HOW BIG IS THE SET OF INFINITELY RENORMALIZABLE QUADRATICS?

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1. Introduction

The Mandelbrot set M is the bifurcation diagram of the complex quadratic family $P_c : z \mapsto z^2 + c$ viewed dynamically (see Figure 1). Despite the one line definition (see below), this set has quite a complicated structure, so that there is a little chance to understand it in full. Still, it is one of the simplest models to test some fundamental dynamical issues.

One of the central problems is to understand the "geometry" of M. As this set is "fractal", this requires special geometric terms. There are some "macroscopic" quantities which (like temperature or pressure in thermodynamics) can be measured and characterize the global geometry of a set. The best established quantities of such kind are associated to the notion of Hausdorff dimension. In particular, a "fractal set" is usually defined as a set whose Hausdorff dimension is greater than the topological dimension (see Mandelbrot [Man]). (Keep in mind a plane Jordan curve of positive Lebesgue measure).

It is still unknown whether the boundary of the Mandelbrot set has zero Lebesgue measure. However, by a remarkable result of Shishikura [Sh1], ∂M has Hausdorff dimension 2, so that it is indeed fractal according to the previous definition.

The quadratic family is extremely rich. There is a great combinatorial diversity in this family, and one wishes to estimate which combinatorial types prevail. A basic classification of quadratics is motivated by the renormalization theory. At the computer pictures one can see inside of M little copies of itself, all over the place (see Figure 2). A quadratic polynomial P_c is said to be renormalizable if c belongs to such a copy (different from the whole set). It is said to be twice renormalizable if it belongs to a nest of two copies, etc. Let $\mathcal{I} \subset M$ stand for the set of infinitely renormalizable quadratics (which belong to infinite nests of little Mandelbrot sets), and $\mathcal{F} = M \setminus \mathcal{I}$.

Shishikura's Theorem actually shows that the set $\mathcal{F}\partial M$ has dimension two but does not tackle the problem of how big the set $\mathcal{I} \cap \partial M$ is. This problem has been floating around for a while. All kind of guesses has been made in the whole range from naive 0 to 2. In this paper we develop a method which shows that the dimension of $\mathcal{I}\cap\partial M$ is at least 1/2 of the dimension of the whole boundary ∂M.

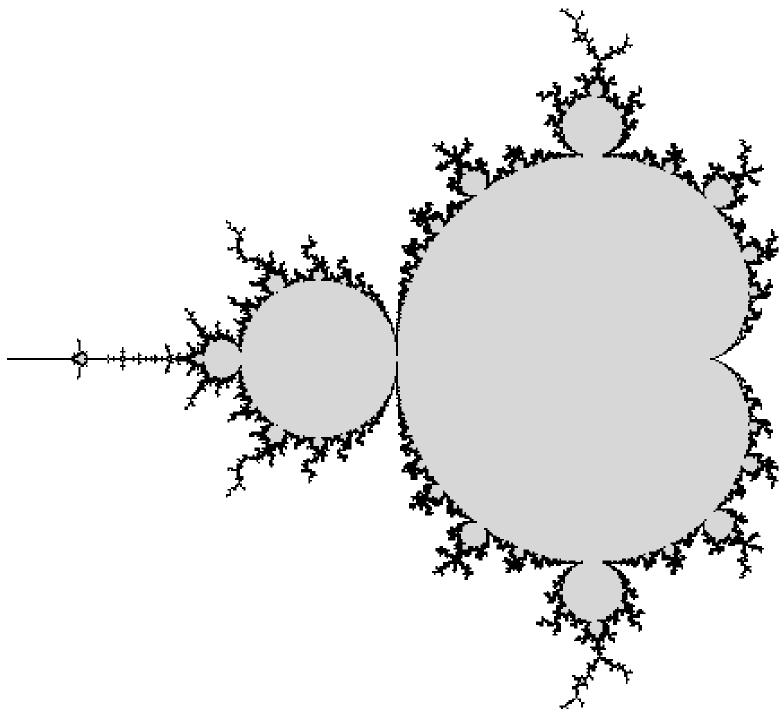

Figure 1. The Mandelbrot set.

Theorem 1.1. *The set $\mathcal{I}\cap\partial M$ of infinitely renormalizable complex parameter values on the boundary of M has Hausdorff dimension at least 1. The set $\mathcal{I}\cap\mathbb{R}$ of infinitely renormalizable real parameter values has Hausdorff dimension at least 1/2.*

Briefly, the proof goes as follows. Given some non-renormalizable quadratic polynomial f, we construct (by means of the "puzzle" machinery) a family of little Mandelbrot sets whose geometry is under control. We estimate the critical exponent for this family by one half of the "hyperbolic Hausdorff dimension" of the Julia set $J(f)$. Taking Shishikura's polynomial f with the Julia set of hyperbolic dimension 2, we obtain the above estimate.

Acknowledgement. Mitsu Shishikura has told me that by means of parabolic bifurcation he can construct a set of infinitely renormalizable complex parameter

values of dimension at least 1/2, and a set of infinitely renormalizable real parameter values of dimension at least 1/3.

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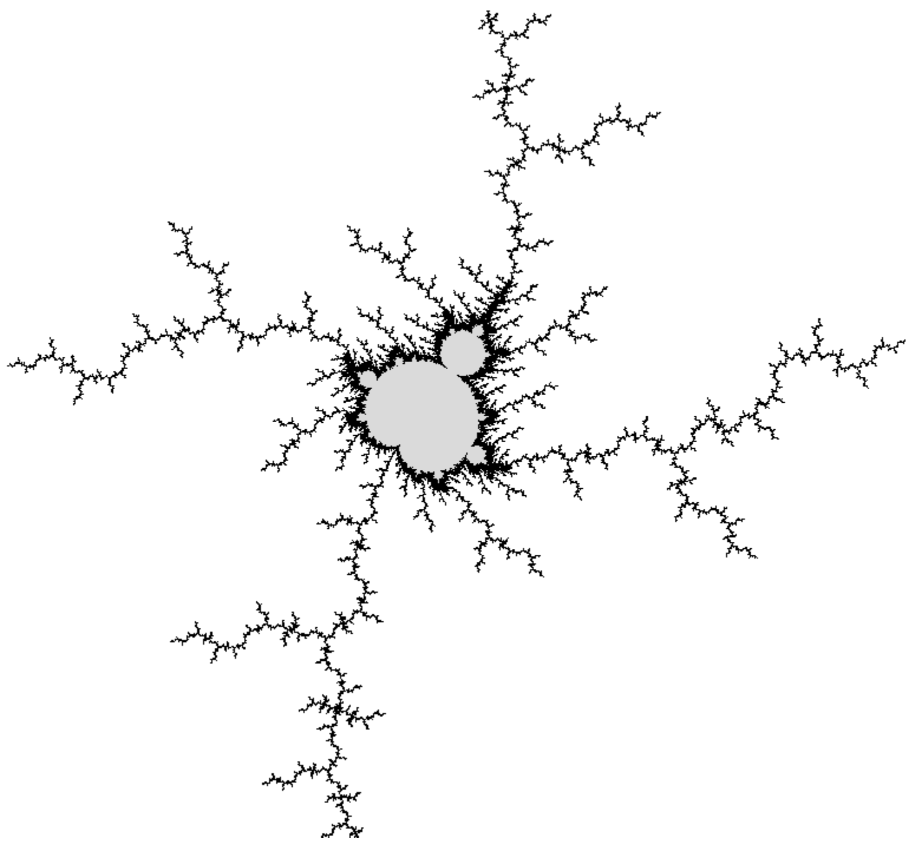

Figure 2. A little copy of the Mandelbrot set.

2. Preliminaries

2.1. General terminology and notations. Let $\mathbb{C}$ denote as usual the complex plane; $\mathbb{N} = \{1, 2, , \dots\}$; $D(z, r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\}$.

A *topological disk* is a simply connected domain in $\mathbb{C}$. A *topological annulus* is a doubly connected domain. The *modulus* of a topological annulus, $\mod A$, is equal to $\log(R/r)$, provided A is conformally equivalent to a round annulus $\{z : r < |z| < R\}$.

We assume that the reader is familiar with the standard theory of quasi-conformal maps (see e.g., [A] for a reference). We will use the abbreviation "qc" for quasi-conformal maps, and "K-qc" for qc maps with dilatation bounded by $K \geq 1$.

2.2. Hausdorff dimension. For a detailed account of the notion of Hausdorff dimension see e.g., [Mat]. To be definite, let everything below happen inside the complex plane $\mathbb{C}$.

Given a $\delta \geq 0$, the Hausdorff δ-measure h_δ is defined as follows

$$h_\delta(X) = \lim_{\epsilon \to 0} \inf \sum (\operatorname{diam} U_i)^\delta,$$

where the infimum is taken over all coverings of X by sets U_i of diameter at most $\epsilon > 0$. For any X there is a unique critical exponent separating infinite and vanishing values of the measure $h_\delta(X)$. This exponent is called the Hausdorff dimension of X and is denoted by $\mathrm{HD}(X)$. Note that one can clearly use only round disks for the sets U_i in the above definition. Note also that if $X = \cup X_i$ then

$$\mathrm{HD}(X) = \sup \mathrm{HD}(X_i). \tag{2.1}$$

Given a Borel measure μ, the Hausdorff dimension $\mathrm{HD}(\mu)$ is defined as the infimum of the $\mathrm{HD}(X)$ when X runs over all measurable sets of full measure.

Let $X \subset \mathbb{C}$ be a compact set invariant under certain analytic map f defined in a neighborhood of X. Let us call such a set "dynamical". A dynamical set X is called *hyperbolic* if there exist $C > 0$ and $\lambda > 1$ such that for all $z \in X$

$$|Df^n(z)| \geq C\lambda^n, \; n = 0, 1, \ldots$$

The *hyperbolic dimension* $\mathrm{HD}_{\mathrm{hyp}}(X)$ of a dynamical set X is defined as the supremum of the dimensions of all invariant hyperbolic subsets of X (see [DU, Sh1]).

2.3. Quadratic family. Let us consider the complex quadratic family $P_c : z \mapsto z^2 + c$. The filled Julia set $K_c \equiv K(P_c)$ is defined as the set of non-escaping points (where "escaping" means convergence to ∞). The Julia set $J_c \equiv J(P_c)$ is the boundary of K_c. It is a compact invariant subset of $\mathbb{C}$. Moreover, J_c is either totally disconnected (Cantor) or connected depending on whether the the critical point 0 is escaping or not.

The Mandelbrot set M is defined as the set of parameter values $c \in \mathbb{C}$ for which the Julia set J_c is connected. Notice that the Mandelbrot set is originated from a big domain containing 0 and bounded by the "main cardioid". It is specified by the property that the maps in this domain have an attracting fixed point. This domain will play a special role for our discussion.

Theorem A (Shishikura [Sh1]). *For a generic $c \in \partial M$, $\mathrm{HD}_{hyp}(J_c) = 2$.*

Here "genericity" is understood in the sense of Baire category as a property which occurs on the complement of a countable union of nowhere dense sets.

Note that, on the other hand, it is well-known that for a generic $c \in \partial M$, the orbit of the critical point is dense in the Julia set J_c. Such maps are non-renormalizable in the sense explained below.

2.4. Quadratic-like maps and little Mandelbrot sets. There is a very fruitful generalization of the notion of a quadratic polynomials due to Douady and Hubbard [DH]. Let $U \Subset V$ be two topological discs. A branched double covering $f : U \to V$ is called a *quadratic-like map*. Unless otherwise is assumed, we normalize such a map to put its critical point at the origin.

A quadratic-like map is usually considered up to choice of domains U and V, so that it should be more carefully called a quadratic-like *germ*. Let $\mod(f) = \sup \mod(V \setminus U)$ where the supremum is taken over all possible choices of U and V. The bigger $\mod(f)$, the better geometric control over f we have (the closer f to being purely quadratic).

The filled Julia set $K(f)$ of a quadratic-like map is also defined as the set of non-escaping points (where now "escaping" means landing at the fundamental annulus $V \setminus U$ under some iterate of f), and the Julia set $J(f)$ is defined as the boundary of $K(f)$. For the Julia set of a quadratic-like map, there is the same dichotomy (either Cantor or connected) as for a polynomial.

Assume now that we have an analytic quadratic-like family $\mathbf{f} = \{f_\lambda : U_\lambda \to V_\lambda\}$ over a parameter domain D. Let us call such a family *proper* if $f_\lambda 0 \in \partial V_\lambda$ for $\lambda \in \partial D$. For a proper quadratic-like family, let us define the *winding number* $w(\mathbf{f})$ as the winding number of the critical value $f_\lambda 0$ around the critical point 0 when λ goes once around ∂D. A proper family is called *unfolded* if it has winding number 1.

Similarly to the polynomial case, the Mandelbrot set $M(\mathbf{f})$ of a quadratic-like family is defined as the set of parameter values $\lambda \in D$ for which the Julia set $J(f_\lambda)$ is connected. Let us now state the following fundamental result:

Theorem B (Douady & Hubbard [DH]**).** *For any unfolded proper holomorphic family* $\mathbf{f}$*, the Mandelbrot set* $M(\mathbf{f})$ *is canonically homeomorphic to the standard Mandelbrot set* M.

The canonical homeomorphism $\tau : M(\mathbf{f}) \to M$ is also called *straightening*.

Theorem B provides a way to construct little Mandelbrot copies inside of M (see Douady [D]). Assume that we managed to find an $n \in \mathbb{N}$, a parameter region $D \subset \mathbb{C}$ and a proper unfolded quadratic-like family $g_c = P_c^n : U_c \to V_c$ over c, where $0 \in U_c \Subset V_c$ are topological disks around the origin. Then by Theorem B this family generates a little homeomorphic copy of the Mandelbrot set. Moreover, the maps P_c for which $c \in M(\mathbf{f})$ are called *renormalizable*, and g_c (considered up to rescaling) is called the *renormalization* of P_c.

Note that for a renormalizable map f, the orbit of the critical point cannot accumulate on the both fixed point. In particular, it is not dense in $J(f)$.

Given a quadratic-like family $\mathbf{f} = \{f_\lambda\}$, $\lambda \in D$, let

$$\text{mod}\ (\mathbf{f}) = \inf_{\lambda \in D} \quad \text{mod}\ (f_\lambda).$$

Theorem C [L4]. *For a proper unfolded quadratic-like family* $\mathbf{f}$ *over* D*, the straightening* $\tau : M(\mathbf{f}) \to M$ *admits a* K*-qc extension to* D*, with dilatation* K *depending only on* mod $(\mathbf{f})$*. Moreover* $K \to 1$ *as* mod $(\mathbf{f}) \to \infty$.

2.5. Puzzle and parapuzzle. "Puzzle" is a powerful technique which opens many locks of holomorphic dynamics. For complex quadratic polynomials it was introduced by Yoccoz in 1990 (see [H] and [M]) as a tool to the celebrated MLC Conjecture ("the Mandelbrot set is locally connected"), and has been successfully applied to many other problems since then (see the survey [L5]). Let us start with a simple but important consequence:

Proposition D [L1]. *Let* f *be a non-renormalizable quadratic-like map, and* $Q \subset J(f)$ *be a compact invariant set which does not contain the critical point. Then* Q *is hyperbolic.*

(Note that, vice versa, obviously hyperbolic sets may not contain the critical point).

Below we will outline the author's treatment on the puzzle whose detailed account is given in [L2] - [L3]. Our approach is based on the idea of a generalized quadratic-like map and a generalized renormalization. Let us consider a finite or infinite family of topological disks U_i compactly contained in a topological disk V. A map $g : \cup U_i \to V$ is called *generalized quadratic-like* if it is quadratic-like on one of these disks, labeled as U_0, and a conformal diffeomorphism $U_i \to V$ on the others. The filled Julia set $K(g)$ is defined as the set of points which never escape $\cup U_i$.

Given a quadratic-like map, we construct (by means of the puzzle) a "principal nest" $V^0 \supset V^1 \supset \dots$ of topological disks, and a sequence of generalized quadratic-like maps $g_n : \cup V_i^n \to V^{n-1}$ (where $V_0^n \equiv V^n$) called *generalized renormalizations.* The map g_n is constructed inductively as the first return map to V^{n-1} (and a quite special initial construction is needed for V^0; note that this initial construction is not quite canonical but involves some choice). The domains V_i^n will also be called *puzzle pieces.*

Let us now consider a proper unfolded quadratic-like family f_λ over a region D. Let $\lambda_0 \in D$. Then for each n, there is a parameter domain $P^n = P^n(\lambda_0)$ ("parapuzzle piece") such that all the maps f_λ, $\lambda \in P^n$ have "the same combinatorics" up to level n. This means that the domains of the generalized renormalizations $g_{n,\lambda} : V_{i,\lambda}^n \to V_\lambda^{n-1}$ move continuously with $\lambda \in P^n$ (after an appropriate choice of the initial domains V_λ^0).

We call a holomorphic function ψ *γ-linear* if it is univalent and has non-linearity bounded by γ:

$$\sup_{z,\zeta} \log \left| \frac{\psi'(z)}{\psi'(\zeta)} \right| \le \gamma.$$

We call it *γ-quadratic* if it is a composition of a purely quadratic and a γ-linear function.

Theorem E [L3]. *Let $\mathbf{f} = \{f_\lambda,\ \lambda \in D\}$, be a proper unfolded quadratic-like family. Let $\lambda_0 \in D$ be a non-renormalizable parameter value. Then there is a subsequence of levels n with the following properties:*

- *The quadratic-like family $\mathbf{g}_n = \{g_{n,\lambda} : V_\lambda^n \to V_\lambda^{n-1},\ \lambda \in P^n(\lambda_0) \equiv P^n\}$, is proper and unfolded;*
- $\mod(\mathbf{g}_n) \to \infty$;
- *The maps $g_{n,\lambda}$ are γ_n-quadratic, where $\gamma_n \to 0$ as $n \to \infty$;*
- *The critical value $\phi_n(\lambda) = g_{n,\lambda}(0)$ is γ_n-linear on P^n.*

The last statement will allow us to compare the sizes of certain Julia and Mandelbrot sets (according to the general philosophy of correspondence between dynamical and parameter objects).

3. Linear model.

Let us start with a piecewise linear model which will provide us good heuristics of the phenomenon (see [PW] for a detailed discussion of this kind of models). Let $g : \cup U_i \to V$ be a piecewise linear map defined on the union of (perhaps infinitely many) topological disks U_i with pairwise disjoint closures compactly contained in V. Let $Q(g) = \{z : g^n z \in \cup U_i,\ n = 0, 1, \dots\}$ stand for the "Julia set" of g. Let λ_i denote the derivative of $g|U_i$.

Let δ_{cr} denote the *critical exponent* for the sequence $(\lambda_i)^{-1}$, defined by the following equation:

$$\sum_i \frac{1}{\lambda_i^{\delta_{cr}}} = 1. \tag{3.1}$$

Lemma 3.1. *For a piecewise linear map $g : \cup U_i \to V$, the Hausdorff dimension and the critical exponent are equal:* $\mathrm{HD}(Q(g)) = \delta_{cr}$.

Proof. Let us start with the upper estimate for the dimension. Let

$$P(\delta) = \sum \frac{1}{\lambda_i^\delta}.$$

Then for any $\delta > \delta_{cr}$, $P(\delta) < 1$. For $\bar{i} = (i_0, \dots i_{n-1})$, let $U_{\bar{i}}^n \equiv U_{i_0 \dots i_{n-1}}^n$ denote the "cylinders" of rank n, that is, the topological disks mapped onto V under g^n, and such that

$$g^k U_{i_0 \dots i_{n-1}}^n \subset U_{i_k},\ k = 0, \dots n-1.$$

Normalizing V in such a way that $\operatorname{diam} V = 1$, we have

$$\operatorname{diam} U^n_{i_1\dots i_n} = \frac{1}{\lambda_{i_0}\dots\lambda_{i_{n-1}}}.$$

Hence

$$\sum (\operatorname{diam} U^n_{i_0\dots i_{n-1}})^\delta = P(\delta)^n \to 0 \quad \text{as} \quad n\to\infty.$$

Hence $h_\delta(Q(g)) = 0$, and $\mathrm{HD}(Q(g)) \le \delta$. As this holds for any $\delta > \delta_{cr}$, we conclude that $\mathrm{HD}(K(g)) \le \delta_{cr}$.

To estimate the dimension of $Q(g)$ from below we will use a well-known Billingsley's trick exploiting measures with certain local properties (see [B]).

Take any $\delta < \delta_{cr}$. Then $P(\delta) > 1$. Let us keep finitely many sets U_i which still satisfy this property. It is enough to show that the Hausdorff dimension of the corresponding Julia set is at least δ. So keeping the same notations let us work further with this truncated system.

Let $\mu_{n,\delta}$ denote a probability measure which assigns to every cylinder $U^n_{\bar i}$ the mass $(\operatorname{diam} U^n_{\bar i})^\delta / P(\delta)^n$ (the particular choice of the measure with this property is not important). Let μ be any limit measure for sequence $\mu_{n,\delta}$. Clearly μ is supported on $Q(g)$.

Let us show that for any cylinder $U^l_{\bar j}$,

$$\mu(U^l_{\bar j}) \le (\operatorname{diam} U^l_{\bar j})^\delta. \tag{3.2}$$

As $\operatorname{supp} \mu = Q(g)$ does not intersect $\partial U^l_{\bar j}$, it is enough to show that for any n,

$$\mu_{l+n,\delta}(U^l_{\bar j}) \le (\operatorname{diam} U^l_{\bar j})^\delta.$$

This means by definition that

$$\frac{\sum_{\bar i} (\operatorname{diam} U^{l+n}_{\bar j,\bar i})^\delta}{P(\delta)^{n+l}} \le (\operatorname{diam} U^l_{\bar j})^\delta,$$

or $P(\delta)^n \le P(\delta)^{n+l}$. This is certainly true as $P(\delta) > 1$, and (3.2) follows.

Let us now prove a similar estimate for disks $D(z,r)$ centered at $z\in Q(g)$:

$$\mu(D(z,r)) \le C r^\delta. \tag{3.3}$$

Let $U = U^l_{\bar i}$ be the smallest cylindrical set circumscribed about $D(z,r)$. Let us first show that

$$r \ge q \operatorname{diam} U, \tag{3.4}$$

with a q independent of $D(z,r)$. Indeed, $f^l D(z,r) \equiv D(f^l z, R)$ does not belong to any U_i (since otherwizse there would be a smaller cylinder circumscribed about

$D(z,r)$). Hence

$$\frac{R}{\operatorname{diam} V} \geq \inf_i \frac{\operatorname{dist}(\partial U_i, Q(g))}{\operatorname{diam} V} \equiv q > 0.$$

Since the map f^l is linear, (3.4) follows.

Now (3.3) follows from (3.2) and (3.4):

$$\mu(D(z,r)) \leq \mu(U) \leq (\operatorname{diam} U)^\delta \leq q^{-\delta} r^\delta.$$

Finally, let $\{X_i = D(z_i, r_i)\}$ be a covering of $Q(g)$ by disks. Then

$$\sum (\operatorname{diam} X_i)^\delta \geq C^{-1} \sum \mu(X_i) \geq C^{-1}.$$

Hence $h_\delta(Q(g)) > 0$, and $\operatorname{HD}(Q(g)) \geq \delta$. □

4. Proof of Theorem 1.1

4.1. Basic construction. Let $\mathbf{f} = \{f_\lambda\}$ be a proper unfolded quadratic-like family with $M(\mathbf{f}) \equiv M$. Let us start with a Shishikura's non-renormalizable map $f \equiv f_{\lambda_0} \equiv f_0$ with $\operatorname{HD}_{\mathrm{hyp}}(J(f)) = 2$ (see Theorem A). Let us consider the principle nest of puzzle pieces $V^0 \supset V^1 \supset \dots$, and the corresponding sequence of the generalized renormalizations $g_n : \cup_{i=0}^{\infty} V_i^n \to V^{n-1}$.

The map $g_n : \cup_{i\neq 0} V_i^n \to V^{n-1}$ generates a Cantor set

$$Q^n = \{z \in V^{n-1} : g_n^m z \in \bigcup_{i\neq 0} V_i^n,\ m = 0, 1, \dots\}. \tag{4.1}$$

Proposition D easily implies that

$$\limsup \operatorname{HD}(Q^n) = \operatorname{HD}_{\mathrm{hyp}}(J(f)) = 2 \tag{4.2}$$

(and moreover, the subsequence for which $\operatorname{HD}(Q^n) \to \infty$ can be selected from the subsequence provided by Theorem E).

Given a small ρ, let us fix a level n for which $\operatorname{HD}(Q^n) > 2 - \rho$. Let us use the following notation:

$$g \equiv g_n : \bigcup_{i=0}^{\infty} U_i \to V,$$

where U_0 is the critical puzzle piece. We will also skip the label n in all associated objects: $Q^n \equiv Q$ etc.

Given a finite string $\bar{i} = (i_0, \dots, i_{l-1})$, $i_m \in \mathbb{N}$, consider the cylinder

$$U_{\bar{i}} = \{z : g^m z \in U_{i_m},\ m = 0, \dots, l-1\}.$$

Let $W_{\bar{i}} \subset U_{\bar{i}}$ be the pull-back of U_0 under the map $g^l : U_{\bar{i}} \to V$.

Furthermore, let us consider the parapuzzle piece $P \ni \lambda_0$ given by Theorem E, so that for $\lambda \in P$ all the maps f_λ admit a generalized renormalization $g_\lambda : \cup U_{i,\lambda} \to V_\lambda$ with the same combinatorics.

The generalized quadratic-like family $\{g_\lambda\}$ produces the following families $\mathcal{M}^{n,l}$ of little Mandelbrot sets $M_{\bar{i}}$ labeled by strings $\bar{i} = (i_0, \ldots, i_{l-1})$, $i_m \in \mathbb{N}$. Let us consider a proper unfolded quadratic-like family

$$g_{\bar{i},\lambda} \equiv g_\lambda^{l+1} : W_{\bar{i},\lambda} \to V_\lambda, \ \lambda \in P. \tag{4.3}$$

Let $M_{\bar{i}} \subset P$ be the Mandelbrot set of this family.

Let us consider the straightening map on the union of these Mandelbrot sets:

$$\tau : \cup M_{\bar{i}} \to M. \tag{4.4}$$

Let $\mathcal{Y} \equiv \mathcal{Y}^{n,l} \subset M$ be the set of infinitely renormalizable parameter values whose "combinatorial type" is encoded by the sets $M_{\bar{i}}$:

$$\mathcal{Y} = \{\lambda : \tau^k(\lambda) \in \cup M_{\bar{i}}, \ k = 0, 1 \ldots\}. \tag{4.5}$$

We will show that $\mathrm{HD}(\mathcal{Y}^l)$ is at least $1 - \epsilon$ provided l is sufficiently big.

4.2. Heuristic argument. Take a small quantifier $\gamma > 0$. Let us pretend that the maps $g|U_i$, $i \neq 0$, are γ-linear. Then the linear model suggests that the dimension of the corresponding Julia set $Q = Q(g)$ is almost equal to the critical exponent δ_{cr} for the series $\sum(\mathrm{diam}(U_i))^\delta$ (we normalize g_0 so that $\mathrm{diam}\, V_0 = 1$). As $\mathrm{HD}(Q) \geq 2 - \rho$,

$$\sum (\mathrm{diam}\, U_i)^{2-\epsilon} > 1,$$

for some $\epsilon = \epsilon(\gamma, \rho) \to 0$ as $\gamma \to 0$, $\rho \to 0$. Hence for any A, there is a level l such that

$$\sum_{|\bar{i}|=l} (\mathrm{diam}\, U_{\bar{i},\lambda})^{2-\epsilon} \geq A. \tag{4.6}$$

But $\mathrm{diam}\, W_{\bar{i},\lambda} / \mathrm{diam}\, U_{\bar{i},\lambda} = \mathrm{diam}\, U_{0,\lambda}(1 + O(\gamma))$. From here and (4.6) we conclude that

$$\sum_{|\bar{i}|=l} (\mathrm{diam}\, W_{\bar{i},\lambda})^{2-\epsilon} > A', \tag{4.7}$$

where A' is arbitrary big for sufficiently big l.

Note now that the Julia set $J(g_{\bar{i},\lambda})$ has size of order $(\mathrm{diam}\, W_{\bar{i},\lambda})^2$. Indeed, let $\mathrm{diam}\, W_{\bar{i},\lambda} \approx \exp(-\mu_\lambda)$, where $\mu_\lambda = \mod (V_\lambda \setminus W_{\bar{i},\lambda})$. Then

$$\mathrm{diam}\, J(g_{\bar{i},\lambda}) \approx \exp \sum_{k \geq 0} \mu_\lambda / 2^k \approx (\mathrm{diam}\, W_{\bar{i},\lambda})^2.$$

Hence the critical exponent for the sequence of the $\mathrm{diam}\, J(g_{\bar{i},\lambda})$ is at least $1 - \epsilon/2$.

But since the critical value function $\phi(\lambda) = g_\lambda(0)$ has bounded distortion, the Mandelbrot sets $M_{\bar{i}}$ have the comparable diameters (rel $\mathrm{diam}\, P$), so that that their critical exponent is also at least $1 - \epsilon/2$. Pretending that the map τ is almost piecewise linear we conclude from the linear model that the Hausdorff dimension of $\mathcal{Y}$ is not much smaller than 1 as well. (Warning: one should be really careful here since τ is not even smooth. However, all the iterates τ^n turn out to be uniformly $(1+\gamma)$-qc, with a small γ, which is good enough.)

4.3. Formal argument. Let us start with an estimate of the size of the Mandelbrot set of a quadratic-like family (actually we will estimate the size of the domain bounded by the main cardioid):

Lemma 4.1. *Let $g : W \to V$ be a quadratic-like map which is of the form $z \mapsto \phi \circ \psi(z)^2$, where ϕ and ψ are univalent maps with distortion bounded by D, and $\psi(0) = 0$. Let $c = g(0)$ be the critical value of g. Then there is a constant $L = L(D)$ such that for $c \le L(\operatorname{diam} W)^2/\operatorname{diam} V$, g has an attracting fixed point.*

Proof. Since the desired inequality is scaling invariant, we can assume that $\operatorname{diam} V = 1$. Then

$$a|z| \le |g'(z)| \le A|z| \quad \text{and} \quad az^2 \le |g(z) - c| \le Az^2,$$

with the ratio A/a depending on D only. It follows that

$$\frac{1}{4}a(\operatorname{diam} W)^2 \le \operatorname{diam} V + |c| \le 2,$$

and hence

$$(\operatorname{diam} W)^2 \le \frac{C}{A}, \tag{4.8}$$

with $C = C(D)$.

Let now α denote the fixed point of g with the smaller multiplier. If $|\alpha| < 1/A$ then $|g'(\alpha)| < 1$, so that α is attracting. On the other hand, for $|\alpha| = 1/(2A)$,

$$|c| \ge |\alpha| - A|\alpha|^2 \ge \frac{1}{4A}.$$

Thus for $|c| \le 1/(4A)$ the map g has an attracting fixed point. But by (4.8), this happens once

$$|c| \le \frac{(\operatorname{diam} W)^2}{4C},$$

and we are done. $\square$

We have selected above a generalized renormalization g so that $\mathrm{HD}(Q(g)) > 2 - \epsilon$, and thus $h_{2-\epsilon}(Q(g)) = \infty$. By the definition of the Hausdorff measure, for any A there is a level l such that (4.6) holds.

Furthermore, Theorem E implies that the maps (4.3) are quadratic up to a bounded distortion, and hence

$$\operatorname{diam} W_{\bar{i},\lambda} / \operatorname{diam} U_{\bar{i},\lambda} \ge K^{-1} \operatorname{diam} U_{0,\lambda},$$

so that (4.7) follows.

Let $\psi_{\bar{i}}(\lambda)$ be the critical point of the quadratic-like map $g_{\bar{i},\lambda} : W_{\bar{i},\lambda} \to V_\lambda$, so that $g^l_\lambda(\psi_{\bar{i}}(\lambda)) = 0$. This is a non-vanishing function with bounded (uniformly in n and $\bar{i}$) derivative (see [L3]). Hence it can be normalized so that $\psi_{\bar{i}}(\lambda) \equiv 1$, and after this normalization the critical value function $\phi(\lambda) = g_\lambda(0)$ will still have bounded distortion.

Note also that for the fixed level n, all the domains V_λ has the size of order 1. By Lemma 4.1, the little Mandelbrot set $M_{\bar{i}}$ contains the following set:

$$X_{\bar{i}} = \{\lambda : |\phi(\lambda) - 1| < L(\operatorname{diam} W_{\bar{i},\lambda})^2\}. \tag{4.9}$$

As ϕ has a bounded distortion, $\operatorname{diam} X_{\bar{i}} \geq L_1(\operatorname{diam} W_{\bar{i},\lambda})^2$. All the more $\operatorname{diam} M_{\bar{i}} \geq L_1(\operatorname{diam} W_{\bar{i},\lambda})^2$. Together with (4.7) this implies that $\sum(\operatorname{diam} M_{\bar{i}})^{1-\epsilon/2} > L_2$ with a big L_2. Let $\mathcal{J}$ be a finite subfamily of this family of little Mandelbrot sets which still satisfies the last inequality. ¿From now on we will drop the "bar" and use simple labeling M_j, $j \in \mathcal{J}$, for the sets of this family:

$$\sum_{\mathcal{J}} (\operatorname{diam} M_j)^{1-\epsilon/2} > L_2. \tag{4.10}$$

Let us consider the straightening map (4.4) on this family. It naturally produces a family $N_{\bar{j}} \equiv N^m_{j_0,\dots j_{m-1}}$ of the Mandelbrot sets such that

$$\tau^k(N^m_{j_0,\dots j_{m-1}}) \subset M_{j_k},\ k = 0, 1\dots, m-1.$$

If n (selected in §4.1) is big then the little Mandelbrot sets M_j have a "big combinatorial type" in the sense of [L2]. Theorem IV′ of [L2] implies that the Mandelbrot sets $N^m_{\bar{j}}$ are generated by proper unfolded quadratic-like families with big modulus over disjoint parameter domains $\Delta^m_{\bar{j}}$. Moreover, the Mandelbrot sets $N^m_{\bar{j}}$ are tiny inside these domains (i.e., the topological annuli $\Delta^m_{\bar{j}} \setminus N^m_{\bar{j}}$ have big moduli).

By Theorem C, for appropriate choice of the domains, the straightenings $\tau^m|\Delta^m_{\bar{j}}$ are $(1+\gamma)$-qc, provided n is big enough.

Since the Mandelbrot sets $N^m_{\bar{j}}$ are well inside the $\Delta^m_{\bar{j}}$, there is a neighborhood Δ of M covered by all the images $\tau^m\Delta^m_{\bar{j}}$. Shrinking the domains $\Delta^m_{\bar{j}}$, we can assume that $\tau^m\Delta^m_{\bar{j}} = \Delta$. Moreover, the maps $\tau^m : \Delta^m_{\bar{j}} \to \Delta$ are Hölder continuous with an exponent $1-\kappa$ such that $\kappa = \kappa(\gamma) \to 0$ as $\gamma \to 0$, and a uniform constant. Together with (4.10) this implies that for $\delta = (1-\epsilon/2)(1-\kappa)$

$$\sum_{j_m} (\operatorname{diam} N^{m+1}_{j_0,\dots,j_m})^\delta \geq 2(\operatorname{diam} N^m_{j_0,\dots,j_{m-1}})^\delta. \tag{4.11}$$

Let $\mathcal{Y}$ be the set of infinitely renormalizable parameter values whose combinatorial type is encoded by the sets of family $\mathcal{J}$ as in (4.5). We are ready to show that $\mathrm{HD}(\mathcal{Y}) \geq \delta$. The argument imitates the linear model. Let

$$P_m = \sum_{|\bar{j}|=m} (\operatorname{diam} N^m_{\bar{j}})^\delta,$$

where the summation is taken over all multi-indices $\bar{j} = (j_0, \dots, j_{m-1})$, $j_k \in \mathcal{J}$, of length m. Let us consider a probability measure μ_m which assigns to each $N^m_{\bar{j}}$ mass $(\operatorname{diam} N^m_{\bar{j}})^\delta P_m^{-1}$. Then for any little Mandelbrot set $N^m_{\bar{i}}$ and any $s \geq 0$,

$$\mu_{m+s}(\Delta^m_{\bar{j}}) \leq (\operatorname{diam} N^m_{\bar{j}})^\delta. \tag{4.12}$$

Indeed, this inequality can be rewritten as

$$\frac{\sum_{|\bar{k}|=s} (\operatorname{diam} N^{m+s}_{\bar{j},\bar{k}})^\delta}{(\operatorname{diam} N^m_{\bar{j}})^\delta} \leq P_{m+s}.$$

Since the straightening $N^m_{\bar{j}} \to M$ is $(1+\gamma)$-qc, it follows from the estimate $2P_s \leq P_{m+s}$, which is true by (4.11).

Let μ be any limit measure for the sequence μ_m. Since $\mathcal{Y}$ is compactly contained in $\Delta^m_{\bar{j}}$, $\mu(\partial \Delta^m_{\bar{j}}) = 0$. Hence (4.12) yields

$$\mu(\Delta^m_{\bar{j}}) = \lim_{s\to\infty} \mu_{s+m}(\Delta^m_{\bar{j}}) \leq (\operatorname{diam} N^m_{\bar{j}})^\delta. \tag{4.13}$$

Let us now prove a similar estimate for round disks $D(\lambda, r)$ with $\lambda \in \mathcal{Y}$:

$$\mu(D(\lambda, r)) \leq r^\delta. \tag{4.14}$$

Let $\Delta^t_{\bar{j}}$ be the smallest domain of our family containing $D(\lambda, r)$. Since the straightening $\tau^t : \Delta^t_{\bar{j}} \to \Delta$ is $(1+\gamma)$-qc, the image $D = \tau^t D(\lambda, r)$ is an "ellipse" with bounded shape. Since D is not contained in any domain Δ^1_j and $\mathcal{Y} \ni \tau^t \lambda$ is compactly contained in $\cup \Delta^1_j$, D is commensurable with M. As τ^t is $(1+\gamma)$-qc, $D(\lambda, r)$ is commensurable with $N^t_{\bar{j}}$. Hence

$$\mu(D(\lambda, r)) \leq \mu(\Delta^t_{\bar{j}}) \leq (\operatorname{diam} N^t_{\bar{j}})^\delta \leq K r^\delta,$$

and we are done.

This proves the complex theorem. To obtain the corresponding estimate for real parameter values, start with, say, the Ulam-Neumann map $z \mapsto z^2 - 2$. It is easy to see that its Julia set $[-2, 2]$ has hyperbolic dimension 1. Then for any nearby quadratic $z \mapsto z^2 - c$, $c < 2$, the hyperbolic dimension of its invariant interval is close to 1. Take a nearby quadratic which is non-renormalizable and has a recurrent critical point. Then the real trace of the above construction produces a parameter set $\mathcal{Y} \subset \mathbb{R}$ of dimension at least $1/2 - \epsilon$.

5. Concluding Remarks.

Let us finish by mentioning a few related results on the geometry of the Mandelbrot set and quadratic Julia sets.

It is known that if $c \in \mathcal{F}$ is at most finitely renormalizable and does not have neutral periodic points then $\operatorname{meas}(J(P_c)) = 0$ ([L1, Sh2]). Moreover, these Julia sets are actually "removable" which is a stronger property [K]. Many infinitely renormalizable quadratics also have the Julia set of zero measure [Y] but it is unknown whether it is always so.

Moreover, the boundary of the set $\mathcal{F}$ has zero Lebesgue measure as well (though Hausdorff dimension 2!) [Sh2] . It is not known whether the complementary set of infinitely renormalizable maps, $\mathcal{I}$, has zero measure. The MLC Conjecture would assert that at least this set has empty interior.

Note that the class $\mathcal{T} \subset \mathcal{I}$ of infinitely renormalizable maps of "big type" considered in [L2] (for which MLC is established to be true) has zero measure. Indeed, by [L2] and Theorem C, any $c \in \mathcal{T}$ is *the intersection of a nest of little Mandelbrot sets* $M_i(c)$ *of bounded shape* (i.e., uniformly K-qc equivalent to M). Hence the domain bounded by the main cardioid of $M_i(c)$ occupies a definite proportion of M_i. As the parameters in this domain don't belong to $\mathcal{I}$, c may not be a density point of $\mathcal{I}$. Still, $\mathcal{T}$ is quite a big set: $\mathrm{HD}(\mathcal{T}) \geq 1$ by the result of this paper.

Note also that [L2, L4] (see also [DD]) imply that all maximal Mandelbrot copies centered at the real line have indeed a bounded shape (a copy is called *maximal* if it is not contained in any other copy except the whole set M). We are confident that all little Mandelbrot sets centered at the real line actually have bounded shape.

Let us finally state our most recent result:

Theorem 5.1. *The set* $\mathcal{I} \cap \mathbb{R}$ *of real infinitely renormalizable parameter values has zero linear measure.*

We derive this result from the Renormalization Conjecture which we have recently proven for arbitrary combinatorial types ([L6]). We have also proven that the Hausdorff dimension of the set of real infinitely renormalizable maps of bounded type (by some N) is strictly less than 1 [L4]. We don't know yet whether the Hausdorff dimension of the whole set $\mathcal{I} \cap \mathbb{R}$ is strictly less than 1.

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