent of n), and converge to $D\phi(f) : \mathcal{B}_V \to \mathcal{B}_U$ in the operator topology.

Let us now discuss a notion of a submanifold in a complex space $\mathcal{S}$. We will deal with two situations.

1) *Finite dimensional submanifold* (more generally, a Banach submanifold) is a subset in $\mathcal{S}$ which locally sits in some Banach slice $\mathcal{B}_U$ and is submanifold therein. By P2, this definition is independent of the choice of the slice $\mathcal{B}_U$ (up to a slight shrinking of U).

2) *Regular parametrized submanifold.* Let $\mathcal{M}$ be a complex manifold modeled on a family of Banach spaces. An analytic map $i : \mathcal{M} \to \mathcal{S}$ into a regular part of $\mathcal{S}$ is called *immersion* if for any $m \in \mathcal{M}$ the differential $Di(m)$ is a linear homeomorphism onto its image. The image $\mathcal{X}$ of an injective immersion i is called an *immersed submanifold.* It is called an *(embedded) submanifold* if additionally i is a homeomorphism onto $\mathcal{X}$ supplied with the induced topology. For example, if there is an analytic projection $\pi : \mathcal{S} \to \mathcal{M}$ such that $\pi \circ i = \mathrm{id}$ then $\mathcal{X}$ is a submanifold in $\mathcal{M}$.

If $i : (\mathcal{M}, m) \to (\mathcal{X}, f) \subset (\mathcal{S}, f)$ is an embedding, then the tangent space $\mathrm{T}_f\mathcal{X}$ is defined as the image of the differential $Di(m)$. It is a closed linear subspace in $\mathrm{T}_f\mathcal{Q}$. Its codimension is called the codimension of $\mathcal{X}$ at f. We say that a submanifold $\mathcal{X}$ has codimension d if it has codimension d at all its points.

Two submanifolds $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{S}$ are called *transverse* at a point $g \in \mathcal{X} \cap \mathcal{Y}$ if $\mathrm{T}_g\mathcal{X} \oplus \mathrm{T}_g\mathcal{Y} = \mathrm{T}_g\mathcal{S}$.

Let is finally make a remark on the space $\mathcal{E}$ of real analytic circle double coverings $f : V \to V'$ (see the definition in §2.1). This space is modeled on the family of Banach spaces $\mathcal{B}^s_V$ of even analytic functions on $f : V \to \mathbb{C}$ which commute with the involution about the circle. This latter condition turns $\mathcal{B}^s_V$ into the real (rather than complex) Banach space. Hence the above discussion allows us to supply $\mathcal{E}$ only with topology and real analytic structure modeled on the family of real Banach spaces. However, it is shown in [L5, Lemma 4.3] that $\mathcal{E}$ is naturally homeomorphic to the hybrid class $\mathcal{H}_0$ and thus inherits the complex structure from that space.

7. Appendix 3: Basins of attraction

In this appendix we will proof a modified version of Lemma 2.1 from [L5], which is the key to the construction of shadowing orbits (see §3.7). Consider a direct decomposition of a complex Banach space $\mathcal{B}$ into two subspaces E^h and E^v called respectively "horizontal" and "vertical". Let $p_{h/v} : \mathcal{B} \to E^{h/v}$ stand for the horizontal/vertical projections, and let $u^{h/v}$ stand for the corresponding components of a vector $u \in \mathcal{B}$. As in (3.4), C_f^θ will stand for the vertical tangent θ-cones in $\mathcal{B}$.

Take two domains $\Lambda \subset E^h$ and $S \subset E^v$ containing 0. Consider a domain $Q \subset \mathcal{B}$ containing 0 which is foliated by codimension-one complex analytic submanifolds $\mathcal{H}_c$ represented as graphs of analytic functions $\phi_c : \Lambda \to \mathbb{C}$, $c \in S$. (In other words, we consider a holomorphic motion of S over Λ which "fills in" Q.) We call $Q \approx \Lambda \times S$ a *foliated bidisk* over Λ with zero section S. Let $\partial^h Q = \partial Q \smallsetminus p_h^{-1} \partial \Lambda$.

Lemma 7.1. *Let* $0 < \eta < \theta < \pi/2$ *and*

$$0 < \varepsilon < \operatorname{tg} \theta - \operatorname{tg} \eta. \tag{7.1}$$

Consider the following data:

(i) *A decomposed complex Banach space* $\mathcal{B} = E^h \oplus E^v$ *with* $\dim E^v = d < \infty$*;*
(ii) *A complex analytic operator* $T : (D, 0) \to (\mathcal{B}, 0)$ *defined in a neighborhood* D *of 0;*
(iii) *A foliated bidisk* $Q \approx \Lambda \times S \subset D$ *over the unit ball* Λ *in* E^h *such that* $\operatorname{dist}(0, \partial S) < \varepsilon$*;*
(iv) *A forward invariant domain* $A \subset Q$ *such that* $\bar{A} \subset D$*.*

Assume the following properties:

H1. *The spectrum of the differential* $DT(0)$ *is contained in the open unit disk;*
H2. A *is contained in a basin of 0; moreover, if* $g \in \partial A \cap Q$ *then* $Tg \in \partial A$*;*
H3. *The cone field* $\{C_g^\theta\}$ *is invariant:* $DT(g)(C_g^\theta) \subset C_{Tg}^\theta$*;*
H4. *The slopes of the leaves* L_c*,* $c \in S$*, are bounded by* $\operatorname{tg} \eta$*.*

Then there exists a point $g \in \partial A \cap \partial^h Q$*.*

Proof. Let $\partial^v A = \partial A \cap p_h^{-1}\partial\Lambda$ and $\partial^h A = \partial A \setminus \partial^v A$. We will show that $\partial^h A \cap \partial Q \neq \emptyset$. Assume the contrary:

$$\partial^h A \cap \partial Q = \emptyset. \tag{7.2}$$

Let us consider a family $\mathcal{G}$ of immersed analytic manifolds $\psi : (\Omega, 0) \to (\Gamma, 0)$, where $\Omega = \Omega_\psi$ is a bounded domain in $\mathbb{C}^d$ and $\Gamma = \Gamma_\psi \subset A$, with the following properties:

A1. For any $z \in \Omega$, $f = \psi(z)$, the tangent space $\mathrm{T}_f\Gamma \equiv DT_z(\mathbb{C}^d)$, belongs to the cone C_f^θ.

A2. The manifolds are properly immersed into A in the sense that if a curve $\gamma(t) \subset \Omega$, $0 < t < \infty$, tends to $\partial\Omega$ as $t \to \infty$ then $\psi(\gamma(t))$ tends to the boundary ∂A.

Let us define $\partial\Gamma$, $\Gamma \in \mathcal{G}$ as the set of limit points of all curves $\gamma(t)$ as above. Then

$$\partial\Gamma \subset \partial^h A, \tag{7.3}$$

since assumptions H4, A1, (7.1) and (iii) imply that $\partial\Gamma$ does not intersect $\partial^v A$.

Note that the family $\mathcal{G}$ is non-empty: just let Γ be the connected component of $E^v \cap A$.

Furthermore, by Property A1, the projection $p_v : \Gamma \to E^v$ of any $\Gamma \in \mathcal{G}$ onto the vertical subspace is non-singular. Moreover, for any tangent vector $u \in \mathrm{T}_f\Gamma$,

$$\|u^v\| \asymp \|u\|. \tag{7.4}$$

Take now two balls $D^h \equiv D_r^h \subset E^h$ and $D^v \equiv D_\rho^v \subset E^v$ centered at the origin where the radii r and ρ satisfy the property

$$\rho < r \operatorname{tg} \theta \tag{7.5}$$

and are so small that the bidisk $\tilde{A} = D^h \times D^v$ is contained in A. Let $\tilde{\mathcal{G}}$ denote the family of immersed holomorphic submanifolds $\tilde{\psi} : \tilde{\Omega} \to \tilde{\Gamma}$ in $\tilde{A}$ satisfying properties Ã1 and Ã2 which are obtained from properties A1 and A2 by replacing the domain A with the bidisk $\tilde{A}$. Moreover, if $\tilde{\Gamma} \in \tilde{\mathcal{G}}$, then by (7.5) $\partial\tilde{\Gamma} \subset \partial^h \tilde{A}$.

Given a manifold $\tilde{\psi} : \tilde{\Omega} \to \tilde{\Gamma}$ of class $\tilde{\mathcal{G}}$, the map $p_v \circ \tilde{\psi} : \tilde{\Omega} \to D^v$ is non-singular and proper. Hence it is a diffeomorphism. It follows that the curve $\tilde{\Gamma}$ is a graph of a holomorphic function $D_\rho^v \to D_r^h$.

There is a natural restriction operator $\mathcal{G} \to \tilde{\mathcal{G}}$. Namely, given a curve $\psi : \Omega \to \Gamma$ in family $\mathcal{G}$, let $\tilde{\Omega}$ be the connected component of $\psi^{-1}\tilde{A}$ containing 0, and let $\tilde{\psi} : \tilde{\Omega} \to \tilde{\Gamma}$ be the restriction of ψ to it.

Let us supply the manifolds $\Gamma \in \mathcal{G}$ with the Kobayashi metrics. Recall that the Kobayashi norm of a tangent vector $v \in T_f\Gamma$ is defined as

$$\|v\|_\Gamma = \inf_\gamma \|w\|_P,$$

where $\|w\|_P$ stands for the Poincaré norm of a vector $w \in \mathrm{T}_0\mathbb{D}$, and the infimum is taken over all holomorphic curves $\gamma : (\mathbb{D}, w) \to (\Gamma, v)$ that factor via the parametrization $\psi : \Omega \to \Gamma$. The Kobayashi metric is invariant under holomorphic coverings and increases under shrinking the manifold.

Remark. A covering map between immersed manifolds is defined as a local homeomorphism satisfying the curve lifting property (where local homeomorphisms and curves are understood in terms of parametrizations of the manifolds). A holomorphic covering of finite degree can be also defined as a proper non-singular holomorphic map.

For a tangent vector $u \in \mathrm{T}_0\tilde{\Gamma}$, $\Gamma \in \mathcal{G}$, its Kobayashi norm is uniformly subordinate to the Banach one:

$$\|u\|_\Gamma \le C\|u\|, \tag{7.6}$$

where the constant C is independent of u. Indeed

$$\|u\|_\Gamma \le \|u\|_{\tilde{\Gamma}} = \|p_v u\|_{D^v}.$$

On the other hand, by (7.4),

$$\|u\| \asymp \|p_v u\| \asymp \|p_v u\|_{D^v}.$$

Let us now consider a manifold transformation $T_* : \Gamma_\psi \mapsto \Gamma_{T\circ\psi}$, $\Gamma \in \mathcal{G}$. By the invariant cone field assumption H3, $T \circ \psi : \Omega_\psi \to A$ is an immersion satisfying property A1. By assumption H3 and (7.2), it satisfies A2 as well. Thus T_* transforms $\mathcal{G}$ into itself. Moreover, the map $T : \Gamma \to T_*\Gamma$ is proper and non-singular, and hence is a holomorphic covering of finite degree.

Since the Kobayashi metric is invariant under holomorphic coverings, for any tangent vector $u \in \mathrm{T}_0\tilde{\Gamma}$ we have:

$$\|DT^n(u)\|_{T_*^n\Gamma} = \|u\|_\Gamma.$$

On the other hand, since 0 is an attracting point,

$$\|DT^n(u)\| \to 0 \quad \text{as} \quad n \to \infty.$$

These last two estimates contradict (7.6). $\square$

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