Lectures on the Dynamics of Quadratic Polynomials

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Abstract.

Contents

CHAPTER 0

Introduction

1. General terminology and notations

As usual, $\mathbb{N} = \{1, 2, \dots\}$ stands for the set of natural numbers;
$\mathbb{R}$ stands for the real line;
$\mathbb{C}$ stands for the complex plane,
and $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ stands for the Riemann sphere.
For $a \in \mathbb{C}$, $r > 0$, let

$$\mathbb{D}(a, r) = \{z \in \mathbb{C} : \ |z - a| < r\}; \quad \bar{\mathbb{D}}(a, r) = \{z \in \mathbb{C} : \ |z - a| \le r\}.$$

Let $\mathbb{D}_r \equiv \mathbb{D}(0, r)$, and let $\mathbb{D} \equiv \mathbb{D}_1$ denote the unit disk.
Let $\mathbb{T}_r = \partial \mathbb{D}_r$, and let $\mathbb{T} \equiv \mathbb{T}_1$ denote the unit circle;
$\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$, $\mathbb{D}^* = \mathbb{D} \smallsetminus \{0\}$.
$\mathbb{A}(r, R) = \{z : \ r < |z| < R\}$ is an open round annulus; The notaions $\mathbb{A}[r, R]$ or $\mathbb{A}(r, R]$ for the closed or semi-closed annuli are self-explanatory.
The *equator* of $\mathbb{A}(r, R)$ is the curve $|z| = \sqrt{Rr}$.
$\mathbb{H} = \{z : \ \Im z > 0$ is the upper half plane.

$\mathrm{SL}(2, R)$ is the group of 2×2 matrices over a ring R with determinant 1 (we will deal with $R = \mathbb{C}$, $\mathbb{R}$, or $\mathbb{Z}$);
$\mathrm{PSL}(2, R) = \mathrm{SL}(2, R)/\{\pm I\}$, where I is the unit matrix.

$\bar{X}$ denotes the closure of a set X; $\operatorname{int} X$ denotes its interior.
$U \Subset V$ means that U is *compactly contained in* V, i.e., $\bar{U}$ is a compact set contained in V.
A compact metrizable space is called *perfect* if it does not have isolated points.
A *Cantor set* is a totally disconnected perfect set. All compact sets are homeomorphic.

For two sets X and Y in a metric space with metric d, let

$$\operatorname{dist}(X, Y) = \inf_{x \in X, \, y \in Y} d(x, y).$$

If one of these sets is a singleton, say $X = \{x\}$, then we use notation $\operatorname{dist}(x, Y)$ for the distance from X to Y.

$$\operatorname{diam} X = \sup_{x,y \in X} d(x, y).$$

Notation (X, Y) stands for the pair of spaces such that $X \supset Y$. A pair (X, a) of a space X and a "preferred point" $a \in X$ is called a *pointed* space.
Notation $f : (X, Y) \to (X', Y')$ means a map $f : X \to X'$ such that $f(Y) \subset Y'$. In the particular case of pointed spaces $f : (X, a) \to (X', a')$ we thus have: $f(a) = a'$.
Similar notations apply to triples, (X, Y, Z), where $X \supset Y \supset Z$, etc.

Part 1

Conformal and quasi-conformal geometry

CHAPTER 1

Conformal geometry

1. Riemann surfaces

1.1. Topological surfaces.

1.1.1. *Definitions and examples.*

Definition 1.1. A *(topological) surface* S (without boundary) is a two-dimensional topological manifold with countable base. It means that S is a topological space with a countable base and any $z \in S$ has a neighborhood $U \ni z$ homeomorphic to an open subset V of $\mathbb{R}^2$. The corresponding homeomorphism $\phi : U \to V$ is called a *(topological) local chart* on S. Such a local chart assigns to any point $z \in U$ its *local coordinates* $(x, y) = \phi(z) \in \mathbb{R}^2$.

A family of local charts whose domains cover S is called a *topological atlas* on S.

Given two local charts $\phi : U \to V$ and $\tilde{\phi} : \tilde{U} \to \tilde{V}$, the composition

$$\tilde{\phi} \circ \phi^{-1} : \phi(U \cap \tilde{U}) \to \tilde{\phi}(U \cap \tilde{U})$$

is called the *transition map* from one chart to the other.

A surface is called *orientable* if it admits an atlas with orientation preserving transition maps. Such a surface can be oriented in exactly two ways. *In what follows we will only deal with orientable (and naturally oriented) surfaces.*

Unless otherwise is explicitly said, we will assume that the surfaces under consideration are *connected.* The simplest (and most important for us) surfaces are:

• The whole plane $\mathbb{R}^2$ (homeomorphic to the open unit disk $\mathbb{D} \subset \mathbb{R}^2$).

• The unit *sphere* S^2 in $\mathbb{R}^3$ (homeomorphic via the stereographic projection to the one-point compactification of the plane); it is also called a "closed surface of genus 0" (in this context "closed" means "compact without boundary").

• A *cylinder* or *topological annulus* $C(a, b) = \mathbb{T} \times (a, b)$, where $-\infty \leq a < b \leq +\infty$. It can also be represented as the quotient of the strip $P(a, b) = \mathbb{R} \times (a, b)$ modulo the cyclic group of translations $z \mapsto z + 2\pi n$,

$n \in \mathbb{Z}$. All the cylinders $C(a,b)$ are homeomorphic to any annulus $\mathbb{A}(r,R)$, to the punctured disk $\mathbb{D}^*$ and to the punctured plane $\mathbb{C}^*$).

• The torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$, also called a "closed surface of genus 1". It can also be represented as the quotient of $\mathbb{R}^2$ modulo the action of a rank 2 abelian group $z \mapsto z + \alpha m + \beta n$, $(m,n) \in \mathbb{Z}^2$, where α and β is an arbitrary basis in $\mathbb{R}^2$.

It is intuitively obvious that (up to a homeomorphism) *there are only two simply connected surfaces*: the plane and the sphere.

If we have a certain standard surface S (say, the unit disk or the unit sphere), a "topological S" (say, a "topological disk" or a "topological sphere") refers to a surface homeomorphic to the standard one.

One can also consider surfaces with boundary. The local model of a surface near a boundary point is given by a relative neighborhood of a point $(x,0)$ in the closed upper half-plane $\bar{\mathbb{H}}$. The orientation of a surface naturally induces an orientation of its boundary (locally corresponding to the positively oriented real line).

For instance, we can consider cylinders with boundary: $C[a,b] = \mathbb{T} \times [a,b]$ or $C[a,b) = \mathbb{T} \times [a,b)$. They will be still called "cylinders" or "topological annuli". Cylinders $C(a,b)$ without boundary will be also called "open", while cylinders $C[a,b]$ will be called "closed" (according to the type of the interval involved).

Cylinders (with or without boundary) are the only topological surfaces whose fundamental group is $\mathbb{Z}$.

A *Jordan curve* $\gamma \subset S$ on a surface is a topologically embedded unit circle. A *Jordan disk* $D \subset S$ is a topological disk bounded by a Jordan curve. Both open and closed Jordan disks are allowed.

1.1.2. *New surfaces from old ones.* There are two basic ways of building new surfaces out of old ones: making holes and gluing their boundaries. Of course, any open subset of a surface is also a surface. In particular, one can make a (closed) *hole* in a surface, that is, remove a closed Jordan disk. A topologically equivalent operation is to make a *puncture* in a surface. By removing an open Jordan disk (open hole) we obtain a surface with boundary.

If we have two open holes (on a single surface or two different surfaces S_i) bounded by Jordan curves γ_i, we can *glue* these boundaries together by means of an orientation reversing homeomorphism $h : \gamma_1 \to \gamma_2$. (It can be also thought as attaching a cylinder to these curves.) We denote this operation by $S_1 \sqcup_h S_2$. For instance, by gluing together two closed disks we obtain a topological sphere: $\mathbb{D} \sqcup_h \mathbb{D} \approx \S^2$.

Combining the above operations, we obtain operations of taking *connected sums* and attaching a *handle.* To take a connected sum of two surfaces S_1 and S_2, make an open hole in each of them and glue together the boundaris of these holes. To attach a handle to a surface S, make two open holes in it and glue together their boundaries.

If we attach a handle to a sphere, we obtain a topological torus. If we attach g handles to a sphere, we obtain a "closed surface of genus g". It turns out that any closed orientable surface is homeomorphic to one of those. Thus closed orientable surfaces are topologically classified by a single number $g \in \mathbb{Z}_+$, its genus.

One says that a surface S (with or without boundary) has a *finite topological type* if its fundamental group $\pi(S)$ is finietly generated (e.g., any compact surface is of finite type). It turns out that it is equivalent to saying that S is homeomorpic to a closed surface with finitely many open or closed holes. Clearly such a surface admits a decomposition

$$S = K \bigcup_i \sqcup_{h_i} C_i \,,$$

where K is a compact surface and $C_i \approx \mathbb{T} \times [0,1)$ are half-open cylinders. The set $K = K_S$ is called the *compact core* of S. Note that it is obviously a deformation retract for S. We say that the cylinders C_i represent the *ends* of S.

Each end can be compactified in two ways, by adding a missing boundary curve to the cylinder, or by adding one point. In the former case, the added boundary curve is called the *ideal boundary* of the end. Let $\hat{S}$ denote the compactification of S by adding ideal boundaries to *all* ends.

1.1.3. *Euler characteristic.* Let S be a compact surface (with or without boundary) Its *Euler characteristic* is defined as

$$\chi(S) = f - e + v,$$

where f, e and v are respectively the numbers of faces, edges and vertices in any triangulation of S.

The Euler characteristic is obviously additive:

$$\chi(S_1 \sqcup_h S_2) = \chi(S_1) + \chi(S_2).$$

Since the cylinder $\mathbb{T} \times [0,1]$ has zero Euler characteristic, $\chi(\hat{S}) = \chi(K_S)$ for a surface S of finite type. We can use this as a definition of $\chi(S)$ in this case.

Making a hole in a surface drops its Euler characteristic by one; attaching a handle does not change it. Hence $\chi(S) = 2 - 2g - n$ for a surface of genus g with n holes.

Note that the above list of simplest surfaces is the full list of sufaces of finite type without boundary with *non-negative* Euler characteristic:

$$\chi(\mathbb{R}^2) = 1, \quad \chi(S^2) = 2, \quad \chi(\mathbb{T} \times (0,1)) = \chi(\mathbb{T}^2) = 0.$$

1.1.4. *Marking.* A surface S can be *marked* with an extra topological data. It can be either several marked points $a_i \in S$, or several closed curves $\gamma_i \subset S$ up to homotopy (usually but not always they form a basis of $\pi_1(S)$), or a parametrization of several boundary components $\Gamma_i \subset \partial S$, $\phi_i : \mathbb{T} \to \Gamma_i$.

The marked objects may or may not be distinguished. (for instance, two marked points or the generators of π_1 may be differently colored). Accordingly, the marking is called *colored* or *uncolored*.

A homeomorphism $h : S \to \tilde{S}$ between marked surfaces should respect the marked data: marked points should go to the corresponding points ($h(a_i) = \tilde{a}_i$), marked curves γ_i should go to the corresponding curves $\tilde{\gamma}_i$ up to homotopy ($h(\gamma_i) \simeq \tilde{\gamma}_i$), and the boundary parametrizations should be naturally related ($h \circ \phi_i = \tilde{\phi}_i$).

1.2. Analytic and geometric structures on surfaces. Rough topological structure can be refined by requiring that the transition maps belong to a certain "structural pseudo-group", which often means: "have certain regularity". For example, a *smooth* structure on S is given by a family of local charts $\phi_i : U_i \to V_i$ such that all the transition maps are smooth (with a prescribed order of smoothness). A surface endowed with a smooth structure is naturally called a *smooth surface.* A local chart $\phi : U \to V$ smoothly related to the charts ϕ_i (i.e., with smooth transition maps) is referred to as a "smooth local chart". A family of smooth local charts covering S is called a "smooth atlas" on S. A smooth structure comes together with affiliated notions of smooth functions, maps and diffeomorphisms.

There is a smooth version of the connected sum operation in which the boundary curves are assumed to be smooth and the boundary gluing map h is assumed to be an orientation reversing diffeomorphism. To get a feel for it, we suggest the reader to do the following exercise:

EXERCISE 1.1. *Consider two copies D_1 and D_2 of the closed unit disk $\mathbb{D} \subset \mathbb{R}^2$. Glue them together by means of a diffeomorphism $h : \partial D_1 \to \partial D_2$ of the boundary circles. You obtain a topological sphere S^2. Show that it can be endowed with a unique smooth structure compatible with the smooth structures on D_1 and D_2 (that is, such that the tautological embeddings $D_i \to S^2$ are smooth). The boundary circles ∂D_i become smooth Jordan curves on this smooth sphere. Show that this sphere is diffeomorphic to the standard "round sphere" in $\mathbb{R}^3$.*

Real analytic structures would be the next natural refinement of smooth structures.

If $\mathbb{R}^2$ is considered as the complex plane $\mathbb{C}$ with $z = x + iy$, then we can talk about *complex analytic* $\equiv$ *holomorphic* transition maps and corresponding complex analytic structures and surfaces. Such surfaces are known under a special name of *Riemann surfaces.* A holomorphic diffeomorphism between two Riemann surfaces is often called an *isomorphism.* Accordingly a holomorphic diffeomorphism of a Riemann surface onto itself is called its *automorphism.*

Connected sum operation still works in the category of Riemann surfaces. In its simplest version the boundary curves and the gluing diffeomorphism should be taken real analytic. Here is a representative statement:

EXERCISE 1.2. *Assume that in Exercise 1.1* $\mathbb{R}^2 \equiv \mathbb{C}$ *and the gluing diffeomorphism* h *is real analytic. Then* S^2 *can be supplied with a unique complex analytic structure compatible with the complex analytic structure on the disks* $D_i \subset \mathbb{C}$*. The boundary circles* ∂D_i *become real analytic Jordan curves on this "Riemann sphere".*

More generally, we can attach handles to the sphere by means of real analytic boundary map, and obtain an example of a Riemann surface of genus g. It is remarkable that, in fact, it can be done with only smooth gluing map, or even with a singular map of a certain class. This operation (with a singular gluing map) has very important applications in Teichmüller theory, theory of Kleinian groups and dynamics (see **??**).

If $\mathbb{R}^2$ is supplied with the standard Euclidean metric, then we can consider *conformal* transition maps, i.e., diffeomorphisms preserving angles between curves. The first thing students usually learn in complex analysis is that the class of orientation preserving conformal maps coincides (in dimension 2!) with the class of invertible complex analytic maps. Thus the notion of a *conformal structure* on an oriented surface is equivalent to the notion of a complex analytic structure (though it is worthwhile to keep in mind their conceptual difference: one comes from geometry, the other comes from analysis).

One can go further to *projective*, *affine*, *Euclidean* or *hyperbolic* structures. We will specify this discussion in a due course.

One can also go in the opposite direction and consider *rough structures* on a topological surface whose structural pseudo-group is bigger then the pseudo-group of diffeomorphisms, e.g., "bi-Lipschitz structurs". Even rougher, *quasi-conformal*, structures will play an important role in our discussion.

To comfort a rigorously-minded reader, let us finish this brief excursion with a definition of a *pseudo-group* on $\mathbb{R}^2$ (in the generality adequate to the above discussion). It is a family of local homeomorphisms $f : U \to V$ between open subsets of $\mathbb{R}^2$ (where the subsets depend on f) which is closed under taking inverse maps and taking compositions (on the appropriately restricted domains). The above structures are related to the pseudo-groups of all local (orientation preserving) homeomorphisms, local diffeomorphisms, locally biholomorphic maps, local isometries (Euclidean or hyperbolic) etc.

1.3. Three geometries.

1.3.1. *Affine geometry.* Consider the complex plane $\mathbb{C}$. Holomorphic automorphisms of $\mathbb{C}$ are complex affine maps $A : z \mapsto az + b$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$. They form a group Aff bi-transitively acting on the plane: any pair of points can be moved (in a unique way) to any other pair of points. Thus the complex plane $\mathbb{C}$ is endowed with the *affine structure* canonically affiliated with its complex analytic structure.

Of course, the plane can be also endowed with a Euclidean metric $|z|^2$. However, this metric can be multiplied by any scalar $t > 0$, and there is no way to normalize it in terms of the complex structure only. All these Euclidean structures have the same group Euc of *Euclidean motions* $A : z \mapsto az + b$ with $|a| = 1$. This group acts transitively on the plane with the group of rotations $z \mapsto e^{2\pi i\theta} z$, $0 \le \theta < 1$, stabilizing the origin.

The group Aff has very few discrete subgoups acting freely on $\mathbb{C}$: rank 1 cyclic group actions $z \mapsto z + an$, $n \in \mathbb{Z}$, and rank 2 cyclic group actions $z \mapsto an + bm$, $(m, n) \in \mathbb{Z}^2$, where (a, b) is an arbitrary basis in $\mathbb{C}$ over $\mathbb{R}$. All rank 1 actions are conjugate by an affine transformation, so that the quotients modulo these actions are all isomorphic. Taking $a = 2\pi$ we realize these quotients as the bi-infinite cylinder $\mathbb{C}/2\pi\mathbb{Z}$. It is isomorphic to the puncured plane $\mathbb{C}^*$ by means of the exponential map $\mathbb{C}/2\pi\mathbb{Z} \to \mathbb{C}^*$, $z \mapsto e^{2\pi z}$. The quotients of rank 2 are all homeomorphic to the torus. However, they may represent different Riemann surfaces (see below 1.4.2).

Note that the above discrete groups preserve the Euclidean structures on $\mathbb{C}$. Hence these structures can be pushed down to the quotient Riemann surface. Moreover, now they can be canonically normalized: in the case of the cylinder we can normalize the lengths of the closed geodesics to be 1 (or 2π). In the case of the torus we can normalize its total area. Thus, complex tori and the bi-infinite cylinder are endowed with a canonical Euclidean structure. For this reason, they are called *flat*.

1.3.2. *Spherical (projective) geometry.* Consider now the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Its bi-holomorphic automorphisms are *Möbius* transformations

$$f : z \mapsto \frac{az+b}{cz+d}; \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

We will denote this Möbius group by Möb($\mathbb{C}$). It acts triply transitive on the sphere: any three points (a, b, c) on the sphere can be moved by a unique Möbius transformation to any other three points (a', b', c').

Note that the Riemann sphere is isomorphic to the complex projective line $\mathbb{CP}^1$. For this reason Möbius transformations are also called *projective*. Algebraicly the Möbius group is isomorphic to the linear projective group $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm I\}$ of 2×2 matrices M with $\det M = 1$ modulo reflection $M \mapsto -M$.

Any Möbius transformation has a fixed point on the sphere. Hence *there are no Riemann surfaces whose universal covering is* $\bar{\mathbb{C}}$. In fact, any non-identical Möbius transformations has either one or two fixed points, and can be classified depending on their nature. The multiplier of a fixed point α is the derivative $f'(\alpha)$ calculated in *any* local chart around α.

We wish to bring a Möbius transformation to a simplest normal form by means of a conjugacy $\phi^{-1} \circ f \circ \phi$ by some $\phi \in$ Möb($\mathbb{C}$). Since Möb($\mathbb{C}$) acts double transitively, we can find some ϕ which sends one fixed point of f to ∞ and the other (if exists) to 0. This leads to the following classification:

(i) A *hyperbolic* Möbius transformation has an attracting and repelling fixed points with multipliers λ amd λ^{-1}, where $|\lambda| < 1$. Its normal form is a global linear contraction $z \to \lambda z$ (with possible spiralling if λ is unreal).

(ii) An *elliptic* Möbius transformation has two fixed points with multipliers λ and λ^{-1} where $\lambda = e^{2\pi i\theta}$, $\theta \in [0, 1)$. Its normal form is the rotation $z \to e^{2\pi i\theta} z$.

(iii) (ii) A *parabolic* Möbius transformation has a single fixed point with multiplier 1. Its normal form is a translation $z \mapsto z + a$ (which can be further normalized so that $a = 1$.

Exercise 1.3. *Prove all the above statements which look new to you.*

1.3.3. *Hyperbolic geometry.*

1.4. Annulus and torus.

1.4.1. *Modulus of an annulus.* Consider an open topological annulus A. Let us endow it with a complex structure. Then A can be represented as the quotient of either $\mathbb{C}$ or $\mathbb{H}$ modulo an action of a cyclic group $< \gamma >$. As we have seen above, in the former case A is isomorphic to the flat cylider $\mathbb{C}/\mathbb{Z} \approx \mathbb{C}^*$. In the latter case, we obtain either the punctured disk $\mathbb{D}^*$ (if γ is parabolic) or an annulus $\mathbb{A}(r, R)$ (if γ is hyperbolic). In the hyperbolic case we call A a *conformal annulus.*

EXERCISE 1.4. *Write down explicitly the covering maps* $\mathbb{H} \to \mathbb{D}^*$ *and* $\mathbb{H} \to A(r, R)$.

EXERCISE 1.5. *Prove that two round annuli* $\mathbb{A}(r, R)$ *and* $\mathbb{A}(r', R')$ *are conformally equivalent if and only if* $R/r = R'/r'$.

Let

$$\operatorname{mod}(A) = \frac{1}{2\pi} \log \frac{R}{r}$$

for a round annulus $A = \mathbb{A}(r, R)$. For an arbitrary conformal annulus A, define its *modulus*, $\operatorname{mod}(A)$, as the modulus of a round annulus $\mathbb{A}(r, R)$ isomorphic to A. According to the above exercise, this definition is correct and, moreover, $\operatorname{mod}(A)$ is the only conformal invariant of a conformal annulus.

If A is isomorphic to $\mathbb{C}^*$ or $\mathbb{D}^*$ then we let $\operatorname{mod}(A) = \infty$.

If A is a topological annulus with boundary whose interior is endowed with a complex structure, then $\operatorname{mod}(A)$ is defined as the modulus of the $\operatorname{int}(A)$.

The *equator* of a conformal annulus A is the image of the equator of the round annulus (see §1) under the uniformization $A(r, R) \to A$.

EXERCISE 1.6. (i) *Write down the hyperbolic metric on a conformal annulus represented as the quotient of the strip* $S_h = \{0 < \Im z < h\}$ *modulo the action of the cyclic group generated by* $z \mapsto z + 2\pi$. *(What is the relation between* h *and* $\operatorname{mod} A$*?)*

(ii) *Prove that the equator is the unique closed hypebolic geodesic of a conformal annulus* A *in the homotopy class of the generator of* $\pi_1(A)$.

(iii) *Show that the hyperbolic length of the equator is equal to* $1/\operatorname{mod}(A)$. *Relate it to the multiplier of the deck transformation of* $\mathbb{H}$ *covering* A.

Even if A is a hyperbolic annulus, it is possible to endow it with a flat, rather than hyperbolic, metric. To this end realize A as the quotient of a strip S_h modulo the cyclic group of translations (see the above exercise). Since the flat metric on S_h is translation invariant, it descends to A. In this case we call A a *flat cylinder.*

1.4.2. *Modulus of the torus.* Let us take a closer look at the actions of the group $\Gamma \approx \mathbb{Z}^2$ on the (oriented) affine plane $P \approx \mathbb{C}$ by translations (see §1.3.1). We would like to classify these actions up to affine conjugacy, i.e., two actions T and S are considered to be equivalent if there is an (orientation preserving) affine automorphism $A : P \to P$ and an algebraic automorphism $i : \Gamma \to \Gamma$ such that for any $\gamma \in \Gamma$ the following diagram is commutative:

$$\begin{array}{ccc} P & \xrightarrow[T^\gamma]{} & P \\ A \downarrow & & \downarrow A \\ P & \xrightarrow[S^{i(\gamma)}]{} & P \end{array} \tag{1.1}$$

This is equivalent to classifying the quotient tori P/T^Γ up to conformal equivalence (since a conformal isomorphism between the quotient tori lifts to an affine isomorphism between the universal covering spaces conjugating the actions of the covering groups).

The conjugacy A in the above definition will also be called *equivariant* with respect to the corresponding group actions.

The problem becomes easier if to require first that $i = \mathrm{id}$ in (1.1). Fix an *uncolored pair* of generators α and β of Γ. Since T acts by translations and since P is affine, the ratio

$$\tau = \tau(T) = \frac{T^\beta(z) - z}{T^\alpha(z) - z}$$

makes sense and is independent of $z \in P$. Moreover, by switching the generators α and β we replace τ with $1/\tau$. Thus, we can color the generators in such a way that $\Im\tau > 0$. (With this choice, the basis of P corresponding to the generators $\{\alpha, \beta\}$ is positively oriented.)

EXERCISE 1.7. *Show that two actions T and S of $\Gamma = <\alpha, \beta>$ are affinely equivalent with $i = \mathrm{id}$ if and only if $\tau(T) = \tau(\tilde{T})$.*

According to the discussion in §1.1.4, the choice of generators of Γ means *(uncolored) marking* of the corresponding torus. Thus, the marked tori are classified by a single complex modulus $\tau \in \mathbb{H}$.

Forgetting the marking amounts to replacement one basis $\{\alpha, \beta\}$ in Γ by another, $\{\tilde{\alpha}, \tilde{\beta}\}$. If both bases are positively oriented then there exists a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

such that $\tilde{\alpha} = a\,\alpha + b\,\beta$, $\tilde{\beta} = c\,\alpha + d\,\beta$. Hence

$$\tilde{\tau} = \frac{a\tau + b}{c\tau + d}.$$

Thus, the unmarked tori are parametrized by a point $\tau \in \mathbb{H}$ modulo the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ by Möbius transformations. The kernel of this action consists of two matrices, $\pm I$, so that the quotient group of Möbius transformations is isomorphic to $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\bmod\{\pm I\}$. This group is called *modular*. (In what follows, the modular group is identified with $\mathrm{PSL}(2,\mathbb{Z})$.)

Remark. Passing from $\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{PSL}(2,\mathbb{Z})$ has an underlying geometric reason. All tori $\mathbb{C}/\Gamma$ have a conformal symmetry $z \mapsto -z$. It change marking $\{\alpha, \beta\}$ by $-I\{\alpha, \beta\}$. Thus, remarking by $-I$ acts trivially on the space of marked tori.

The modular group has two generators, the translation $z \mapsto z + 1$ and the second order rotation $z \mapsto -1/z$. The intersection of their fundamental domains gives the standard fundamental domain Δ for this action.

EXERCISE 1.8. *a) Verify the last statement.*

b) Find all points in Δ *that are fixed under some transformation of* $\mathrm{PSL}(2,\mathbb{Z})$. *What are the orders of their stabilizers?*

c) What is the special property of the tori corresponding to the fixed points?

d) Show that by identifying the sides of Δ *according to the action of the generators we obtain a topological plane* $Q \approx \mathbb{R}^2$.

e) Endow the above plane with the complex structure so that the natural projection $\mathbb{H} \to Q$ *is holomorphic. Show that* $Q \approx \mathbb{C}$. *(The corresponding holomorphic function* $\mathbb{H} \to \mathbb{C}$ *is called modular).*

Thus, the unmarked tori are parametrized by a single modulus $\mu \in \mathbb{H}/\mathrm{PSL}(2,\mathbb{Z}) \approx \mathbb{C}$.

In the dynamical context we will be dealing with the intermadiate case of *partially marked* tori, i.e., tori with one marked generator α of the fundamental group. This space can be viewed as the quotient of the space of fully marked tori by means of forgetting the second generator, β. If we have two bases $\{\alpha, \beta\}$ and $\{\alpha, \tilde{\beta}\}$ in Γ with the same α, then $\tilde{\beta} = \beta + n\alpha$ for some $n \in \mathbb{Z}$. Hence $\tilde{\tau} = \tau + n$.

Thus, the space of partially marked tori is parametrized by $\mathbb{H}$ modulo action of the cyclic group by translations $\tau \mapsto \tau + n$. The quotient space is identified with the punctured disk $\mathbb{D}^*$ by means of the exponential map $\mathbb{H} \to \mathbb{D}^*$, $\tau \mapsto \lambda = e^{2\pi i \tau}$. So, the partially marked tori are parametrized by a single modulus $\lambda \in \mathbb{D}^*$. We will denote such a torus by $\mathbb{T}^2_\lambda$.

This modulus λ makes a good dynamical sense. Consider the covering $p : S \to \mathbb{T}^2_\lambda$ of the partially marked torus corresponding to the marked cyclic group. Its covering space S is obtained by taking the quotient of $\mathbb{C}$ by the action of the marked cyclic group $z \mapsto z + n$, $n \in \mathbb{Z}$. By means of the exponential map $z \mapsto e^{2\pi i z}$, this quotient is identified with $\mathbb{C}^*$. Moreover, the action of the complementary cyclic group $z \mapsto z + n\tau$, $n \in \mathbb{Z}$, descends to the action $\zeta \mapsto \lambda^n \zeta$ on $\mathbb{C}^*$, where the *multiplier* $\lambda = e^{2\pi i \tau}$ *is exactly the modulus of the torus*!

Thus, the partially marked torus $\mathbb{T}^2_\lambda$ with modulus $\lambda \in \mathbb{D}^*$ can be realized as the quotient of $\mathbb{C}^*$ modulo the cyclic action generated by the hyperbolic Möbius transformation $\zeta \mapsto \lambda\zeta$ with multiplier λ.

2. Uniformization Theorem

2.0.3. The following theorem of Riemann and Koebe is the most fundamental result of complex analysis:

THEOREM 1.1. *Any simply connected Riemann surface is conformally equivalent to either the Riemann sphere* $\bar{\mathbb{C}}$*, or to the complex plane* $\mathbb{C}$*, or the unit disk* $\mathbb{D}$*.*

2.0.4. *Classification of Riemann surfaces.* Consider now any Riemann surface S. Let $\pi : \hat{S} \to S$ be its universal covering. Then the complex structure on S naturally lifts to $\hat{S}$ turning S into a simply connected Riemann surface which holomorphically covers S. Thus, we come up with the following classification of Riemann surfaces:

THEOREM 1.2. *Any Riemann surface* S *is conformally equivalent to one of the following surfaces:*

- *The Riemann sphere* $\bar{\mathbb{C}}$ *(spherical case);*
- *The complex plane* $\mathbb{C}$*, or the punctured plane* $\mathbb{C}^*$*, or a torus* $\mathbb{T}^2_\tau$*,* $\tau \in \mathbb{H}$ *(parabolic case);*
- *The quotient of the hyperbolic plane* $\mathbb{H}^2$ *mmodulo a discrete group of isometries (hyperbolic case).*

Thus, *any* Riemann surface comes endowed with one of the three geometries described in §1.3: projective, affine, or hyperbolic.

2.0.5. *Simply connected plane domains.*

3. Principles of the hyperbolic metric

3.1. Schwarz Lemma.

3.2. Normal families and Montel's Theorem. Let U be an open subset of $\mathbb{C}$, and let $\mathcal{M}(U)$ be the space of meromorphic functions $\phi : U \to \bar{\mathbb{C}}$. Supply the *target* Riemann sphere $\bar{\mathbb{C}}$ with the spherical metric d_s and the space $\mathcal{M}(U)$ with the topology of uniform convergence on compact subsets of U. Thus $\phi_n \to \phi$ if for any compact subset $K \subset U$, $d_s(\phi_n(z), \phi(z)) \to 0$ uniformly on U.

EXERCISE 1.9. *Endow $\mathcal{M}(U)$ with an invariant metric compatible with the above convergence (invariance means that* $\text{dist}(\phi, \psi) = \text{dist}(\phi - \psi, 0)$*). Show that $\mathcal{M}(U)$ is complete with respect to this metric. Thus $\mathcal{M}(U)$ is closed in the space $C(U)$ of continuous functions $\phi : U \to \bar{\mathbb{C}}$ (endowed with the topology of uniform convergence on compact subsets of U).*

It is worthy to remember that the target should be supplied with the *spherical* rather than *Euclidean* metric even if the original family consists of *holomorphic* functions. In the limit we can still obtain a meromorphic function, though of a very special kind:

EXERCISE 1.10. *Let $\phi_n : U \to \mathbb{C}$ be a sequence of holomorphic functions converging to a meromorphic function $\phi : U \to \bar{\mathbb{C}}$ such that $\phi(z) = \infty$ for some $z \in U$. Then $\phi(z) \equiv \infty$, and thus $\phi_n(z) \to \infty$ uniformly on compact subsets of U. (Recall the Hurwitz Theorem.)*

formulate beforehand

A family of meromorphic functions on U is called *normal* if it is precompact in $\mathcal{M}(U)$.

EXERCISE 1.11. *Show that normality is the local property: If a family is normal near each point $z \in U$, then it is normal on U.*

EXERCISE 1.12. *If the domain $U \subset \mathbb{C}$ is supplied with the Euclidean metric $|dz|$ while the target $\bar{\mathbb{C}}$ is supplied with the spherical metric $|dz|/(1 + |z|^2)$, then the corresponding "ES norm" of the differential $D\phi(z)$ is equal to $|\phi'(z)|/(1+|\phi(z)|^2)$, $z \in U$. Show that a family of meromorphic functions $\phi_n : U \to \bar{\mathbb{C}}$ is normal if and only if the ES norms $\|D\phi_n(z)\|$ are uniformly bounded on compact subsets of U.*

THEOREM 1.3 (Little Montel). *Any bounded family of holomorphic functions is normal.*

PROOF. It is because the derivative of a holomorphic function can be estimated via the function itself. Indeed by the Cauchy formula

$$|\phi'(z)| \leq \frac{\max_{\zeta \in U} |\phi(\zeta)|}{\text{dist}(z, \partial U)}.$$

Thus if a family of holomorphic functions ϕ_n is uniformly bounded, their derivatives are uniformly bounded on compact subsets of U. By the Arzela-Ascoli Criterion, this family is precompact in the space $C(U)$ of continuous functions. Since uniform (on compact subsets) limits of holomorphic functions are holomorphic (compare with Exercise 1.9), the original family is precompact in the space $\mathcal{M}(U)$. □

PROPOSITION 1.4. *A sequence of holomorphic functions is normal if and only if from any subsequence one can extract a further subsequence which is either bounded or divergent to* ∞.

THEOREM 1.5 (Montel). *If a family of meromorphic functions* $\phi_n : U \to \bar{\mathbb{C}}$ *does not assume three values then it is normal.*

THEOREM 1.6 (Refined Montel). *Let* $\{\phi_n : U \to \bar{\mathbb{C}}\}$ *be a family of meromorphic functions. Assume that there exists three meromorphic functions* $\psi_i : U \to \bar{\mathbb{C}}$ *such that for any* $z \in U$ *and* $i \neq j$ *we have:* $\psi_i(z) \neq \psi_j(z)$ *and* $\phi_n(z) \neq \psi_i(z)$. *Then the family* $\{\phi_n\}$ *is normal.*

EXERCISE 1.13. *What would happen if we allowed* $\psi_i(z) = \psi_j(z)$ *for some* $z \in U$*?* ??

Given a family $\{\phi_n\}$ of meromorphic functions on U, we can define its *set of normality* as the maximal open $F \subset U$ set on which this family is normal.

EXERCISE 1.14. *Show that the set of normality is well-defined.*

3.3. Koebe Distortion Theorem. We are now going to discuss one of the most beautiful and important theorems of the classical geometric functions theory.

The *inner radius* $r_{D,a} \equiv \text{dist}(a, \partial D)$ of a pointed disk (D, a) is as the biggest round disk $\mathbb{D}(a, \rho)$ contained in D. The *outer radius* $R_{D,a} \equiv$ H-dist$(a, \partial D)$ is the radius of the smallest disk $\bar{\mathbb{D}}(a, \rho)$ containing D. (If $a = 0$, we will simply write r_D and R_D.) The *eccentricity* of a pointed disk (D, a) is the ratio $R_{D,a}/r_{D,a}$.

THEOREM 1.7 (Koebe Distortion). *Let* $\phi : (\mathbb{D}, 0) \to (D, a)$ *be a conformal map, and let* $k \in (0, 1)$, $D_k = \phi(\mathbb{D}_k)$. *Then there exist constants* $C = C(k)$ *and* $L = L(k)$ *(independent of a particular* ϕ*!) such that*

$$\frac{|\phi'(z)|}{|\phi'(\zeta)|} \leq C(k) \text{ for all } z, \zeta \in \mathbb{D}_k \tag{3.1}$$

and

$$L(k)^{-1}|\phi'(0)| \leq r_{D_k,a} \leq R_{D_k,a} \leq L(k)\,|\phi'(0)|. \tag{3.2}$$

In particular, the inner radius of the image $\phi(\mathbb{D})$ around a is bounded from below by an absolute constant times the derivative at the origin:

$$r_{\phi(D),a} \geq \rho|\phi'(0)| > 0. \tag{3.3}$$

The expression in (3.1) is called the *distortion* of ϕ, its logarithm is called the *non-linearity* of ϕ. Thus estimate (3.1) tells us that the function ϕ restricted to $\mathbb{D}_k$ has a *uniformly bounded distortion.* Estimate (3.2) tells that the eccentricity of the domain D_k around a is uniformly bounded. Note that since *any* topological disk in $\mathbb{C}$, except $\mathbb{C}$ itself, can be uniformized by $\mathbb{D}$, there could be no possible bounds on the distortion and eccentricity in the whole unit disk $\mathbb{D}$. However, once the disk is shrunk a little bit, the bounds appear!

The Koebe Distortion Theorem is equivalent to the normality of the space of normalized univalent functions:

THEOREM 1.8. *The space $\mathcal{U}$ of univalent functions $\phi : (\mathbb{D}, 0) \to (\mathbb{C}, 0)$ with $|\phi'(0)| = 1$ is compact (in the topology of uniform convergence on compact subsets of $\mathbb{D}$).*

PROOF. Note first that the image $\phi(\mathbb{D})$ cannot contain the whole unit circle $\mathbb{T}$. Otherwise the inverse map ϕ^{-1} would be well-defined on some disk $\mathbb{D}_r$ with $r > 1$, and by the Schwarcz Lemma, $|D\phi^{-1}(0)| \leq 1/r < 1$ contrary to the normalization assumption.

Hence for any $\phi \in \mathcal{U}$ there is a $\theta \in \mathbb{R}$ such that the rotated function $e^{i\theta}\phi$ does not assume value 1. Since the group of rotation is compact, it is enough to prove that the space $\mathcal{U}_0 \subset \mathcal{U}$ of univalent functions $\phi \in \mathcal{U}$ which do not assume value 1 is compact.

Let us puncture $\mathbb{D}$ at the origin, and restrict all the functions $\phi \in \mathcal{U}_0$ to the punctured disk $\mathbb{D}^*$. Since all the ϕ are univalent, they do not assume value 0 in $\mathbb{D}^*$. By the Montel Theorem, the family $\mathcal{U}_0$ is normal on $\mathbb{D}^*$.

Let us show that it is normal at the origin as well. Take a Jordan curve $\gamma \subset \mathbb{D}^*$ around 0, and let Δ be the disk bounded by γ. Restrict all the functions $\phi \in \mathcal{U}_0$ to γ. By normality in $\mathbb{D}^*$, the family $\mathcal{U}_0$ is either uniformly bounded on γ, or admits a sequence which is uniformly going to ∞. But the latter is impossible since all the curves $\phi_n(\gamma)$ intersect the interval $[0, 1]$ (as they go once around 0 and do not go around 1). Thus, the family $\mathcal{U}_0$ is uniformly bounded on γ. By the Maximum Principle, it is is uniformly bounded, and hence normal, on Δ as well.

Thus, the family $\mathcal{U}_0$ is precompact. What is left, is to check that it contains all limiting functions. By the Argument Principle, limits of univalent functions can be either univalent or constant. But the latter is not possible in our situation because of normalization $|\phi'(0)| = 1$. $\square$

EXERCISE 1.15. *(a) Show that a family $\mathcal{F}$ of univalent functions $\phi : \mathbb{D} \to \mathbb{C}$ is precompact in the space of all univalent functions if and only if there exists a constant $C > 0$ such that*

$$|\phi(0)| \le C \quad \text{and } C^{-1} \le |\phi'(0)| \le C \text{ for all } \phi \in \mathcal{F}.$$

b) Let (Ω, a) be a pointed domain in $\mathbb{C}$ and let $C > 0$. Consider a family $\mathcal{F}$ of univalent functions $\phi : \Omega \to \mathbb{C}$ such that $|\phi(a)| \le C$. Show that this family is normal if and only if there exists $\rho > 0$ such that each function $\phi \in \mathcal{F}$ omits some value ζ with $|\zeta| < \rho$.

Proof of the Koebe Distortion Theorem. Compactness of the family $\mathcal{U}$ immediately yields that functions $\phi \in \mathcal{U}$ and their derivatives are uniformly bounded on any smaller disk $\mathbb{D}_k$, $k \in (0, 1)$. Combined with the fact that all functions of $\mathcal{U}$ are univalent, compactness also implies a lower bound on the inner radius $r_{\phi(D_k)}$ and on the derivative $\phi'(z)$ in $\mathbb{D}_k$. These imply estimates (3.1) and (3.2) on the dsitortion and eccentricity by normalizing a univalent function $\phi : \mathbb{D} \to \mathbb{C}$, i.e., considering

$$\tilde{\phi}(z) = \frac{\phi(z) - a}{f'(0)} \in \mathcal{U}.$$

(Note that this normalization does not change either distortion of the function, or the eccentricity of the image.)

Estimate (3.3) is an obvious consequence of the left-hand side of (3.2). □

We have given a qualitative version of the Koebe Distortion Theorem, which will be sufficient for all our purposes. The quantitative version provides sharp constants $C(k)$, $L(k)$, and ρ, all attained for a remarkable extremal *Koebe funcion* $f(z) = z/(1 - z)^2 \in \mathcal{U}$. The sharp value of the constant ρ is particularly famous:

Koebe 1/4-Theorem. *Let $\phi : (\mathbb{D}, 0) \to (\mathbb{C}, 0)$ be a univalent function with $\phi'(0) = 1$. Then $\phi(\mathbb{D}) \supset \mathbb{D}_{1/4}$, and this estimate is attained for the Koebe function.*

We will sometimes refer to the Koebe 1/4-Theorem rather than its qualitatve version (3.3), though as we have mentioned, the sharp constants never matter for us.

EXERCISE 1.16. *Find the image of the unit disk under the Koebe function.*

Let us finish with an invariant form of the Koebe Distortion Theorem:

THEOREM 1.9. *Consider a pair of conformal disks $\Delta \Subset D$. Let $\operatorname{mod}(D \smallsetminus \Delta) \geq \mu > 0$. Then any univalent function $\phi : D \to \mathbb{C}$ has a bounded (in terms of μ) distortion on Δ:*

$$\frac{|\phi'(z)|}{|\phi'(\zeta)|} \leq C(\mu) \text{ for all } z, \zeta \in \Delta.$$

The proof will make use of one important property of the modulus of an annulus: if an annulus is getting pinched, then its modulus is vanishing:

LEMMA 1.10. *Let $0 \in K \subset \mathbb{D}$, where K is compact. If*

$$\operatorname{mod}(D \smallsetminus K) \geq \mu > 0$$

then $K \subset \mathbb{D}_k$ where the radius $k = k(\mu) < 1$ depends only on μ.

PROOF. Assume there exists a sequence of compact sets K_i satisfying the assumptions but such that $R_i \to 1$, where R_i is the outer radius of K_i around 0. Let us uniformize $D \smallsetminus K_i$ by a round annulus, $h_i : \mathbb{A}(\rho_i, 1) \to \mathbb{D} \smallsetminus K_i$. Then $\rho_i \leq \rho \equiv e^{-\mu} < 1$. Thus, the maps h_i are well-defined on a common annulus $A = \mathbb{A}(\rho, 1)$. By the Little Montel Theorem, they form a normal family on A, so that we can select a converging subsequence $h_{i_n} \to h$.

Let $\gamma \subset A$ be the equator of A. Then $h(\gamma)$ is a Jordan curve in $\mathbb{D}$ which separates the sets K_{i_n} (with sufficiently big n) from the unit circle - contradiction. □

Remark. The extremal compact sets in the above lemma (minimizing k for a given μ) are the straight intervals $[0, ke^{i\theta}]$.

Proof of Theorem 1.9 Let us uniformize D by the unit disk, $h : \mathbb{D} \to D$, in such a way that $h(0) \in \Delta$. Let $\tilde{\Delta} = h^{-1}\Delta$ and $\tilde{\phi} = \phi \circ h$. By Lemma **??**, $\tilde{\Delta} \subset \mathbb{D}_k$, where $k = k(\mu) < 1$. By the Koebe Theorem, the distortion of the functions h and $\tilde{\phi}$ on $\tilde{\Delta}$ is bounded by some constant $C = C(k)$. Hence the distortion of ϕ is bounded by C^2. □

We will often use the following informal formulation of Theorem 1.9: "If $\phi : D \to \mathbb{C}$ is a univalent function and $\Delta \subset D$ is *well inside* D, then ϕ has *a bounded distortion* on Δ".

Or else: "If a univalent function $\phi : \Delta \to \mathbb{C}$ has a *definite space* around Δ, then it has a bounded distortion on Δ".

3.4. Relation between hyperbolic and Euclidean metrics.

LEMMA 1.11. *Let $D \subset \mathbb{C}$ be a conformal disk endowed with the hyperbolic metric ρ_D. Then*

$$\frac{1}{4}\frac{|dz|}{\operatorname{dist}(z,\partial D)} \le d\rho_D(z) \le \frac{|dz|}{\operatorname{dist}(z,\partial D)}.$$

Remark. Of course, particular constants in the above estimates will not matter for us.

PROOF. Let $r = \operatorname{dist}(z,\partial D)$; then $\mathbb{D}(z,r) \subset D$. Consider a linear map $h : \mathbb{D} \to \mathbb{D}(z,r)$ as a map from $\mathbb{D}$ into D. By the Schwarz Lemma, it contracts the hyperbolic metric. Hence

$$d\rho_D(z) \le h_*(d\rho_{\mathbb{D}}(0)) = h_*(|d\zeta|) = |dz|/r.$$

To obtain the opposite inequality, consider the Riemann mapping $\psi : (\mathbb{D}, 0) \to (D, z)$. By definition of the hyperbolic metric,

$$d\rho_D(z) = \psi_*(d\rho_{\mathbb{D}}(0)) = \psi_*(|d\zeta|) = \frac{|dz|}{|\psi'(0)|}.$$

But by the Koebe 1/4-Theorem, $r \le |\psi'(0)|/4$, so that $d\rho_D(z) \ge |dz|/4r$. □

4. Proper maps and branched coverings

A continuous map $f : S \to T$ between two topological spaces is called *proper* if for any compact set $K \subset T$, its full preimage $f^{-1}K$ is compact. In other words, $fz \to \infty$ in T as $z \to \infty$ in S (where the neighborhoods of "∞" are defined as complements of compact subsets). Full preimages of points under a proper map will also be called its *fibers*. Note that discrete fibers are finite. If a proper map $f : S \to T$ is injective then we say that S *is properly embedded into* T.

EXERCISE 1.17. *Assume that S & T are precompact domains in some ambient surfaces and that f admits a continuos extension to the closure $\bar{S}$. Then f is proper if and only if $f(\partial S) \subset \partial T$.*

EXERCISE 1.18. *Let $V \subset T$ be a domain and $U \subset S$ be a component of $f^{-1}V$. If $f : S \to T$ is proper, then the restriction $f : U \to V$ is proper as well.*

Let now S and T be topological surfaces, and f be a *topologically holomorphic map*. The latter means that for any point $a \in S$, there exist local charts $\phi : (U, a) \to (\mathbb{C}, 0)$ and $\psi : (V, fa) \to (\mathbb{C}, 0)$ such that $\psi \circ f \circ \phi^{-1}(z) = z^d$, where $d \in \mathbb{N}$. This number $d \equiv \deg_a f$ is called the *(local) degree* of f at a. If $\deg_a f > 1$, then a is called a *branched* or *critical* point of f, and $f(a)$ is called a branched or critical value of f. We also say that d is the multiplicity of a as a preimage of $b = f(a)$.

Exercise 1.19. *Show than any non-constant holomorphic map between two Riemann surfaces is topologically holomorphic.*

A basic property of topologically holomorphic proper maps is that they have a global degree:

Proposition 1.12. *Let $f : S \to T$ be a topologically holomorphic proper map between two surfaces. Assume that T is connected. Then all points $b \in T$ have the same (finite) number of preimages counted with multiplicities. This number is called the degree of f, $\deg f$.*

Proof. Since the fibers of a topologically holomorphic map are discrete, they are finite. Take some point $b \in T$, and consider the fiber over it, $f^{-1}b = \{a_i\}_{i=1}^{l}$. Let $d_i = \deg_{a_i} f$. Then there exists a neighborhood V of b and neighborgood U_i of a_i such that any point $z \in V$, $z \neq b$, has exactly d_i preimages in U_i, and all of them are regular.

Let us show that if V is sufficiently small then all preimages of $z \in V$ belong to $\cup U_i$. Otherwise there would exist sequences $z_n \to b$ and $\zeta_n \in S \smallsetminus \cup U_i$ such that $f(\zeta_n) = z_n$. Since f is proper, the sequence $\{\zeta_n\}$ would have a limit point $\zeta \in S \smallsetminus \cup U_i$. Then $f(\zeta) = b$ while ζ would be different from the a_i - contradiction.

Thus all points close to b have the same number of preimages counted with multiplicities as b, so that this number is locally constant. Since T is connected, this number is globally constant. □

Corollary 1.13. *Topologically holomorphic proper maps are surjective.*

The above picture for proper maps suggests the following generalization. A topologically holomorphic map $f : S \to T$ between two surfaces is called a *branched covering* of degree $d \in \mathbb{N} \cup \{\infty\}$ if any point $b \in T$ has a neighborhood V with the following property. Let $f^{-1}b = \{a_i\}$ and let U_i be the components of $f^{-1}V$ containing a_i. Then these components are pairwise disjoint, and there exist maps $\phi_i : (U_i, a_i) \to (\mathbb{C}, 0)$ and $\psi : (V, b) \to (\mathbb{C}, 0)$ such that $\psi \circ f \circ \phi_i^{-1}(z) = z_i^d$. Moreover, $\sum d_i = d$. (A branched covering of degree 2 will be also called a *double branched covering*.)

We see that *a topologically holomorphic map is proper if and only if it is a branched covering of finite degree.* All terminilogy developed above for proper maps immediately extends to arbitrary branched coverings.

Note that if $V \subset T$ is a domain which does not contain any critical values, then the "map f is unbranched over V", i.e., its restriction $f^{-1}V \to V$ is a covering map. In particular, if V is simply connected, then $f^{-1}V$ is the union of d disjoint domains U_i each of which homeomorphically projects onto V. In this case we have d well-defined branches $f_i^{-1} : V \to U_i$ of the inverse map. (We will often use the same notation f^{-1} for the inverse branches.)

Let us finish with a beautiful relation between topology of the surfaces S and T, and branching properties of f.

Riemann - Hurwitz formula. *Let $f : S \to T$ be a branched covering of degree d between two topological surfaces of finite type. Let C be the set of branched points of f. Then*

$$\chi(S) = \deg f \cdot \chi(T) - \sum_{a \in C} (\deg_a f - 1).$$

Let us define the *multiplicity* of $a \in C$ as a critical point to be equal to $\deg_a f - 1$ (in the holomorphic case it is the multiplicity of a as the root of the equation $f'(a) = 0$). Then the sum in the right-hand side of the Riemann-Hurwitz formula is equal to the *number of critical points of f counted with multiplicities.*

EXERCISE 1.20. *A double branched covering between two topological disks has a single branched point of degree 2.*

4.1. Topological Argument Principle. Consider the punctured plane $\mathbb{R}^2 \smallsetminus \{b\}$. If $\gamma : S^1 \to \mathbb{R}^2 \smallsetminus \{b\}$ is a smooth oriented Jordan curve then one can define the winding number of γ around b as

$$w_b(\gamma) = \int_\gamma d(\arg(x - b)).$$

Since the 1-form $d(\arg(x-b))$ is closed, the winding number is the same for homotopic curves. Hence we can define the winding number $w_b(\gamma)$ for any continuous Jordan curve $\gamma : S^1 \to \mathbb{R}^2 \smallsetminus \{b\}$ by approximating it with a smooth Jordan curves.

Furthermore, the winding number can be linearly extended to any simplicial 1-cycle in $\mathbb{R}^2 \smallsetminus \{b\}$ with integer coefficients (i.e., a formal combination of oriented Jordan curves in $\mathbb{R}^2 \smallsetminus \{b\}$) and then factored to the first homology group. It gives an isomorphism

$$w : H_1(\mathbb{R}^2 \smallsetminus \{b\}) \to \mathbb{Z}, \quad [\gamma] \mapsto w_b(\gamma). \tag{4.1}$$

EXERCISE 1.21. *Prove the last statement.*

Let $x \in D$ be an isolated preimage of $b = fx$. Then one can define the $\operatorname{ind}_x(f)$ as follows. Take a disk $V \subset D$ around x that does not contain other preimages of $b = fx$. Take a positively oriented Jordan loop $\gamma \subset V \setminus \{x\}$ around x whose image does not pass through b, and calculate the winding number of the curve $f : \gamma \to \mathbb{R}^2 \setminus \{b\}$ around b:

$$\mathrm{ind}_x(f) = w_{fx}(f \circ \gamma).$$

Clearly it does not depend on the loop γ, since the curves corresponding to different loops are homotopic without crossing b.

PROPOSITION 1.14. *Let $D \subset \mathbb{R}^2$ be a domain bounded by a Jordan curve Γ, and let $f : \bar{D} \to \mathbb{R}^2$ be a continuous map such that the curve $f \circ \Gamma$ does not pass through some point $b \in \mathbb{R}^2$. Assume that the preimage of this point $f^{-1}b$ is discrete in D. Then*

$$\sum_{x \in f^{-1}b} \mathrm{ind}_x(f) = w_b(f \circ \Gamma),$$

provided Γ is positively oriented.

PROOF. Note first that since $f^{-1}b$ is a discrete subset of a compact set $\bar{D}$, $f^{-1}x$ is actually finite, so that the above sum makes sense.

Select now small Jordan loops γ_i around points $x_i \in f^{-1}b$, and orient them anti-clockwise. Since Γ and these loops bound a 2-cell, $[\Gamma] = \sum [\gamma_i]$ in $H_1(\bar{D} \setminus f^{-1}b)$. Hence $f_*[\Gamma] = \sum f_*[\gamma_i]$ in $H_1(\mathbb{R}^2 \setminus \{b\})$. Applying the isomorphism (4.1), we obtain the desired formula. ☐

EXERCISE 1.22. *Let $f : D \to \mathbb{R}^2$ be a continuous map, and let $a \in D$ be an isolated point in the fiber $f^{-1}b$, where $b = f(a)$. Assume that $\mathrm{ind}_a(f) \neq 0$. Then f is locally surjective near a, i.e., for any $\epsilon > 0$ there exists a $\delta > 0$ such that $f(\mathbb{D}_\epsilon(a)) \supset \mathbb{D}_\delta(b)$.*

Hint: For a small ϵ-circle γ around a, the curve $f \circ \gamma$ stays some positive distance δ from b. Then for any $b' \in \mathbb{D}_\delta(b)$ we have: $\mathrm{ind}_b(f \circ \gamma) = \mathrm{ind}_b(f \circ \gamma) \neq 0$. But if $b' \notin f(\mathbb{D}_\epsilon(a))$ then the curve $f \circ \gamma$ could be shrunk to b without crossing b'.

4.1.1. *Degree of proper maps.*

4.2. Lifts.

LEMMA 1.15. *Let $f : (S, a) \to (T, b)$ and $\tilde{f} : (\tilde{S}, \tilde{a}) \to \tilde{T}, \tilde{b})$ be two double branched between topological disks (with or without boundary) coverings branched at points a and $\tilde{a}$ respectively. Then any homeomorphism $h : (T, b) \to (\tilde{T}, \tilde{b})$ lifts to a homeomorphism $H : (S, a) \to (\tilde{S}, \tilde{a})$ which makes the diagram*

$$\begin{array}{ccc} (S,a) & \xrightarrow[H]{} & (\tilde{S},\tilde{a}) \\ f\downarrow & & \downarrow\tilde{f} \\ (T,b) & \xrightarrow[h]{} & (\tilde{T},\tilde{b}) \end{array}$$

commutative. Moreover, the lift H is uniquely determined by its value at any unbranched point $z \neq a$. Hence there exists exactly two lifts.

If the above surfaces are Riemann and the map h is holomorphic then then the lifts H are holomorphic as well.

PROOF. Puncturing all the surfaces at their preferred points, we obtain four topological annuli. The maps f and $\tilde{f}$ restrict to the unbranched double coverings between respective annuli, while h restricts to a homeomorphism. The image of the fundamental group $\pi_1(S \smallsetminus \{a\})$ under f consist of homotopy classes of curves with winding number 2 around b, and similar statement holds for $\tilde{f}$. Since the winding number is preserved under homeomorphisms,

$$h_*(f_*(\pi_1(S \smallsetminus \{a\})) = \tilde{f}_*(\pi_1(\tilde{S} \smallsetminus \{\tilde{a}\})). \tag{4.2}$$

By the general theory of covering maps, h admits a lift

$$H : S \smallsetminus \{a\} \to \tilde{S} \smallsetminus \{\tilde{a}\}$$

which makes the "punctured" diagram (4.2) commutative. Moreover, this lift is uniquely determined by the value of H at any point $z \in S \smallsetminus \{a\}$.

Extend now H at the branched point by letting $H(a) = \tilde{a}$. It is clear from the local structure of branched coverings that this extension is continuous (as well as the inverse one), so that it provides us with the desired lift.

If all the given maps are holomorphic then the lift H is also holomorphic on the punctured disk $S \smallsetminus \{a\}$. Since isolated singularities are removable for bounded holomorphic maps, the extension of H to the whole disk is also holomorphic. □

EXERCISE 1.23. *Similar statement holds for branched coverings f and $\tilde{f}$ with a single branched point (of any degree). Analyse the situation with two branched points.*

EXERCISE 1.24. *Assume that all the topological disks in the above lemma are $\mathbb{R}$-symmetric and that all the maps commute with the reflection σ with respect to $\mathbb{R}$. Assume also that $h(f(T \cap \mathbb{R})) = \tilde{f}(\tilde{T} \cap \mathbb{R})$. Then both lifts H also commute with σ (in particualar, they preserve the real line).*

5. Conformal Invariants

5.1. Extremal length. Given a family Γ of plane Jordan arcs, we will define a conformal invariant $\lambda(\Gamma)$ called the extremal length of

Γ. Consider a measurable conformal metric $\rho|dz|$ on $\mathbb{C}$ with finite total mass

$$m_\rho = \int \rho^2 |dz|^2$$

(such metrics will be called *admissible*). Let

$$l_\rho(\gamma) = \int_\gamma \rho|dz|,$$

stand for the length of $\gamma \in \Gamma$ in this metric (with the convention that it is infinite if γ is non-rectifiable, or ρ is non-integrable over γ). Let

$$l_\rho(\Gamma) = \inf_{\gamma\in\Gamma} l_\rho(\gamma).$$

Normalize the length in the scaling invariant way:

$$\lambda_\rho(\Gamma) = \frac{l_\rho(\Gamma)^2}{m_\rho},$$

and define the *extremal length* of Γ as follows:

$$\lambda(\Gamma) = \sup_\rho \lambda_\rho(\Gamma),$$

where the supremum is taken over all admissible metrics.

EXERCISE 1.25. *If a family of curves Δ contains a family Γ, then $\lambda(\Delta) \le \lambda(\Gamma)$.*

5.2. Modulus of an annulus.

5.2.1. *Modulus and extremal length.* We will now calculate the modulus of an annulus in terms of the extremal length. Consider a flat cylinder $A = S^1 \times (0,h)$ where $S^1 \times (0,1)$ with the circumferance equal to a. Vertical arcs γ_θ, $\theta \in S^1$, joining the top to the bottom of A will be called *vertical sections* of A.

try it

If A is a conformal annulus, then it is isomorphic to a flat cylinder, $A \approx S^1 \times (0,h)$, and we will freely identify them. In particular, curves in A corresponding to vertical/horizontal curves in the cylinder will be also referred to as vertical/horizontal. By saying that an arc $\gamma : (0,1) \hookrightarrow A$ "joins the bottom of A to its top" we mean that it happens in the cylinder model (so that this arc does not necessarily land at some points of ∂A).

PROPOSITION 1.16. *Let Γ be any family of Jordan arcs in the annulus A joining its bottom to the top and containing almost all vertical sections. Then $\lambda(\Gamma) = \mathrm{mod}(A)$.*

Proof. We will identify A with the cylinder $S^1 \times (0, h)$. Take first the flat metric E on the cylinder. Then $l_E(\gamma) \geq h$ for any $\gamma \in \Gamma$, so that $l_E(\Gamma) = h$. On the other hand, $m_E(\Gamma) = ah$. Hence $\lambda_E(\Gamma) = h^2/ah = \operatorname{mod}(A)$.

Take now any admissible metric ρ on A. Then for any vertical section $\gamma_\theta \in \Gamma$, $\theta \in S^1$, we have: $\lambda_\rho(\Gamma) \leq l_\rho(\gamma)$. Integrating this over S^1 (using that $\gamma_\theta \in \Gamma$ for a.e. $\theta \in S^1$) and applying the Cauchy-Schwarz inequality, we obtain:

$$(a \cdot \lambda(\Gamma))^2 \leq \left(\int_A \rho \, dm \right)^2 \leq ahm_\rho(A).$$

Hence $\lambda_\rho(A) \leq \operatorname{mod}(A)$, and the statement follows. □

To evaluate the modulus, one can also consider the "dual" family of curves:

Exercise 1.26. *Let Γ be a family of homotopic closed Jordan curves in A containing almost all horizontal curves. Then* $\operatorname{mod}(A) = 1/\lambda(\Gamma)$.

5.2.2. *Euclidean geometry of an annulus.* The length-area method allows one to relate $\operatorname{mod}(A)$ to the Euclidean geometry of A. As a simple illustration, let us show that $\operatorname{mod}(A)$ is bounded by the "distance between the inner and the outer complements of A rel the size of the inner complement":

Lemma 1.17. *Consider a topological annulus $A \subset \mathbb{C}$. Let K and Q stand for its inner and outer complements respectively. Then* *define*

$$\operatorname{mod}(A) \leq C \operatorname{dist}(K, Q) / \operatorname{diam} K.$$

Proof. Let Γ be the family of horizontal curves in A. According to the last Exercise, we need to bound $\lambda(\Gamma)$ from below.

Take points $a \in K$ and $c \in Q$ on minimal distance $\operatorname{dist}(K, Q)$, and then select a point $b \in K$ such that $\operatorname{dist}(a, b) > \operatorname{diam} K/2$. Consider a family Δ of closed Jordan curves $\gamma \subset \mathbb{C} \setminus \{a, b, c\}$ with winding number 1 around a and b and winding number 0 around c. Since $\Gamma \subset \Delta$, $\lambda(\Gamma) \geq \lambda(\Delta)$.

Let us estimate $\lambda(\Delta)$ from below. Rescale the configuration $\{a, b, c\}$ (without changing notations) so that $|a - b| = 1$ and $|a - c| = d$, where

$$\frac{1}{2} \operatorname{dist}(K, Q) / \operatorname{diam} K \leq d \leq \operatorname{dist}(K, Q) / \operatorname{diam} K.$$

Consider a unit neighborhood B of the union $[a, b] \cup [a, c]$ of two intervals, and endow it with the Euclidean metric E (extended by 0 outside B). Then $l_E(\Delta) \geq 1$ while $m_E(B) \leq Ad$. Hence $\lambda_E(\Delta) \geq 1/Ad$, and we are done. □

EXERCISE 1.27. *For an annulus A as above, prove a lower bound:*

$$\operatorname{mod}(A) \geq \mu(\operatorname{dist}(K, Q)/\operatorname{diam}(K)) > 0.$$

5.3. Gröztsch Inequality.

5.3.1. The following inequality plays an outstanding role in holomorphic dynamics:

THEOREM 1.18 (Gröztsch Inequality). *Consider a conformal annulus A containing two disjoint conformal annuli A_1 and A_2 homotopically equivalent to A. Then*

$$\operatorname{mod}(A) \geq \operatorname{mod} A_1 + \operatorname{mod} A_2.$$

PROOF. Consider the family Γ of Jordan curves in A representing a generator of $\pi_1(A)$ and its subfamilies Γ_i of curves contained in A_i, $i = 1, 2$. Recall that $\operatorname{mod}(A) = 1/\lambda(\Gamma)$ (Exercise 1.26) and similarly for annuli A_i.

For any admissible conformal metric ρ, we have:

$$m_\rho(A) \geq m_\rho(A_1) + m_\rho(A_2)$$

and

$$l_\rho(\Gamma) \leq l_\rho(\Gamma_i), \;\; i = 1, 2.$$

Hence

$$\frac{m_\rho(A)}{l_\rho(\Gamma)} \geq \frac{m_\rho(A_1)}{l_\rho(\Gamma_1)} + \frac{m_\rho(A_2)}{l_\rho(\Gamma_2)} \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

Minimizing the left-hand side over all admissible metrics, we obtain the desired inequality. □

5.3.2. *Shrinking nests of annuli.* Let $X \subset \mathbb{C}$ be a compact connected set. Let us say that a sequence of disjoint annuli $A_n \subset \mathbb{C}$ is *nested around* X if for any any n, A_n separates both A_{n+1} and X from ∞. (We will also call it a "nest of annuli around X".)

COROLLARY 1.19. *Consider a nest of annuli A_n around X. If $\sum \operatorname{mod} A_n = \infty$ then X is a single point.*

PROOF. Only the first annulus, A_1, can be unbounded in $\mathbb{C}$. Take some disk $D = \mathbb{D}_R$ containing A_2, and consider the annulus $D \setminus X$. By the Gröztsch Inequality,

$$\operatorname{mod}(D \setminus X) \geq \sum_{n \geq 2} \operatorname{mod} A_n = \infty.$$

Hence X is a single point. □

5.4. Dirichlet integral.

5.4.1. *Definition.* Consider a Riemann surface S endowed with a smooth conformal metric ρ. The *Dirichlet integral (D.I.)* of a function $\chi : S \to \mathbb{C}$ is defined as

$$D(\chi) = \int \|\nabla \chi\|_\rho \, dm_\rho,$$

where the norm of the gradient and the area form are evaluated with respect to ρ. However:

EXERCISE 1.28. *The Dirichlet integral is independent of the choice of the conformal metric ρ. In particular, it is invariant under conformal changes of variable.*

In the local coordinates, the Dirichlet integral is expressed as follows:

$$D(h) = \int (|h_x|^2 + |h_y|^2) dm = \int (|\partial h|^2 + |\bar{\partial} h|^2) dm.$$

In particular, for a conformal map $h : U \hookrightarrow \mathbb{C}$ we have the area formula:

$$D(h) = \int |h'(z)|^2 dm = h(U).$$

5.4.2. *D.I. of a harmonic function.*

EXERCISE 1.29. *Consider a flat cylinder $A = S^1 \times (0, h)$ with the unit circumference. Let $\chi : A \to (0, 1)$ be the projection to the second coordinate (the "height" function) divided by h. Then $D(\chi) = 1/h$.*

Note that the function χ in the exercise is a harmonic function with boundary values 0 and 1 on the boundary components of the cylinder (i.e., the solution of the Dirichlet problem with such boundary values).

EXERCISE 1.30. *Such a harmonic function is unique up to switching the boundary components of A, which leads to replacement of χ by $1 - \chi$.*

Due to the conformal invariance of the Dirichlet integral (as well as the modulus of an annulus and harmonicity of a function), these trivial remarks immediately yield a non-trivial formula:

PROPOSITION 1.20. *Let us consider a conformal annulus A. Then there exist exactly two proper harmonic function $\chi_i : A \to (0, 1)$ (such that $\chi_1 + \chi_2 = 1$) and $D(\chi_i) = 1/\operatorname{mod}(A)$.*

5.4.3. *Multi-connected case.* Let S be a compact Riemann surface with boundary. Let $\partial S = (\partial S)_0 \sqcup (\partial S)_1$, where each $(\partial S)_i \neq \emptyset$ is the union of several boundary components of ∂S. Let us consider two families of curves: the "vertical family" Γ^v consisting of arcs joining $(\partial S)_0$ to $(\partial S)_1$, and the "horizontal family" Γ^h consisting of Jordan multi-curves separating $(\partial S)_0$ from $(\partial S)_1$. (A *multicurve* is a finite union of Jordan curves.)

Let $\chi : S \to [0, 1]$ be the solution of the Dirichlet problem equal to 0 on $(\partial S)_0$ and equal to 1 on $(\partial S)_1$.

THEOREM 1.21.

$$\lambda(\Gamma^v) = \frac{1}{\lambda(\Gamma^h)} = \frac{1}{D(h)}.$$

The *modulus of S rel the boundaries* $(\partial S)_0$ and $(\partial S)_1$, is defined as the above extremal length:

$$\operatorname{mod}((\partial S)_0, (\partial S)_1) = \lambda(\Gamma^v).$$

Remark. Physically, we can think of the pair $(\partial S)_0$ and $(\partial S)_1$ in S as an electric condensator. The harmonic function χ represents the potential of the electric field created by the uniformly distributed charge on $(\partial S)_1$. The Dirichlet integral $D(\chi)$ is the *energy* of this field. Thus, $\operatorname{mod}((\partial S)_0, (\partial S)_1) = 1/D(\chi)$ is equal to the ratio of the charge to the energy, that is, to the *capacity* of the condensator.

6. Carathéodory topology

6.1. Hausdorff convergence. Let (X, d) be a metric space. The *Hausdorff distance* between two subsets Y and Z in X is defined as follows:

$$\text{H-dist}(Y, Z) = \max(\sup_{y \in Y} d(y, Z), \sup_{z \in Z} d(Y, z))$$

Note that $\text{H-dist}(Y, Z) < \epsilon$ means that Z is contained in an ϵ-neighborhood of Y and vice versa.

Let $\mathcal{X}$ be the space of closed subsets in X.

EXERCISE 1.31. *(i) Show that that* H-dist *defines a metric on $\mathcal{X}$;*
(ii) If X is complete then $\mathcal{X}$ is complete as well;
(iii) If X is compact then $\mathcal{X}$ is compact as well.

6.2. Carathéodory convergence. Let us consider the space $\mathcal{D}$ of all pointed conformal disks (D, a) in the complex plane. This space can be endowed with a natural topology called *Carathéodory*. We will describe it it terms of convergence:

DEFINITION 1.2. A sequence of pointed disks $(D_n, a_n) \in \mathcal{D}$ converges to a disk $(D, a) \in \mathcal{D}$ if:
(i) $a_n \to a$;
(ii) Any compact subset $K \subset D$ is eventually contained in all disks D_n:

$$\exists\, N : \ K \subset D_n \ \forall\, n \geq N;$$

(iii) If K is a topological disk contained in infinitely many domains D_n then K is contained in D.

Note that this definition allows one to pinch out big bubbles from the domains D_n (see Figure ...).

EXERCISE 1.32. *a) Describe a topology on $\mathcal{D}$ which generates the Carathéodory convergence.*

b) Show that if ∂D_n converges to ∂D in the Hausdorff metric then the disks D_n converge to D in the Carathéodory sense.

The above purely geometric definition can be reformulated in terms of the uniformizations of the disks under consideration. Let us uniformize any pointed disk $(D, a) \in \mathcal{D}$ by a conformal map $\phi : \mathbb{D} \to D$ *normalized* so that $\phi(0) = a$ and $\phi'(0) > 0$.

PROPOSITION 1.22. *A sequence of pointed disks $(D_n, a) \in \mathcal{D}$ converges to a pointed disk $(D, a) \in \mathcal{D}$ if the corresponding sequence of normalized uniformizations $\phi : D_n \to D$ converges to D uniformly on compact subsets of $\mathbb{D}$.*

PROOF. ☐

Recall that $r_{D,a}$ stands for the inner radius of the domain D with respect to $a \in D$ (see §3.3). For $r \in (0, 1)$, let $\mathcal{D}_r$ stand for the family of pointed disks $(D, a) \in \mathcal{D}$ with $r \leq r_{D,a} \leq 1/r$.

COROLLARY 1.23. *The space $\mathcal{D}_r$ is compact.*

PROOF. Let $\phi_D : (\mathbb{D}, 0) \to (D, a)$ be the normalized uniformization of D. Then

$$r \leq \phi'_D(0) \leq \frac{1}{4r}$$

(The left-hand estimate follows from the Schwarz Lemma applied to $\phi^{-1} : \mathbb{D}(a, r) \to \mathbb{D}$. The right-hand estimate follows from the Koebe

1/4-Theorem applied to ϕ_D itself.) By the Koebe Distortion Theorem, the family of univalent functions ϕ_D, $D \in \mathcal{D}_r$, is compact. By Proposition 1.22, the space $\mathcal{D}_r$ is compact as well.

□

CHAPTER 2

Quasi-conformal geometry

7. Definition and regularity properties

7.1. Linear discussion. Let us cosider an R-linear automorphism A of the complex plane $\mathbb{C}_{\mathbb{R}} \approx \mathbb{R}^2$ viewed as the two-dimensional oriented real Euclidean space with the positively oriented orthonormal basis $\{1, i\}$. According to a well-known structural theorem for linear maps, A can be decomposed into a product of a self-adjoint operator S and an orthogonal operator O, $A = O \cdot S$. This decomposition is unique up to multiplying O by -1. We can normalize it so that the bigger eigenvalue λ_+ of S is positive.

Let λ_- stands for the smaller eigenvalues of S; it is positive or negative depending on whether A preserves or reverses the orientation. Let e_+ and e_- stand for the corresonding eigenvectors. We can select this basis to be orthonormal and positively oriented. Then $E \equiv A^{-1}\mathbb{D} = S^{-1}O^{-1}\mathbb{D}$ is the ellipse with big axis along u_- of length $1/|\lambda_-|$ and small axis along e_+ of length $1/\lambda_+$. The accentricity of this ellipse, i.e., the ratio of the axes, is equal to $\lambda_+/|\lambda_-|$. This accentricity will be also called the *dilatation* of A, $\operatorname{Dil} A$. picture

This ellipse E determines a new Euclidean structure in $\mathbb{C}_{\mathbb{R}}$. If A is replaced by a proportional linear map A', the ellipse is scaled and the Euclidean structure is replaced by a conformally equivalent (i.e., proportional). Thus an operator A up to a scalar factor determines a conformal structure on $\mathbb{C}_{\mathbb{R}}$, and vice versa.

Let us calculate the above quantities in coordinates $z, \bar{z}$ of $\mathbb{C}_{\mathbb{R}}$. The operator A can be represented as $z \mapsto az + b\bar{z} = a(z + \mu\bar{z})$, where $\mu = b/a$ is called the *Beltrami coefficient* of A. Let $\mu = e^{i\theta}$. Then the maximum of A on the unit circle $\mathbb{T} = \{e^{i\phi}\}$ is attained at the direction $\phi = -\theta/2 \mod \pi$, while the minimum is attained at the orthognal direction $-\theta/2 + \pi/2 \mod \pi$. These are the eigendirections of S coinciding with the small and big axes of the ellipse $A^{-1}\mathbb{D}$ respectively. The corresponding eigenvalues are equal to $\lambda_+ = |a|(1 + |\mu|) = |a| + |b|$ and $\lambda_- = |a|(1 - |\mu|) = |a| - |b|$. Thus

$$\operatorname{Dil}(A) = \frac{1 + |\mu|}{1 - |\mu|}, \quad \det(A) = |a|^2 - |b|^2. \tag{7.1}$$

So the shape and orientation of the ellipse E is controlled by $|\mu|$ and $\arg\mu$ respectively. We also see that A is orientation preserving if and only if $|b| < |a|$, i.e., $|\mu| < 1$, and A is conformal (i.e., proportional to an orthogonal operator) if and only if $\mu = 0$.

Consider now a non-linear map $f : U \to \mathbb{C}$ on a domain $U \subset \mathbb{C}$ differentiable at a point $z \in U$, and apply the above discussion to its differential $Df(z) = \partial f(z)dz + \bar{\partial} f(z)d\bar{z}$, where $\partial f \equiv \partial f/\partial z$, $\bar{\partial} f \equiv \partial f/\partial \bar{z}$. Assume that $Df(z)$ is non-singular. The Beltrami coefficient of this map is equal to $\mu_f(z) = \bar{\partial} f(z)/\partial f(z)$. We conclude that the infinitesimal ellipse

$$E_f(z) \equiv Df(z)^{-1}\mathbb{T}_{fz} \Subset T_z U \tag{7.2}$$

(where $\mathbb{T}_{fz}$ is a round circle in the tangent space T_{fz}) has a small axis in the direction $-\arg(\mu(z))/2 \ mod\, \pi$ and the eccentricity

$$\operatorname{Dil} E_f(z) \equiv \operatorname{Dil} Df(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}. \tag{7.3}$$

Moreover,

$$\operatorname{Jac}(f, z) \equiv \det Df(z) = |\partial f(z)|^2 - |\bar{\partial} f(z)|^2,$$

and f is orientation preserving at z if and only if $|\mu_f(z)| < 1$. It is conformal at z if and only if $\mu_f(z) = 0$, which is equivalent to the Cauchy-Riemann equation $\bar{\partial} f(z) = 0$.

7.2. Conformal structures. A (measurable) *conformal structure* on a domain $U \subset \mathbb{C}$ is a measurable family of conformal structures in the tangent planes $T_z U$, $z \in U$. In other words, it is a measurable family of infinitesimal ellipses $E(z) \subset T_z U$ defined up to scaling by a measurable function $\rho(z) > 0$, $z \in U$. (As always in the measurable category, all the above objects are defined almost everywhere.) According to the linear discussion, any conformal structure is determined by its *Beltrami coefficient* $\mu(z)$, $z \in U$, a measurable function in z assuming its values in $\mathbb{D}$, and vice versa. Thus conformal structures on U are described analytically as elements μ from the unit ball of $L^\infty(U)$. We say that a conformal structure has a *bounded dilatation* if the eccentricities of the ellipses $E(z)$ are bounded almost everywhere. In terms of Beltrami coefficients, it means that $\|\mu\|_\infty < 1$. The *standard* conformal structure σ is given by the family of infinitesimal circles. The corresponding Beltrami coefficient vanishes almost everywhere: $\mu = 0$.

Denote by $\mathrm{DH}^+(U, V)$ (standing for "differentiable homeomorphisms") the space of orientation preserving homeomorphisms $f : U \to V$, which are differentiable almost everywhere (with respect to the Lebesgue measure) with a non-singular differential $Df(z)$ measurably depending on

z. (If we do not need to specify the domain and the range of f we write simply $f \in \mathrm{DH}^+$; if we do not assume that f is orientation preserving, we skip "+"). Consider some homeomorphism $f \in \mathrm{DH}^+(U, V)$ between two domains in $\mathbb{C}$. Then by the above discussion we obtain a measurable family $\mathcal{E}$ of infinitesimal ellipses $E_f(z) = Df(z)^{-1}\mathbb{T}_{fz} \subset T_z U$. If f is postcomposed with a conformal map $\phi : V \to \mathbb{C}$, then the family of ellipses is scaled by a real factor (depending on z). Thus any homeomorphism $f \in DH^+(U, V)$ (defined up to a postcomposition with a conformal map) determines a (measurable) conformal structure $\mathcal{E}_f = f^*\sigma$ on U. The Beltrami coefficient of this structure is equal to picture $\mu_f(z) = \bar{\partial} f(z)/\partial f(z)$. It is also called the Beltrami coefficient of f. We say that f has a bounded dilatation if the corresponding conformal structure $\mathcal{E}_f$ does. In this case we set

$$\mathrm{Dil}(f) = \mathrm{Dil}(\mathcal{E}_f) = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}.$$

What happens with conformal structures under conformal changes of variable? Let us consider a conformal map $\phi : \tilde{U} \to U$. Let $E(z)$ be an infinitesimal ellipse in $T_z U$ and $\tilde{E}(\tilde{z}) = D\phi^{-1}E(z)$ be the corresponding ellipse in $T_{\tilde{z}}\tilde{U}$. Then the dilatations of these ellipses are equal, while the small axis of $E(z)$ is obtained from the small axis of $\tilde{E}(\tilde{z})$ by rotation through the angle $\arg f'(z)$. It follows that $\tilde{\mu}(\tilde{z})/\mu(z) = f'(z)/\bar{f'}(z)$, so that the differential (-1,1)-form $\mu(z)dz/d\bar{z}$ is invariant under the above change of variable.

This allows us to generalize the above discussion to arbitrary Riemann surfaces. A (measurable) conformal structure on a Riemann surface S is a measurable family of infinitesimal ellipses defined up to scaling. Analytically it is described as a measurable *Beltrami differential* (i.e., (1,-1)-differential form) μ with $\|\mu\|_\infty < 1$. To any homeomorphism $f \in \mathrm{DH}^+(S, S')$ between two Riemann surfaces corresponds a conformal structure $\mathcal{E}_f = f^*\sigma$ on S with the Beltrami differential $\mu_f = \bar{\partial} f/\partial f$ (where $\bar{\partial} f$ and ∂f are now understood as differential 1-forms). Note that the ellipses $E_f(z)$ are well-defined only up to scaling since the round circles on S' are well-defined only up to scaling (as there is no preferred metric on S').

Remark. A key problem is whether any conformal structure $\mathcal{E}$ is associated to a certain map f. This problem has a remarkable positive solution in the category of quasi-conformal maps.

Let us consider a smaller class $\mathrm{AC}^+(U, V) \subset \mathrm{DH}^+(U, V)$ of absolutely continuous orientation preserving homeomorphisms from U onto V. (Reminder: f is absolutely continuous if for any set X of zero

Lebesgue measure, the preimage $f^{-1}X$ has also zero measure.) Then we can naturally pull back any measurable conformal structure $\mathcal{E}'$ on S' to obtain a structure $\mathcal{E} = f^*(\mathcal{E}')$ on S. If f^{-1} is also absolutely continuous then we can push forward the structures: $\mathcal{E}' = f_*(\mathcal{E})$. We will use similar notations for pull-backs and push-forwards of Beltrami differentials. *In fact, in what follows we will not make notational differences between conformal structures and Beltrami differentials.*

EXERCISE 2.1. *Calculate the Beltrami differential $f * \mu$ in terms of μ and Df. Show that* $\mathrm{Dil}(f^*\mu(z)) \leq \mathrm{Dil}\, Df(z) \cdot \mathrm{Dil}\, \mu(f(z))$. *Moreover, dilatation behaves submultiplicatively under compositions:*

$$\mathrm{Dil}(f \circ g) \leq \mathrm{Dil}(f) \cdot \mathrm{Dil}(g).$$

Thus, if a conformal structure ν on S' has a bounded dilatation and f has a bounded dilatation, then the pull-back structure $f^*\nu$ has a bounded dilatation as well.

More generally, let us consider a (non-invertible) map $f : U \to V$ which locally belongs to class AC^+ outside a finite set of "critical points". For such maps the push-forward operation is not well-defined, but the pull-back $\nu = f^*\mu$ is still well-defined. The fact that f has critical points does not cause any troubles since we need to know μ only almost everywhere. The property that $\mathrm{Dil}(f^*\mu) \leq \mathrm{Dil}(f) \cdot \mathrm{Dil}(\mu)$ is obviously valid in this generality.

7.3. Distributional derivatives and absolute continuity on lines. Let U be a domain in $\mathbb{C} \equiv \mathbb{C}_{\mathbb{R}}$. All functions below are assumed to be complex valued. A *test function* ϕ on U is an infinitely differentiable function with compact support. One says that a locally integrable function $f : U \to \mathbb{C}$ has *distributional partial derivatives of class* L^1_{loc} if there exist functions h and g of class L^1_{loc} on U such that for any test function ϕ,

$$\int_U f \cdot \partial\phi dm = -\int_U h\phi dm; \quad \int_U f \cdot \bar{\partial}\phi dm = -\int_U g\phi dm,$$

where m is the Lebesgue measure. In this case h and g are called ∂ and $\bar{\partial}$ derivatives of f in the sense of distributions. Clearly this notiona is invariant under smooth changes of variable, so that it makes sense on any smooth manifold (and for all dimensions).

EXERCISE 2.2. *Prove that a function f on the interval* $(0, 1)$ *has a destributional derivative of class L^1_{loc} if and only if it is absolutely continuous. Moreover, its classical derivative $f'(x)$ coincides with the distributional derivative.*

There is a similar criterion in the two-dimensional setting. A continuous function $f : U \to \mathbb{C}$ is called *absolutely continuous on lines* if for any family of parallel lines in any disk $D \Subset U$, f is absolutely continuous on almost all of them. Thus, taking a typical line l of the above family, the curve $f : l \to \mathbb{C}$ is rectifiable. Clearly such functions have classical partial derivatives almost everywhere.

PROPOSITION 2.1. *Consider a homeomorphism $f : U \to V$ between two domains in the complex plane. It has distributional partial derivatives of class L^1_{loc} if and only if it is absolutely continuous on lines.*

In fact, in the proof of existence of distributional partial derivatives (the easy direction of the above Proposition), just two transversal families of parallel lines are used. Thus one can relax the definition of absolutele continuity on lines by taking any two directions ("horizontal" and "vertical").

PROPOSITION 2.2. *Consider a homeomorphism $f : U \to V$ which is absolutely continuous on lines. Then for almost any $z \in U$, f is differentiable at z in the classical sense, i.e., $f \in \mathrm{DH}$.*

This result can be viewed as a measurable generalization of the elementary fact that existence of continuous partial derivatives implies differentiability.

7.4. Definition. We are now ready to give a definition of quasi-conformality. An orientation preserving homeomorphism $f : S \to S'$ between two Riemann surfaces is called quasi-conformal if

- It has locally integrable distributional partial derivatives;
- It has bounded dilatation.

Note that the second property makes sense because the first property implies that f is differentiable a.e. in the classical sense (by the results of §7.3).

We will often abbreviate "quasi-conformal" as "qc". A qc map f is called K-qc if $\mathrm{Dil}(f) \leq K$.

A map $f : S \to S'$ is called K- *quasi-regular* if for any $z \in S$ there exist K-qc local charts $\phi : (U, z) \to (\mathbb{C}, 0)$ and $\psi : (V, f(z)) \to (\mathbb{C}, 0)$ such that $\psi \circ f \circ \phi^{-1} :\mapsto z^d$. Sometimes we will abbreviate K-quasi-regular maps as "K-qr". A map is called quasi-regular if it is K-qr for some K.

EXERCISE 2.1. *Show that any quasi-regular map $f : S \to S'$ can be decomposed as $g \circ h$, where $h : S \to T$ is a qc map to some Riemann* *surface T and $g : T \to S'$ is holomorphic. In particular, if $S = S' = \bar{\mathbb{C}}$ then also $T = \mathbb{C}$ and $g : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a rational map.*

7.5. Absolute continuity and Sobolev class H. We will now prove several important regularity properties of quasi-conformal maps. Let us define a Sobolev class $H = H(U)$ as the space of uniformly continuous functions $f : U \to \mathbb{C}$ whose distributional partial derivatives on U belong to $L^2(U)$. The norm on H is the maximum of the uniform norm of f and L^2-norm of its partial derivatives. Infinitely smooth functions are dense in H. This can be shown by the standard regularization procedure: convolute f with a sequence of functions $\phi_n(x) = n^2\phi(n^{-1}x)$, where ϕ is a non-negative test function on U with $\int \phi \, dm = 1$ (see [**?**, Ch V, §2.1]).

Proposition 2.3. *Quasiconformal maps are absolutely continuous with respect to the Lebesgue measure, and thus for any Borel set $X \subset U$,*

$$m(fX) = \int_X \operatorname{Jac}(f, z)\, dm.$$

The partial derivatives ∂f and $\bar{\partial} f$ belong to L^2_{loc}.

Proof. Since both statements are local, we can restrict ourselves to homeomorphisms $f : U \to U'$ between domains in the complex plane. Consider the pull-back of the Lebesgue measure on U', $\mu = f^* m$. It is a Borel measure defined as follows: $\mu(X) = m(fX)$ for any Borel set $X \subset U$. Let us decompose it into absolutely continuous and singular parts: $\mu = h \cdot m + \nu$. By the Lebesgue Density Points Theorem, for almost all $z \in U$, we have:

$$\frac{1}{\pi\epsilon^2} \int_{\mathbb{D}(z,\epsilon)} h \; dm \to h(z); \quad \frac{1}{\pi\epsilon^2} \nu(\mathbb{D}(z,\epsilon)) \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Summing up we obtain:

$$\frac{m(f(\mathbb{D}(z,\epsilon))}{m(\mathbb{D}(z,\epsilon))} = \frac{\mu(\mathbb{D}(z,\epsilon)}{m(\mathbb{D}(z,\epsilon)} \to h(z) \quad \text{as} \quad \epsilon \to 0.$$

But if f is differentiable at z then the left hand-side of the last equation goes to $\operatorname{Jac}(f, z)$. Hence $\operatorname{Jac}(f, z) = h(z)$ a.e. It follows that for any Borel set X,

$$\int_X \operatorname{Jac}(f, z)\, dm = \int_X h \, dm \le \mu(X) = m(fX). \tag{7.4}$$

But $\text{Jac}(f,z) = |\bar\partial f(z)|^2 - |\partial f(z)|^2 \geq (1-k^2)\,|\partial f(z)|^2$, where $k = \|\mu_f\|_\infty$. Thus

$$\int_X |\partial f|^2\, dm \leq \frac{1}{1-k^2}\, m(fX); \quad \int_X |\bar\partial f|^2\, dm \leq \frac{k^2}{1-k^2}\, m(fX), \tag{7.5}$$

and we see that the partial derivatives of f are locally square integrable.

What is left is to prove the opposite to (7.4). As we have just shown, f locally belongs to the Sobolev class H. Without loss of generality we can assume that it is so on the whole domain U, i.e., $f \in H(U)$. Let us approximate f in $H(U)$ by a sequence of C^∞ functions f_n. Take a domain $D \Subset U$ with piecewise smooth boundary (e.g., a rectangle).

Let $V_n \subset f_n D$ be the set of regular values of f_n. By Sard's Theorem, it has full measure in $f_n D$. Let $R = f_n^{-1} V_n \cap D$. Note that the $\int_{R_n} \text{Jac}\, f_n\, dm$ is equal to the area of the image of $f_n | R_n$ counted with multiplicities:

$$\int_{R_n} \text{Jac}(f_n, z)\, dm = \int_{V_n} \text{card}(f_n^{-1}\zeta)\, dm \geq m(V_n) = m(f_n D).$$

Since $f_n \to f$ uniformly on D, $\liminf m(f_n D) \geq m(fD)$. Since $\text{Jac}(f_n) \to \text{Jac}(f)$ in $L^1(U)$,

$$\int_R \text{Jac}(f_n, z)\, dm \to \int_R \text{Jac}(f, z)\, dm \leq \int_D \text{Jac}(f, z)\, dm.$$

Putting the last estimates together, we obtain the desired estimate for D.

For an arbitrary Borel set $X \subset U$, the result follows by a simple approximation argument using a covering of X by a union of rectangles D_i with disjoint interiors such that $m(\cup D_i \smallsetminus X) < \epsilon$. □

7.6. Weil's Lemma. This lemma asserts that a 1-qc map is conformal. In other words, if a qc map is infiniesimally conformal on the set of full measure (i.e., $\bar\partial f(z) = 0$ a.e.), then it is conformal in the classical set. Since $\bar\partial f(z) = 0$ is just the Cauchy-Riemann equation, this statement is classical for smooth maps.

Let us formulate a more general version of Weil's Lemma:

LEMMA 2.4 (Weil). *Assume that a continuous function $f : U \to \mathbb{C}$ has distributional derivatives of class L^1_{loc}. If $\bar\partial f(z) = 0$ a.e., then f is holomorphic.*

PROOF. By approximation, Weil's Lemma can be reduced to the classical statement. Since the statement is local, we can assume without loss of generality that the partial derivatives of f belong to $L^1(U)$. Convoluting f with smooth bump-functions we obtain a sequence of smooth functions $f_n = f * \theta_n$ converging to f uniformly on U with derivatives converging in $L^1(U)$. Let us show that $\bar\partial f_n = 0$. For a test function ϕ on U, we have:

$$\int \bar\partial f_n(z)\, \phi(z)\, dm(z) = -\int f_n(z)\, \bar\partial\phi(z)\, dm(z) =$$

$$-\int f(\zeta)\, dm(\zeta) \int \theta_n(z-\zeta) \bar\partial\phi(z)\, dm(z) = \int f(\zeta)\, dm(\zeta) \int \bar\partial\theta_n(z-\zeta)\phi(z)\, dm(z) =$$

$$\int \phi(z)\, dm(z) \int f(\zeta)\, \bar\partial\theta(n-\zeta)\, dm(\zeta) = \int \phi(z)\, \bar\partial f_n(z)\, dm(z).$$

Here the first and the third equalities are the classical integration by parts, the last one expresses the property that $\bar\partial(f * \theta_n) = f * \bar\partial\theta_n$ (which we leave to the reader as an exercise), and the rest is the Fubini Theorem.

It follows that the smooth functions f_n satisfy the Cauchy-Riemann equations and hence holomorphic. Since uniform limits of holomorphic functions are holomorphic, f is holomorphic as well.

□

7.7. Devil Staircase. The following example shows that Weil's Lemma is not valid for homeomorphisms of class DH (i.e., differentiable a.e.). The technical assumption that the classical derivative can be understood in the sense of distributions (which allows us to integrate by parts) is thus crucial for the statement.

Take the standard Cantor set $K \subset [0,1]$ and construct a devil staircase $h : [0,1] \to [0,1]$, i.e., a continuous monotone function which is constant on the complementary gaps to K.

EXERCISE 2.3. *Do the construction. (Topologically it amounts to showing that by collapsing the gaps to points we obtain a space homeomorphic to the interval.)*

Consider a strip $S = [0,1] \times \mathbb{R}$ and let $f : (x,y) \mapsto (x, y + h(x))$. This is a homeomorphism on S which is a rigid translation on every strip $G \times R$ over a gap $G \subset [0,1] \smallsetminus K$. Since $m(K \times \mathbb{R}) = 0$, this map is conformal a.e. However it is obviously not conformal on the whole strip P.

Clearly f in not absolutely continuous on the horizontal lines: it translates them to devil staircases.

8. Further important properties of qc maps

8.1. Qc Removability and Gluing. A closed set $K \subset \mathbb{C}$ is called *qc removable* if any homeomorphism $h : U \to \mathbb{C}$ defined on an neighborhood U of K, which is quasiconformal on $U \setminus K$, is quasiconformal on U.

Remark. We will see later on (§**??**) that qc removable sets have zero measure and hence $\mathrm{Dil}(f|U) = \mathrm{Dil}(f|U \setminus K)$.

Exercise 2.4. *Show that isolated points are removable.*

Proposition 2.5. *Smooth Jordan arcs are removable.*

Proof. Let us consider a smooth Jordan arc $\Gamma \subset U$ and a homeomorphism $f : U \to \mathbb{C}$ which is quasi-conformal on $U \setminus \Gamma$. We should check that f is absolutely continuous on lines near any point $z \in \Gamma$. Take a small box B centered at z whose sides are not parallel to $T_z\Gamma$. Then any interval l in B parallel to one of its sides intersects Γ at a sinle point ζ. Since for a typical l, f is absolutely continuous on the both sides of $l \setminus \{\zeta\}$, it is absolutely continuous on the whole interval l as well.

Moreover, $\mathrm{Dil}(f)$ is obviously bounded since it is so on $U \setminus \Gamma$ and Γ has zero measure. ☐

The above statement is simple but important for holomorphic dynamics. It will allow us to construct global qc homeomorphisms by gluing together different pieces without spoiling dilatation.

Let us now state a more delicate gluing property:

Lemma 2.6 (Bers). *Consider a closed set $K \subset \bar{\mathbb{C}}$ and two its neighborhoods U and V. Assume that we have two quasi-conformal maps $f : U \setminus K \to \bar{\mathbb{C}}$ and $g : V \to \bar{\mathbb{C}}$ that match on ∂K, i.e., the map*

$$h(z) = \begin{cases} f(z), & z \in U \setminus K \\ g(z), & z \in K \end{cases}$$

is continuous. Then h is quasi-conformal and $\mu_h(z) = \mu_g(z)$ for a.e. $z \in K$.

Proof. Consider a map $\phi = f^{-1} \circ h$. It is well-defined in a neighborhood Ω of K, is identity on K and is quasi-conformal on $\Omega \setminus K$. Let us show that it is quasi-conformal on Ω. Again, the main difficulty is to show that h is abosultely continuous on lines near any point $z \in K$.

Take a little box near some point $z \in K$ with sides parallel to the coordinate axes. Without loss of generality we can assume that $z \neq \infty$ and ϕB is a bounded subset of $\mathbb{C}$. Let ψ denote the extension of $\partial\phi/\partial x$ from $B \setminus K$ onto the whole box B by 0. By (7.5), ψ is square integrable on B and hence it is square integrable on almost all horizontal sections of B. All the more, it is integrable on almost all horizontal sections. Take such a section I, and let us show that ϕ is absolutely continuous on it.

Let $I_j \subset I$ be a finite set of disjoint intervals; $\Delta\phi_j$ denote the increment of ϕ on I_j. We should show that

$$\sum |\Delta\phi_j| \to 0 \quad \text{as} \quad \sum |I|_j \to 0. \tag{8.1}$$

Take one interval I_j and decompose it as $L \cup J \cup R$ where $\partial J \subset K$ and $\operatorname{int} L$ and $\operatorname{int} R$ belong to $B \smallsetminus K$. Then

$$|\Delta\phi_j| \leq |J| + \int_{L\cup R} g\, dx \leq |I_j| + \int_{I_j} g\, dx.$$

Summing up the last estimates over j and using integrability of g on I_j, we obtain (8.1).

Absolute continuity on the vertial lines is treated in exactly the same way. □

8.2. Quasi-invariance of modulus. Next we will show by the length-area method that the modulus of an annulus is a quasi-conformal quasi-invariant.

PROPOSITION 2.7. *Consider a K-qc map $f : A \to \tilde{A}$ between two topological annuli. Then*

$$K^{-1} \operatorname{mod}(\tilde{A}) \leq \operatorname{mod}(A) \leq K \operatorname{mod}(\tilde{A}).$$

PROOF. We will use the notations of (§**??**) for objects related to the extremal length. Take any measurable conformal metric $\rho|dz|$ on the annulus A with finite total mass, and transfer it to $\tilde{A}$ by the rule: $\rho = (\tilde{\rho} \circ f)(|\partial f| - |\bar{\partial} f|)$. Let $\tilde{\Gamma}$ be the family of *vertical* segments joining the bottom to the top of $\tilde{A}$ on which f^{-1} is absolutely continuous; while let Γ be the family of *all rectifiable* curves joining the bottom to the top of A (by Proposition 1.16, either family can be used to evaluate the modulus of an annulus). Take any curve $\tilde{\gamma} \in \tilde{\Gamma}$ and let $\gamma = f^{-1}\tilde{\gamma}$. Since $|Df| \geq |\partial f| - |\bar{\partial} f|$ at the points of differentiability of f, we have the following length estimate:

$$l_{\tilde{\rho}}(\tilde{\Gamma}) = l_{\tilde{\rho}}(\tilde{\gamma}) = \int_{\tilde{\gamma}} \tilde{\rho}|d\zeta| \geq$$

$$\int_{\gamma} (\tilde{\rho} \circ f)(|\partial f| - |\bar{\partial} f|)|dz| = l_\rho(\gamma) \geq l_\rho(\Gamma).$$

Since f is absolutely continuous with respect to the Lebesgue measure (Proposition 8.1), we have the following area estimate:

$$m_{\tilde{\rho}}(\tilde{A}) = \int_{\tilde{A}} \tilde{\rho}^2 \, dm = \int_A (\tilde{\rho} \circ f)^2 \, (|\partial f|^2 - |\bar{\partial} f|^2) \, dm =$$

$$\int \rho^2 \frac{|\partial f| + |\bar{\partial} f|}{|\partial f| - |\bar{\partial} f|} \, dm \le K m_\rho(A).$$

(We see by the way that $\tilde{\rho}$ has also finite total mass.) Dividing the square of the former estimate by the latter, we obtain:

$$\lambda_{\tilde{\rho}}(\tilde{\Gamma}) \ge K^{-1} \lambda_\rho(\Gamma).$$

Taking the supremum over ρ, we conclude that $\operatorname{mod}(\tilde{A}) \ge K^{-1} \operatorname{mod}(A)$. The opposite inequality is obtained by changing the roles of A and $\tilde{A}$. □ invertibility of qc !

Exercise 2.2. *Prove that $\mathbb{C}$ and $\mathbb{D}$ are not qc equivalent.*

8.3. Weak topology in L^2. Before going further, let us briefly recall some background in functional analysis. Consider the space $L^2 = L^2(X)$ on some measure space (X, m). A sequence of functions $h_n \in L^2$ weakly converges to some function $h \in L^2$, $h_n \underset{w}{\to} h$, if for any $\phi \in L^2$, $\int h_n \phi \, dm \to \int h \phi \, dm$. The main advantage of this topology is the property that the balls of L^2 are weakly compact (see e.g., [**?**,]). Note also that vice versa, any weakly convergent sequence belongs to some ball in L^2 (Banach-Schteinhaus [**?**,]).

However, one should handle the weak topology with caution: for instance, product is not a weakly continuous operation:

Exercise 2.3. *Show that $\sin nx \underset{w}{\to} 0$ in $L^2[0, 2\pi]$, while $\sin^2 nx \underset{w}{\to} 1/2$.*

At least, the weak topology respects the order:

Exercise 2.4. *Let $h_n \underset{w}{\to} h$.*

- *If $h_n \ge 0$ then $h \ge 0$;*
- *If $h_n = 0$ a.e. on some subset $Y \subset X$, then $h = 0$ a.e. on Y;*
- *After selecting a further subsequence,*

$$(h_n)_+ \underset{w}{\to} h_+ \text{ and } (h_n)_- \underset{w}{\to} h_-, \quad \text{so that } |h_n| \underset{w}{\to} |h|.$$

Here $h_+(z) = \max(h(z), 0)$, $h_(z) = \min(h(z), 0)$.

8.4. Compactness. We will proceed with the following fundamental property of qc maps:

THEOREM 2.8. *The space of K-qc maps $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ fixing $0, 1$ and ∞ is compact in the topology of uniform convergence on $\bar{\mathbb{C}}$*

PROOF. Denote the space in question by $\mathcal{X}$. First, we will show that the family of maps $f \in \mathcal{X}$ is equicontinuous. Otherwise we would have an $\epsilon > 0$, a sequence of maps $f_n \in \mathcal{X}$, and a sequence of points $z_n, \zeta_n \in \bar{\mathbb{C}}$ such that such that $d(z_n, \zeta_n) \to 0$ while $d(f_n z_n, f_n \zeta_n) \geq \epsilon$, where d stands for the sperical metric. By compactness of $\bar{\mathbb{C}}$, we can assume that the $z_n, \zeta_n \in \bar{\mathbb{C}}$ converge to some point a and the $f_n a$ converge to some b. Postcomposing or/and precomposing the f_n's with $z \mapsto 1/z$ if necessary, we can assume make $|a| \leq 1$, $|b| \leq 1$.

Consider a sequence of annuli $A_n = \{z : r_n < |z - a| < 1/2\}$ where $r_n = \max(|z_n - a|, |\zeta_n - a|) \to 0$. Since the disk $\mathbb{D}(a, 1/2)$ does not contain one of the points 0 or 1, its images $f_n \mathbb{D}(a, 1/2)$ have the same property. Hence the Euclidean distance from the point $f_n a$ (belonging to the inner complement of $f_n A_n$) to the outer complement of that annulus is eventually bounded by 3. On the other hand, the diameter of the inner complement of $f_n A_n$ is bounded from below by $\epsilon > 0$. By Lemma 1.17, $\mathrm{mod}(f_n A_n)$ is bounded from above. But $\mathrm{mod}(A_n) = 1/r_n \to 0$ contradicting quasi-invariance of the modulus (Proposition 2.7).

Hence $\mathcal{X}$ is precompact in the space of continuous maps $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$. Since $\mathcal{X}$ is invariant under taking the inverse $f \mapsto f^{-1}$, and the composition is a continuous operation in the uniform topology, $\mathcal{X}$ is precompact in $\mathrm{Homeo}(\bar{\mathbb{C}})$. Since $\mathrm{Homeo}^+(\bar{\mathbb{C}})$ is closed in $\mathrm{Homeo}(\bar{\mathbb{C}})$, $\mathcal{X}$ is precompact in the former space as well.

To complete the proof, we should show that the limit functions are also K-qc homeomorphisms. Let a sequence $f_n \in \mathcal{X}$ uniformly converges to some f. Given a point $a \in \bar{\mathbb{C}}$, we will show that in some neighborhood of a, f has distributional derivatives of class L^2. Without loss of generality we can assume that $a \in \mathbb{C}$. Take a neighborhood $B \ni a$ such that fB is a bounded subset of $\mathbb{C}$. Then the neighborhoods $f_n B$ are eventually uniformly bounded. By (**??**), the partial derivatives ∂f_n and $\bar{\partial} f_n$ eventually belong to a ball of $L^2(D)$. Hence they form weakly precompact sequences, and we can select limits along subsequences (without changing notations):

$$\partial f_n \underset{w}{\to} h \in L^2(D); \qquad \bar{\partial} f_n \underset{w}{\to} g \in L^2(D).$$

It is straightforward to show that h and g are the distributional partial derivatives of f. Indeed, for any test functions ϕ we have:

$$\int f\,\partial\phi\,dm = \lim \int f_n\,\partial\phi\,dm = -\lim \int \partial f_n\,\phi\,dm = -\int h\phi\,dm, \tag{8.2}$$

and the similarly for the $\bar\partial$-derivative.

What is left is to show that $|h(z)| \leq k|g(z)|$ for a.e. z, where $k = (K+1)/(K-1)$. To see it, select a further subsequence in such a way that $|\partial f_n| \underset{w}{\to} |h|$, $|\bar\partial f_n| \underset{w}{\to} g$ and use the fact that the weak topology respects the order (see Exercise 2.4). ☐

EXERCISE 2.5. *Fix any three points a_1, a_2, a_3 on the sphere $\mathbb{C}$. A family $\mathcal{X}$ of K-qc maps $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is precompact in the space of all K-qc homeomorphisms of the sphere (in the uniform topology) if and only if the reference points are not moved close to each other (or, in formal words: there exists a $\delta > 0$ such that $d(ha_i, ha_j) \geq \delta$ for any $h \in \mathcal{X}$ and $i \neq j$, where d is the spherical metric). Consider first the case $K = 0$.*

We will also need a disk version of the above Compactness Theorem:

COROLLARY 2.9. *The space of K-qc homeomorphisms $f : \mathbb{D} \to \mathbb{D}$ fixing 0 is compact in the topology of uniform convergence on $\mathbb{D}$.*

PROOF. Let $\mathcal{Y}$ be the space of K-qc homeomorphisms $h : \mathbb{D} \to \mathbb{D}$ fixing 0, and $\mathcal{X}$ be the space of $\mathbb{T}$-symmetric K-qc homeomorphisms $H : \mathbb{C} \to \mathbb{C}$ fixing 0 and ∞. (To be $\mathbb{T}$-symmetric means to commute with the involution $\tau : \mathbb{C} \to \mathbb{C}$ with respect to the circle.) Clearly maps $H \in \mathcal{X}$ preserve the unit circle (the set of fixed points of τ); in particular, they do not move 1 close to 0 and ∞. By Theorem 2.8 (and the Exercise following it), $\mathcal{X}$ is compact.

Let us show that $\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic. The restriction of a map $H \in \mathcal{X}$ to the unit disk gives a continuous map $i : \mathcal{X} \to \mathcal{Y}$. The inverse map $i^{-1} : \mathcal{Y} \to \mathcal{X}$ is given by the following extension procedure. First, extend $h \in \mathcal{Y}$ continuously to the closed disk $\mathbb{D}$ (Theorem **??**), and then reflect it symmetrically to the exterior of the disk, i.e., let $H(z) = \tau \circ h \circ \tau(z)$ for $z \in \bar{\mathbb{C}} \smallsetminus \bar{\mathbb{D}}$. Since τ is an (orientation reversing) conformal map, H is K-qc on $\bar{\mathbb{C}} \smallsetminus \mathbb{T}$. By Lemma 2.5, it is K-qc everywhere, and hence belongs to $\mathcal{X}$.

Hence $\mathcal{Y}$ is compact as well. ☐

9. Measurable Riemann Mapping Theorem

We are now ready to prove one of the most remarkable facts of analysis: any measurable conformal structure with bounded dilatation is generated by a quasi-conformal map:

THEOREM 2.10 (Measurable Riemann Mapping Theorem). *Let μ be a measurable Beltrami differential on $\bar{\mathbb{C}}$ with $\|\mu\|_\infty < 1$. Then there is a quasi-conformal map $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ which solves the Beltrami equation: $\bar{\partial}h/\partial h = \mu$. This solution is unique up to post-composition with a Möbius automorphism of $\bar{\mathbb{C}}$. In particular, there is a unique solution fixing three points on $\bar{\mathbb{C}}$ (say, $0, 1$ and ∞).*

The local version of this result sounds as follows:

THEOREM 2.11 (Local integrability). *Let μ be a measurable Beltrami differential on a domain $U \subset \mathbb{C}$ with $\|\mu\|_\infty < 1$. Then there is a quasi-conformal map $h : U \to \mathbb{C}$ which solves the Beltrami equation: $\bar{\partial}h/\partial h = \mu$. This solution is unique up to post-composition with a conformal map.*

The rest of this section will be occupied with a proof of these two theorems.

9.1. Uniqueness. Uniqueness part in the above theorems is a consequence of Weil's Lemma. Indeed, if we have two solutions h and g, then the composition $\psi = g \circ h^{-1}$ is a qc map with $\bar{\partial}\psi = 0$ a.e. on its domain. Hence it is conformal.

9.2. Local vs global. Of course, the global Riemann Measurable Riemann Theorem immediately yields the local integrability (e.g., by zero extantion of μ from U to the whole sphere). Vice versa, the global result follows from the local one and the classical Uniformization
ref Theorem for the sphere . Indeed, by local integrability, there is a finite covering of the sphere $S^2 \equiv \bar{\mathbb{C}}$ by domains U_i and a family of qc maps $\phi_i : U_i \to \mathbb{C}$ solving the Beltrami equation on U_i. By Weil's Lemma, the gluing maps $\phi_i \circ \phi_j^{-1}$ are conformal. Thus the family of maps $\{\phi_i\}$ can be interpreted as a complex analytic atlas on S^2, which endows it with a new complex analytic structure m (compatible with the original qc structure). But by the Uniformization Theorem, all complex analytic structures on S^2 are equivalent, so that there exists a biholomorphic isomorphism $h : (S^2, m) \to \bar{\mathbb{C}}$. It means that the maps $h \circ \phi_i^{-1}$ are conformal on $\phi_i U_i$. Hence h is quasi-conformal on each U_i and $h_*(\mu) = (h \circ \phi_i^{-1})_* \sigma$ over there. Since the atlas is finite, h is a global quasi-conformal solution of the Beltrami equation.

9.3. Strategy. The further strategy of the proof will be the following. First, we will solve the Beltrami equation locally assuming that the coefficient μ is real analytic. It is a classical (and elementary) piece of the PDE theory. By the Uniformization Theorem, it yields a global solution in the real analytic case. Approximating a measurable Beltrami coefficient by real analytic ones and using compactness of the space of normalized K-qc maps, we will complete the proof.

9.4. Real analytic case. Assume that μ is a real analytic Beltrami coefficient in a neighborhood of 0 in $\mathbb{R}^2 \equiv \mathbb{C}_{\mathbb{R}}$ with $|\mu(0)| < 1$. Then it admits a complex analytic extension to a neighborhood of 0 in the complexification $\mathbb{C}^2$. Let (x, y) be the standard coordinates in $\mathbb{C}^2$, and let $u = x + iy,\ v = x - iy$. In these coordinates the complexified Beltrami equation assumes the form:

$$\frac{\partial h}{\partial v} - \mu(u, v)\frac{\partial h}{\partial u} = 0. \tag{9.1}$$

This is a linear equation with variable coefficients, which can be solved by the standard method of characteristics. Namely, let us consider a vector field $W(u, v) = (1, -\mu(u, v))$ near 0 in $\mathbb{C}^2$. Since the left-hand side of (9.1) is the derivative of h along X, we come to the equation $Wh = 0$. Solutions of this equation are the first integrals of the ODE $\dot{w} = W$. But since W is non-singular at 0, this ODE has a non-singular local first integral $h(u, v)$. Restricting h to $\mathbb{R}^2$, we obtain a local solution $h : (\mathbb{R}^2, 0) \to \mathbb{C}$ of the original Beltrami equation. Since h is non-singular at 0, it is a local (real analytic) diffeomorphism.

By means of the Uniformization Theorem, we can now pass from local to global solutions of the Beltrami equation with a real analytic Beltrami differential $\mu(z)d\bar{z}/dz$ on the sphere (see §9.2). Note that the global solution is real analytic as well since the complex structure generated by the local solutions is compatible with the original real analytic structure of the sphere (as local solutions are real analytic).

Exercise 2.6. *For a real analytic Beltrami coefficient*

$$\mu(z) = \sum a_{n,m} z^n \bar{z}^m$$

on $\mathbb{C}$, find the condition of its real analyticity at ∞.

There is also a "semi-local" version of this result:

If μ is a real analytic Beltrami differential on the disk $\mathbb{D}$ with $\|\mu\|_\infty < 1$, then there is a quasi-conformal (real analytic) diffeomorphism $h : \mathbb{D} \to \mathbb{D}$ solving the Beltrami equation $\bar{\partial}h/\partial h = \mu$.

To see it, consider the complex structure m on the disk generated by the local solutions of the Beltrami equation. We obtain a simply connected Riemann surface $S = (\mathbb{D}, m)$. By the Uniformization Theorem, it is conformally equivalent to either the standard disk $(\mathbb{D}, \sigma)$ or to the complex place $\mathbb{C}$. But S is quasi-conformally equivalent to the standard disk via the identical map $\mathrm{id} : (\mathbb{D}, m) \to (\mathbb{D}, \sigma)$. By Exercise 2.2, it is then conformally equivalent to the standard disk, and this equivalence $h : (\mathbb{D}, m) \to (\mathbb{D}, \sigma)$ provides a desired solution of the Beltrami equation.

By §9.1 Such a solution is unique up to a postcomposition with a Möbius automorphism of the disk.

9.5. Approximation. Let us consider an arbitrary measurable Beltrami coefficient μ on a disk $\mathbb{D}$ with $\|\mu\| < \infty$. Select a sequence of real analytic Beltrami coefficients μ_n on $\mathbb{D}$ with $\|\mu_n\|_\infty \leq k < 1$, converging to μ a.e.

Exercise 2.7. *Construct such a sequence (first approximate μ with continuous Beltrami coefficients).*

Applying the results of the previous section, we find a sequence of quasi-conformal maps $h_n : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ solving the Beltrami equations $\bar\partial h_n / \partial h_n = \mu_n$. The dilatation of these maps is bounded by $K = (1 + k)/(1 - k)$. By Corollary 2.9, they form a precompact sequence in the topology of uniform convergence on the disk. Any limit map $h : \mathbb{D} \to \mathbb{D}$ of this sequence is a quasi-conformal homeomorphism of $\mathbb{D}$. Let us show that its Beltrami coefficient is equal to μ.

By (7.5), the partial derivatives of the h_n belong to some ball of the Hilbert space $L^2(\mathbb{D})$. Hence we can select weakly convergent subsequences $\partial h_n \to \phi$, $\bar\partial h_n \to \psi$. We have checked in (8.2) that $\phi = \partial h$ and $\psi = \bar\partial h$. What is left is to check that $\psi = \mu\phi$. To this end, it is enough to show that $\mu_n \, \partial h_n \to \mu\phi$ weakly (to appreciate it, recall that the product is not weakly continuous, see Exercise 2.3). For any test function $g \in L^2(\mathbb{D})$, we have:

$$\left| \int (g\mu\phi - g\mu_n \, \partial h_n) \, dm \right| \leq$$

$$\leq \left| \int g\mu(\phi - \partial h_n) \, dm \right| + \int |g(\mu - \mu_n) \, \partial h_n| \, dm.$$

The first term in the last line goes to 0 since the ∂h_n weakly converge to ϕ. The second term is estimated by the Cauchy-Schwarz inequality by $\|g(\mu - \mu_n)\|_2 \|\partial h_n\|_2$, which goes to 0 since $\mu_n \to \mu$ a.e. and the ∂h_n belong to some Hilbert ball. This yields the desired.

It proves the Measurable Riemann Mapping Theorem on the disk $\mathbb{D}$, which certainly implies the local integrability. Now the global theorem on the sphere follows from the local integrability by §9.2. This completes the proof.

9.6. Conformal and complex structures. Let us discuss the general relation between the notions of complex and conformal structures. Consider an oriented surface S endowed with a *qs structure*, i.e., supplied with an atlas of local charts $\psi_i : V_i \to \mathbb{C}$ with uniformly qc transit maps $\psi_i \circ \psi_j^{-1}$ ("uniformly qc" means "with uniformly bounded dilatation"). Note that a notion of a measurable conformal structure with bounded dilatation makes perfect sense on such a surface (in what follows we call it just a "conformal structure").

Endow S with a complex structure compatible with its qs structure. By definition, it is determined by an atlas $\phi_i : U_i \to \mathbb{C}$ on S of uniformly qc maps such that the transit maps are complex analytic. Then the conformal structures $\mu_i = \phi_i^*(\sigma)$ on U_i coincide on the intersections of the local charts and have uniformly bounded dilatations. Hence they glue into a global conformal structure on S.

Vice versa, any conformal structure μ determines by the Local Integrability Theorem a new complex structure on the surface S compatible with its qc structure (see §9.2).

Thus the notions of conformal and complex structures on a qc surface are equivalent. In what follows we will not distinguish them either conceptually or notationally.

Fixing a reference complex structure on S (so that S becomes a Riemann surface), complex/conformal structures on S get parametrized by measurable Beltrami differentials μ on S with $\|\mu\|_\infty < 1$.

9.7. Moduli spaces. Consider some qc surface S (with or without boundary, possibly marked or partially marked).

The *moduli space* $\mathcal{M}(S)$, or the *deformation space of* S is the space of all conformal structures on S compatible with the underlying qc structure, up to the action of qc homeomorphisms perserving the marked data. In other words, $\mathcal{M}(S)$ is the space of all Riemann surfaces qc equivalent to S, up to conformal equivalence relation (respecting the marked data).

If we fix a reference Riemann surface S_0, then its deformations are represented by qc homeomorphisms $h : S_0 \to S$ to various Riemann surfaces S. Two such homeomorphisms h and $\tilde{h}$ represent the same point of the moduli space if there exists a conformal isomorphism $A : S \to \tilde{S}$ such that the composition $H = \tilde{h}^{-1} \circ A \circ h : S_0 \to S_0$ respects all the marked data. In particular, $H = \mathrm{id}$ on the marked boundary.

In the case when the whole fundamental group is marked, H must be homotopic to the id relative to the marked boundary.

For instance, if S has a finite conformal type, i.e., S is a Riemann surface of genus g with n punctures (without marking), then $\mathcal{M}(S)$ is the classical moduli space $M^{g,n}$. If S is fully marked then $\mathcal{M}(S)$ is the classical Teichmüller space $T^{g,n}$. This space has a natural complex structure of complex dimension $3g-3+n$ for $g>1$. For $g=1$ (the torus case), $\dim T^{1,0}=1$ (see §1.4.2) and $\dim T^{1,n}=n-1$ for $n\geq 1$. For $g=0$ (the sphere case), $\dim T^{0,n}=0$ for $n\leq 3$ (by the Riemann-Koebe Uniformization Theorem and 3-transitivity of the Möbius group action) and $\dim T^{0,n}=n-3$ for $n>3$.

Exercise 2.8. *What is the complex modulus of the four punctured sphere?*

There is a natural projection (fogetting the marking) from $T^{g,n}$ onto $M^{g,n}$. The fibers of this projection are the orbits of the so called "Teichmüller modular group" acting on $T^{g,n}$ (it generalizes the classical modular group $\mathrm{PSL}(2,\mathbb{Z})$, see §1.4.2).

By the Riemann Mapping Theorem, the disk $\mathbb{D}$ does not have moduli. However, if we mark its boundary $\mathbb{T}$, then the space of moduli, $\mathcal{M}(\mathbb{D},\mathbb{T})$, becomes infinitely dimensional! By definition, $\mathcal{M}(\mathbb{D},\mathbb{T})$ is the space of all Beltrami differentials μ on $\mathbb{D}$ up to the action of the group of qc homeomorphisms $h:\mathbb{D}\to\mathbb{D}$ whose boundary restrictions are Möbius: $h|\mathbb{T}\in\mathrm{PSL}(2,\mathbb{R})$. It is called the *universal Teichmüller space*, since it contains all other deformation spaces. This space has several nice descriptions, which will be discussed later on. It plays an important role in holomorphic dynamics.

9.8. Dependence on parameters. It is important to know how the solution of the Beltrami equation depends on the Beltrami differential. It turns out that this dependence is very nice. Below we will formulate three statements of this kind (on continuous, smooth and holomorphic dependence).

Proposition 2.12. *Let μ_n be a sequence of Beltrami differentials on $\mathbb{C}$ with uniformly bounded dilatation, converging a.e. to a differential μ. Consider qc solutions $h_n:\bar{\mathbb{C}}\to\bar{\mathbb{C}}$ and $h:\bar{\mathbb{C}}\to\bar{\mathbb{C}}$ of the corresponding Beltrami equations fixing $0,1$ and ∞. Then the h_n converge to h uniformly on $\mathbb{C}$.*

Proof. By Theorem 8.4, the sequence h_n is precompact. Take any limit map g of this sequence. By the argument of §9.5, its Beltrami differential is equal to μ. By uniqueness of the normalized solution of the Beltrami equation, $g=h$. The conclusion follows. □

Consider a family of Beltrami differentials μ_t depending on parameters $t = (t_1, \dots, t_n)$ ranging over a domain $U \subset \mathbb{R}^n$. This family is said to be *differentiable* at some $t \in U$ if there exist Beltrami differentials α_t^i of class $L^\infty(\mathbb{C})$ (but not necessarily in the unit ball of this space) such that for all sufficiently small $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}^n$, we have:

$$\mu_{t+\epsilon} - \mu_t = \sum_{i=1}^{n} \alpha_t^i \epsilon_i + |\epsilon|\, \beta(t, \epsilon),$$

where the norm $\|\beta_{t,\epsilon}\|_\infty$ stays bounded and $\beta_{t,\epsilon}(z) \to 0$ a.e. on $\mathbb{C}$ as $\epsilon \to 0$.

Assume additionally that the family μ_t is differentiable at all points $t \in U$, that the norms $\|\alpha_t^i\|$ are locally bounded, and that the $\alpha_t^i(z)$ continuously depend on t in the sense of the convergence a.e. Then the family μ_t is said to be *smooth*.

Let us now consider a family of qc maps $h_t : \mathbb{C} \to \mathbb{C}$ depending on parameters $t \in U$. Considering these maps as elements of the Sobolev space H, we can define differentiabilty and smoothness in the usual way. This family is differentiable at some point $t \in U$ if there exist vector fields v_t^i on $\mathbb{C}$ of Sobolev class H such that

$$h_{t+\epsilon} - h_t = \sum_{i=1}^{n} \epsilon_i v_t^i + |\epsilon| g_{t,\epsilon},$$

where $g_{t,\epsilon} \to 0$ in the Sobolev norm as $\epsilon \to 0$ (in particular $g_{t,\epsilon} \to 0$ uniformly on the sphere). If additionally the v_t^i depend continuously on t (as elements of H), then one says that h_t smoothly depends on t. Of course, in this case, any point $z \in \mathbb{C}$ smoothly moves as parameter t changes, i.e., $h_t(z)$ smoothly depends on t.

THEOREM 2.13. *If μ_t, $t \in U \subset \mathbb{R}^n$, is a smooth family of Beltrami differentials, then the normalized solutions $h_t : \mathbb{C} \to \mathbb{C}$ of the corresponding Beltrami equations smoothly depend on t.*

Let us finally discuss the holomorphic dependence on parameters. Let U be a domain in $\mathbb{C}^n$ and let $\mathcal{B}$ be a complex Banach space. A function $f : U \to \mathcal{B}$ is called *holomorphic* if for any linear functional $\phi \in \mathcal{B}^*$, the function $\phi \circ f : U \to \mathbb{C}$ is holomorphic. Beltrami differentials are elements of the complex Banach space L^∞, while qc maps $h : \mathbb{C} \to \mathbb{C}$ are elements of the complex Sobolev space H. So, it makes sense to talk about holomorphic dependence of these objects on complex parameters $t = (t_1, \dots, t_n) \in U$. Note that if h_t depends holomorphically on t, then any point $z \in \mathbb{C}$ moves holomorphically as t changes (in fact, holomorphic dependence on parameters is often understood in this weak sense).

be careful here!

THEOREM 2.14. *If the Beltrami differential μ_t holomorphically depends on parameters $t \in U$, then so do the normalized solutions $h_t : \mathbb{C} \to \mathbb{C}$ of the corresponding Beltrami equations.*

The proofs of the last two theorems can be found in [**AB**].
9.8.1. *Simple conditions.*

LEMMA 2.15. *Let $\mathcal{B}$ be a Banach space, and let $\{f_\lambda\}$, $\lambda \in \mathbb{D}_\rho$, be a uniformly bounded family of linear functionals on $\mathcal{B}$ such that for a dense linear subspace L of points $x \in \mathcal{B}$, the function $\lambda \mapsto f_\lambda(x)$ is holomorphic in λ. Then $\{f_\lambda\}$ as an element of the dual space $\mathcal{B}^*$ depends holomorphically on λ.*

PROOF. For $x \in L$, we have a power series expansion

$$f_\lambda(x) = \sum a_n(x)\lambda^n$$

convergent in $\mathbb{D}_\rho$. By the Cauchy estimate,

$$|a_n(x)| \leq \frac{C\|x\|}{\rho^n}, \quad x \in L,$$

where C is an upper bound for the norms $\|f_\lambda\|$, $\lambda \in \mathbb{D}_\rho$. Clearly, the $a_n(x)$ linearly depend on $x \in L$. Hence, a_n are bounded linear functionals on L; hence they admit an extension to bounded linear functionals on $\mathcal{B}$. Moreover, $\|a_n\| \leq C\rho^{-n}$. It follows that the power series $\sum a_n\lambda^n$ converges in the dual space $\mathcal{B}^*$ uniformly in λ over any disk $\mathbb{D}_r$, $r < \rho$. Hence it represents a holomorphic function $D_\rho \mapsto \mathcal{B}^*$, which, of course, coincides with $\lambda \mapsto f_\lambda$. □

For further applications, let us formulate one simple condition of holomorphic dependence:

LEMMA 2.16. *Let $\rho > 0$ and let $U \subset \mathbb{C}$ be an open subset in $\bar{\mathbb{C}}$ of full measure. Let $\mu_\lambda \in L^\infty(\mathbb{C})$, $\lambda \in \mathbb{D}_\rho$, be a family of Beltrami differentials with $\|\mu_\lambda\|_\infty \leq 1$ whose restriction to U is smooth in both variables (λ, z) and is holomorphic in λ. Then $\{\mu_\lambda\}$ is a holomorphic family of Beltrami differentials.*

PROOF. Let us first assume that $U = \bar{\mathbb{C}}$. Then

$$\mu_\lambda(z) = \sum a_n(z)\lambda^n, \quad \lambda \in \mathbb{D}_\rho,$$

where the a_n are smooth functions on $\bar{\mathbb{C}}$, and the series converges uniformly over $\bar{\mathbb{C}} \times \mathbb{D}_r$ for any $r < \rho$. It follows that the series $\sum a_n\lambda^n$ in L^∞ converges uniformly over $\mathbb{D}_r$ and hence represents a holomorphic function $\mathbb{D}_r \to L^\infty$.

Let us now consider the general case; put $K = \bar{\mathbb{C}} \setminus U$. Consider a sequence of smooth functions $\chi_l : \bar{\mathbb{C}} \to [0,1]$ such that $\chi_l = 0$ on K and for any $z \in U$, $\chi_l(z) \to 1$ as $l \to \infty$.

Consider smooth Beltrami differentials $\mu_\lambda^l = \chi_l \mu_\lambda$. By the above consideration, they depend holomorphically on λ. Moreover, since K has zero area, $\chi_l \mu_\lambda \to \mu_\lambda$ a.e. as $l \to \infty$. Note also that $\|\mu_\lambda^l\|_\infty \leq 1$.

Take any smooth test function ϕ on $\bar{\mathbb{C}}$ and let

$$g_l(\lambda) = \int \mu_\lambda^l \phi dA; \quad g(\lambda) = \int \mu_\lambda \phi dA,$$

where dA is the (normalized) area element on $\bar{\mathbb{C}}$. The family $\{g_l\}$ is uniformly bounded: $|g_l(\lambda)| \leq \|\phi\|_\infty$ By the Lebesgue Bounded Convergence Theorem, $g_l(\lambda) \to g(\lambda)$ as $l \to \infty$

By the previous discussion, functions g_l are holomorphic functions on $\mathbb{D}_\rho$. By the Little Montel Theorem, this family is normal. Hence we can select a subsequence conveging to g uniformly on compact subsets of $\bar{\mathbb{C}}$. It follows that g is holomorphic on $\mathbb{D}_\rho$.

Since smooth functions are dense in L^1, Lemma 2.15 can be applied. It implies the assertion. ☐

EXERCISE 2.9. *Let $f : S \to T$ be a holomorphic map between two Riemann surfaces, and let $\{\mu_\lambda\}$ be a holomorphic family of Beltrami differentials on T. Then $f^*(\mu_\lambda)$ is a holomorphic family of Beltrami differentials on S.*

10. Quasi-symmetric maps

DEFINITION 2.1. A map $h : X \to Y$ between two metric spaces is called *κ-quasi-symmetric* ("κ-qs") if for any triple of points a, b, c with $\operatorname{dist}(a,c) \leq \operatorname{dist}(a,b)$ we have: $\operatorname{dist}(h(a), h(c)) \leq \kappa \operatorname{dist}(h(a), h(b))$. A map is called quasi-symmetric ("qs") if it is κ-qs for some κ. The *dilatation* of a qs map is the smallest κ with this property.

EXERCISE 2.10. *A metric space is called geodesic if any two points in it can be joined with an isometric image of a real interval $[x, y]$. Assume that X is geodesic and $h : X \to Y$ is κ-qs. Then for any $L > 0$ there exists an $M = M(\kappa, L) > 0$ such that*

$$\operatorname{dist}(a,c) \leq L \operatorname{dist}(a,b) \Rightarrow \operatorname{dist}(h(a), h(c)) \leq M \operatorname{dist}(h(a), h(b)).$$

On the plane, the class of orientation preserving quasi-symmetric maps in fact coincides with the class of quasi-conformal maps. In one direction, it is a simple consequence of the Compactness Theorem:

PROPOSITION 2.17. *Any K-quasi-conformal map $h : \mathbb{C} \to \mathbb{C}$ is $\kappa(K)$-quasi-symmetric in the Euclidean metric of the plane.*

PROOF. Otherwise there would exist a sequence of K-qc maps $h_n : \mathbb{C} \to \mathbb{C}$ and a sequence of triples of points a_n, b_n, c_n in $\mathbb{C}$ such that

$$|a_n - c_n| \le |a_n - b_n| \quad \text{but} \quad |h_n(a_n) - h_n(c_n)|/|h_n(a_n) - h_n(b_n)| \to \infty. \tag{10.1}$$

Consider two sequences of affine maps S_n and T_n such that

$$S_n(0) = a_n,\ T_n(h_n(a_n)) = 0 \quad \text{and} \quad S_n(1) = b_n,\ T_n(h_n(b_n)) = 1.$$

Then the normalized maps $H_n = T_n \circ h_n \circ S_n$ fix 0 and 1. By the Compactness Theorem 2.8, they are uniformly bounded on the unit disk $\mathbb{D}$. On the other hand, (10.1) implies that the points $x_n = S_n^{-1} c_n$ belong to $\mathbb{D}$, while $H_n(x_n) = T_n(h_n(c_n)) \to \infty$ - contradiction. □

In particular, if we consider a quasi-conformal map $h : \mathbb{C} \to \mathbb{C}$ preserving the real line $\mathbb{R}$, it restricts to a quasi-symmetric map on the latter. Remarkably, the inverse is also true:

THEOREM 2.18 (Ahlfors-Boerling Extension). *Any κ-qs orientation preserving map $h : \mathbb{R} \to \mathbb{R}$ extends to a $K(\kappa)$-qc map $H : \mathbb{C} \to \mathbb{C}$.*

PROOF. □

Note that in the Ahlfors-Boerling extension is obviously affinely equivariant (that is, commutes with the action of the complex affine group $z \mapsto az + b$).

It looks at first glance that the class of 1D quasi-symmetric maps is a good analogue of the class of 2D quasi-conformal maps. However, this impression is superficial: two-dimensional qc maps are fundamentally better than one-dimensional qs maps. For instance, qc maps can be glued together without any loss of dilatation (Lemma 2.5), while qs maps cannot:

EXERCISE 2.11. *Consider a map $h : \mathbb{R} \to \mathbb{R}$ equal to* id *on the negative axis, and equal to $x \mapsto x^2$ on the positive one. This map is not quasi-symmetric, though its restrictions to the both positive and negative axes are.*

Another big defficiency of one-dimensional qs maps is that they can well be singular (and typically are in the dynamical setting - see **??**), while 2D qc maps are always absolutely continuous (Proposition 8.1).

These advantages of qc maps makes them much more efficient tool for dynamics than one-dimensional qs maps. This is a reason why complexification of one-dimensional dynamical systems is so powerful.

Let us now state an Extension Lemma in an annulus which will be usefull in what follows:

LEMMA 2.19 (Interpolation). *Let us consider two round annuli* $A = \mathbb{A}[1,r]$ *and* $\tilde{A} = \mathbb{A}[1,\tilde{r}]$, *with* $0 < \epsilon \le \mathrm{mod}\, A \le \epsilon^{-1}$ *and* $\epsilon \le \mathrm{mod}\, \tilde{A} \le \epsilon^{-1}$. *Then any* κ*-qs map* $h : (\mathbb{T}, \mathbb{T}_r) \to (\tilde{\mathbb{T}}, \tilde{\mathbb{T}}_{\tilde{r}})$ *admits a* $K(\kappa, \epsilon)$*-qc extension to a map* $H : A \to \tilde{A}$.

PROOF. Since A and $\tilde{A}$ are ϵ^2-qc equivalent, we can assume without loss of generality that $A = \tilde{A}$. Let us cover A by the upper half-plane, $\theta : \mathbb{H} \to A$, $\theta(z) = z^{\frac{-\log ri}{\pi}}$, where the covering group generated by the dilation $T : z \mapsto \lambda z$, with $\lambda = e^{\frac{2\pi^2}{\log r}}$. Let $\bar{h} : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be the lift of h to $\mathbb{R}$ such that $\bar{h}(1) \in [1, \lambda) \equiv I_\lambda$ and $\bar{h}(1) \in (-\lambda, -1]$ (note that $\mathbb{R}_+$ covers $\mathbb{T}_r$, while $\mathbb{R}_-$ covers $\mathbb{T}$). Moreover, since $\deg h = 1$, it commutes with the deck transformation T.

A direct calculation shows that the dilatation of the covering map θ on the fundamental intervals I_λ and $-I_\lambda$ is comparable with $(\log r)^{-1}$. Hence $\bar{h}$ is $C(\kappa, r)$-qs on this interval. By equivariance it is $C(\kappa, r)$-qc on the rays $\mathbb{R}_+$ and $\mathbb{R}_-$.

It is also quasi-symmetric near the origin. Indeed, by the equivariance and normalization,

$$(1+\lambda)^{-1}|J| \le |\bar{h}(J)| \le (1+\lambda)|J|$$

for any interval J containing 0, which easily implies quasi-symmetry.

Since the Ahlfors-Börling extension is affinely equivariant, the map $\bar{h}$ extends to a $K(\kappa, r)$-qc map $\bar{H} : \mathbb{H} \to \mathbb{H}$ commuting with T. Hence $\bar{H}$ descends to a $K(\kappa, r)$-qc map $H : A \to A$. □

****

Note that the Gluing Lemma makes a difference between complex qc and real qs maps which is crucial for the pull-back argument.

Let D be a simply connected domain conformally equivalent to the hyperbolic plane $\mathbb{H}^2$. Given a family of subsets $\{S_k\}_{k=1}^n$ in D, let us say that a family of disjoint annuli $A_k \subset D \setminus \cup S_i$ is *separating* if A_k surrounds S_k but does not surround the S_i, $i \ne k$. The following lemma is used in the present paper uncountably many times:

Moving Lemma. • *Let* $a, b \in D$ *be two points on hyperbolic distance* $\rho \le \bar{\rho}$. *Then there is a diffeomorphism* $\phi : (D, a) \to (D, b)$, *identical near* ∂D, *with dilatation* $\mathrm{Dil}(\phi) = 1 + O(\rho)$, *where the constant depends only on* $\bar{\rho}$.

• *Let* $\{(a_k, b_k)\}_{k=1}^n$ *be a family of pairs of points which admits a family of separating annuli* A_k *with* $\mathrm{mod}\, A_k \ge \mu$. *Then there is a diffeomorphism* $\phi : (D, a_1, \dots a_n) \to (D, b_1, \dots, b_n)$, *identical near* ∂D, *with dilatation* $\mathrm{Dil}(\phi) = 1 + O(e^{-\mu})$.

PROOF. As the statement is conformally equivalent, we can work with the unit disk model of the hyperbolic plane, and can also assume that $a = 0$, $b > 0$. Also, it is enough to prove the statement for sufficiently small ρ.

There is a smooth function $\psi : [0,1] \to [b,1]$ such that $\psi(x) \equiv b$ near 0, $\psi(x) \equiv 0$ near 1, and $\psi'(x) = O(\rho)$, with a constant depending only on $\bar{\rho}$.

Let us define a smooth map $\phi : (\mathbb{D}, 0) \to (\mathbb{D}, b)$ as $z \mapsto z + \psi(|z|)$. Then

$$\partial\phi(z) = 1 + \psi'(|z|)\frac{\bar{z}}{2|z|} = 1 + O(\rho), \quad \bar{\partial}\phi(z) = \psi'(|z|)\frac{z}{2|z|} = O(\rho). \tag{10.2}$$

Thus

$$\mathrm{Jac}(f) = \partial\phi(z)|^2 - |\bar{\partial}\phi(z)|^2 = 1 + O(\rho).$$

Hence for sufficiently small $\rho > 0$, f is a local orientation preserving diffeomorphism. As $f : \partial\mathbb{D} \to \partial\mathbb{D}$, f is a proper map. Hence it is a diffeomorphism.

Finally, (10.2) yields that the Beltrami coefficient $\mu_f = O(\rho)$, so that the dilatation $\mathrm{Dil}(f) = 1 + O(\rho)$. □

Let $Q \subset \mathbb{C}$, $h : Q \to \mathbb{C}$ be a homeomorphism onto its image. It is called *quasi-symmetric (qs)* if for any three points $a, b, c \in Q$ such that $q^{-1} \le |a-b|/|b-c| \le q$, we have: $\kappa(q)^{-1} \le |h(a) - h(b)|/|h(b) - h(c)| \le \kappa(q)$. It is called κ-quasi-symmetric if $\kappa(1) \le \kappa$. It follows from the Compactness Lemma that any K-qc map is κ-quasi-symmetric, with a κ depending only on K.

Let us discuss quasi-symmetric maps of the circle $\mathbb{T} = \{z : |z| = 1\}$. Given an interval $J \subset \mathbb{T}$, let $|J|$ denote its length. An orientation preserving map $h : \mathbb{T} \to \mathbb{T}$ is called *κ-quasi-symmetric* (κ-qs) if for any two adjacent intervals $I, J \subset \mathbb{T}$, $|hI|/|hJ| \le \kappa$.

Let $\mathbb{T}_r = \{z : |z| = r\}$, $\mathbb{T} \equiv \mathbb{T}_1$. Let $\mathbb{A}(r, R) = \{z : r < |z| < R\}$. Similar notations are used for a closed annulus $\mathbb{A}[r, R]$ (or semi-closed one).

proclaim Ahlfors-Börling Extension Theorem. Any κ-quasi-symmetric map $h : \mathbb{T} \to T$ extends to a $K(\kappa)$-qc map $H : \mathbb{C} \to \mathbb{C}$. Vice versa: The restriction of any K-qc map $H : (\mathbb{A}(r^{-1}, r), \mathbb{T}) \to (U, \mathbb{T})$ (where $U \subset \mathbb{C}$) to the circle $\kappa(K, r)$-quasi-symmetric.

Let us note that in the upper half-plane model, the Ahlfors-Börling extension of a qs map $\mathbb{R} \to \mathbb{R}$ is affinely equivariant (that is, commutes with the action of the complex affine group $z \mapsto az + b$).

10.1. Quasicircles. Let us start with an intrinsic geometric definition of quasicircles:

Definition 2.2. A Jordan curve $\gamma \subset \mathbb{C}$ is called a κ-*quasicircle* if for any two points $x, y \in \gamma$ there is an arc $\delta \subset \gamma$ bounded by these points such that

$$\operatorname{diam} \delta \leq \kappa |x - y|.$$

A curve is called a quasicircle if it is a κ-quasicircle for some κ. The best possible κ in the above definition is called the *dilatation* of the quasicircle. A Jordan disk is called (κ-)*quasidisk* if it is bounded by a (κ-)quasicircle.

Exercise 2.12. *Let D be a κ-quasidisk, $\partial D = \gamma$. Show that*

$$\sup_{z \in D} \operatorname{dist}(z, \gamma) \geq c \operatorname{diam} D$$

for some constant $c > 0$ depending only on κ.

On the other hand, quasicircles can also be characterized as qc images of the circle (which explains the importance of this class of curves). Recall from §**??** that $r_{D,a}$ denote the inner radius of a pointed disk (D, a).

Theorem 2.20. *Let (D, a) a pointed κ-quasidisk, and let $\phi : (\mathbb{D}, 0) \to (D, a)$ be the normalized Riemann mapping. Assume that $r_{D,a} \geq c \operatorname{diam} D$, where $c > 0$. Then ϕ admits a K-qc extension to the whole complex plane, where K depends only on κ and c.*

Vice versa, let (D, a) be a Jordan disk such that there exists a K-qc map $h : (\mathbb{C}, \mathbb{D}, 0) \to (\mathbb{C}, D, a)$. Then D is a κ-quasidisk and $r_{D,a} \geq c \operatorname{diam} D$, where the constants κ and $c > 0$ depend only on K.

Recall the definition of the inner and the outer radia, $r_{D,a}$ and $R_{D,a}$ of a pointed domain (D, a). Let $\mathcal{QD}_{\kappa,r}$, $r > 0$, denote the space of pointed κ-quasidisks $(D, 0)$ with $r \leq r_{D,0} \leq R_{D,0} \leq 1/r$, endowed with the Carathéodory topology.

Proposition 2.21. *The space $\mathcal{QD}_{\kappa,r}$ is compact.*

Proof. Consider a quasidisk $(D, 0) \in \mathcal{QD}_{\kappa,r}$. By Theorem 2.20, the normalized Riemann mapping $h : (\mathbb{D}, 0) \to (D, 0)$ admits a K-qc extension to the whole complex plane $\mathbb{C}$, where K depends only on κ and r. Moreover, $r \leq |h(1)| \leq 1/r$. By the Compactness Theorem (see Exercise 2.5), this family of qc maps is compact in the uniform topology on $\mathbb{C}$. Since uniform limits of κ-quasidisks are obviously κ-quasidisks, the conclusion follows. □

A set is called "0-symmetric" if it is invariant under the reflection with respect to the origin.

Exercise 2.13. *Let γ be a 0-symmetric κ-quasicircle. Then the eccentricity of γ around 0 is bounded by $2\kappa + 1$.*

11. Removability

11.1. Conformal vs quasiconformal. Similarly to the notion of qc removability introduced in §8.1 we can define conformal removability:

DEFINITION 2.3. A compact subset $X \subset \mathbb{C}$ is called *conformally removable* if for any open sets $U \supset X$ in $\mathbb{C}$, any homeomorphic embedding $h : U \hookrightarrow \mathbb{C}$ which is conformal on $U \smallsetminus X$ is conformal/qc on U.

It is classical that isolated points and smooth Jordan curves are conformally removable. By §8.1 of Ch. 2, they are qc removable as well. In fact, these two properties are equivalent:

PROPOSITION 2.22. *Conformal removability is equivalent to qc removability.*

Thus, we can unambiguously call a set "removable".

11.2. Removability and area. The Measurable Riemann Mapping Theorem yields:

PROPOSITION 2.23. *Removable sets have zero area.*

PROOF. Assume that $m(X) > 0$. Then there exists a non-trivial Beltrami differential μ supported on X. Let $h : \mathbb{C} \to \mathbb{C}$ be a solution of the corresponding Beltrami equation. Then h is conformal outside X but is not conformal on X. □

The reverse is false:

EXAMPLE 2.1.

11.3. Divergence property.

DEFINITION 2.4. Let us say that a compact set $X \subset \mathbb{C}$ satisfies the *divergence property* if for any point $z \in X$ there exists a nest of annuli $A^n(z)$ around z such that

$$\sum A^n(z) = \infty.$$

Without loss of generality we can assume (and we will always do so) that each annulus in this definition is bounded by two Jordan curves.

LEMMA 2.24. *Compact sets satisfying the divergence property are Cantor.*

PROOF. Consider any connected component X_0 of X, and let $z \in X_0$. Then the annuli $A^n(z)$ are nested around X_0. By Corollary 1.19 of the Grötzsch Inequality, X_0 is a single point. □

LEMMA 2.25. *Let $X \subset \mathbb{C}$ be a compact set satisfying the divergence property. Then for any neighborhood $U \supset X$, any qc embedding $h : U \smallsetminus X \hookrightarrow \mathbb{C}$ admits a homeomorphic extension through X.*

PROOF. Let $h : U \smallsetminus X \hookrightarrow \mathbb{C}$ be a K-qc embedding. If $X \subset U' \Subset U$ then $h(U')$ is bounded in $\mathbb{C}$. So, without loss of generality we can assume that $h(U)$ is bounded in $\mathbb{C}$.

For $z \in X$, let us consider the nest of annuli $h(A^n(z))$. Since h is quasiconformal,

$$\sum \operatorname{mod} h(A^n(z)) \geq K^{-1} \sum \operatorname{mod} A^n(z) = \infty.$$

Let $\Delta^n(z)$ be the bounded component of $\mathbb{C} \smallsetminus h(A^n(z))$, and let

$$\Delta^\infty(z) = \bigcap_n D^n(z).$$

By Corollary 1.19 of the divergence property, $\Delta^\infty(z)$ is a single point $\zeta = \zeta(z)$. Let us extend h through X by letting $h(z) = \zeta$.

This extension is continuous. Indeed, let $D^n(z)$ be the bounded component of $\mathbb{C} \smallsetminus A^n(z)$. Then by Corollary 1.19, $\operatorname{diam} D^n(z) \to 0$, so that $D^n(z)$ is a base of (closed) neighborhoods of z. But

$$\operatorname{diam} h(D^n(z)) = \operatorname{diam} \Delta^n(z) \to 0,$$

which yields continuity of h at z.

Switching the roles of (U, X) and $(h(U), h(X))$, we conclude that h^{-1} admits a continuous extension through $h(X)$. Hence the extension of h is homeomorphic. ☐

It is worthwhile to note that, in fact, general homeomorphisms extend through Cantor sets:

EXERCISE 2.14. *(i) Let us consider two Cantor sets X and $\tilde{X}$ in $\mathbb{C}$ and their respective neighborhoods U and $\tilde{U}$. Then any homeomorphism $h : U \smallsetminus X \to \tilde{U} \smallsetminus \tilde{X}$ admits a homeomorphic extension through X.*

(ii) It was essential to assume that both *sets X and $\tilde{X}$ are Cantor! For* any *compact set $X \subset \mathbb{C}$, give an example of an embedding $h : C \smallsetminus X \hookrightarrow \mathbb{C}$ which does not admit a continuous extension through X.*

LEMMA 2.26. *Compact sets satisfying the divergence property have zero area.*

We will show now that sets satisfying the divergence property are removable, and even in the following stronger sense:

THEOREM 2.27. *Let $X \subset \mathbb{C}$ be a compact set satisfying the divergence property. Then for any neighborhood $U \supset X$, any conformal/qc* *embedding* $h : U \setminus X \hookrightarrow \mathbb{C}$ *admits a conformal/qc extension through* X.

PROOF. Let $h : U \setminus X \hookrightarrow \mathbb{C}$ be a K-qc embedding. By Lemma 2.25, h extends to an embedding $U \hookrightarrow \mathbb{C}$, which will be still denoted by h. Let us show that h belongs to the Sobolev class $H(U)$.

Since X is a Cantor set, it admits a nested base of neighborhoods U^n such that each U^n is the union of finitely many disjoint Jordan diks. Take any $\mu > 0$. By the Grẗzsch Inequality, for any $n \in \mathbb{N}$ there is $k = k(\mu, l) > 0$ such that $\operatorname{mod}(\partial U^{n+k}, \partial U^n) \geq \mu > 0$. Let χ_n be the solution of the Dirichlet problem in $U^n \setminus U^{n+k}$ vanishing on ∂U^{n+k} and equal to 1 on ∂U^n. By Theorem 1.21, $D(\chi_n) \leq 1/\mu$.

Let us continuously extend χ to the whole plane in such a way that it vanishes on U^{n+k} and identically equal to 1 on $\mathbb{C} \setminus U^n$. We obtain a piecewice smooth function $\chi : \mathbb{C} \to [0, 1]$, with the jump of the derivative on the boundary of the domains U^n and U^{n+k}.

Let $h_n = \chi_n h$. These are piecewise smooth functions with bounded Dirichlet integral. Indeed,

$$D(h_n) = \int (|\nabla \chi_n|^2 |h|^2 + |\chi_n|^2 |\nabla h|^2) dm \leq \operatorname{diam}(h(U))/\mu + C(K) m(h(U)),$$

where $C(K) = (1 + k^2)/(1 - k^2)$ comes from the area estimate (area estimate). By weak compactness of the unit ball in $L^2(U)$, we can select a converging subsequence $\partial h_n \to \phi$, $\bar{\partial} h_n \to \psi$. But $h_n \to h$ pointwise on $U \setminus X$, so that by Lemma 2.26, $h_n \to h$ a.e. It follows that ϕ and ψ are distributional partial derivatives of h (see (8.2)).

Finally, if h is conformal on $U \setminus X$ then by Weil's Lemma it is conformal on U. ☐

Compactness in H of functions with bounded D.I. - formulate as a lemma?

Part 2

Complex quadratic family

CHAPTER 3

Dynamical plane

12. Glossary of topological dynamics

This glossary collects some basic notions of dynamics. Its purpose is to fix terminology and notations and to comfort a reader who has no experience with dynamics.

Consider a continuous endomorphism $f : X \to X$ of a topological space X. The n-fold iterate of f is denoted by f^n, $n \in \mathbb{N}$. A *topological dynamical system* (with discrete positive time) is the $\mathbb{N}$-action generated by f, $n \mapsto f^n$. The *orbit* or *trajectory* of a point $x \in K$ is $\text{orb}(x) = \{f^n x\}_{n\in\mathbb{N}}$. The subject of topological dynamics is to study qualitative behavior of orbits of a topological dynamical system.

Here is the simplest possible behavior: a point x is called *fixed* if $fx = x$. More generally, a point x is called *periodic* if it has a finite orbit, i.e., there exists a $p \in \mathbb{N}$ such that $f^p x = x$. The smallest p with this property is called the *period* of x. The orbit of x (consisting of p permutted points) is naturally called a *periodic orbit* or a *cycle* (of period p).

The asymptotic behavior of an orbit can be studied in terms of its limit set. The ω-limit set $\omega(x)$ of a point x is the set of all accumulation points of $\text{orb}(x)$. If X is compact then $\omega(x)$ is a non-empty compact subset of X. We say that the orbit of x *converges to a cycle* (of a periodic point α) if $\omega(x) = \text{orb}(\alpha)$.

A point x is called *recurrent* if $\omega(x) \ni x$. Existence of non-periodic recurrrent points is a feature of non-trivial dynamics.

Two dynamical systems $f : X \to X$ and $g : Y \to Y$ are called *topologically conjugate* (or topologically equivalent) if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, i.e., the following commutative diagram holds:

$$\begin{array}{ccc} X & \xrightarrow[f]{} & X \\ h \downarrow & & \downarrow h \\ Y & \xrightarrow[g]{} & Y \end{array}$$

Classes of topologically equivalent dynamical systems (within an *a priori* specified family) are called *topological classes*. If X and Y are endowed with an extra structure (smooth, conformal, quasi-conformal etc.) respected by h, then f and g are called *smoothly/conformally/quasi-conformally conjugate* (or equivalent). The corresponding equivalence classes are called smooth/conformal/quasi-conformal classes.

Topological conjugacies respect all properties which can be formulated in terms of topological dynamics: orbits go to orbits, cycles go to cycles of the same period, ω-limit sets go to ω-limit sets, converging orbits go to converging orbits etc.

A homeomorphism $h : X \to X$ commuting with a dynamical system $f : X \to X$ (i.e., conjugating f to itself) is called an *automorphism* of f.

A continuous map which makes the above diagram commutative is called *equivariant* (with respect to the actions of f and g). A *surgective* equivariant map is called a *semi-conjugacy* between f and g. In this case g is also called a *quotient* of f.

It will be very convenient to extend the above terminology to *partially defined* maps. Let f and g be partially defined maps on the spaces X and Y respectively (i.e., f maps its domain $\mathrm{Dom}(f) \subset X$ to X, and similarly does g). Let $A \subset X$. A map $h : A \to Y$ is called equivariant (with respect to the actions of f and g) if for any $x \in A \cap \mathrm{Dom}(f)$ such that $fx \in A$ we have: $hx \in \mathrm{Dom}(g)$ and $h(fx) = g(hx)$. (Briefly speaking, the equivariance equation is satisfied whenever it makes sense.)

13. Holomorphic dynamics: basic objects

Below

$$f \equiv f_c : z \mapsto z^2 + c$$

unless otherwise is stated. Dynamical objects will be labelled by either f or c whatever is more convenient in a particular situation (for instance, $D_f(\infty) \equiv D_c(\infty)$ by default). Moreover, the label can be skipped altogether if f is not varied.

13.1. Critical points and values. First note that f^n is a branched covering of $\mathbb{C}$ over itself of degree 2^n. Its critical points and values have a good dynamical meaning:

EXERCISE 3.1. *The set of finite critical points of f^n is $\cup_{k=0}^{n-1} f^{-k}(0)$. We let*

$$C_f = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{n-1} f^{-k}(0)$$

be the set of critical points of iterated f.

The set of critical values of f^n *is* $\{f^k 0\}_{k=1}^n$. *(There are much fewer critical values than critical points!)*

Thus, f^n is an unbranced covering over the complement of $\{f^k 0\}_{k=1}^n$.

COROLLARY 3.1. *Let* V *be a topological disk which does not contain points* $f^k 0$, $k = 1, 2, \dots, n$. *Then the inverse function* f^{-n} *has* 2^n *single-values branches* f_i^{-n} *which univalently map* V *onto pairwise disjoint topological disks* U_i, $i = 1, 2, \dots, 2^n$.

These simple remarks explain why the forward orbit of 0 plays a very special role. We will have many occasions to see that this one orbit is responsible for the diversity of the global dynamics of f.

However, f has one more critical point overlooked so far:

13.2. Looking from infinity. Extend f to an endomorphism of the Riemann sphere $\bar{\mathbb{C}}$. This extension has a critical point at ∞ fixed under f. We will start exploring the dynamics of f from there. The first observation is that $\bar{\mathbb{C}} \smallsetminus \mathbb{D}_R$ is f-invariant for a sufficiently big R, and moreover $f^n z \to \infty$ as $n \to \infty$ for $z \in \bar{\mathbb{C}} \smallsetminus \mathbb{D}_R$. This can be expressed by saying that $\mathbb{C} \smallsetminus \mathbb{D}_R$ belongs to the *basin of infinity* defined as the set of all escaping points:

$$D_f(\infty) = \{z : f^n z \to \infty,\ n \to \infty\} = \bigcup_{n=0}^{\infty} f^{-n}(\mathbb{C} \smallsetminus \mathbb{D}_R).$$

PROPOSITION 3.2. *The basin of infinity* $D_f(\infty)$ *is a completely invariant domain containing* ∞. *def*

PROOF. The only non-obvious statement to check is connectivity of $D_f(\infty)$. To this end let us show inductively that the sets $U_n = f^{-n}(\bar{\mathbb{C}} \smallsetminus \bar{\mathbb{D}}_R)$ are connected. Indeed, assume that U_n is connected while U_{n+1} is not. Consider a bounded component V of U_{n+1}. Then the restriction $f : V \to U_n$ is proper and hence surjective (see §4). In particular f would have a pole in V - contradiction. □

Let $\bar{D}_f(\infty) = D_f(\infty) \cup \{\infty\}$.

13.3. Basic Dichotomy for Julia sets. We can now introduce the fundamental dynamical object, the *filled Julia set* $K(f) = \bar{\mathbb{C}} \smallsetminus D_f(\infty)$. Proposition 3.2 implies that $K(f)$ is a completely invariant compact subset of $\mathbb{C}$. Moreover, it is *full*, i.e., it does not separate the plane (since $D_f(\infty)$ is connected).

EXERCISE 3.2. *The filled Julia set consists of more than one point. (Consider fixed points of* f *and their preimages.)*

The filled Julia set and the basin of infinity have a common boundary, which is called the *Julia set*, $J(f) = \partial K(f) = \partial D_f(\infty)$. Figure .... shows several pictures of the Julia sets $J(f_c)$ for different parameter values c. Generally, topology and geometry of the Julia set is very complicated, and it is hard to put a hold on it. However, there is the following rough classification:

picture

THEOREM 3.3 (Basic Dichotomy). *The Julia set (and the filled Julia set) is either connected or Cantor. The latter happens if and only if the critical point escapes to infinity: $f^n(0) \to \infty$ as $n \to \infty$.*

PROOF. As in the proof of Proposition 3.2, let us consider the increasing sequence of domains $U_n = f^{-n}(\bar{\mathbb{C}} \setminus \bar{\mathbb{D}}_R)$ exhausting the basin of infinity. Assume first that the critical point does not escape to ∞. Then $f : U_{n+1} \to U_n$ is a branched double covering with the only branched point at ∞. By the Riemann-Hurwitz formula, if U_n is simply connected then U_{n+1} is simply connected as well. We conclude inductively that all the domains U_n are simply connected. Hence their union, $D_f(\infty)$, is simply connected as well, and its complement, $K(f)$, is connected. But the boundary of a full connected compact set is connected. Hence $J(f)$ is connected.

Assume now that the critical point escapes to infinity. Then 0 belongs to some domain U_n. Take the smallest n with this property. Adjust the radius R in such a way that the orbit of 0 does not pass through $\mathbb{T}_R = \partial U_0$. Then $0 \notin \partial U_{n-1}$, and hence ∂U_{n-1} is a Jordan curve. Let us consider the complimentary Jordan disk $D \equiv D^0 = \mathbb{C} \setminus \bar{U}_{n-1}$. Since $f(0) \in U_{n-1}$, f is unbranched over D. Hence $f^{-1}D = D_0^1 \cup D_1^1$, where the $D_i^1 \Subset D$ are disjoint topological disks conformally mapped onto D.

Take now the f-preimages of $D_0^1 \cup D_1^1$ in D_0^1. We obtain two Jordan disks D_{00}^2 and D_{01}^2 with disjoint closures conformally mapped by f onto D_0^1 and D_1^1 repsectively. Similar disks, D_{10}^2 and D_{11}^2, we find in D_1^1 (see Figure ....).

picture

Iterating this procedure, we will find that $f^{-n}D$ is the union of 2^n Jordan disks $D^n_{i_0 i_1 \dots i_n}$ such that $D^n_{i_0 \dots i_n}$ is compactly contained in $D^{n-1}_{i_0 \dots i_{n-1}}$ and is conformally mapped by f onto $D^{n-1}_{i_1 \dots i_n}$.

Since $D_0^1 \cup D_1^1$ is compactly contained in D, the branches of the inverse map, $f^{-1} : D_i^1 \to D_{ij}^2$, are uniformly contracting in the hyperbolic metric of D (by the Schwarz-Pick Lemma). Since the domains $D^n_{i_0 i_1 \dots i_n}$ are obtained by iterating these branches, they uniformly exponentially shrink as $n \to \infty$. Hence the filled Julia set $K(f) = \cap f^{-n} D$ is a Cantor set. Of course, the Julia set $J(f)$ coincides with $K(f)$ in this case. $\square$

The Basic Dichotomy is the first example of how the behavior of the critical point influences the global dynamics. In fact, at least on the philosophical level, the dynamics is completely determined by the behavior of this single point. We will see many confirmations of this principle.

13.4. Bernoulli shift. When the Julia set is Cantor, there is an explicit symbolic model for the dynamics of f on it. Consider the space $\Sigma \equiv \Sigma_2^+$ of one-sided sequences $(i_0 i_1 \dots)$ of zeros and ones. Supply it with the weak topology (convergence in this topology means that all coordinates eventually stabilize). We obtain a Cantor set. Define the shift β on this space as the map of forgetting the first coordinate,

$$\beta : (i_0 i_1 \dots) \mapsto (i_1 i_2 \dots).$$

It is called the *(one-sided) Bernoulli shift* with two states.

EXERCISE 3.3. *Show that:*

- *For any open set $U \subset \Sigma$, there exists an $n \in \mathbb{N}$ such that $\beta^n(U) = \Sigma$;*
- *β is topologically transitive;*
- *Periodic points of β are dense in Σ.*

EXERCISE 3.4. *Show that the only non-trivial automorphism of the one-sided Bernoulli shift with two states is induced by the relabeling $0 \leftrightarrow 1$.*

If some endomorphism $f : X \to X$ of a compact space is topologically conjugate to a one-sided Bernoulli shift with two states, then X can be partitioned into two pieces X_0 and X_1 corresponding to sequences which begin with 0 and 1 respectively. This partition is called a *Bernoulli generator* for f. The statement of Exercise 3.4 is equivalent to saying that a Bernoulli generator is unique. For a Cantor Julia set $J(f_c)$, the Bernoulli generator was constructed in the course of the proof of Theorem 3.3:

EXERCISE 3.5. *If $J(f)$ is a Cantor set, then the restriction of f onto $J(f)$ is topologically conjugate to the one-sided Bernoulli shift with two states.*

13.5. Real dichotomy. In the case of real parameter values c, the Bernoulli coding of $J(f_c)$ becomes particularly nice:

EXERCISE 3.6. *Consider a quadratic polynomial $f_c : z \mapsto z^2 + c$ with a real c. Let $J \equiv J(f_c)$.*

- *If $c < -2$ then J is a Cantor set on the real line. In this case the Bernoulli generator for f_c consists of*

$$J_0 = J \cap \{z : \Re z < 0\} \text{ and } J_1 = J \cap \{z : \Re z > 0\}.$$

picture

- *If $c > 1/4$ then J is a Cantor set disjoint from the real line. In this case the Bernoulli generator for f_c consists of*

$$J_0 = J \cap \{z : \Im z > 0\} \text{ and } J_1 = J \cap \{z : \Im z < 0\}.$$

The boundary parameter values $c = 1/4$ and $c = -2$ play a special role in one-dimensional dynamics (both real and complex).

The former map ($c = 1/4$) is specified by the property that it has a multiple fixed point $\alpha = \beta = 1/2$, i.e., $f_c(\alpha) = \alpha$, $f_c'(\alpha) = 1$. The Julia set of this map is a Jordan curve depicted on Figure ... (see §**??** for an explanation of some features of this picture). It is called *cauliflower*, and the map $f_c : z \mapsto z^2 + 1/4$ itself is sometimes called the *cauliflower map*.

The latter map ($c = -2$) is specified by the property that the second iterate of the critical point is fixed under f_c: $0 \mapsto -2 \mapsto 2 \mapsto 2$ (see Figure ...). This map is called *Chebyshev* or *Ulam-Neumann*. The Julia set of this map is unusually simple:

Exercise 3.7 (Chebyshev map). *Let $f \equiv f_{-2} : z \mapsto z^2 - 2$.*

- *The interval $I = [-2, 2]$ is completely invariant under f, i.e., $f^{-1}I = I$.*
- *$J(f) = I$. (To show that all points in $\mathbb{C} \setminus I$ escape to ∞, use Montel's Theorem.)*
- *Consider the the sawlike map*

$$g : [-1, 1] \to [-1, 1], \quad g : x \mapsto 2|x| - 1.$$

Show that the map $h : x \mapsto 2 \sin \frac{\pi}{2} x$ conjugates g to $f|I$.

- *The map $f|I$ is nicely semi-conjugate to the one-sided Bernoulli shift $\sigma : \Sigma \to \Sigma$. Namely, there exists a natural semi-conjugacy $h : \Sigma \to I$ such that $\operatorname{card} f^{-1}x = 1$ for all $x \in I$ except countable many points (preimages of the fixed point $\beta = 2$ under iterates of f). For these special points, $\operatorname{card} f^{-1}(x) = 2$.*

Let us finish with a statement which will complete our discussion of the Basic Dichotomy for real parameter values:

Exercise 3.8. (i) *For $c \in (-\infty, 1/4)$, the map f_c has two real fixed points $\alpha_c < \beta_c$. (We have already observed that these two points collide at $1/2$ when $c = 1/4$.) Point β_c is always repelling.*

(ii) *For* $c \in [-2, 1/4]$, *the interval* $I_c = [-\beta_c, \beta_c]$ *is invariant under* f_c, *and it is the maximal* f_c*-invariant interval on the real line.*
(iii) *For* $c \in [-2, 1/4]$, *the critical point is non-escaping and hence the Julia set* $J(f_c)$ *is connected.*

The above fixed points, α_c and β_c, will be called α- and β-fixed points respectively. As one can see from the second item of the above Exercise, they play quite a different dynamical role. In §**??** a similar classification of the fixed points will be given for any quadratic polynomial with connected Julia set.

Let us summarize Exercises 3.6 and 3.8:

PROPOSITION 3.4. *For real c, the Julia set* $J(f_c)$ *is connected if and only if* $c \in [-2, 1/4]$.

13.6. Fatou set. The Fatou set is defined as the complement of the Julia set:

$$F(f) = \bar{\mathbb{C}} \setminus J(f) = D_f(\infty) \cup \operatorname{int} K(f).$$

Since $K(f)$ is full, all components of $\operatorname{int} K(f)$ are simply connected. Only one of them can contain the critical point. Such a component (if exists) is called *critical*.

Let U be one of the components of $\operatorname{int} K$. Since $\operatorname{int} K$ is invariant, it is mapped by f to some other component V. Moreover, $f(\partial U) \subset \partial V$ since the Julia set is also invariant. Hence $f : U \to V$ is proper, and thus surjective. Moreover, since V is simply connected, $f : U \to V$ is either a conformal isomorphism (if U is not critical), or is a double branched covering (if U is critical).

The Fatou set can be also characterized as *the set of normality* (and was actually classically defined in this way):

PROPOSITION 3.5. *The Fatou set* $F(f)$ *is the maximal set on which the family of iterates* f^n *is normal.*

PROOF. On $D_f(\infty)$, the iterates of f locally uniformly converge to ∞, while on $\operatorname{int} K(f)$ they are uniformly bounded. Hence they form a normal family on $F(f)$. On the other hand, if $z \in J(f)$, then the orbit of z is bounded while there are nearby points escaping to ∞. Hence the family of iterates is not normal near z. ☐

14. Periodic motions

14.1. Periodic points: rough classification. Poincaré said that ...

Consider a periodic point α of period p. The local dynamics near its cycle depends first of all on its *multiplier* $\lambda = (f^p)'(z)$.

If $|\lambda| < 1$ then α is called *attracting*. The orbits of all nearby points exponentially fast converge to $\boldsymbol{\alpha}$ and, in particular, are bounded. It follows that attracting cycles belong to $F(f)$.

A particular case of an attracting cycle is a *superattracting* one when $|\lambda| = 0$. Nearby points converge to a superattracting cycle at a superexponential rate.

The *basin of attraction* of an attracting cycle $\boldsymbol{\alpha}$ is the set of all points whose orbits converge $\boldsymbol{\alpha}$:

$$D_f(\boldsymbol{\alpha}) = \{z : \ f^n z \to \boldsymbol{\alpha} \text{ as } n \to \infty.\}$$

EXERCISE 3.9. *Show that the basin $D_f(\boldsymbol{\alpha})$ a completely invariant union of components of* $\operatorname{int} K(f)$*.*

The union of components of $D_f(\boldsymbol{\alpha})$ containing the points of $\boldsymbol{\alpha}$ is called the *immediate basin* of attraction of the cycle $\boldsymbol{\alpha}$. We will denote it by $D_f^0(\boldsymbol{\alpha})$.

EXERCISE 3.10. *Show that the immediate basin of an attracting cycle consists of exactly p components, where p is the period of α.*

We will now state one of the most important facts of the classical holomorphic dynamics:

THEOREM 3.6. *The immediate basin of attraction $D_f^0(\boldsymbol{\alpha})$ of an attracting cycle $\boldsymbol{\alpha}$ contains the critical point 0.*

PROOF. Otherwise f^p would conformally map each component U of the immediate basin onto itsef. Hence it would be a hyperbolic isometry of U, despite the fact that $|f'(\alpha)| < 1$. ☐

COROLLARY 3.7. *A quadratic polynomial can have at most one attracting cycle.*

Of course, the period of this cycle can be arbitrary big. If a quadratic polynomial does indeed have an attracting cycle, it is called *hyperbolic*. For instance, polynomials $z \mapsto z^2$, $z \mapsto z^2 - 1$,... (see Figure ...) are hyperbolic. Though dynamically non-trivial, it is a well understood class of quadratic polynomials.

Note that quadratic polynomials with Cantor Julia set are also called hyperbolic. A reason is that in this case the orbit of the critical point still converges to an attracting fixed point (at ∞). Quadratic polynomials with connected Julia set but without attracting periodic points are not hyperbolic (by definition).

If $|\lambda| > 1$ then α (and its cycle) is called *repelling*. Nearby points exponentially fast escape from a neighborhood of a repelling cycle. Since $(f^{np})'(\alpha) = \lambda^n \to \infty$, the family of iterates is not normal near a repelling point, see **??**. Hence repelling periodic points belong to the Julia set. In fact, they are dense in the Julia set (see §**??**), so that the Julia can be alternatively defined as the closure of repelling cycles. It is a view of the Julia set "from inside".

If $|\lambda| = 1$ then α (and its cycle) is called *neutral.* Local dynamics near a neutral point delicately depends on the arithmetic of the rotation number $\theta = \frac{1}{2\pi} \arg \lambda$. If θ is rational then α is called *parabolic*; otherwise it is called *irrational.* Parabolic points belong to the Julia set:

EXERCISE 3.11. *Show that if* $\theta = q/l$, *then* $(f^{pln})'(\alpha) \to \infty$.

The basin of attraction of a parabolic cycle $\boldsymbol{\alpha}$ is defined as follows:

$$D_f(\boldsymbol{\alpha}) = \{z : f^n z \to \boldsymbol{\alpha} \text{ as } n \to \infty \text{ but } f^n z \notin \boldsymbol{\alpha} \text{ for any } n \in \mathbb{N}^*.\}$$

It turns out that with this definition, $D_f(\boldsymbol{\alpha})$ is a completely invariant union of components of $\operatorname{int} K$. Moreover, among these components there are *pl* components cyclically permuted by f, while all others are preimages of these. The union of these *pl* components is called the the *immediate basin* of attraction of $\boldsymbol{\alpha}$. It will also be denoted as $D^0_f(\boldsymbol{\alpha})$.

As in the attracting case, the immediate basin of a parabolic cycle also must contain the critical point. Hence a quadratic polynomial can have at most one parabolic cycle and in this case it cannot have attracting cycles. A polynomial with a parabolic cycle is naturally called *parabolic.* (It is a preview to a more detailed discussion in §**??**).

Irrational periodic points may or may not belong to the Julia set (depending primarily on the Diophantine properties of its rotaion number). Irrational periodic points lying in the Fatou set are called *Siegel*, and those lying in the Julia set are called *Cremer.* The component of $F(f)$ containing a Siegel point is called a *Siegel disk.* Local dynamics on a Siegel disk is quite simple:

PROPOSITION 3.8. *Let* U *be a Siegel disk of period* p *containing a periodic point* α *with rotation number* θ. *Then* $f^p|U$ *is conformally conjugate to the rotation of* $\mathbb{D}$ *by* θ.

PROOF. Consider the Riemann map $\phi : (U, \alpha) \to (\mathbb{D}, 0)$. Then $g = \phi \circ f^p \circ \phi^{-1}$ is a holomorphic endomorphism of the unit disk fixing 0, with $|g'(0)| = |\lambda| = 1$. By the Schwarz Lemma, $g(z) = \lambda z$. □

We will see later on that *a quadratic polynomial can have at most one non-repelling cycle* (see theorem 3.20). If it has one, it can be non-contradictory classified as either hyperbolic, or parabolic, or Siegel, or Cremer.

14.2. Periodic components. The notions of a periodic component of $F(f)$ and its cycle are self-explanatory. It is classically known that such a component is always associated with a non-repelling periodic point:

THEOREM 3.9. *Let* $\mathbf{U} = \{U_i\}_{i=1}^p$ *be a cycle of periodic components of* $\operatorname{int} K(f)$. *Then one of the following three possibilities can happen:*

- $\mathbf{U}$ *is the immediate basin of an attracting cycle;*
- $\mathbf{U}$ *is the immediate basin of a parabolic cycle* $\boldsymbol{\alpha} \subset \partial\mathbf{U}$ *of some period* $q|p$;
- $\mathbf{U}$ *is the cycle of Siegel disks.*

Proof Take a component U of the cycle $\mathbf{U}$, and let $g = f^p$. By the Schwarz-Pick Lemma, $g|U$ is either a conformal automorphism of U, or it strictly contracts the hyperbolic metric dist_h on U. In the former case, it is either elliptic, or otherwise. If g is elliptic then U is a Siegel disk. Otherwise the orbits of g converge to the boundary of U.

Let us show that if an orbit $\{z_n = g^n z\}$, $z \in U$, converges to ∂U, then it converges to a g-fixed point $\beta \in \partial U$. Join z and $g(z)$ with a smooth arc γ, and let $\gamma_n = f^n\gamma$. By the Schwarz-Pick Lemma, the hyperbolic length of the arcs γ_n stays bounded. Hence they uniformly escape to the boundary of U. Moreover, by the relation between the hyperbolic and Euclidean metrics (Lemma 1.11), the Euclidean length of the γ_n shrinks to 0. In particular,

$$|g(z_n) - z_n| = |z_{n+1} - z_n| \to 0 \tag{14.1}$$

as $n \to \infty$. By continuity, all limit points of the orbit $\{z_n\}$ are fixed under g. But g being a polynomial has only finitely many fixed points. On the other hand, (14.1) implies the ω-limit set of the orbit $\{z_n\}$ is connected. Hence it consists of a single fixed point β.

Moreover, the orbit $\{\zeta_n\}$ of any other point $\zeta \in U$ must converge to the same fixed point β. Indeed, the hyperbolic distance between z_n and ζ_n stays bounded and hence the Euclidean distance between these points shrink to 0.

Thus either U is a Siegel disk, or the g-orbits in U converge to a g-fixed point β, or the map $g : U \to U$ strictly contracts the hyperbolic metric and its orbits do not escape to the boundary ∂U. Let us show that in the latter case, g has an attracting fixed point α in U.

Take a g-orbit $\{z_n\}$, and let $d_n = \operatorname{dist}_h(z_0, z_n)$. Since g is strictly contracting,

$$\operatorname{dist}_h(z_{n+1}, z_n) \le \rho(d_n) \operatorname{dist}_h(z_n, z_{n-1}),$$

where the contraction factor $\rho(d_n) < 1$ depends only on $\text{dist}_h(z_n, z_0)$. Since the orbit $\{z_n\}$ does not escape to ∂U, this contraction factor is bounded away from 1 for infinitely many moments n, and hence $\text{dist}_h(z_{n+1}, z_n) \to 0$. It follows that any ω-limit point of this orbit in U is fixed under g.

By strict contraction, g can have only one fixed point in U, and hence any orbit must converge to this point. Strict contraction also implies that this point is attracting.

We still need to prove the most delicate property: in the case when the orbits escape to the boundary point $\beta \in \partial U$, this point is parabolic. In fact, we will show that $g'(\beta) = 1$. Of course, this point cannot be either repelling (since it attracts some orbits) or attracting (since it lies on the Julia set). So it is a neutral point with some rotation number $\theta \in [0, 1)$. The following lemma will complete the proof.

LEMMA 3.10 (Necklace Lemma). *Let $f : z \mapsto \lambda z + a_2 z^2 + \dots$ be a holomorphic map near the origin, and let $|\lambda| = 1$. Assume that there exists a domain $\Omega \subset \mathbb{C}^*$ such that all iterates f^n are well-defined on Ω, $f(\Omega) \cap \Omega \neq \emptyset$, and $f^n(\Omega) \to 0$ as $n \to \infty$. Then $\lambda = 1$.*

PROOF. Consider a chain of domains $\Omega_n = f^n \Omega$ convergin to 0. Without loss of generality we can assume that all the domains lie in a small neighborhood of 0 and hence the iterates $f^n | \Omega$ are univalent. Fix a base point $a \in \Omega$ such that $f(a) \in \Omega$, and let

$$\phi_n(z) = \frac{f^n(z)}{f^n(a)}.$$

These functions are univalent, normalized by $\phi_n(a) = 1$, and do not have zeros. By the Koebe Distortion Theorem (the version given in Exercise 1.15,b), they form a normal family. Moreover, any limit function ϕ of this family is non-constant since $\phi(fa) = \lambda \neq 1 = \phi(a)$. Hence the derivatives $\phi_n' | \Omega$ are bounded away from 0 and $\text{dist}(1, \partial \Omega_n) \geq \epsilon > 0$ for all $n \in \mathbb{N}$. It follows that

$$\text{dist}(f^n a, \partial \Omega_n) \geq \epsilon \, r_n, \quad n \in \mathbb{N},$$

where $r_n = |f^n a|$. On the other hand, f acts almost as the rotation by θ near 0, where $\theta = \arg \lambda \in (0, 1)$. Since this rotation is recurrent and $\theta \neq 0$, there exists an $l > 0$ such that

$$\text{dist}(f^{n+l} a, f^n a) = o(r_n) \quad \text{as } n \to \infty$$

The last two estimates imply that $\Omega_{n+l} \cap \Omega_n \neq \emptyset$ for alll sufficiently big n.

Hence the chain of domains $\Omega_n, \dots, \Omega_{n+l}$ closes up, and their union form a "necklace" around 0. Take a Jordan curve γ in this necklace, and let D be the disk bounded by γ. Then $f^n(\gamma) \to 0$ as $n \to \infty$. By the Maximum Principle, $f^N(D) \Subset D$ for some N. By the Schwarz Lemma, $|\lambda| < 1$ – contradiction. □

15. Quasi-conformal deformations

15.1. Idea of the method.

15.1.1. *Pullbacks.* Consider a K-quasi-regular branched covering $f : S \to S'$ between Riemann surfaces (see §7.4). Then any conformal structure μ on S' can be *pulled back* to a structure $\nu = \mathbf{f}^*(\mu)$ on S. Indeed, quasi-regular maps are differentiable a.e. on S with non-degenerate derivative so that we can let $\nu(z) = (Df(z)^{-1})_*(\mu)$ for a.e. $z \in S$. This structure has a bounded dilatation:

$$\frac{\|\nu\|_\infty + 1}{\|\nu\|_\infty - 1} \le K \frac{\|\mu\|_\infty + 1}{\|\mu\|_\infty - 1}.$$

If f is holomorphic then in any conformal local charts near z and $f(z)$ we have:

$$f^*\mu(z) = \frac{\overline{f'(z)}}{f'(z)} \mu(fz)$$

(since the critical points of f are isolated, this expression makes sence a.e.). An obvious (either from this formula or geometrically) but crucial remark is that *holomorphic pull-backs preserve dilatation of conformal structures.*

15.1.2. *Qc surgeries and deformations.* Consider now a qr map $f : \mathbb{C} \to \mathbb{C}$ *preserving* some conformal structure μ on $\bar{\mathbb{C}}$. By the Measurable Riemann Mapping Theorem, there is a qc homeomorphism $h_\mu : \mathbb{C} \to \mathbb{C}$ such that $(h_\mu)_*(\mu) = \sigma$. Then $f_\mu = h_\mu \circ f \circ h_\mu^{-1}$ is a quasi-regular map preserving the standard structure σ on $\bar{\mathbb{C}}$. By Weil's Lemma, f_μ is holomorphic outside its critical points. Since the isolated singularities are removable, f_μ is holomorphic everywhere, so that it is a rational endormorphism of the Riemann sphere. Of course, $\deg(f_\mu) = \deg(f)$. Since h_μ is unique up to post-composition with a Möbius map, $f = f_\mu$ *is uniquely determined by* μ *up to conjugacy by a Möbius map.*

Thus, a qc invariant view of a rational map of the Riemann sphere is a *quasi-regular endomorphism* $f : (S^2, \mu) \to (S^2, \mu)$ *of a qc sphere* S^2 *which preserves some conformal structure* μ. This provides us with a powerful tool of holomorphic dynamics: the method of qc surgery. The recepie is to cook by hands a quasi-regular endomorphism of a qc sphere with desired dynamical properties. If it admits an invariant conformal structure, then it can be realized as a rational endomorphism of the Riemann sphere.

It may happen that f itself is a rational map preserving a non-trivial conformal structure μ. Then f_μ is called a *qc deformation of* f. If f is polynomial, then let us normalize h_μ so that it fixes ∞. Then $f_\mu^{-1}(\infty) = \infty$ and hence the deformation f_μ is polynomial as well. If $f : z \mapsto z^2 + c$ is quadratic then let us additionally make h_μ fix 0. Then 0 is a critical point of f_μ, so that

$$f_\mu(z) = t(\mu)z^2 + b(\mu), \quad t \in \mathbb{C}^*. \tag{15.1}$$

Composing h_μ with complex scaling $z \mapsto t(\mu)z$, we turn this quadratic polynomial to the normal form $z \mapsto z^2 + c(\mu)$.

Assume now that $\mu = \mu_\lambda$ depends holomorphically on parameter λ. By Theorem 2.14, the map $h_\lambda \equiv h_{\mu(\lambda)}$ is also holomorphic in λ. However, the inverse map h_λ^{-1} is not necessarily holomorphic in λ.

EXERCISE 3.12. *Give an example.*

It is a miracle that despite it, the deformation $f_\lambda \equiv f_{\mu(\lambda)}$ is still holomorphic in λ!

LEMMA 3.11. *Let $f_\lambda = h_\lambda \circ f \circ h_\lambda^{-1}$, where f and f_λ are holomorphic functions and h_λ is a holomorphic family of qc maps. Then f_λ holomorphically depends on λ.*

PROOF. Taking $\bar\partial$-derivative of the expression $h_\lambda \circ f_0 = f_\lambda \circ h_\lambda$, we obtain:

$$0 = \bar\partial h_\lambda \circ f_0 = f'_\lambda \circ \bar\partial h_\lambda + \bar\partial f_\lambda \circ h_\lambda = \bar\partial f_\lambda \circ h_\lambda.$$

□

COROLLARY 3.12. *Consider a quadratic map $f : z \mapsto z^2 + c_0$. Let μ_λ be a holomorphic family of f-invariant Beltrami differentials on $\mathbb{C}$. Normalize the solution $h_\lambda : \mathbb{C} \to \mathbb{C}$ of the corresponding Beltrami equiation so that the qc deformation $f_\lambda = h_\lambda \circ f \circ h_\lambda^{-1}$ has a normal form $f_\lambda : z \mapsto z^2 + c(\lambda)$. Then the parameter $c(\lambda)$ depends holomorphically on λ.*

PROOF. Consider first the solution $H_\lambda : \mathbb{C} \to \mathbb{C}$ of the Beltrami equation which fixes 0 and 1. It conjugates f to a quadratic polynomial of form (15.1). By Lemma 3.11, its coefficients $t(\lambda)$ and $b(\lambda)$ depend holomorphically on λ. The complex rescaling $T_\lambda : z \mapsto t(\lambda)z$ reduces this polynomial to the normal form with $c(\lambda) = t(\lambda)b(\lambda)$, and we see that $c(\lambda)$ depends holomorphically on λ as well. □

15.2. Sullivan's No Wandering Domains Theorem.

16. Remarkable functional equations

Study of certain functional equations was one of the main motivations for the classical work in holomorphic dynamics. By means of these equations the local dynamics near periodic points of different types can be reduced to the simplest normal form. But it turns out that the role of the equations goes far beyond local issues: global solutions of the equations play a crucial role in understanding the dynamics.

We will start with the local analysis and then globalize it (though sometimes one can go the other way around). For the local analysis we put the fixed point at the origin and consider a holomorphic map

$$f : z \mapsto \lambda z + a_2 z^2 + \dots \tag{16.1}$$

near the origin.

16.1. Attracting points and linearizing coordinates. Let us start with the symplest case of an attracting fixed point. In turns out that such a map can always be linearized near the origin:

THEOREM 3.13. *Consider a holomorphic map (16.1) near the origin. Assume* $0 < |\lambda| < 1$*. Then there exists an* f*-invariant Jordan disk* $V \ni 0$*, an* $r > 0$*, and a conformal map* $\phi : (V, 0) \to \mathbb{D}_r$ *with* $\phi'(0) = 1$ *satisfying the equation:*

$$\phi(fz) = \lambda\phi(z) \tag{16.2}$$

The above properties determine uniquely the germ of ϕ *at the origin.*

The above function ϕ is called the *linearizing coordinate* for f near 0. It locally conjugates f to the linear map $z \mapsto \lambda z$.

EXERCISE 3.13. *Show that if a holomorphic germ* f *near the origin commutes with the linear germ* $z \mapsto \lambda z$, $0 < |\lambda| < 1$*, then* f *is itself linear.*

16.2. Superattractng points and Böttcher coordinates.

THEOREM 3.14. *Let* $f : z \mapsto z^d + a_{d+1} z^{d+1} + \dots$ *be a holomorphic map near the origin,* $d \geq 2$*. Then there exists an* f*-invariant Jordan disk* $V \ni 0$*,* $r \in (0, 1)$*, and a conformal map* $\phi : (V, 0) \to \mathbb{D}_r$ *satisfying the equation:*

$$\phi(fz) = \phi(z)^d. \tag{16.3}$$

The above properties determine uniquely the germ of ϕ *at the origin. Moreover,* $\phi'(0) = 1$*.*

The map ϕ is called the *Böttcher map*, or the *Böttcher coordinate* near 0. Equation (16.3) is called the *Böttcher equation.* In the Böttcher coordinate the map f assumes the normal form $z \mapsto z^d$.

EXERCISE 3.14. *Let $d \geq 2$. Show that there are no holomorphic germs commuting with $g : z \mapsto z^d$ near the origin, except g itself.*

16.3. Parabolic points and Écale-Voronin cylinders.

16.4. Global leaf of a repelling point.

16.5. Böttcher vs Riemann. Let us now consider a quadratic polynomial f_c near ∞. Since ∞ is a superattracting fixed point of f of degree 2, the map f_c near ∞ can be reduced in the Böttcher coordinate to the map $z \mapsto z^2$ (Theorem 3.14). Thus, there is a Jordan disk $V_c \subset \mathbb{C}$ whose complement $\mathbb{C} \smallsetminus V_c$ is f_c–invariant, some $R > 1$, and a conformal map $\phi_c : \mathbb{C} \smallsetminus V_c \to \mathbb{C} \smallsetminus \mathbb{D}_R$ satisfying the Böttcher equation:

$$\phi_c(f_c z) = \phi_c(z)^2. \tag{16.4}$$

Moreover, $\phi_c(z) \sim z$ as $z \to \infty$.

We will now globalize the Böttcher function.

16.5.1. *Connected case.*

THEOREM 3.15. *Let $f_c : z \mapsto z^2 + c$ be a quadratic polynomial with connected Julia set. Then the Böttcher function admits an analytic extension to the complement of the filled Julia set. Moreover, it conformally maps $\mathbb{C} \smallsetminus K(f)$ onto the complement of the unit disk.*

PROOF. We will skip label c from the notations. Let, as usual, $f_0(z) = z^2$.

Let $U^n = \bar{\mathbb{C}} \smallsetminus f^{-n}\bar{V}$. Then $U^0 \subset U^1 \subset U^2 \subset \dots$ and $\cup U^n = \bar{D}_f(\infty)$. Since the filled Julia set $K(f)$ is connected, the domains U^n are topological disks and the maps $f : U^{n+1} \to U^n$ are double coverings branched point at ∞ (recall the proof of Theorem 3.3).

Let $\Delta^n = \bar{\mathbb{C}} \smallsetminus \bar{\mathbb{D}}_{R^{1/2^n}}$. By Lemma 1.15, the Böttcher map $\phi : U^0 \to \Delta^0$ admits a lift $\Phi : U^1 \to \Delta^1$ such that $f_0 \circ \Phi = \phi \circ f$. But the Böttcher equation tells us that $\phi : U^0 \to \Delta^0$ is a lift of its restriction $\phi : f(U^0) \to f_0(\Delta^0)$. If we select Φ so that $\Phi(z) = \phi(z)$ at some finite point $z \in U^0$, then these two lifts must coincide on U^0: $\Phi|U^0 = \phi$. Thus, Φ is the analytic extension of ϕ to U^1. Obviously, it satisfies the Böttcher equation as well.

In the same way, the Böttcher map can be consecutively extended to all the domains U^n and hence to their union $\bar{D}_f(\infty)$. □

Thus, the Böttcher map gives the uniformization of $\mathbb{C} \smallsetminus K(f)$ by the unit disk. Given the intricate fractal structure of the Julia set, this is quite remarkable that its complement can be uniformized in this explixit way!

One can also go the other way around and costruct the Böttcher map by means of uniformization:

Exercise 3.15. *Let $f = f_c$ be a quadratic polynomial with connected Julia set. Then the basin of infinity $\bar{D}_f(\infty)$ is a conformal disk. Uniformize it by the complement of the unit disk; $\psi : (\mathbb{D}, \infty) \to (D_f(\infty), \infty)$, normalized at ∞ so that $\psi(z) \sim \lambda z$ with $\lambda > 0$. Prove (without using the Böttcher theorem) that ψ conjugates $f_0 : z \mapsto z^2$ on $\mathbb{C} \smallsetminus \mathbb{D}$ to f on the basin of ∞.*

Let us finish with a curious consequence of Theorem 3.15:

Corollary 3.16. *Let $f_c : z \mapsto z^2 + c$. Then the conformal capacity of the filled Julia set $K(f_c)$ is equal to 1.*

16.5.2. *Cantor case.* In the disconnected case the Böttcher function ϕ_c cannot be any more extended to the whole basin of ∞, as it starts to branch at the critical point 0. However, ϕ_c can still be extended to a big invariant region Ω_c containing 0 on its boundary.

Theorem 3.17. *Let $f_c : z \mapsto z^2 + c$ be a quadratic polynomial with disconnected Julia set. Then the Böttcher function ϕ_c admits the analytic extension to a domain Ω_c bounded by a "figure eight" curve branched at the critical point 0. Moreover, ϕ_c maps Ω_c conformally onto the complement of some disk $\bar{\mathbb{D}}_R$ with $R > 1$.*

Proof. Again, we skip the label c.

Since $0 \in D_f(\infty)$, the orb(0) lands at the domain V of the Böttcher function near ∞. By shrinking V, we can make $f^n 0 \in \partial V$ for some $n > 0$. Then there are no obstructions for consecutive extensions of ϕ to the domains $U^k = \bar{\mathbb{C}} \smallsetminus f^{-k}\bar{V}$, $k = 0, 1, \dots, n$ (in the same way as in the connectef case). All these domains are bounded by real analytic curves except the last one, U^n, which is bounded by a figure eight curve branched at 0. This is the desired domain Ω. □

Important Remark. Since the critical value $c \in \partial U^{n-1}$ belongs to the domain of ϕ_c, the expression $\phi_c(c)$ is well-defined (provided the Julia set $J(f_c)$ is disconnected). It gives the *Böttcher coordinate of the critical value* as a function of the parameter c. This function will play a crucial role in what follows.

16.6. External rays and equipotentials. The map $f_0 : z \mapsto z^2$ on $\mathbb{C} \smallsetminus \mathbb{D}$ has two invariant foliations: foliation by the straight rays going to ∞ and foliation by round circles centered at the origin. By means of the Böttcher map, these two foliations can be transferred to the basin of infinity of f, supplying us with a basic dynamical structure.

16.6.1. *Connected case.* Let us first assume that the Julia set $J(f)$ is connected. Then the Böttcher map $\phi : D_f(\infty) \to \mathbb{C} \setminus \mathbb{D}$ gives a global uniformization of the basin of infinity, so that we obtain two orthogonal invariant foliations therein. The leaves of these foliations are called *external rays* and *equipotentials*:

$$\mathcal{R}^\theta = \mathcal{R}^\theta_f = \phi^{-1}\{re^{2\pi i\theta} : \ 1 < r < \infty\}, \quad \theta \in \mathbb{R};$$

$$\mathcal{E}^t = \mathcal{E}^t_f = \phi^{-1}\{e^{t+2\pi i\theta} : \ 0 \le \theta < 1\}, \quad t > 0,$$

where θ is called the *external angle* of the ray $\mathcal{R}^\theta$ and t is called the *level* of the equipotential $\mathcal{E}^t$. Moreover, $f(\mathcal{R}^\theta) = \mathcal{R}^{2\theta}$ and $f(\mathcal{E}^t) = \mathcal{E}^{2t}$.

We will use notation $\mathcal{R}^\theta(t)$ for the point on the ray $\mathcal{R}^\theta$ whose equipotential level is equal to t.

16.6.2. *Disconnected case.* In the disconnected case we can still define the rays and equipotentials in the domain Ω of analyticity of the Böttcher function. Pulling these two foliations back under iterates of f, we extend them to singular foliations on the whole basin of ∞. They have singularities at the critical points of iterated f, i.e., at 0 and all its preimages under the iterates of f. (Recall from §13.1 that this set is called C_f.)

In this context *external rays* will be understood as leaves of these foliaitons which go to ∞ (i.e., the maximal non-singular extensions of the rays in Ω). Countably many rays land at the preimages of 0. All other rays are properly embedded into the basin; they will be called *proper rays*. Two (improper) rays landing at the critical point 0 will be called the *critical rays*. The particularly important ray going through the critical value will be called the *principal ray* (its external angle will be also called *principal*). Of cource, it contains the (coinciding) images of two critical rays.

16.7. Green function. The *Green function* of a quadratic polynomial $f = f_c$ is defined as follows:

$$G_c(z) = \log|\phi_c(z)|, \tag{16.5}$$

where ϕ_c is the Böttcher map of f_c. The Green function is harmonic wherever the Böttcher function is defined (since the Böttcher function never vanishes) and has a logarithmic singularity at ∞:

$$G(z) = \log|z| + o(1).$$

In the connected case, (16.5) defines the Green function in the whole basin $D(\infty)$. In the disconnected case definition (16.5) can be used only in the domain Ω. However, in either case the Green function satisfies the equation:

$$G(fz) = 2G(z). \tag{16.6}$$

This equation can be obviously used in order to extend the Green function harmonically to the *whole basin of* ∞. Let us summarize simple properties of this extension:

EXERCISE 3.16. *a) In the connected case the Green function does not have critical points. In the disconnected case, its critical points coincide with the critical points of iterated f.*

b) Equipotentials are the level sets of the Green function, while external rays (and their preimages) are its gradient curves.

c) The Brolin formula holds:

$$G(z) = \lim_{n\to\infty} \frac{1}{2^n} \log |f^n z|, \quad z \in D(\infty).$$

d) Extention of the Green function by 0 through the filled Julia set $K(f)$, gives a continuous subharmonic function on the whole complex plane.

From the physical point of view, one should imagine that the filled Julia set K is a conductor of electric charge put in the electric field of the unit charge at ∞. Let the charges in K settle down in the equilibrium state (according to the "harmonic distribution" on the Julia set). Then the Green function is the electric potential in the space $\mathbb{R}^2$ created jointly by these charges on K and the charge at ∞. (That is why the name "equipotentials").

EXERCISE 3.17. *Assume that the Julia set $J(f)$ is connected. Endow its basin $D(\infty)$ with the hyprbolci metric ρ. Then for any external ray $\mathcal{R}^\theta$ we have:*

$$\rho(z,\zeta) = \left| \log \frac{G(z)}{G(\zeta)} \right|, \quad z, \zeta \in \mathcal{R}^\theta .$$

17. Quadratic-like maps

17.1. The concept.

17.1.1. *Definition and first properties.* The notion of a quadratic-like map is a fruitful generalization of the notion of a quadratic polynomial.

DEFINITION 3.1. A *quadratic-like map* $f : U \to U'$ is a holomorphic
define double branched covering between two conformal disks U and U' such

that $U \Subset U'$. The annulus $A = U' \smallsetminus \bar{U}$ is called the *fundamental annulus* of f.

By the Riemann-Hurwitz Theorem, any quadratic-like map has a single critical point, which is of course non-degenerate. We normalize f so that the critical point sits at 0 (unless otherwise is explicitly stated). Note that any quadratic polynomial $f = f_c$ restricts to a quadratic-like map $f : f^{-1}(\mathbb{D}_R) \to \mathbb{D}_R$ whose range is a round disk with sufficiently big radius R. need?

Technical Conventions: In what follows we will consider only *even* quadratic-like maps, i.e, such that $f(z) = f(-z)$ for all $z \in U$, with 0-*symmetric* domains U and U'. Moreover, we will assume that both domains are bounded by *piecewice smooth Jordan curves.*

The notion of a quadratic-like map does not fit to a canonical dynamical framework, where the phase space is assumed to be invariant under the dynamics. In the quadratic- like case, some orbits *escape* through the fundamental annulus (i.e., $f^n z \in A$ for some $n \in \mathbb{N}$), and we cannot iterate them any further. However, there are still a plenty of non-escaping points, which form a dynamically significant object. The set of all non-escaping points is called the *filled Julia set* of f and is denoted in the same way as for polynomials:

$$K(f) = \{z : f^n z \in U,\ n = 0, 1, \dots .\}$$

By definition, the *Julia set* of f is the boundary of the filled Julia set: $J(f) = \partial K(f)$. Dynamical features of quadratic-like maps are very similar to those of quadratic maps (in §17.3 we will see a good reason for it):

EXERCISE 3.18. *Check that all dynamical properties of quadratic polynomials established in in §§13 - 14 are still valid for quadratic-like maps. In particular,*

(i) The filled Julia set $K(f)$ is a completely invariant full compact subset of U.

(ii) Basic dichotomy: $J(f)$ and $K(f)$ are either connected or Cantor; the former holds if and only if the critical point is non-escaping: $0 \in K(f)$.

(iii) Any periodic component of $\operatorname{int} K(f)$ *is either in the immediate basin of an attracting/parabolic cycle, or is a Siegel disk.*

(iv) f can have at most one attracting cycle.

17.1.2. *Adjustments.* In fact, the notion of a quadratic-like map with the fixed domain is too rigid. We want to allow some adjustment of the domains which does not effect the essential dynamics of the map.

Let us say that a quadratic-like map $g : V \to V'$ is an *adjustment* of another quadratic-like map $f : U \to U'$ if $V \subset U$, $g = f|V$, and $\partial V' \subset \bar{U}' \smallsetminus U$. (In particular, we can restrict f to $V = f^{-1}U$, provided $f(0) \in U$.)

EXERCISE 3.19. *(i) Show that adjustments do not change the Julia set.*

(ii) Consider a topological disk $V' \subset U'$ containing the critical value $f(0)$ and such that $\partial V' \subset \bar{U}' \smallsetminus \bar{U}$. Let $V = f^{-1}V'$. Then the restriction $f : V \to V'$ is a quadratic-like map.

An appropriate adjustment allows one to improve the geometry of a quadratic-like map:

LEMMA 3.18. *Consider a quadratic-like map $f : U \to U'$ with*

$$\operatorname{mod}(U' \smallsetminus \bar{U}) \geq \mu > 0 \tag{17.1}$$

and $f(0) \in U$. Then there is an adjustment $g : V \to V'$ such that:

(i) *The new domains V and V' are bounded by real analytic κ-quasicircles γ and γ' with κ depending only on μ. Moreover, these curves have a bounded (in terms of μ) eccentricity around the origin.*

(ii) $\operatorname{mod}(V' \smallsetminus \bar{V}) \geq \mu/2 > 0$.

(iii) *g admits a decomposition*

$$g = h \circ f_0, \tag{17.2}$$

where $f_0(z) = z^2$ and h is a univalent function on $W = f_0(V)$ with distortion bounded by some constant $C(\mu)$.

PROOF. Let us uniformize the fundamental annulus A of f by a round annulus, $\phi : \mathbb{A}(1/r, r) \to A$, where $r \geq e^{\mu/2} \equiv r_0$. Then $\gamma' = \phi(\mathbb{T})$ is the equator of A. Consider the disk V' bounded by γ', and let $V = f^{-1}V'$. Since $f(0) \in V'$, V is a conformal disk and the restriction $f : V \to V'$ is a quadratic-like adjsutment of f (see Exerecise 3.19).

Restrict ϕ to the annulus $\mathbb{A}(1/r_0, r_0)$. Take an arc $\alpha = [a, b]$ on $\mathbb{T}$ of length at most $\delta = (1 - 1/r_0)/2$. By the Koebe Distortion and 1/4 Theorems in the disk $\mathbb{D}_{2\delta}(u)$,

$$|\phi(b) - \phi(a)| \geq \frac{\delta}{2}|f'(a)|; \quad l(\phi(\alpha)) \leq K(r_0)\,|f'(a)|,$$

where l stands for the arc length. Hence $\gamma' = \phi(\mathbb{T})$ is a quasi-circle with the dilatation depending only on r_0.

Applying the same argument to the uniformization of $f^{-1}A$, we conclude that its equator $\gamma = \partial V$ is a quasicircle with bounded dilatation as well.

Since γ and γ' are 0-symmetric κ-quasicircle, the eccentricity of these curves around 0 is bounded by some constant $C(\kappa)$ (see Exercise 2.13). This proves (i).

Property (ii) is obvious since $\operatorname{mod}(V' \setminus \bar{V}) \geq \operatorname{mod} \mathbb{A}(1, r_0) = \log r_0$.

Since g is assumed to be even, it admits decomposition (17.2). Moreover, h admits a univalent extension to the disk $\tilde{W} = f_0(U)$, and

$$\operatorname{mod}(\tilde{W} \setminus W) = 2 \operatorname{mod}(U \setminus V) \geq \mu/2.$$

The Koebe Distortion Theorem (in the invariant form 1.9) completes the proof. □

If some map g admits decomposition (17.2), we say that "it is a quadratic map up to bounded distortion".

17.1.3. *Quadratic-like germs.* Let us say that two quadratic-like maps f and $\tilde{f}$ represent the same *quadratic-like germ* if there is a sequence of quadratic-like maps $f = f_0, f_1, \ldots, f_n = \tilde{f}$, such that f_{i+1} is obtained by an ajustment of f_i or the other way around. We will not make notational differences between maps and germs.

According to Exercise 3.19, a quadratic-like germ f have a well-defined Julia set $J(f)$ (the notations for the dynamical objects of the germs will be the same as for the maps).

We will usually consider quadratic-like maps/germs up to *affine conjugacy* or *rescaling*. Thus, we allow ourselves to replace $f(z)$ by $\lambda^{-1} f(\lambda z)$ with some $\lambda \in \mathbb{C}^*$. This allows us to *normalize* f in different convenient ways. For example, we can select the normal form

$$f(z) = c + z^2 + \ldots \qquad (17.3)$$

with the second order Taylor coefficient at the origin equal to 1.

Let us refine Lemma 3.18 a bit:

LEMMA 3.19. *Let $f : U \to U'$ be a quadratic-like map with connected Julia set satisfying (17.1). Then the germ of f can be represented with a quadratic-like map $g : V \to V'$ satisfying the following properties:*

(i) *The same as in Lemma 3.18;*

(ii) $\min(\mu/2, 1/4) \leq \operatorname{mod}(V' \setminus V) \leq 1$*;*

(iii) *If f is normalized by (17.3) then*

$$\rho \leq r_V \leq R_{V'} \leq 1/\rho$$

for some constant $\rho \in (0, 1)$ depending only on μ.

PROOF. Let $U^n = f^{-n}U'$ and let $A^n = U^{n-1} \setminus U^n$. Since the Julia set is connected, the restrictions $f : U^n \to U^{n-1}$ are quadratic-like maps obtained by consecutive adjustments of $f : U \to U'$. Hence they represent the same germ. Since $\mathrm{mod}\, A^n = \mathrm{mod}\, A^1/2^{n-1}$, we can select n in such a way that $\tilde{\mu} \equiv \min(\mu, 1/2) \leq \mathrm{mod}\, A^n \leq 1$. Let us now adjust $f|U^n$ once more as in Lemma 3.18. We obtain a quadratic-like map $g : V \to V'$ representing the same germ and satisfying properties (i)-(ii). Moreover, both domains have eccentricity bounded by some $e = e\mu)$.

Assume now that f is normalized by (17.3), so is g. Then in representation (17.2), $g = h \circ f_0$, the univalent map $h : (W, 0) \to (V', c)$ is also normalized: $h'(0) = 1$. Since $W = f_0(V)$,

$$0 < C^{-1} r_W \leq r_{V',c} \leq R_{V'c} \leq C R_W$$

for some constant $C = C(\mu)$ depending only on μ. Hence

$$C^{-1} r_V^2 \leq r_{V',c} \leq R_{V',c} \leq C R_V^2. \tag{17.4}$$

But since $V' \supset V$, we have: $R_{V',c} \geq R_V/2$. By the right-hand side of (17.4), $R_V \geq 1/2C$. Since V has a bounded eccentricity, the inner radius r_V is also bounded away from 0: $r_V \geq 1/2Ce$.

On the other hand, if $r_V = L >> 1$ then the left-hand side of (17.4) (and bounded eccentricity of V) implies that the annulus $V' \setminus V$ contains the round annulus whose inner radius is of order L and the outer radius is of order L^2, so that $\mathrm{mod}(V' \setminus V) \geq \gamma \log L$, where $\gamma = \gamma(\mu) > 0$. Since the modulus of $V' \setminus V$ is bounded, we conclude that L is bounded as well. ☐

17.2. Uniqueness of a non-repelling cycle. We will now give the first illustration of how useful the notion of a quadratic-like map is. It exploits the flexibility of this class of maps: small perturbations of a quadratic-like map are still quadratic-like (on a slightly adjusted domain):

EXERCISE 3.20. *Let $f : U \to U'$ be a quadratic-like map with the fundamental annulus A. Take a Jordan curve $\gamma' \subset A$ generating $\pi_1(A)$, and let V' be the domain bounded by γ'. Let ϕ be a bounded holomorphic function on U with $\|\phi\|_\infty < \mathrm{dist}(\gamma, \partial U')$. Let $g = f + \phi$ and $V = g^{-1}V'$. Then $g : V \to V'$ is a quadratic-like map. (Hint: Take a Jordan curve Γ close to ∂U with winding number 1 around the origin and, look at the curve $g : \Gamma \to \mathbb{C}$, and apply the Argument Principle.)*

THEOREM 3.20. *Any quadratic-like map (in particular, any quadratic polynomial) has at most one non-repelling cycle.*

Proof. Assume that a quadratic-like map $f : U \to U'$ has two non-repelling cycles $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^{p}$ and $\boldsymbol{\beta} = \{\beta_k\}_{k=0}^{q}$. Let μ and ν be their multipliers. Take two numbers a and b to be specified below.

Using the Interpolation formulas, find a polynomial ϕ (of degree $2p + 2q - 1$) vanishing at points α_k and β_k, such that $\phi'(\alpha_0) = a$, $\phi'(\beta_0) = b$, while the derivatives at all other points α_k and β_k $(k > 0)$ vanish.

Let $g = f + \epsilon\phi$, where $\epsilon > 0$. Then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are periodic cycles for g with multipliers

$$\lambda' = \lambda + a\epsilon \prod_{k>0} f'(\alpha_k) \quad \text{and} \quad \mu' = \mu + b\epsilon \prod_{k>0} f'(\beta_k)$$

respectively. Since $|\lambda| \leq 1$ and $|\mu \leq 1$, parameters a and b can be obviously selected in such a way that $|\lambda'| < 1$ and $|\mu'| < 1$ for all sufficiently small $\epsilon > 0$. Thus the cycles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ become attracting under g. But for a sufficiently small ϵ, g is a quadratic-like map on a slightly adjusted domain containing both cycles (see Exercise 3.20). As such, it is allowed to have at most one attracting cycle (Exercise 3.18) - contradiction. □

This result together with Theorem 3.9 immediately yields:

Corollary 3.21. *A quadratic polynomial can have at most one cycle of components of* $\operatorname{int} K(f)$.

17.3. Straightening Theorem. If the reader tried to extend the basic dynamical theory from quadratic polynomials to quadratic-like maps, quite likely he was stuck with the No Wandering Domains Theorem. The only known proof of this theorem crucially uses the fact that a polynomial of a given degree depends on finitely many parameters. The flexibility offered by the infinitely dimensional space of quadratic-like maps looks at this moment like a big disadvantage. It turns out, however, that the theorem is still valid for quadratic-like maps, and actually there is no need to prove it independently (as well as to repeat any other pieces of the topological theory). In fact, quadratic-like maps do not exibit any new features of topological dynamics, since all of them are topologically equivalent to polynomials (restricted to appropriate domains)!

The proof of this theorem was historically the first application of the so called *quasi-conformal surgery* technique. The idea of this technique is to cook by hands a quasi-regular map with desired dynamical properties which topologically looks like a polynomial. If you then manage to find an invariant conformal structure for this map, then by the Measurable Riemann Mapping Theorem it can be realised as a true polynomial.

To state the result precisely, we need a few definitions. Two quadratic-like maps f and g are called *topologically conjugate* if they become such after some adjustments of their domains. Thus there exist adjustments $f : U \to U'$ and $g : V \to V'$ and a homeomorphism $h : (U', U) \to (V', V)$ such that the following diagram is commutative:

$$\begin{array}{ccc} U & \xrightarrow[f]{} & U' \\ h\downarrow & & \downarrow h \\ V & \xrightarrow[g]{} & V' \end{array}$$

In case when one of the maps is a global polynomial, we allow to take any quadratic-like restriction of it.

If the homeomorphism h in the above definition can be selected quasi-conformal (respectively conformal or affine) then the maps f and g are called quasi-conformally (respectively conformally or affinely) conjugate. Two quadratic-like maps are called *hybrid* equivalent if they are qc conjugate by a map h with $\bar{\partial} h = 0$ a.e. on the filled Julia set $K(f)$.

Remark. The last condition implies that h is conformal on the $\operatorname{int} K(f)$. On the Julia set $J(f)$ it gives an extra restriction only if $J(f)$ has positive measure (and so far there are no examples of Julia sets of positive measure).

The equivalence classes of topologically (respectively qc, hybrid etc.) conjugate quadratic-like maps are called topological (respectively qc, hybrid etc.) classes.

THEOREM 3.22. *Any quadratic-like map g is hybrid conjugate to a quadratic polynomial f_c. If $J(f)$ is connected then the corresponding polynomial f_c is unique.*

This polynomial f_c is called the *straightening* of g.

COROLLARY 3.23. *If g is a quadratic-like map, then:*
- (i) *There are no wandering components of* $\operatorname{int} K(g)$*;*
- (ii) *Repelling periodic points are dense in $J(g)$;*
- (iii) *If all periodic points of g are repelling then $K(g)$ is nowhere dense.*

Remark. If $J(g)$ is a Cantor set, then the straightening is not unique. Indeed, by **??**, all quadratic polynomials f_c, $c \in \mathbb{C} \smallsetminus M$, are qc equivalent. Since their filled Julia sets have zero measure, they are actually hybrid equivalent. Hence all of them are going to be the "straightenings" of g. We will see however that sometimes there is a preferred choice (see §**??**).

Existence of the straightening will be proven in the next section, while uniqueness will be postponed until the end of §18.

17.4. Construction of the straightening. The idea is to "mate" g near $K(g)$ with $f_0 : z \mapsto z^2$ near ∞.

First let us adjust $g : U \to U'$ by Lemma 3.18 so that U and U' are bounded by real analytic curves. Take some $r > 1$. Consider two closed disks: the disk $\bar{U}'$ endowed with the map $g : \bar{U} \to \bar{U}'$ and the disk $\bar{\mathbb{C}} \setminus \mathbb{D}_r$ endowed with the map $f_0 : \bar{\mathbb{C}} \setminus \mathbb{D}_r \to \bar{\mathbb{C}} \setminus \mathbb{D}_{r^2}$. Think of them as two hemi-spheres $S^2_+ \equiv U'$ and $S^2_- \equiv \bar{\mathbb{C}} \setminus \bar{\mathbb{D}}_r$ (see Fugure ...) and glue them together by an orientation preserving diffeomorphism $h : \bar{U}' \setminus U \to \mathbb{A}[r, r^2]$ between the closed fundamental annnuli *respecting the boundary dynamical relation*, i.e., such that

$$h(gz) = f_0(hz) \quad \text{for } z \in \partial U. \tag{17.5}$$

Exercise 3.21. *Construct such a diffeomorphism. To this end first consider any diffeomorphism $h_1 : \partial U' \to \mathbb{T}_{r^2}$, then lift it to a diffeomorphism $h_2 : \partial U \to \mathbb{T}_r$ satisfying (17.5), and finally interpolate in between h_1 and h_2.*

In this way we obtain a smooth oriented sphere

$$S^2 = \bar{S}^2_+ \sqcup_h \bar{S}^2_- \equiv \bar{U}' \sqcup_h (\mathbb{C} \setminus \mathbb{D}_r)$$

with the atlas of two local charts given by the identical maps $\phi_+ : S^2_+ \to U'$ and $\phi_- : S^2_- \to \bar{\mathbb{C}} \setminus \bar{\mathbb{D}}_r$. Moreover, the hemi-shperes S^2_+ and S^2_- are bounded by smooth Jordan curves. For instance,

$$\gamma \equiv \partial S^2_- = \phi_+^{-1} h^{-1}(\mathbb{T}_2) = \phi_+^{-1} \partial U.$$

Define now a map $F : S^2 \to S^2$ by letting

$$F(z) = \begin{cases} \phi_+^{-1} \circ g \circ \phi_+(z) & \text{for } z \in \phi_+^{-1}\bar{U} \\ \phi_-^{-1} \circ f_0 \circ \phi_-(z) & \text{for } z \in \bar{S}^2_- \end{cases}$$

(It is certainly quite a puritan way of writing since the maps ϕ_- and ϕ_+ are un fact identical.) By (17.5), these two formulas match on $\gamma \equiv \partial S^2_- = \phi_+^{-1} \partial U$. Hence F is a continuous endomorphism of S^2. Moreover, it is a double branched covering of the sphere onto itself (with two simple branched points at "0"$\equiv \phi_-^{-1}(0)$ and "∞"$\equiv \phi_+^{-1}(\infty)$).

Exercise 3.22. *Check the last statement.*

Since F is holomorphic in the local charts $\phi_\pm$, it is a smooth quasi-regular map on $S^2 \setminus \gamma$. By Lemma 2.5, F is quasi-regular on the whole sphere.

EXERCISE 3.23. *Show that the gluing diffeomorphism h can be chosen in such a way that the map F is smooth.*

use t for h; include a proof of this exercise with bounds for the extension

We will now construct an F-invariant conformal structure μ on S^2 (with a bounded dilatation with respect to the qc structure of the *smooth* sphere S^2). Start in a neighborhood of ∞: $\mu|S^2_- = (\phi_-)^*\sigma$. Since σ is f_0-invariant, $\mu|S^2_-$ is F-invariant. Since ϕ_- admits a smooth extension to $\gamma = \partial S^2_-$, it has a bounded dilatation. Hence $\mu|S^2_-$ has a bounded dilatation as well.

Next, pull-back this structure from the fundamental annulus $A = S^2_+ \cap S^2_-$ to its preimages $A_n = F^{-n}A$, $\mu|A_n = (F^n)^*(\mu|A)$. (We do not bother to define the structure on the union of smooth curves, $\cup\partial A_n$, since it is a set of measure zero.) Since F is holomorphic in the local chart ϕ_+ (namely, equal to g), all these structures have the same dilatation as $\mu|A$. Hence they form a single F-invariant measurable conformal structure with bounded dilatation on $S^2 \setminus \phi_+^{-1}K(g)$.

Finally, let $\mu = (\phi_+)^*\sigma$ on $\phi_+^{-1}K(g)$. We obtain an F-invariant measurable conformal structure μ with bounded dilatation on the whole sphere S^2. By the Measurable Riemann Mapping Theorem, there exists a qc map $H : (S^2, \mu) \to \mathbb{C}$ normalised so that $H(0) = 0$, $H(\infty) = \infty$ and $H\phi_-^{-1}(z) \sim z$ as $z \to \infty$. Then the map $f = H \circ F \circ H^{-1}$ is a quadratic polynomial (see §**??**) with the critical point at the origin and asymptotic to z^2 at ∞. Hence $f = f_c : z \mapsto z^2 + c$ for some c.

EXERCISE 3.24. *Show that $K(f) = H(\phi_+^{-1}K(g))$.*

The qc map $H \circ \phi_+^{-1}$ conjugates $g : U \to U'$ to a quadratic-like restriction of f. Moreover, restricting it to $K(g)$, we see that

$$(H \circ \phi_+^{-1})_*\sigma = H_*\mu = \sigma,$$

so that H is a hybrid conjugacy between g and a restriction of f. Thus f is a straightening of g.

17.4.1. *Comments on the straightening construction.* Note first that the map $B \equiv \phi_- \circ H^{-1}$ in the above construction is the Böttcher coordinate for f on $\Omega \equiv H(S^2_-)$. Indeed:

- B is conformal on Ω since both ϕ_- and H transfer the conformal structure $\mu|S^2_-$ to σ, and
- B conjugates $f_0 : z \mapsto z^2$ to f.

Since $B(\partial\Omega) = \mathbb{T}_r$, $\partial\Omega = E_r$ is the equipotential of radius r for f. Thus *we have conjugated* $f : U \to U'$ *to* $f : D_r \to D_{r^2}$ *where* D_r *is the disk bounded by the equipotential* E_r *of radius* r.

extend the tubing up to the critical point and improve correspondingly the above statement

Second, note that the above construction of f was uniquely determined by the choice of the gluing diffeomorphism $h : \bar{U}' \setminus U \to \mathbb{A}[r, r^2]$ satisfying (17.5). Such a diffeomorphism will be called *tubing.* Thus *tubing determines the straightening uniquely.* In fact, in the case of connected Julia set, the straightening is independent even of the choice of tubing (see the next section).

Finally, let us dwell on an important issue of a bound on the dilatation of the qc homeomorphism conjugating g to f.

LEMMA 3.24. *Let* $g : U \to U'$ *be a quadratic-like map with* $\operatorname{mod}(U' \setminus U) \geq \delta > 0$. *Then* g *is hybrid conjugate to a straightening* f_c *by a* K*-qc map, where the dilatation* K *depends only on* δ.

PROOF. Let us first adjust g according to Lemma 3.18 (keeping the same notations for the domains U and U').

Let us now follow the proof of the Straightening Theorem. Look at the conformal structure $\mu = (\phi_-)^*\sigma$ on the fundamental annulus A in the local chart ϕ_+, i.e., consider

$$\nu = (\phi_+)_*(\mu|A) = h^*\sigma.$$

Its dilatation is equal to the dilatation of h. The pull-backs of ν by the iterates of g (corresponding to the pull-backs of μ by the iterates F) do not change its dilatation. The final extension to the filled Julia set $K(g)$ has zero dilatation. Thus *the dilatation of* $\nu|U' = \phi_+(\mu|S^2_+)$ *is equal to the dilatation of the tubing diffeomorphism* h.

The qc map $H \circ \phi_+^{-1}$ conjugating $g : U \to U'$ to $f_c : D_r \to D_{r^2}$ transfers $\nu|U'$ to σ. Hence its dilatation is also equal to $\operatorname{Dil}(h)$. Thus we only need to argue that h can be selected so that its dilatation depends only on δ.

Let us conformally uniformize the fundamental annulus $R = U' \setminus \bar{U}$, $\Phi : \mathbb{A}(1, \rho) \to R$. Since the boundary curves of R are $\kappa(\delta)$-quasi-circles, Φ admits a $\kappa_1(\delta)$-quasi-symmetric extension to the boundary (**??**). Let us select the map $h : \partial U' \to \mathbb{T}_{r^2}$ on the outer boundary of R in such a way that $h \circ \Phi$ is the homothety of $\mathbb{T}_\rho$ onto T_{r^2}. Following the strategy of Exercise 3.21, lift h to the inner boundary ∂U via the covering map $g : \partial U \to \partial U'$. Since by Lemma 3.18 this covering is $\kappa(\delta)$-quasi-symmetric, $h : \partial U \to \mathbb{T}_r$ is $\kappa_2(\delta)$-quasi-symmetric and hence $h \circ \Phi : \mathbb{T}_1 \to \mathbb{T}_r$ is $\kappa_3(\delta)$-quasi-symmetric.

By Lemma **??**, $h \circ \Phi$ admits a qc extension to R with the dilatation depending only on $\kappa_3(\delta)$ and $\mathrm{mod}(R)/\log r$. Selecting r in such a way that the latter ratio is bounded (for instance, take $\log r = \mathrm{mod}\, R$), we obtain a map $h \circ \Phi$ with dilatation depending only on δ. Since $\mathrm{Dil}(h) = \mathrm{Dil}(h \circ \Phi)$, we are done. □

18. Expanding circle maps

Before passing to the uniquenss part of the Straightening Theorem, let us dwell on an important relation between quadratic-like and circle maps.

18.1. Definition. Recall that $\mathbb{T} \subset \mathbb{C}$ stands for the unit circle (endowed with the induced real analytic structure and Riemannian metric). *Symmetry* with respect to $\mathbb{T}$ is understood in the sense of the anti-holomorphic reflection $\tau : z \mapsto 1/\bar{z}$.

Let us say that $g : \mathbb{T} \to \mathbb{T}$ is an *expanding circle map of class $\mathcal{E}$* if it satisfies the following properties:

(i) g is an orientation preserving double covering of the circle over itself;

(ii) g is real analytic;

(iii) g is expanding, i.e, there exist constants $C > 0$ and $\lambda > 1$ such that for any $z \in \mathbb{T}$,

$$\|Dg^n(z)\| \geq C\lambda^n, \quad n = 0, 1, \ldots. \tag{18.1}$$

The simplest example is provided by the *quadratic circle map* $f_0 : z \mapsto z^2$. A little more generally, we have the Blyaschke circle maps:

Exercise 3.25. *Let $g : \mathbb{D} \to \mathbb{D}$ be a holomorphic double covering of the unit disk over itself which has a fixed point in $\mathbb{D}$. By Exercise **??**, g admits a continuous extension to the unit circle T. Show that this extension is an expanding circle map of class $\mathcal{E}$.*

*Hint: By Exercise **??**, g actually extends to the whole sphere. To show that it is expanding on $\mathbb{T}$, use the hyperbolic metric in $\bar{\mathbb{C}} \setminus (\mathrm{orb}(a) \cup \mathrm{orb}(1/\bar{a})$, where $a \in \mathbb{D}$ is the critical point of g.*

To state some results in adequately general form, we will also consider a bigger class $\mathcal{E}^1$ of C^1-smooth expanding circle maps and a class $\mathcal{E}^{1+\alpha}$ of C^1-smooth maps whose derivative satisfies the Hölder condition with exponent $\alpha \in (0, 1)$. (However, for applications to holomorphic dynamics we will only need real analytic maps, so that the reader can always assume it.)

EXERCISE 3.26. *(i) For any $g \in \mathcal{E}^1$, there exists a smooth Riemannian metric ρ on $\mathbb{T}$ such that*

$$\|Dg(z)\|_\rho \geq \lambda > 1 \text{ for all } z \in \mathbb{T}.$$

This metric is called Lyapunov. Hint: Consider $\rho = \dots$

EXERCISE 3.27. *Show that any expanding circle map $g \in \mathcal{E}^1$ has a unique fixed point $\beta \equiv \beta_g \in \mathbb{T}$.*

Hint: Lifting g to the universal covering, you obtain an orientation preserving diffeomorphism $G : \mathbb{R} \to \mathbb{R}$ satisfying the properties: a)$G(x+1) = G(x)+2$; b) all fixed points of G is repelling. Or, use the Lefschetz formula instead.

18.2. Symbolic model. Let us consider a symbolic sequence $\bar{k} = (k_0, k_1, \dots) \in \Sigma$ of zeros and ones. Each such a sequence represents some number

$$\theta(\bar{k}) = \sum_{n=0}^{\infty} \frac{k_n}{2^{n+1}} \in [0,1]$$

in its diadic expansion. As everybody learns in the school (in the context of decimal expansions), all numbers except those of the form $m/2^n$ admit a unique diadic expansion. The numbers of the form $m/2^n$ with odd m admit exactly two diadic expansions:

$$\frac{k_0}{2} + \dots + \frac{k_{n-2}}{2^{n-1}} + \frac{1}{2^n} = \frac{k_0}{2} + \dots + \frac{k_{n-2}}{2^{n-1}} + \sum_{m=n+1}^{\infty} \frac{1}{2^m}.$$

Thus the corresponding symbolic sequences viewed as representations of numbers should be identified. If we consider the numbers mod 1, then we should also identify the sequence **0** of all zeros to the sequence **1** of all ones. Let us call these identifications on Σ "arithmetic" and the space Σ modulo these identifications *arithmetic quotient* of Σ. Of course, this quotient is in a natural one-to-one correspondence with the unit interval with identified endpoints, i.e., with the circle.

EXERCISE 3.28. *Show that the projection*

$$\pi_0 : \Sigma \to \mathbb{T}, \quad \bar{k} \mapsto \exp(2\pi i\, \theta(\bar{k}))$$

(continuously) semi-conjugates the Bernoulli shift $\sigma : \Sigma \to \Sigma$ (see §13.4) to the circle endomorphism $f_0 : z \mapsto z^2$. Thus $f_0 : \mathbb{T} \to \mathbb{T}$ is topologically conjugate to the arithmetic quotient of the Bernoulli shift.

It turns out that the same is true for all expanding circle maps $g \in \mathcal{E}^1$:

LEMMA 3.25. *Any circle expanding map $f \in \mathcal{E}^1$ is topologically conjugate to the arithmetic quotient of the Bernoulli shift.*

Proof. Let $g \in \mathcal{E}^1$. Consider its fixed point β. It has a single perimage β^1 different from $\beta \equiv \beta^0$. These two points, β and β^0, divide the circle into two (open) intervals intervals, I^1_0 and I^1_1 (counting anti-clockwise starting from β). Moreover, g homeomorphically maps each I^1_k onto $\mathbb{T} \smallsetminus \beta$. Hence each I^1_k contains a preimage β^2_k of β^1. This point divides I^1_k into two open intervals, I^2_{k0} and I^2_{k1} (counting anti-clockwise). We obtain four intervals, I^2_{kj}, $k, j \in \{0,1\}$ such that g homeomorphically maps each I^2_{kj} onto I^1_k.

Continuing inductively, we see that

$$\mathbb{T} \smallsetminus g^{-n}\beta = \bigcup_{k_s \in \{0,1\}} I^n_{k_0\, k_1\, \dots k_{n-1}},$$

where:

(i) the anti-clockwise order of the intervals $I^n_{\bar{k}}$ (starting from β) corresponds to the lexicographic order on the symbolic strings $\bar{k} = (k_0\, k_1\, \dots k_{n-1})$;

(ii) the map g homeomorphically maps $I^n_{\bar{k}}$ onto $I^{n-1}_{\sigma(\bar{k}}$), where the strimg $\sigma(\bar{k}) = (k_1 \dots k_{n-1})$ is obtained from $\bar{k}$ by erasing the first symbol.

(iii) any interval $I^n_{\bar{k}}$ contains a point $\beta^{n+1}_{\bar{k}} \in g^{-(n+1)}\beta$ which divides it into two intervals $I^{n+1}_{\bar{k}\,0}$ and $I^{n+1}_{\bar{k}\,1}$ of the next level.

Thus g^n homeomorphically maps each interval $I^n_{\bar{k}}$ onto the punctured circle $\mathbb{T} \smallsetminus \{\beta\}$. Since g is expanding, the lengths of these intervals shrink exponentially fast:

$$|I^n_{\bar{k}}| \le \frac{2\pi}{C} \lambda^{-n},$$

where $C > 0$ and $\lambda > 1$ are constants from (18.1). It follows that for any infinite sequence $\bar{k} = (k_0 k_1 \dots) \in \Sigma$ of zeros and ones, the closed intervals $\bar{I}^n_{k_0 \dots k_{n-1}}$ form a nest shrinking to a single point $z = \pi(\bar{k})$. Thus we obtain a map $\pi : \Sigma \to \mathbb{T}$.

Under this map, the cylinders of rank n are mapped to the intervals of rank n. Since the latter shrink, π is continuous.

The above property (ii) implies that π is equivariant. Thus g is a quotient of the Bernoulli shift.

We only need to describe the fibers of π. If z is not an iterated preimage of β, then it belongs to a *single* interval of any rank. Hence $\text{card}(\pi^{-1}(z)) = 1$. Obviously the fiber $\pi^{-1}(\beta)$ consists of two extremal sequences, $(\mathbf{0})$ and $\mathbf{1}$. Otherwise $z = \beta^{n+1}_{k_0 \dots k_{n-1}} \in g^{-(n+1)}\beta$ for some $n \ge 0$ (except that for $n = 0$, the point β^1 does not have subsripts). Then it is a boundary point for exactly two intervals of each order $m \ge n+1$. For $m = n+1$, the corresponding symbolic sequences differ by the last symbol only: $(k_0 \dots k_{n-1}\, 0)$ and $(k_0 \dots k_{n-1}\, 1)$. For all further levels, we should add symbol 1 to the first sequence and symbol 0 to the second one. Thus:

$$\pi(k_0 \dots k_{n-1}\, 0\,1\,1\,1 \dots) = z = \pi(k_0 \dots k_{n-1}\, 1\,0\,0\,0\, \dots),$$

which are exactly the arithmetic identifications on Σ. □

Thus all expanding circle maps of class $\mathcal{E}^1$ are topologically the same:

PROPOSITION 3.26. *Any two expanding circle maps of class $\mathcal{E}^1$ are topologically conjugate by a unique orientation preserving circle homeomorphism. In particular, expanding circle maps do not admit non-trivial orientation preserving automorphisms.*

PROOF. Lemma 3.25 gives the same standard model for any expanding circle map of class $\mathcal{E}^1$. In this model, the anti-clockwise order on $\mathbb{T} \smallsetminus \{\beta\}$ corresponds to the lexicographic order on Σ. Hence the corresponding conjugacy h between two circle maps, g and $\tilde{g}$, is orientation preserving.

Such a conjugacy is unique. Indeed, it must carry the points of $g^{-n}(\beta)$ to $\tilde{g}^{-1}(\tilde{\beta})$ preserving their anti-clockwise order starting from the corresponding fixed points, β and $\tilde{\beta}$. Hence h is uniquely determined on the iterated preimages of β. Since these preimages are dense in $\mathbb{T}$ (by the previous lemma), h is uniquely determined on the whole circle. □

Remarks. 1. Expanding circle maps have one orientation reversing automorphism. In the case of $z \mapsto z^2$ it is just $z \mapsto \bar{z}$ (compare with Exercise 3.4).

2. The above discussion can be generalized in a straightforward way to expanding circle maps of degree $d > 2$. There is one difference though: if $d > 2$ then the group of orientation preserving automorphisms of g is not trivial any more but rather the cyclic group of order $d - 1$ (consider $z \mapsto z^d$).

18.3. Equivariant liftings. Let us describe a lifting construction which will find numerous applications in what follows.

Consider two open conformal annuli $\Omega \subset \Omega' \subset \mathbb{C}$ with a common inner boundary. Assume that $A = \overline{\Omega' \smallsetminus \Omega}$ is a (closed) annulus whose boundary components are smooth Jordan curves. Let $g : \Omega \to \Omega'$ be a holomorphic double covering map. A point $z \in \Omega$ is called *escaping* if $f^n z \in A$ for some $n \in \mathbb{N}$.

Consider also another map $\tilde{g} : \tilde{\Omega} \to \tilde{\Omega}'$ with the same properties (all corresponding objects for $\tilde{g}$ will be marked with "tilde").

Lemma 3.27. *Under the circumstances just described, assume that all points of* Ω *and* Ω' *are escaping. Then any equivariant homeomorphism* $H : A \to \tilde{A}$ *admits a unique homeomorphic extension* $h : \Omega' \to \tilde{\Omega}'$ *conjugating* g *to* $\tilde{g}$*. If* H *is quasi-conformal then so is* h*, and* $\mathrm{Dil}(h) = \mathrm{Dil}(H)$*. Moreover, the Beltrami differential* $\mu_h = h^*\sigma$ *is obtained by pulling back the Beltrami differential* $\mu_H = H^*\sigma$ *by the iterates of* g*:* $\mu_h|A^n = (g^n)^*\mu_H$*.*

Proof. Let $A^n = g^{-n}A$, and let Γ^n be the outer boundary of A^n (coinciding for $n \geq 1$ with the inner boundary of A^{n-1}). Consider an equivariant homeomorphism $H : A \to \tilde{A}$. This map admits a lift $H_1 : A^1 \to \tilde{A}^1$ such that $\tilde{g} \circ H_1 = H \circ g|A^1$. In fact, there are exacly two such lifts determined by a value of H_1 at a single point.

The restriction of H_1 to the outer boundary Γ^1 is a lift of $H : \Gamma^0 \to \tilde{\Gamma}^0$. But since H is equivariant on ∂A, its restriction to Γ^1 is also a lift of $H : \Gamma^0 \to \tilde{\Gamma}^0$. Hence the lift H_1 can be chosen in such a way that $H_1|\Gamma^1 = H|\Gamma^1$. With this choice, H and H_1 glue together to an equivarinat homeomorphism $h_1 : A \cup A^1 \to \tilde{A} \cup \tilde{A}^1$. Now equivariance means that $\tilde{g} \circ h_1|A^1 = h_1 \circ g|A^1$. In particular, h_1 is equivariant on the boundary of A^1, so that we can apply to it the above construction. It provides us with an equivariant extension $h_2 : A \cup A^1 \cup A^2 \to \tilde{A} \cup \tilde{A}^1 \cup \tilde{A}^2$ of h_1.

Proceeding in this way we will obtain a sequence of equivariant liftings $H_n : A^n \to \tilde{A}^n$ which glue together to equivariant homeomorphisms

$$h_n : \bigcup_{k=0}^{n} A^k \to \bigcup_{k=0}^{n} \tilde{A}^k$$

extending one another. Since all the points in Ω escape, the annuli A^k exhaust Ω', and similarly for $\tilde{\Omega}'$. Hence the direct limit of equivariant extensions h_n is a homeomeorhism $h : \Omega' \to \tilde{\Omega}'$ conjugating g to $\tilde{g}$.

It shows existence of a conjugacy h for any given H. Uniqueness is obvious: $h|A$ consecutively determines the lifts $h|\, A^n$ by requirements of equivarience and continuous matching.

Finally, assume that H is quasi-conformal with dilatation K. Since g and $\tilde{g}$ are conformal, all the consecutive lifts of H to the annuli A^n are qc maps with the same dilatation K. By Proposition 2.5, their gluings (maps h_n) are K-qc maps as well. The direct limit h of K-qc extensions h_n is obviously K-qc as well.

The last statement is obvious due to the natural behavior of the Beltrami differentials under conformal liftings: $\mu_{H_n} = (g^{\circ n})^*\mu_H$ since $\tilde{g}^{\circ n} \circ H_n = H \circ g^{\circ n}$ where $g^{\circ n}$ and $\tilde{g}^{\circ n}$ are conformal. □

Remark. We do not need to assume that the annuli Ω and Ω' are embedded to $\mathbb{C}$.

PROBLEM 3.28. *Is the assumption that all points in Ω escape automatically satisfied if* $\mathrm{mod}(\Omega) < \infty$*?*

18.4. Complex extensions of circle maps. In this section we will take a closer look at the holomorphic extensions of expanding cicle maps of class $\mathcal{E}$.

EXERCISE 3.29. *(i) For any $g \in \mathcal{E}$, there exist two $\mathbb{T}$-symmetric topological annuli $V \Subset V'$ (bounded by smooth Jordan curves) such that g admits a holomorphic extension to V and maps it onto V' as a double covering.*

Hint: Extend the Lyapunov metric from Exercise 3.26 to a neighborhood of $\mathbb{T}$.

(ii) Show that vice versa, property (i) imlies that $g \in \mathcal{E}$. Hint: Use the hyperbolic metric in V'.

(iii) Show that all points $z \in V \setminus \mathbb{T}$ escape, i.e., $g^n z \in V' \setminus V$ for some $n \in \mathbb{N}$.

Hints should go to an Appendix.

Thus property (i) can be used as a definition of an expanding circle map of class $\mathcal{E}$. In fact, only exterior part of the above extension is needed to reconstruct the circle map (it will be useful in what follows):

LEMMA 3.29. *Let $\Omega \subset \Omega' \subset \mathbb{C}$ be two open conformal annnuli whose inner boundaries coincide with the unit circle $\mathbb{T}$. Let $g : \Omega \to \Omega'$ be a holomorphic double covering. Then g admits an extension to a holomorphic double covering $G : V \to V'$, where $V \Subset V'$ are $\mathbb{T}$-symmetric annuli such that $\Omega = V \setminus \bar{D}$ and $\Omega' = V' \setminus \bar{D}$. If the outer boundary of Ω is contained in Ω', then $V \Subset V'$ and the restriction $G|\mathbb{T}$ is an expanding cicle map of class $\mathcal{E}$.*

PROOF. First show that g continuously extends to $\mathbb{T}$ (apply boundary properties of confomal maps to inverse branches of g **??**). Then use the Schwarz Reflection Principle. $\square$

Consider a holomorphic extension $g : V \to V'$ of a map $g \in \mathcal{E}$ given by Exercise **??**. Thus $V \Subset V'$ are two $\mathbb{T}$-symmetric annuli neighborhoods of the circle. Let $A = (\bar{V}' \setminus V) \setminus \mathbb{D}$ be the "outer" fundamental annulus for g.

Given another map $\tilde{g} : \tilde{V} \to \tilde{V}'$ as above, we will mark the corresponding objects with "tilde".

PROPOSITION 3.30. *Any two expanding circle maps $g : V \to V'$ and $\tilde{g} : \tilde{V} \to \tilde{V}'$ are conjugate by a qc map $h : (V', V, \mathbb{T}) \to (\tilde{V}', \tilde{V}, \mathbb{T})$ commuting with the reflection τ about the circle. In fact, any equivariant qc map $H : A \to \tilde{A}$ between the fundamental annuli admits a unique extension to a qc conjugacy h as above. Moreover* $\operatorname{Dil}(h) = \operatorname{Dil}(H)$.

PROOF. Consider an equivariant qc map H as above with dilatation K. By Lemma 3.27 it can be uniquely lifted to an equivariant K-qc homeomorphism $h : V' \setminus \bar{\mathbb{D}} \to \tilde{V}' \setminus \bar{D}$. By **??**, h admits a continuous extension to the unit circle. Reflecting it to the interior of the circle (and then exploiting Proposition 2.5) we obtain a desired K-qc conjugacy $h : V' \to \tilde{V}'$. □

Let us endow the exterior $\mathbb{C} \setminus \bar{\mathbb{D}}$ of the unit disk, with the hyperbolic metric $\rho \equiv \rho_{\mathbb{C} \setminus \bar{D}}$. The hyperbolic length of a curve γ will be denoted by $l_\rho(\gamma)$, while it Euclidean length will be denoted by $|\gamma|$.

LEMMA 3.31. *Let $g : V \to V'$ be an expanding circle map of class $\mathcal{E}$. Let Ω and Ω' be two (open) annuli whose inner boundary is the circle $\mathbb{T}$. Let $h : \Omega \to \Omega'$ be a homeomorphism commuting with g. Then h admits a continuous extension to a map $\Omega \cup \mathbb{T} \to \tilde{\Omega} \cup \mathbb{T}$ identical on the circle.*

*********************************** unedited

PROOF. Given a set $X \subset A$, let $\tilde{X}$ denote its image by ω. Let us take a configuration consisting of a round annulus $L^0 = \mathbb{A}[r, r^2]$ contained in A, and an interval $I_0 = [r, r^2]$. Let $L^n = P_0^{-n} L^0$, and I_k^n denote the components of $P_0^{-n} I^0$, $k = 0, 1, \dots, 2^n - 1$. The intervals I_k^n subdivide the annulus L^n into 2^n "Carleson boxes" Q_k^n.

Since the (multi-valued) square root map P_0^{-1} is infinitesimally contracting in the hyperbolic metric, the hyperbolic diameters of the boxes $\tilde{Q}_k^n$ are uniformly bounded by a constant C.

Let us now show that ω is a hyperbolic quasi-isometry near the circle, that is, there exist $\epsilon > 0$ and $A, B > 0$ such that

$$A^{-1} \rho(z, \zeta) - B \le \rho(\tilde{z}, \tilde{\zeta}) \le A \rho(z, \zeta) + B, \tag{18.2}$$

provided $z, \zeta \in \mathbb{A}(1, 1 + \epsilon)$, $|z - \zeta| < \epsilon$.

Let γ be the arc of the hyperbolic geodesic joining z and ζ. Clearly it is contained in the annulus $\mathbb{A}(1, r)$, provided ϵ is sufficiently small. Let $t > 1$ be the radius of the circle $\mathbb{T}_t$ centered at 0 and tangent to γ. Let us replace γ with a *combinatorial geodesic* Γ going radially up from z to the intersection with $\mathbb{T}_t$, then going along this circle, and then radially down to ζ. Let N be the number of the Carleson boxes intersected by Γ. Then one can easily see that

$$\rho(z,\zeta) = l_\rho(\gamma) \asymp l_\rho(\Gamma) \asymp N,$$

provided $\rho(z,\zeta) \geq 10\log(1/r)$ (here $\log(1/r)$ is the hyperbolic size of the boxes Q_k^n).

On the other hand

$$\rho(\tilde{z},\tilde{\zeta}) \leq l_\rho(\tilde{\Gamma}) \leq CN,$$

so that $\rho(\tilde{z},\tilde{\zeta}) \leq C_1\rho(z,\zeta)$, and (18.2) follows.

But quasi-isometries of the hyperbolic plane admit continuous extensions to $\mathbb{T}$ (see, e.g., [**Th**]). Finally, it is an easy exercise to show that the only homeomorphism of the circle commuting with P_0 is identical. ☐

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We will show next that "outer automorphisms" of circle maps move points bounded hyperbolic distance:

Lemma 3.32. *Let $g : V \to V'$ be a map of class $\mathcal{E}$. Let Ω and Ω' be two open annuli in $V \smallsetminus \bar{\mathbb{D}}$ with inner boundary $\mathbb{T}$, and let $h : \Omega \to \Omega'$ be an automorphism of g. Then for any $\delta > 0$ there exists an $R = R(\delta) > 0$ such that $\rho(z, hz) \leq R$ for all points $z \in \Omega$ whose distance from the outer boundary of Ω is at least δ.*

Proof. By Proposition 3.30, g is qc conjugate to the quadratic circle map $f_0 : z \mapsto z^2$. Of course, this conjugacy can be extended to a global qc homeomorphism of $\bar{C}$ (e.g., by **??**). Since qc homeomorphisms of $\mathbb{C} \smallsetminus \bar{\mathbb{D}}$ are hyperbolic quasi-isometries (**??**), it is enough to prove the assertion for f_0. So, let us assume from now on that $g = f_0$.

Of course, the assertion is true for any compact subset of Ω. Hence we need to check it only near to the unit circle.

By 3.31, h admits a continuous extension to the unit circle. Of course, it still commutes with g on the circle. By Proposition 3.26, $h|\mathbb{T} = \text{id}$. Hence for any $\epsilon > 0$ there exists an $r > 1$ such that $\mathbb{A}(1,r] \Subset \Omega$ and

$$|z - hz| < \epsilon \quad \text{for} \quad z \in \mathbb{A}(1,r].$$

Consider a fundamental annulus A of g compactly contained in $\mathbb{A}(1,r]$. By compactness, there exists an $R > 0$ such that

$$\rho(z, hz) \leq R \quad \text{for} \quad z \in A.$$

Let $A^n = g^{-n}A$. Take some $z \in A^1$. Since $|z - hz| < \epsilon$, these points are obtained by applying the *same local branch* of the square root map g^{-1} to the points gz and $g(hz) = h(gz)$. Since the local branches of g^{-1} preserve the hyperbolic distance on $\mathbb{C} \setminus \bar{\mathbb{D}}$, we have: $\rho(z, hz) = \rho(gz, h(gz)) \le R$.

Replacing A by A^1, we obtain the same bound for any $z \in A^2$, etc. The conclusion follows. ☐

18.5. External map (the connected case). To any quadratic-like map $f : U \to U'$ one can naturally associate an expanding circle map g of class $\mathcal{E}$ which captures dynamics outside the Julia set. For this reason g is called the *external map* of f.

The construction is very simple if the Julia set $J(f)$ is connected. In this case the basin of infinity $D_f(\infty) = \mathbb{C} \setminus K(f)$ is simply connected and can be conformally mapped onto the complement of the unit disk:

$$R : \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus \bar{\mathbb{D}}.$$

Let $\Omega = R(V \setminus K(f))$, $\Omega' = R(V' \setminus K(f))$. These are two conformal annuli with smooth boundary. Moreover, the have a common inner boundary, the unit circle $\mathbb{T}$, while the outer boundary of Ω is contained in Ω'. Conjugating f by R we obtain a holomorphic double covering

$$g : \Omega \to \Omega', \quad g(z) = R \circ f \circ R^{-1}(z) \quad \text{for} \quad z \in \Omega.$$

By Lemma 3.29, g can be extended to an expanding circle map of class $\mathcal{E}$.

In fact, this map is not uniquely defined since the Riemann map R is defined up to post-composition with rotation $z \mapsto e^{2\pi i \theta} z$, $0 \le \theta < 2\pi$. A natural way to *normalize* g is to put its fixed point β to $1 \in \mathbb{T}$.

Note also that if f is replaced by an affinely conjugate map $A^{-1} \circ f \circ A$, where $A : z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$, then the Riemann map R is replaced by $R \circ A$, and the external map g remains the same. Thus, to any quadratic-like map f (with connected Julia set) prescribed *up to an affine conjugacy* corresponds an expanding circle map g well-defined *up to rotation conjugacy.*

We will consider the case of disconnected Julia set in §**??**.

18.6. Uniqueness of the straightening. Let us first show that "external automorphisms" of quadratic-like maps admit a continuous extension to the Julia set by identity (compare with Lemma 3.31).

LEMMA 3.33. *Let $f : U \to U'$ be a quadratic-like map with connected Julia set. Let $W \subset U$ and $W' \subset U$ be two (open) annuli whose inner boundary is $J(f)$. Let $h : W \to W'$ be a homeomorphism commuting with f. Then h admits a continuous extension to a map $W \cup J(f) \to W' \cup J(f)$ identical on the Julia set.*

PROOF. Consider the Riemann mapping $R : \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus \bar{\mathbb{D}}$ and the external circle map $g : V \to V'$, $g|V \setminus \bar{D} = R \circ f \circ R^{-1}$. Transfer the annuli W and W' to the g-plane. We obtain two annuli $\Omega = R(W)$ and $\Omega' = R(W')$ in $V \setminus \bar{\mathbb{D}}$ attached to the unit circle $\mathbb{T}$. Of course, the homeomorphism $k : \Omega \to \Omega'$, $k = R \circ h \circ R^{-1}$, commutes with g.

By Lemma 3.32, k moves points near $\mathbb{T}$ bounded hyperbolic distance: $\rho_{\mathbb{C}\setminus\bar{\mathbb{D}}}(k(z), z) \leq R$. Since the Riemann mapping $R : \mathbb{C} \setminus \bar{D} \to \mathbb{C} \setminus K(f)$ is a hyperbolic isometry, the same is true for h:

$$\rho_{\mathbb{C}\setminus K(f)}(z, h(z) \leq R$$

for $z \in W$ near $J(f)$. By **??**, the Euclidean distance $|z - hz|$ goes to 0 as $z \to J(f)$, $z \in W$. It follows that the extension of h by the identity on the Julia set is continuous. □

COROLLARY 3.34. *Let f and $\tilde{f}$ be two quadratic-like maps, and let a homeomorphism h conjugates f to $\tilde{f}$ in some neighborhoods of the filled Julia sets. Then h is uniquely determined on $J(f)$.*

PROBLEM 3.35. *Assume that quadratic polynomials f and $\tilde{f}$ are conjugate on the Julia sets only. Is the conjugacy unique?*

Let us now summarize the above results:

THEOREM 3.36. *Let us consider two quadratic-like maps $f : U \to U'$ and $\tilde{f} : \tilde{U} \to \tilde{U}'$ with connected Julia sets. Assume that they are topologically conjugate near their Julia sets by a homeomorphism $\psi : V \to \tilde{V}$. Assume also that we are given an equivariant homeomorphism $H : A \to \tilde{A}$ between the (closed) fundamental annuli of f and $\tilde{f}$.*

Then there exists a unique homeomorphism $h : U' \to \tilde{U}'$ conjugating f to $\tilde{f}$, coinciding with ψ on the Julia set $J(f)$, and coinciding with H on A.

If H is qc, then $h|\, U \setminus K(f)$ is also qc with the same dilatation. If both H and ψ are qc, then h is qc, and

$$\mathrm{Dil}(h) = \max(\mathrm{Dil}\, H,\, \mathrm{Dil}(\psi|\, K(f)).$$

In particular, if f and g are hybrid equivalent by means of ψ, then $\mathrm{Dil}(h) = \mathrm{Dil}(H)$.

PROOF. By the Lifting Construction of §18.3, H admits a unique equivariant extension to a homeomorphism $h : U \setminus K(f) \to \tilde{U} \setminus K(\tilde{f})$. This extension continuously matches with ψ on the filled Julia set. Indeed, $\psi^{-1} \circ h$ commutes with f on some exterior neighborhood of $K(f)$. By Lemma 3.33, this map continuously extends to the filled

Julia set as identity. Hence h continuous extends to the filled Julia set as ψ.

If H is qc then $h|\, U \smallsetminus K(f)$ is qc with the same dilatation by Lemma 3.27. All the rest follows from the Bers Lemma 2.6. □

Of course, we can always construct an equivariant qc map H between the fundamental annuli. Hence if two quadratic-like maps are topologically equivalent, then the conjugacy can be selected quasi-conformal outside the filled Julia set. If they are hybrid equivalent, then the dilatation of the conjugacy is completely controlled by the dilatation of H, which is in turn controlled by the geometry of the fundamental annuli (see **??**). In the case of global polynomials we can do even better:

COROLLARY 3.37. *Consider two quadratic polynomials $f : z \mapsto z^2 + c$ and $\tilde{f} : z \mapsto z^2 + \tilde{c}$ with connected Julia sets. If they are topologically conjugate near their filled Julia sets, then there is a global conjugacy $h : \mathbb{C} \to \mathbb{C}$ which is conformal on the basin of ∞. If f and $\tilde{f}$ are hybrid conjugate near their filled Julia sets, then $f = \tilde{f}$.*

PROOF. By §**??**, the Riemann-Bötcher map $B_f : D_f(\infty) \to \mathbb{C} \smallsetminus \bar{\mathbb{D}}$ conjugates f to $z \mapsto z^2$. Hence the composition

$$R : B_{\tilde{f}}^{-1} \circ B_f : \; D_f(\infty) \to D_{\tilde{f}}(\infty) \tag{18.3}$$

conformally conjugates f to $\tilde{f}$ on their basins of ∞. By the previous theorem, this conjugacy matches with the topological conjugacy on the filled Julia set giving us a desired global conjugacy h.

Moreover, If f and $\tilde{f}$ are hybrid equivalent, then $\mathrm{Dil}(h) = 0$ a.e. By Weil's Lemma 2.4, h is conformal and hence affine. But if two quadratic polynomials in the normal form $z^2 + c$ are affinely equivalent, then they are equal. □

The last statement of the above Corollary gives the uniqueness part of the Straightening Theorem.

18.6.1. *Picture.*

CHAPTER 4

Combinatorics of external rays

1. Dynamical rays

1.1. Motivaing problems. Consider a quadratic polynomial $f = f_c$ with connected Julia set. As we know (§**??**), its basin of infinity is uniformized by the Böttcher map $\phi : D_f(\infty) \to \mathbb{C} \setminus \mathbb{D}$, which conjugates f to $z \mapsto z^2$. If the Julia set was locally connected then by the Carathéodory theorem the inverse map would ϕ^{-1} extend continuously to the unit circle $\mathbb{T}$. This would give a representation of $f|\, J(f)$ as a quotient of the the doubling map $\theta \mapsto 2\theta \mod 1$ of the circle $\mathbb{R}/Z \approx \mathbb{T}$. This observation immeadiately leads to the followong problems:

1) Describe explicitly equivalence realtions on the circle corresponding to all possible Julia sets;

2) Study the problem of local conectivity of the Julia sets.

It turns out that the first problem can be addressed in a comprehensive way. The second problem is very delicate. However, even non-locally connected examples can be partially treated due to the fact that many external rays always land at some points of the Julia set. This is the main theme of the following discussion.

1.2. Landing of rational rays. We say that an external ray $\mathcal{R}^\theta$ *lands* at some point z of the Julia set if $\mathcal{R}^\theta(t) \to z$ as $t \to 0$. Two rays $\mathcal{R}^{\theta/2}$ and $\mathcal{R}^{\theta/2+1/2}$ will be called "preimages" of the ray $\mathcal{R}^\theta$. Obviously, if some ray lands, then its image and both its preimages land as well.

An external ray $\mathcal{R}^\theta$ is called *rational* if $\theta \in \mathbb{Q}$, and *irrational* otherwise. Dynamically the rational rays are characterized by the property of being either periodic or preperiodic:

EXERCISE 4.1. *Let* $\mathcal{R} = \mathcal{R}^\theta$.

a) If θ *is irraional then the rays* $f^n(\mathcal{R})$, $n = 0, 1, \dots,$ *are all distinct.*

Assume θ *is rational:* $\theta = q/p$, *where* q *and* p *are mutually prime. Then*

(i) *If* p *is odd then* $\mathcal{R}$ *is periodic: there exists an* l *such that* $f^l(\mathcal{R}) = \mathcal{R}$.

(ii) *If p is even then $\mathcal{R}$ is preperiodic: there are l and $r > 0$ such that $f^r(\mathcal{R})$ is a periodic ray of period l, while the rays $f^k(\mathcal{R})$, $k = 0, 1, \dots, r-1$, are not periodic.*

How to calculate l and r?

THEOREM 4.1. *Let f be a quadratic polynomial with connected Julia set. Then any rational ray $\mathcal{R} = \mathcal{R}_f^\theta$ lands at some repelling or parabolic point of f.*

PROOF. Without loss of generality we can assume that the ray $\mathcal{R}$ is periodic and hence invariant under some iterate $g = f^l$. Let $d = 2^l$. Consider a sequence of points $z_n = \mathcal{R}(1/d^n)$, and let γ_n be the sequence of arcs on $\mathcal{R}$ bounded by the points z_n and z_{n+1}. Then $g(\gamma_n) = \gamma_{n-1}$.

Endow the basin $D = D_f(\infty)$ with the hyperbolic metric ρ. Since $g : D \to D$ is a covering map, it locally preserves ρ. Hence the hyperbolic length of the arcs γ_n are all equal to some L.

But all the rays accumulate on the Julia set as $t \to 0$. By the relation between the hyperbolic and Euclidean metrics (Lemma 1.11), the Euclidean length of these arcs goes to 0 as $n \to \infty$. Hence the limit set of the sequence $\{z_n\}$ is a connected set consisting of the fixed points of g. Since g has only finitely many fixed points, this limit set consists of a single fixed point β. It follows that the ray $\mathcal{R}$ lands at $\beta \in J(f)$ (compare with the proof of Theorem 3.9).

Since $\beta \in J(f)$, it can be either repelling, or parabolic, or Cremer. But the latter case is excluded by the Necklace Lemma 3.10. ☐

CHAPTER 5

Parameter plane (the Mandelbrot set)

20. Definition and first properties

The Mandelbrot set presents at one glance the whole dynamical diversity of the complex quadratic family $f_c : z \mapsto z^2 + c$. Figure ... shows this set and its blow-ups in several places. It is remarkable that all this intricate structure is hidden behind the following one-line definition.

Recall the Basic Dichotomy for the quadratic maps: the Julia set $J(f_c)$ is either connected or Cantor (Theorem 3.3). By definition, the *Mandelbrot set* M consists of those parameter values $c \in \mathbb{C}$ for which the Julia set $J(f_c)$ is connected. It is equivalent to saying that the orbit of the critical point

$$0 \mapsto c \mapsto c^2 + c \mapsto (c^2 + c)^2 + c \mapsto \dots \tag{20.1}$$

is not escaping to ∞. Let us denote the nth polynomial in (20.1) by $\phi_n(c)$, so that $\phi_0(c) \equiv 0$, $\phi_1(c) \equiv c$, and recursively

$$\phi_{n+1}(c) = \phi_n(c)^2 + c. \tag{20.2}$$

Note that $\deg \phi_n = 2^{n-1}$.

Though the polynomials ϕ_n are not iterates of a single polynomial, they behave in many respects similarly to the iterated polynomials:

EXERCISE 5.1 (Simplest properties of M). *Prove the following properties:*

(i) *If $|\phi_n(c)| > 2$ for some $n \in \mathbb{N}$ then $\phi_n(c) \to \infty$ as $n \to \infty$. In particular, $M \subset \mathbb{D}_2$.*
(ii) *$\phi_n(c) \to \infty$ locally uniformly on $\mathbb{C} \smallsetminus M$. Hence M is compact.*
(iii) *$\mathbb{C} \smallsetminus M$ is connected (recall the proof of Theorem 3.3). Hence M is full and all components of* $\operatorname{int} M$ *are simply connected.*
(iv) *The set of normality of the sequence $\{\phi_n\}$ coincides with $\mathbb{C} \smallsetminus \partial M$ (compare with Proposition 3.5).*

One can see a similarity between the Mandelbrot set (representing the whole quadratic family) and a fillied Julia set of a particular quadratic map. It is just the first indication of a deep relation between dynamical and parameter objects.

Note that Proposition 3.4 describes the real slice of the Mandelbrot set:

$$M \cap \mathbb{R} = [-2, 1/4].$$

What immediately catches the eye in the Mandelbrot set is the *main cardioid* C with a cusp at $c = 1/4$. The cardioid bounds a domain of parameter values c such that f_c has an attracting fixed point.

EXERCISE 5.2. *Show that the main cardioid is given by the equation*

$$c = \frac{1}{2}e^{2\pi i\theta} - \frac{1}{4}e^{4\pi i\theta}, \quad 0 \le \theta < 2\pi,$$

where $\lambda = e^{2\pi i\theta}$ *is the multiplier of the neutral fixed point of* f_c.

Let us now take a look at how periodic points move with parameter:

LEMMA 5.1. *Let* f_{c_0} *has a cycle* $\{\alpha_k\}_{k=0}^{p-1}$ *of period* p *with multiplier* $\lambda_0 \ne 1$. *Then for nearby* c, *the maps* f_c *have a cycle* $\{\alpha_k(c)\}_{k=0}^{p-1}$ *holomorphically depending on* c. *Its multiplier* $\lambda(c)$ *holomorphically depends on* c *as well.*

PROOF. Consider an algebraic equation $f_c^p(z) = z$. For $c = c_0$ it has roots $z = \alpha_k$, $k = 0, \dots, p-1$ (and maybe others). Since

$$\left.\frac{\partial(f_c^p(z) - z)}{dz}\right|_{c=c_0,\, z=\alpha_k} = \lambda_0 - 1 \ne 0,$$

the Implicit Function Theorem yields the first assertion. The second assertion follows from the formula for the multiplier:

$$\lambda(c) = 2^p \prod_{k=0}^{p-1} \alpha_k(c).$$

□

Thus periodic points of f_c as functions of the parameter are algebraic functions branched at parabolic points only.

A parameter value $c \in \mathbb{C}$ is called *hyperbolic/parabolic/Siegel* etc. if the corresponding quadratic polynomial f_c is such.

PROPOSITION 5.2 (Hyperbolic components). *The set* $\mathcal{H}$ *of hyperbolic parameter values is contained in* $\operatorname{int} M$. *If* H *is a component of* $\operatorname{int} M$ *intersecting* $\mathcal{H}$ *then* $H \subset \mathcal{H}$.

PROOF. Lemma 5.1 implies that the set of hyperbolic parameter values is open. Since parameters in $\mathbb{C} \smallsetminus M$ are not hyperbolic (according
!!! to our terminology: see §**??**), the boundary parameter values $c \in \partial M$ cannot be hyperbolic either. Thus $\mathcal{H} \subset \operatorname{int} M$.

Take some some hyperbolic parameter value $c_0 \in H_0$, and let $f_0 \equiv f_{c_0}$. This map has an attracting cycle of some period p. By Theorem 3.6, this cycle contains a point α_0 such that

$$\phi_{pn}(c_0) \equiv f_0^{pn}(0) \to \alpha_0 \text{ as } n \to \infty.$$

It is easy to see (Exercise!) that for nearby $c \in H$ we have:

$$\phi_{pn}(c) \equiv f_0^{pn}(0) \to \alpha_0(c) \text{ as } n \to \infty,$$

where $\alpha_0(c)$ is the holomorphically moving attracting periodic point of f_c (Lemma **??**). But the sequence of polynomials $\phi_{pn}(c)$, $n = 0, 1, \ldots,$ is normal in H (Exercise 5.1, (iv)). Hence it must converge in the *whole domain* H to some holomorphic function $\tilde{\alpha}(c)$ coinciding with $\alpha_0(c)$ near c_0. By analytic continuation, $\tilde{\alpha}(c)$ is a a periodic point of f_c with period dividing p.

Moreover, the cycle of this point attracts the critical orbit persistently in H. It is impossible if this cycle is repelling somewhere. Indeed, a repelling cycles can only attract an orbit which eventually lands at it. This property is not locally persistent since otherwise it would hold for *all* $c \in \mathbb{C}$ (while it is violated, say, for $c = 1$).

If $\tilde{\alpha}(c)$ were parabolic for some $c \in H$, then it could be made repelling for a nearby parameter value. Thus $\tilde{\alpha}(c)$ is attracting for all $c \in H$, so that $H \subset \mathcal{H}$. □

A component H of $\operatorname{int} M$ is called *hyperbolic* if it consists of hyperbolic parameter value. Otherwise H is called *queer*. The reason for the last term is that it is generally believed that there are no queer components. In fact, it is a central conjecture in contemporary holomorphic dynamics:

Conjecture 5.3. *The interior of the Mandelbrot set consists of hyperbolic parameter values.*

21. Connectivity of M

21.1. Uniformization of $\mathbb{C} \smallsetminus M$. In this section we will prove the first deep result about the Mandelbrot set established by Douady and Hubbard in early 1980's. The strategy of the proof is quite remarkable: it is based on the explicit uniformization of the complement $\mathbb{C} \smallsetminus M$ by $\mathbb{C} \smallsetminus \mathbb{D}$. Recall from §16.5.2 that for $c \in \mathbb{C} \smallsetminus M$, we have a well-defined function

$$a = \Phi_M(c) = \phi_c(c), \tag{21.1}$$

where ϕ_c is the Böttcher function for f_c extended to the complement of the figure eight equipotential centered at 0. The point a is called the Böttcher position of the critical value of f_c.

THEOREM 5.4. *The Mandelbrot set M is connected. The function Φ_M conformally maps $\mathbb{C} \setminus M$ onto $\mathbb{C} \setminus \mathbb{D}$. Moreover, it is tangent to the identity at ∞: $\Phi_M(z) \sim z$ as $z \to \infty$.*

The proof given below is not the shortest one but it gives a bright illustration of the ideas of qc deformations which can be applied to variety of situations. A shorter path will be outlined in the Exercise 21.6.

21.2. Qc deformation. The idea is to deform the map by moving around the Böttcher position of its critical value. To this end let us consider a two parameter family of diffeomorphisms $\psi_{\omega,q} : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus \mathbb{D}$ written in the polar coordinates as follows:

$$\psi = \psi_{\omega,q}(r,\theta) = (r^\omega, \theta + q \log r), \quad \omega > 0,\ q \in \mathbb{R}.$$

In terms of complex variabe $a = re^{i\theta} \in \mathbb{C} \setminus \mathbb{D}$ and complex parameter $\lambda = \omega + iq$, $\Re\lambda > 0$, this family can be expressed in the following concise form:

$$\psi_\lambda(a) = |a|^{\lambda-1} a. \tag{21.2}$$

This family commutes with $f_0 : a \mapsto a^2$: $\psi(a^2) = \psi(a)^2$, and acts transitively on $\mathbb{C} \setminus \mathbb{D}$, i.e., for any $a_\star$ and a in $\mathbb{C} \setminus \mathbb{D}$, there exists a λ, such that $\psi_\lambda(a_\star) = a$. (Note also that ψ_λ are automorphisms of $\mathbb{C} \setminus \mathbb{D}$ viewed as a multiplicative semigroup.)

Take now a quadratic polynomial $f_\star \equiv f_{c_\star}$ with $c_\star \in \mathbb{C} \setminus M$. Let us consider its Böttcher function $\phi_\star : \Omega_\star \to \mathbb{C} \setminus \mathbb{D}_\star$, where $\Omega_* \equiv \Omega_{c_\star}$ is the complement of the figure eight equipotenial (see §16.5.2) and $\mathbb{D}_\star \equiv \mathbb{D}_{R_\star}$ is the corresponding round disk, $R_\star > 1$. Take the standard conformal structure σ on $\mathbb{C} \setminus \mathbb{D}$ and pull it back by the composition $\psi_\lambda \circ \phi_\star$. We obtain a conformal structure $\mu = \mu_\lambda$ in $\Omega_\star$. Since ψ_λ commute with f_0 while the Böttcher function conjugates $f_\star$ to f_0, the structure μ is invariant under $f_\star$.

Let us pull this structure back to the preimages of $\Omega_\star$:

$$\mu^n \mid \Omega^n = (f_\star^n)^*(\mu),$$

where $\Omega_\star^n = f_\star^{-n}\Omega_\star$. Since μ is invariant on $\Omega_\star$, the structures μ^{n+1} and μ^n coincide on $\Omega_\star^n$, so that they are organized in a single conformal structure on $\cup\Omega_\star^n = \mathbb{C} \setminus J(f_\star)$. Extend it to the Julia set $J(f_\star)$ as the standard conformal structure.

We will keep notation $\mu \equiv \mu_\lambda$ for the conformal structure on $\mathbb{C}$ we have just constructed. By construction, it is invariant under $f_\star$. Moreover, it has a bounded dilatation since holomorphic pullbacks preserve dilatation: $\|\mu_\lambda\|_\infty = \|(\psi_\lambda)^*(\sigma)\|_\infty < 1$.

By the Measurable Riemann Mapping Theorem, there is a qc map $h_\lambda : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $(h_\lambda)_\star(\mu_\lambda) = \sigma$. By Corollary **??**, $h_l a$ can be normalized so that it conjugates f_λ to a quadratic map $f_c \equiv f_{c(\lambda)} : z \mapsto z^2 + c(\lambda)$. Of course, the Julia set f_c is also Cantor, so that $c \in \mathbb{C} \smallsetminus M$.

This family of quadratic polynomials is the desired qc deformation of $f_\star$.

21.3. Analyticity. We have to check three propertices of the map $\Phi_M : \mathbb{C} \smallsetminus M \to \mathbb{C} \smallsetminus \mathbb{D}$: analyticity, surjectivity, and injectivity. Let us take them one by one.

It is obvious from formula (21.2) that the Beltrami differential

$$\nu_\lambda = (\psi_\lambda)^*(\sigma) = \bar{\partial}\psi_\lambda / \partial\psi_\lambda$$

depends holomorphically on λ. Hence the Beltrami differential $(f_\star)^*(\nu_\lambda)$ on $\Omega_\star$ also depends holomorphically on λ (see Exercise 2.9). Pulling it back by the iterates of $f_\star$ and extending it in the standard way to $J(f)$, we obtain by Lemma 2.16 a holomorphic family of Beltrami differentials μ_λ on $\mathbb{C}$. By Corollary 3.11, $c(\lambda)$ is holomorphic on λ as well.

21.4. Surjectivity. Note that the map $\psi_\lambda \circ \phi_\star \circ h_\lambda^{-1}$ conformally conjugates the polynomial $f_c \equiv f_{c(\lambda)}$ near ∞ to $f_0 : z \mapsto z^2$. By Theorem 3.14, these properties determine uniquely the Böttcher map ϕ_c of f_c, so that $\phi_c = \psi_\lambda \circ \phi_\star \circ h_\lambda^{-1}$ with $c = c(\lambda)$. Since h_λ conjugates $f_\star$ to f_c, we have: $h_\lambda(c_*) = c$ and hence

$$\Phi_M(c) = \phi_c(c) = \psi_\lambda \circ \phi_\star(c_\star) = \psi_\lambda(a_\star),$$

where $a_\star$ is the Böttcher position of the critical value of $f_\star$. Since the family $\{\psi_\lambda\}$ acts transitively on $\mathbb{C} \smallsetminus \mathbb{D}$, any point $a \in \mathbb{C} \smallsetminus \mathbb{D}$ can be relasized as $\Phi_M(c)$ for some $c = c(\lambda)$.

21.5. Injectivity. We have to check that if

$$\phi_c(c) = a = \phi_{\tilde{c}}(\tilde{c}) \tag{21.3}$$

for two parameter valus c and $\tilde{c}$ in $\mathbb{C} \smallsetminus M$, then $c = \tilde{c}$. We let $f \equiv f_c$, $\phi \equiv \phi_c$, $\tilde{f} \equiv f_{\tilde{c}}$, $\tilde{\phi} \equiv \phi_{\tilde{c}}$. Similarly, we will mark with "tilde" the dynamical objects associated with $\tilde{f}$ that naturally correspond to dynamical objects associated with f.

Let $R = \sqrt{|a|}$. Then the maps ϕ^{-1} and $\tilde{\phi}^{-1}$ map $\mathbb{C} \smallsetminus \bar{\mathbb{D}}_R$ onto the domains $\Omega \equiv \Omega_c$ and $\tilde{\Omega} \equiv \Omega_{\tilde{c}}$ respectively. Moreover, they extend continuously to the boundary circle mapping it onto the boundary figures eight $\Gamma = \partial\Omega$ and $\tilde{\Gamma} = \partial\tilde{\Omega}$, and this extension if one-to-one except that

$$\phi^{-1}(\pm\sqrt{a}) = 0 = \tilde{\phi}^{-1}(\pm\sqrt{a}).$$

Hence the conformal map $h = \tilde{\phi}^{-1} \circ \phi : \Omega \to \tilde{\Omega}$ admits a homeomorphic extension to the closure of its domain:

$$h : (\mathrm{cl}(\Omega), 0) \to (\mathrm{cl}(\tilde{\Omega}), 0).$$

Consider a domain $\Omega^0 = f(\Omega)$ (exterior of the equipotential passing through c) and the complementary Jordan disk $\Delta^0 = \mathbb{C} \smallsetminus \Omega^0$. We will describe a hierarchical decomposition of Δ^0 into topological annuli A_i^n, $n = 1, \dots,\ i = 1, 2, \dots, 2^n$. Let $\Omega^n = f^{-n}\Omega^0$ (so that $\Omega \equiv \Omega^1$). The boundary $\partial\Omega^n$ consists of 2^{n-1} disjoint figures eight. The loops of these figures eight bound 2^n (closed) Jordan disks Δ_i^n. The map f conformally maps Δ_i^n onto some Δ_j^{n-1}, $n \geq 1$. Let $A_i^n = \Delta_i^n \cap \mathrm{cl}(\Omega^{n+1})$. These are closed topological annuli each of which is bounded by a Jordan curve and a figure eight. They tile $\Delta^0 \smallsetminus J(f)$. The map f conformally maps A_i^n onto some A_j^{n-1}, $n \geq 1$.

Let us lift $h \equiv h_1$ to conformal maps $H_i : A_i^1 \to \tilde{A}_i^1$:

$$H_i \mid A_i^1 = (\tilde{f}|\tilde{A}_i^1)^{-1} \circ h \circ (f|A_i^1). \tag{21.4}$$

Since h is equivariant on the boundary of $\Omega^1 \smallsetminus \Omega^0$, it matches with the H_i on $\partial\Delta_i^1$. Putting these maps together, we obtain an equivariant homeomorphism $h_2 : \mathrm{cl}(\Omega^2) \to \mathrm{cl}(\tilde{\Omega}^2)$ conformal in the complement of the figure eight Γ:

$$h_2(z) = \begin{cases} h(z), & z \in \Omega^1, \\ H_i(z), & z \in A_i^1. \end{cases}$$

Since smooth curves are removable (recall §11), h_2 is conformal in $\Omega^2 \smallsetminus \{0\}$. Since isolated points are removable, h_2 is conformal in Ω^2. Thus h admits an equivariant conformal extension to Ω^2.

In the same way, h_2 can be lifted to four annuli A_i^2. This gives an equivariant conformal extension of h to Ω^3. Proceeding in this way, we will consecutively obtain an equivariant conformal extension of h to all the domains Ω^n and hence to their union $\cup\Omega^n = \mathbb{C} \smallsetminus J(f)$.

Since the Julia set $J(f)$ is removable (Theorem 2.27), this map admits a conformal extension through $J(f)$. Thus, f and $\tilde{f}$ are conformally equivalent, and hence $c = \tilde{c}$.

Theorem 5.4 is now proven.

21.6. A more elementary proof. We will now outline in a series of exercises a more elementary proof of Theorem 5.4 (which was the original proof given by Douady and Hubbard). It is based upon the explicit formula for the B"ottcher coordinate near ∞ (compare §**??**):

$$\phi_c(z) = \lim_{n\to\infty} (f_c^n(z))^{1/2^n}, \tag{21.5}$$

where the root in the right-hand side is selected in such a way that it is tangent to the identity at ∞.

Consider the set $\Omega = \{(c, z) \in \mathbb{C}^2 : z \in \Omega_c\}$, where we let $\Omega_c = \mathbb{C} \smallsetminus K(f_c)$ for $c \in M$.

EXERCISE 5.3. *Show that Ω is open. Prove that the series (21.5) converges locally uniformly on Ω. Conclude that the Böttcher function $(c, z) \mapsto \phi_c(z)$ is holomorphic on Ω, and the function $\Phi_M(c) = \phi_c(c)$ is holomorphic on $\mathbb{C} \smallsetminus M$.*

EXERCISE 5.4. *Prove that the map $\Phi_M : \mathbb{C}\smallsetminus M \to \mathbb{C}\smallsetminus\mathbb{D}$ is proper:*

$$|\Phi_M(c)| \to 1 \text{ as } c \to \partial M.$$

Hence RM has a well-defined degree.

EXERCISE 5.5. *Prove that* $\deg \Phi_M = 1$. *Hence Φ_M is a conformal isomorphism between $\mathbb{C} \smallsetminus M$ and $\mathbb{C} \smallsetminus \mathbb{D}$.*

22. The Multiplier Theorem

22.1. Statement. Let us pick a favorite hyperbolic component H of the Mandelbrot set M. For $c \in H$, the polynomial f_c has a unique attracting cycle $\boldsymbol{\alpha}_c = \{\alpha_k(c)\}_{k=0}^{p-1}$ of period p. By Lemma 5.1, the multiplier $\lambda(c)$ of this cycle holomorphically depends on c, so that we obtain a holomorphic map $\lambda : H \to \mathbb{D}$. It is remarkable that this map gives an explicit uniformization of H by the unit disk:

THEOREM 5.5. *The multiplier map $\lambda : H \to \mathbb{D}$ is a conformal isomorphism.*

This theorem is in many respects analogous to Theorem 5.4 on connectivity of the Mandelbrot set. The latter gives an explicit dynamical uniformization of $\mathbb{C} \smallsetminus M$; the former gives the one for the hyperbolic component. The ideas of the proofs are also similar.

We already know that λ is holomorphic, so we need to verify that it is surjective and injective. The first statement is easy:

EXERCISE 5.6. *The multiplier map $\lambda : H \to \mathbb{D}$ is proper and hence surjective. In particular, H contains a superattracting parameter value.*

22.2. Qc deformation. Let $Z \subset H$ be the set of superattracting parameter values in H. Take some point $c_0 \in H \smallsetminus Z$, and let $\lambda_0 \in \mathbb{D}^*$ be the multiplier of the corresponding attracting cycle. We will produce a qc deformation of $f_0 \equiv f_{c_0}$ by deforming the associated fundamental torus.

22.2.1. *Fundamental torus.* Take a little topological disk $D = \mathbb{D}(a_0, \epsilon)$ around the attracting periodic point a_0 of f_0. It is invariant under $g_0 \equiv f_0^p$ and the quotient of D under the action of f_0 is a conformal torus T_0. Its fundamental group has one marked generater corresponding to a little Jordan curve around α_0.

By the Linearization Theorem (3.13), the action of g_0 on D is conformally equivalent to the linear action of $\zeta \mapsto \lambda_0 \zeta$ on $\mathbb{D}^*$. Hence the partially marked torus T_0 is conformally equivalent to $\mathbb{T}^2_{\lambda_0}$, so that λ_0 is the modulus of T_0 (see §1.4.2).

Let us select a family of deformations $\psi_\lambda : \mathbb{T}^2_{\lambda_0} \to \mathbb{T}^2_\lambda$ of T_{λ_0} to nearby tori. For instance, ψ_λ can be chosen to be linear in the logarithmic coordinates $(x, y) = \log \zeta$, $\tau = \log \lambda$:

$$x + y\tau_0 \mapsto x + y\tau; \quad x \in \mathbb{R},\ y \geq 0.$$

This gives us a complex one-parameter family of Beltrami differentials $\nu_\lambda = \psi_\lambda^*(\sigma)$ on $T_0 \approx \mathbb{T}^2_{\lambda_0}$ (in what follows we identify T_0 with $\mathbb{T}^2_{\lambda_0}$).

EXERCISE 5.7. *Calculate ν_λ explicitly (for the linear deformation).*

22.2.2. *Qc deformation of f_0.* We can lift ν_λ to the disk D and then pull it back by iterates of f_0. This gives us a family of f_0-invariant Beltrami differentials μ_λ on the attracting basin of $\boldsymbol{\alpha}$. These Beltrami differentials have a bounded dilatation since the pull-backs under holomorphic maps preserve dilatation. Extend the μ_λ by 0 outside the attracting basin (keeping the notation). We obtain a family of measurable f_0-invariant conformal structures μ_λ on the Riemann sphere. Solving the Beltrami equation $(h_\lambda)_*(\mu_\lambda) = \sigma$ (with an appropriately normalization) we obtain a qc deformation of f_0 (see Corollary 3.12):

$$f_{c(\lambda)} = h_\lambda \circ f_0 \circ h_\lambda^{-1} : \ z \mapsto z^2 + c(\lambda). \tag{22.1}$$

Moreover, note that this deformation is conformal on the basin of ∞.

Let us show that the multiplier of the attracting fixed point of $f_{c(\lambda)}$ is equal to λ. Consider the torus T_λ associated with the attracting cycle of $f_{c(\lambda)}$. Then h_λ descends to a homeomorphism $H_\lambda : T_0 \to T_\lambda$ such that $(H_\lambda)_*(\nu_\lambda) = \sigma$. Since $(\psi_\lambda)_*(\nu) = \sigma$ as well, the map

$$\psi_\lambda \circ H_\lambda^{-1} : T_\lambda \to \mathbb{T}^2_\lambda$$

is conformal. Hence the partially marked torus T_λ has the same modulus as $\mathbb{T}^2_\lambda$, which is λ. But as we know, this modulus is equal to the multiplier of the corresponding attracting cycle.

This deformation immediately leads to the following important conclusion:

LEMMA 5.6. *All maps f_c, $c \in H \smallsetminus Z$, are qc equivalent (and the conjugacy is conformal on the basin of ∞). Moreover,* $\operatorname{card} Z = 1$.

PROOF. Take some $c_0 \in H \smallsetminus Z$. By Proposition 2.12, the deformation parameter $c(\lambda)$ in (22.1) depends continuously on λ. Hence $c : \lambda \mapsto c(\lambda)$ is the local right inverse to the multiplier function. But holomorphic functions do not have continuous right inverses near their critical points. Consequently, c_0 is not a critical point of the multiplier function λ and, moreover, c is the local inverse to λ. It follows that any c near c_0 can be represented as $c(\lambda)$, and hence f_c is qc equivalent to f_{c_0}.

Let us decompose the domain $H \smallsetminus Z$ into the union of disjoint qc classes (with conformal conjugacy on the basin of ∞). We have just shown that each qc class in this decomposition is open. Since $H \smallsetminus Z$ is connected, it consists of a single qc class.

Furthermore, we have shown that λ does not have critical points in $H \smallsetminus Z$. Hence $\lambda : H \smallsetminus Z \to \mathbb{D}^*$ is an unbranched covering. By the Riemann-Hurwitz formula (for the trivial case of unbranched coverings), the Euler characteristic of $H \smallsetminus Z$ is equal to 0, i.e., $1 - \operatorname{card} Z = 0$. □

Thus, every hyperbolic component H contains a unique superattracting parameter value c_H. It is called the *center* of H.

22.3. Injectivity. The following lemma will complete the proof of the Multiplier Theorem:

LEMMA 5.7. *Consider two parameter values c and $\tilde{c}$ in $H \smallsetminus Z$. If $\lambda(c) = \lambda(\tilde{c})$ then the quadratic maps f_c and $f_{\tilde{c}}$ are conformally equivalent on $\mathbb{C}$.*

The idea is to turn the qc conjugacy from Lemma 5.6 into a conformal conjugacy. To this end we need to modify the conjugacy on the basin of the attracting cycle. Let us start with the component D_0 of the basin containing 0.

CHAPTER 6

Little Mandelbrot copies

1. Quadratic-like families

1.1. Definitions. Let $\Lambda \subset \mathbb{C}$ be a domain in the complex plane. A *quadratic-like family* $\mathbf{g}$ *over* Λ is a family of quadratic-like maps $g_\lambda : U_\lambda \to U'_\lambda$ depending on $\lambda \in \Lambda$ such that:

- The *tube* $\mathbb{U} = \{(\lambda, z) : \ \lambda \in \Lambda, \ z \in U_\lambda\}$ is a domain in $\mathbb{C}^2$;
- $g_\lambda(z)$ is holomorphic in two variables on $\mathbb{U}$.

As usual, we assume that the critical value of each f_λ is located at the origin 0.

We will now formulate several additional assumptions which will make a quadratic-like family nice. First of them is minor. We say that $\mathbf{g}$ *extends beyond* $\mathbb{U}$ if there exists a domain $\Lambda' \Supset \Lambda$ and a quadratic-like family $G_\lambda : V_\lambda \to V'_\lambda$ over Λ' such that for $\lambda \in \Lambda$, g_λ is an adjustment (see §17) of G_λ.

We call a quadratic-like family $\mathbf{g} : U_\lambda \to U'_\lambda$ over Λ *proper* if

- The domains Λ, U_λ and U'_λ are bounded by smooth Jordan curves;
- $\mathbf{g}$ admits an extension beyond $\mathbb{U}$;
- For $\lambda \in \partial\Lambda$, $g_\lambda(0) \in \partial U'_\lambda$.

(The first two assumptions are imposed only for the sake of convenience.) Obviously $g_\lambda(0) \neq 0$ for $\lambda \in \partial\Lambda$, so that we have a well defined winding number of the curve $\lambda \mapsto g_\lambda(0)$, $\lambda \in \partial\Lambda$, around 0. We call it the winding number of $\mathbf{g}$ and denote $w(\mathbf{g})$. A proper family $\mathbf{g}$ is called *unfolded* if $w(\mathbf{g}) = 1$. By the Argument Principle, any proper unfolded quadratic-like family has a unique parameter value $*$ such that f_* has a superattracting fixed point, i.e., $f_*(0) = 0$. We will select $*$ as the *base point* in Λ.

Finally, we want the fundamental annulus $A_\lambda = U'_\lambda \smallsetminus \bar{U}_\lambda$ of g_λ to move holomorphically with λ. So, assume that there is a holomorphic motion $h_\lambda : \bar{A}_* \to \bar{A}_\lambda$ respecting the boundary dynamical relation, i.e., such that

$$h_\lambda(g_* z) = g_\lambda(h_\lambda(z)) \quad \text{for} \quad z \in \partial U_*.$$

For a technical reason, we impose the following boundary assumption on this motion:

Boundary extension. Let $\lambda \in \partial\Lambda$. The homeomorphisms $h_\mu : \bar{U}'_* \smallsetminus \bar{U}_* \to \bar{U}'_\mu \smallsetminus \bar{U}_\mu$, $\mu \in \Lambda$, uniformly converge as $\mu \to \lambda$ to a continuous map $h_\lambda : \bar{U}'_* \smallsetminus U_* \to \bar{U}'_\lambda \smallsetminus U_\lambda$, which is one-to-one everywhere, except that $h_\lambda^{-1}(0)$ consists of two points on ∂U_*. (Note that ∂U_λ is a "figure eight" curve for $\lambda \in \partial\Lambda$.)

Denote this holomorphic motion by $\mathbf{h}$. We say that the quadratic-like family $\mathbf{g}$ is *equipped* with the holomorphic motion $\mathbf{h}$. Sometimes we will use notation $(\mathbf{g}, \mathbf{h})$ for an equipped quadratic-like family.

For equipped families, there is a natural choice of tubing (see §17.4.1) holomorphically depending on λ. Namely, select any tubing $t_* : \bar{A}_* \to \bar{\mathbb{A}}[r, r^2]$ for the base point, and then let

$$t_\lambda = t_* \circ h_\lambda^{-1}. \tag{1.1}$$

These are tubings since the holomorphic motion h_λ respects the boundary dynamical relations.

The Mandelbrot set $M(\mathbf{g})$ of the quadratic-like family is defined as $\{\lambda \in \Lambda : \ J(g_\lambda) \text{ is connected}\}$. If $\mathbf{g}$ is proper, then $M(\mathbf{g})$ is compactly contained in Λ.

Let us finish with a few terminological and notational remarks. Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ stand for the projection onto the first coordinate. We call a set $\mathbb{U} \subset \mathbb{C}^2$ a *tube over* $\Lambda = \pi(\mathbb{U}) \subset \mathbb{C}$ if it is a fiber bundle over Λ whose *fibers* $U_\lambda = \mathbb{U} \cap \pi^{-1}\lambda$ are Jordan disks (either open or closed). For $X \subset \Lambda$, we let $\mathbb{U}|X = \mathbb{U} \cap \pi^{-1}X$.

1.2. Restricted quadratic family. In this section we will show that the quadratic family $\{f_c\}_{c\in\mathbb{C}}$ can be naturally restricted to a proper unfolded equipped quadratic-like family.

Fix some $r > 1$. Restrict the parameter domain $\mathbb{C}$ to the topological disk $D \equiv D_{r^2}$ bounded by the parameter equipotential of radius r^2. According to formula (**??**), this parameter domain is specified by the property that $f_c(0) \in \Omega_c(r^2) \equiv \Omega'_c$, where $\Omega_c(\rho)$ is the domain bounded by the dynamical equipotential of level ρ. Hence for $c \in D$, f_c restricts to a quadratic-like map $f_c : \tilde{\Omega}_c \to \tilde{\Omega}'_c$, where $\tilde{\Omega}_c \equiv \Omega_c(r)$. These quadratic-like maps obviously form a quadratic-like family over D, which we will call a *restricted quadratic family.*

The restricted quadratic family is proper. The first two properties of the definition are obvious. The main property, $f_c(0) \in \partial\tilde{\Omega}'_c$ for $c \in \partial D$, follows from formula (**??**). The winding number of this family is equal to 1. Indeed, when the *parameter* c runs once along the boundary ∂D, the *critical value* $c = f_c(0)$ runs once around $0 \in D$.

The restricted quadratic family is equipped with the holomorphic motion of the fundamental annulus given by the Böttcher maps. Select 0 is the base point in D and let

$$B_c^{-1} : \mathbb{A}[r, r^2] \to \bar{\Omega}_c' \smallsetminus \Omega_c \tag{1.2}$$

(note that $\mathbb{A}[r, r^2] = \Omega_0' \smallsetminus \Omega_0$). Since the Böttcher function $B_c^{-1}(z)$ is holomorphic it two variables (**??**), $\{B_c^{-1}\}_{c \in D}$ is a holomorphic motion. This motion admits the boundary extension (see the previous section), since for $c \in \partial D$, B_c^{-1} homeomorphically maps $\mathbb{C} \smallsetminus \mathbb{D}_r$ onto $\mathbb{C} \smallsetminus \Omega_c(r)$ except that two points on $\mathbb{T}_r$ collapse to 0 (see §**??**).

Thus the restricted quadratic family satisfy all the properties formulated in the previous section.

1.3. Straightening of quadratic-like families. The Mandelbrot set $M(\mathbf{g})$ of any quadratic-like family $\mathbf{g}$ can be canonically mapped into the genuine Mandelbrot set M. Namely, by the Straightening Theorem, for any $\lambda \in M(\mathbf{g})$ there is a unique quadratic polynomial $f_{c(\lambda)} : z \mapsto z^2 + c(\lambda)$, $c(\lambda) \in M$, which is hybrid equivalent to g_λ. The map $\chi : \lambda \mapsto c(\lambda)$ is called the *straightening* of $M(\mathbf{g})$.

We know that the straightening is not canonically defined outsed the Mandelbrot set but rather depends on the choice of the tubing. But for equipped families there is a natural choice given by (1.1). With this choice, the straightening χ admits an extension to the whole parameter domain Λ, which well be still denoted by χ.

Recall that D_r stands for the parameter disk bounded by the parameter equipotential of radius r (in the quadratic family). We can now formulate a fundamental result of the theory of quadratic-like families:

THEOREM 6.1. *Let* $\mathbf{g}$ *be a proper unfolded equipped quadratic-like family over* Λ. *Endow it with a holomorphic tubing given by (1.1). Then the corresponding straightening* χ *is a homeomorphism from* Λ *onto* D_{r^2}.

The proof of this theorem will be split into several pieces which are important on their own right.

1.4. The critical value moves transversally to h. We say that a holomorphic curve $\Gamma \subset \mathbb{C}^2$ is a *global transversal* to a holomorphic motion $\mathbf{h}$ if it transversally intersects each leaf of $\mathbf{h}$ at a single point.

LEMMA 6.2. *Under the assumptions of Theorem 6.1, the graph of the function* $\lambda \mapsto g_\lambda(0)$, $\lambda \in \Lambda$, *is a global transversal to the holomorphic motion* $\mathbf{h}$ *on* $\mathbb{U}' \smallsetminus \mathbb{U}$.

We will also express it by saying that *the critical value moves transversally to* **h**. The moral of this lemma is that *in the complex setting the transversality can come for purely topological reasons.*

Proof. Take a point $z \in U'_* \setminus U_*$ and consider its leaf

$$L_z = \{(\lambda, \zeta) \in \mathbb{C}^2 : \lambda \in \Lambda, \ \zeta = h_\lambda(z)\}.$$

Since the motion **h** admits a continuous extension to the boundary $\partial\Lambda$, the function $\psi : \lambda \mapsto h_\lambda(z)$ is continuous up to the boundary and $\psi(\lambda) \in U'_\lambda \setminus U_\lambda$, $\lambda \in \partial\Lambda$. Since the tube $\mathbb{V} \equiv \mathbb{U}|\partial\Lambda$ is homeomorphic to the solid torus $\partial\Lambda \times \mathbb{D}$ over $\partial\Lambda$, the curve $\lambda \mapsto \psi(\lambda)$, $\lambda \in \partial\Lambda$, is homotopic to the zero curve $\lambda \mapsto 0$ in $\mathbb{V}$, i.e., these two curves can be joined by a continuous family of curves $\psi_t : \partial\Lambda \to \mathbb{V}$, $0 \le t \le 1$.

Consider now the curve $\phi : \lambda \mapsto f_\lambda(0)$, $\lambda \in \partial\Lambda$. Since **f** is proper, $\phi(\lambda) \in \partial\mathbb{V}$. Hence $\phi(\lambda) - \psi_t(\lambda) \neq 0$ for $\lambda \in \partial\Lambda$. It follows that the curves $\lambda \mapsto \phi(\lambda) - \psi(\lambda)$ and $\lambda \mapsto \phi(\lambda)$, $\lambda \in \partial\Lambda$, have the same winding number around 0. But the latter number is equal to 1, since **f** is unfolded. Hence the former number is also equal to 1. By the classical Argument Principle, the graphs of the functions ϕ and ψ have a single transverse intersection, and that is what we need. ☐

1.5. Uniformization of the complement of $M(\mathbf{g})$. In this section we will construct a dynamical (non-conformal) uniformization of $\Lambda \setminus M(\mathbf{g})$ which generalizes the uniformization of $\mathbb{C} \setminus M$ constructed in §**??**. This construction will illustrate how to relate the parameter and dynamical planes by means of holomorphic moions.

Let us consider a set $P = \{\lambda \in \Lambda : \ g_\lambda(0) \in U'_\lambda \setminus U_\lambda\}$ (i.e., the set of parameters for which the critical point escapes under the first iterate through the fundamental annulus $A_\lambda = U'_\lambda \setminus U_\lambda$). Note that all points in Λ sufficiently close to $\partial\Lambda$ obviously belong to P. We will show that P is an annulus naturally homeomorphic to the dynamical annulus $A_* = U'_* \setminus U_*$.

To this end consider the graph of the function $\phi : \lambda \mapsto g_\lambda(0)$,

$$\Gamma = \{(\lambda, z) \in \mathbb{C}^2 : \ \lambda \in \Lambda, \ z = g_\lambda(0)\}.$$

By Lemma 6.2, this graph is a global transversal to the holomorphic motion **h**. Hence there is a well defined holonomy $\gamma : A_* \to \Gamma$ along the leaves of **f** , and it maps A_* homeomorphically onto a topological annulus $B \subset \Gamma$. Obviously, $\pi(B) = P$. Altogether, we have a homeomorphism $\pi \circ \gamma$ from A_* onto P. It follows, in particular that P is a topological annulus, whose inner boundary is a Jordan curve in Λ and the outer boundary is $\partial\Lambda$.

Let us consider the domain $\Lambda' = \Lambda \setminus P$. The restriction of our quadratic-like family to this parameter domain is not proper any more. To restore this property, we have to restrict the dynamical domains as well. Let $V_\lambda = g_\lambda^{-1} U_\lambda$. For any $\lambda \in \Lambda'$, $g_\lambda(0) \in U_\lambda$; hence V_λ is a topological disk and $g_\lambda : V_\lambda \to U_\lambda$ is a quadratic-like map. This gives us a quadratic-like family over Λ'.

It is proper since by construction $g_\lambda(0) \in U_\lambda$ for $\lambda \in \partial\Lambda'$ (other technical properties required in the definition are even more obvious). It has winding number one since the function $\phi : \lambda \mapsto g_\lambda(0)$ does not have zeros in the annulus $\bar{R}$. It follows that the boundary curves $\phi : \partial\Lambda \to \mathbb{C}^*$ and $\phi : \partial\Lambda' \to \mathbb{C}^*$ are homotopic and hence they have the same winding number around 0.

Let us now equip this family with a holomorphic motion $h'_\lambda : A'_* \to A'_\lambda$ of the fundamental annulus $A'_\lambda = U_\lambda \setminus V_\lambda$. This motion is obtained by lifting the motion h_λ by means of the double coverings $g_\lambda : A'_\lambda \to A_\lambda$,

$$\begin{array}{ccc} A'_* & \xrightarrow[h'_\lambda]{} & A'_\lambda \\ g_* \downarrow & & \downarrow g_\lambda \\ A_* & \xrightarrow[h_\lambda]{} & A_\lambda \end{array}$$

We need to check that this lift can be chosen holomorphic in λ. To this end take a point $z \in A_*$ and consider its orbit $\psi : \lambda \mapsto h_\lambda(z)$, $\lambda \in \Lambda'$. Take some $\zeta \in A'_*$ such that $g_*(\zeta) = z$. We want to find a holomorphic function $\psi' : \lambda \to h'_\lambda(\zeta)$ which makes the above diagram commutative, i.e., it should satisfy the equation:

$$g_\lambda(\psi'(\lambda)) = \psi(\lambda).$$

By the Implicit Function Theorem, this equation has a local holomorphic solution if $g'_\lambda(\zeta) \neq 0$, i.e., if ζ is not a critical point of g_λ. This condition is certainly satisfied in our situation.

By the λ-lemma, the original holomorphic motion $\mathbf{h}$ mathches with $\mathbf{h}'$ on the common boundary $\partial^i A_\lambda = \partial^o A'_\lambda$, so that together they provide a single holomorphic motion of the union $A_\lambda \cup A'_\lambda$ over Λ'.

Let $P' = \{\lambda \in \Lambda' : \; g_\lambda(0) \in A'_\lambda\}$. Applying the above result to the restricted quadratic-like family, we obtain a homeomorphism $\pi \circ \gamma' : A'_* \to P'$, where γ' is the holonomy along $\mathbf{h}'$. Since γ' matches with γ on the common boundary of the annuli, they give us a homeomorphism of the union of the dynamical annuli onto the union of parameter annuli, $A \cup A' \to P \cup P'$.

Proceeding in the same way, we will construct:

- A nest of parameter annuli P^n attached one to the next and the corresponding parameter domains $\Lambda^n = \Lambda^{n-1} \setminus P^{n-1}$ (where $\Lambda^0 \equiv \Lambda$, $P^0 \equiv P$, $\Lambda^1 \equiv \Lambda'$). Moreover, $\cup P^n = \Lambda \setminus M(\mathbf{g})$.
- A sequence of proper unfolded quadratic-like families

$$g_{n,\lambda} \equiv g_\lambda : V_\lambda^{n+1} \to V_\lambda^n \text{ over } \Lambda^n,$$

  where $V_\lambda^n = g_\lambda^{-n} U'_\lambda$ (thus $V_\lambda^0 \equiv U'_\lambda$, $V_\lambda^1 \equiv U_\lambda$ and $V_\lambda^2 \equiv V_\lambda$).
- A sequence of holomorphic motions $h_{n,\lambda}$ of the fundamental annulus $A_\lambda^n \equiv \bar{V}_\lambda^n \setminus V_\lambda^{n+1}$ over Λ^n which equip $g_{n,\lambda}$; moreover $h_{n+1,\lambda}$ is obtained by lifting $h_{n,\lambda}$ by means of the coverings $g_\lambda : A_\lambda^n \to A_\lambda^{n-1}$. These holomorphic motions match on the common boundaries of the fundamental annuli.

Let $\gamma_n : A_*^n \to \Gamma$ be the holonomy along $\mathbf{h}_n$ (recall that Γ is the graph of the function $\phi : \lambda \mapsto f_\lambda(0)$. Since the holomorphic motions match on the common boundaries, these holonomies also match, and determine a continuous injection $\gamma : U_* \setminus K(f_*) \to \Gamma$. Composing it with the projection π, we obtain a homeomorphism

$$\pi \circ \gamma : U_* \setminus K(f_*) \to \Lambda \setminus M(\mathbf{g})$$

between the dynamical and parameter annuli. Note that the inverse map is equal to $\gamma^{-1} \circ \phi$.

Composing the above homeomorphism with the tubing (1.1), we obtain a "uniformization" of $\Lambda \setminus M(\mathbf{g})$ by a round annulus:

$$S : t_* \circ (\pi \circ \gamma)^{-1} = t_\lambda \circ \phi : \; \Lambda \setminus M(\mathbf{g}) \to \mathbb{A}(1, r^2), \quad S(\lambda) = t_\lambda(g_\lambda(0)). \tag{1.3}$$

We call $S(\lambda)$ "the tubing position of the critical value of g_λ".

Remark. The above uniformization of $\Lambda \setminus M$ is generally not conformal. However, in the case of a restricted quadratic family, it is the restriction of the conformal uniformization of $\mathbb{C} \setminus M$. Indeed, in this case, the tubing t_λ turns into the Böttcher maps B_c (see (1.2)), the critical value $g_\lambda(0)$ turns into c, and formula (1.3) turns into formula (**??**) for the Riemann map $R : \mathbb{C} \setminus M \to \mathbb{C} \setminus \mathbb{D}$, $R(c) = B_c(c)$.

COROLLARY 6.3. *The Mandelbrot set $M(\mathbf{g})$ is connected and full.*

1.6. Adjustments of quadratic-like families.

Include the "maximal" extension of the leaves up to the critical value

1.7. Quasi-conformality of the uniformization. Given a holomorphic motion $\mathbf{h}$ over Λ, let

$$\operatorname{Dil}(\mathbf{h}) = \sup_{\lambda\in\Lambda} \operatorname{Dil}(h_\lambda)$$

(which can be infinite). We say that the holomorphic motion $\mathbf{h}$ is K-qc if $\operatorname{Dil}(\mathbf{h}) \le K$. In the following statement we will use the notations of §1.5.

LEMMA 6.4. *Under the assumptions of Theorem 6.1, assume that the tubing $t_* : A_* \to \mathbb{A}(r, r^2)$ and the holomorphic motion $\mathbf{h}$ are K-qc. Then the uniformization $S : \Lambda \smallsetminus M(\mathbf{g}) \to \mathbb{A}(1, r^2)$ is K-qc as well.*

In fact, we can make the dilatation depend only on $\operatorname{mod}(A_*)$ and $\operatorname{mod}(\Lambda \smallsetminus \Lambda')$, after an appropriate adjustment of the family $\mathbf{g}$.

LEMMA 6.5. *Let us consider a quadratic-like family $\mathbf{g}$ over Λ satisfying the assumptions of Theorem 6.1. This family can be adjusted to a family $\tilde{\mathbf{g}}$ over $\tilde\Lambda$ in such a way that the dilatation of the straightening $\tilde\chi : \tilde\Lambda \smallsetminus M(\tilde{\mathbf{g}}) \to D \smallsetminus M$ will depend only on* $\operatorname{mod}(A_*)$ *and* $\operatorname{mod}(\Lambda \smallsetminus \Lambda')$.

1.8. Looking from the outside. We are now ready to prove that the straightening is a homeomorphism outside the Mandelbrot sets.

LEMMA 6.6. *Under the assumptions of Theorem 6.1, the straightening $\chi : \Lambda \smallsetminus M(\mathbf{g}) \to D_{r^2} \smallsetminus M$ is a homeomorphism.*

PROOF. Let us consider the uniformizations $S : \Lambda \smallsetminus M(\mathbf{g}) \to \mathbb{A}(1, r^2)$ and $R : D \smallsetminus M \to \mathbb{A}(1, r^2)$ constructed above. Then

$$\chi = R^{-1} \circ S. \tag{1.4}$$

Indeed, let $\lambda \in \Lambda \smallsetminus M(\mathbf{g})$ and $c = \chi(\lambda) \in D \smallsetminus M$. Putting together (**??**) and (1.3), we obtain:

$$S(\lambda) = t_\lambda(g_\lambda(0)) = B_c(c) = R(c),$$

which is exactly (1.4). Since S and R are both homeomorphisms, χ is a homeomorphism as well. □

1.9. Miracle of continuity. We will now show that the straightening is continuous on the boundary of $M(\mathbf{g})$:

LEMMA 6.7. *Under the assumptions of Theorem 6.1, the straightening χ is continuous at any point $\lambda \in \partial M(\mathbf{g})$ and moreover $\chi(\lambda) \in \partial M$.*

PROOF. First we will show that $\chi|\,\partial M(\mathbf{g})$ is a continuous extension of $\chi|\,\Lambda \smallsetminus M(\mathbf{g})$. Let $\lambda_n \in \Lambda \smallsetminus M(\mathbf{g})$ be a sequence of parameter values converging to some $\lambda \in \partial M$. Let $c_n = \chi(\lambda_n)$ and $c = \chi(\lambda) \in M$. We shoud show that $c_n \to c$. Let $g_\lambda : U \to U'$, $f_c : \Omega \to \Omega'$.

By Lemma 6.6, the map $\chi : \Lambda \smallsetminus \operatorname{int} M(\mathbf{g}) \to D \smallsetminus \operatorname{int} M$ is proper, and hence any limit point d of $\{c_n\} \subset D \smallsetminus M$ belongs to ∂M. We assert that $g_\lambda : U \to U'$ is qc conjugate to $f_d : V \to V'$. Indeed, the $g_{\lambda_n} : U_n \to U_n'$ are hybrid equivalent to the $f_{c_n} : \Omega_n \to \Omega_n'$ by means of some qc maps $\psi_n : U_n' \to \Omega_n'$. By the straightening construction (see the proof of Lemma 3.24), the dilatation of ψ_n is equal to the dilatation of the tubing $t_{\lambda_n} = t_* \circ h_\lambda^{-1}$, which is locally bounded by the λ-lemma. By **??**, the sequence ψ_n is pre-compact in the topology of uniform convergence on compact subsets of U'. Take a limit map $\psi : U' \to \Omega'$. Since $g_{\lambda_n} \to g_\lambda$ uniformly on compact subsets of U and $f_{c_n} \to f_d$ (along a subsequence) uniformly on compact subsets of Ω, the map ψ conjugates g_λ to f_d, as was asserted.

But g_λ is also hybrid equivalent to f_c. Thus f_c and f_d are qc conjugate in some neighborhoods of their filled Julia sets. By **??**, they are qc conjugate on the whole complex plane. Since $d \in \partial M$, the Rigidity Theorem **??**(...) implies the desired: $c = d$ (and, in particular, $c \in \partial M$).

The above argument implies that χ continuously maps $\Lambda \smallsetminus \operatorname{int} M(\mathbf{g})$ into $D \smallsetminus \operatorname{int} M$. We still need to show that χ is continuous at any point $\lambda \in \partial M$ even if it is approached from the interior of $M(\mathbf{g})$. The argument is similar to the above except one detail. So, let now $\{\lambda_n\}$ be any sequence in Λ converging to λ. Let c_n, c and d be as above. Then the above argument shows that f_c is qc equivalent to f_d. But now we already know that $c \in \partial M$ (though this time we do not know it for d). Hence by the Rigidity Theorem **??**(...), $c = d$. ☐

"Only by miracle can one ensure the continuity of straightening in degree 2" said Adrien Douady [**D1**]. As we have seen, a reason behind this miracle is quasi-conformal rigidity of the quadratic maps f_c with $c \in \partial M$ (**??**). Another reason is the λ-lemma (see §**??**). All these reasons are valid only for one-parameter families. There are no miracles in the polynomial families with more parameters, see [**DH2**, §...].

1.10. Analyticity of $\chi : \operatorname{int} M(\mathbf{g}) \to \operatorname{int} M$. *The assumptions of Theorem 6.1 will be standing until the end of this section*

1.10.1. *Hyperbolic components.* As in the case of the genuine Mandelbrote set, a component H of $\operatorname{int} M(\mathbf{g})$ is called hyperbolic if it contains a hyperbolic parameter value.

EXERCISE 6.1. *Show that:*

(i) *All parameter values in a hyperbolic component of $M(\mathbf{g})$ are hyperbolic;*
(ii) *Neutral parameter values belong to $\partial M(\mathbf{g})$ (compare Lemma* **??***);*

LEMMA 6.8. *If P is a hyperbolic component of $\operatorname{int} M(\mathbf{g})$ then there exists a hyperbolic component Q of $\operatorname{int} M$ such that $\chi : P \to Q$ is a proper holomorphic map.*

PROOF. Obviously the straightening of a hyperbolic map is hyperbolic. Hence $\chi(P)$ belongs to some hyperbolic component Q of $\operatorname{int} M$. Moreover, since the hybrid conjugacy is conformal on the interior of the filled Julia set, it preserves the multiplies of attracting cycles. Hence

$$\mu_P(\lambda) = \mu_Q(c) \text{ for } c = \chi(\lambda),$$

where μ_P and μ_Q are the multiplier functions on the domains P and Q respectively. By the Implicit Function Theorem, both these functions are holomorphic. Moreover, by Theorem **??**, μ_Q is a conformal isomorphism onto $\mathbb{D}$. Hence $\chi = \mu_Q^{-1} \circ \mu_P$ is holomorphic as well.

By Lemma 6.7, the map $\chi : P \to Q$ is continuous up to the boundary and $\chi(\partial P) \subset \partial Q$. Hence it is proper. $\square$

1.10.2. *Queer components.* As in the quadratic case, a non-hyperbolic component of $\operatorname{int} M(\mathbf{g})$ is called queer. Let us first extend Lemma **??** to quadratic-like families:

LEMMA 6.9. *Let P be a queer component of $M(\mathbf{g})$. Take a base point $* \in P$. Then there is a holomorphic motion $H_\lambda : U'_* \to U'_\lambda$ conjugating g_* to g_λ.*

PROOF. Since $M(\mathbf{g})$ is equipped, there is a holomorphic motion $h_\lambda : A_* \to A_\lambda$. Let $A^n_\lambda = g_\lambda^{-n} A_\lambda$. Since the critical point is non-escaping under the iterates of g_λ, A^n_λ is an annulus and $g^n_\lambda : A^n_\lambda \to A_\lambda$ is a covering map. By **??**, $\mathbf{h}$ can be consequtively lifted to holomorphic motions $h_{n,\lambda} : A^n_* \to A^n_\lambda$. By the λ-lemma (**??**), they automatically match on the common boundaries of the annuli, so that we have a single holomorphic motion $H_\lambda : U'_* \smallsetminus K(g_*) \to U'_\lambda \smallsetminus K(g_\lambda)$ conjugating g_* to g_λ. Since the sets $K(g_\lambda)$ are nowhere dense (see Corollary 3.23), the λ-lemma extension of H_λ to the whole domain U'_* still conjugates g_* to g_λ. $\square$

LEMMA 6.10. *Let H_λ be the holomorphic motion constructed in the previous lemma. Then the Beltrami differential*

$$\mu_\lambda(z) = \begin{cases} \frac{\bar\partial H_\lambda(z)}{\partial H_\lambda(z)}, & z \in K(g_*), \\ 0, & z \in \mathbb{C} \smallsetminus K(g_*), \end{cases}$$

holomorphically depends on $\lambda \in P$.

We can now prove an analogue of Lemma 6.8 for queer components.

LEMMA 6.11. *The straightening* χ *is holomorphic on the queer components of* $\operatorname{int} M(\mathbf{g})$.

PROOF. Consider a queer component $P \subset \operatorname{int} M(\mathbf{g})$ with a base point $*$. For $\lambda \in P$, let $h_\lambda : U'_\lambda \to \Omega'_\lambda$ denote the hybrid conjugacy between $g_\lambda : U_\lambda \to U'_\lambda$ and its straightening $f_\lambda \equiv f_{\chi(\lambda)} : \Omega_\lambda \to \Omega'_\lambda$, and let $h \equiv h_*$. Then $f_* : \Omega_* \to \Omega'_*$ is qc equivalent to $f_\lambda : \Omega_\lambda \to \Omega'_\lambda$ by means of the map $\psi_\lambda : h_\lambda \circ H_\lambda \circ h^{-1}$, where $\{H_\lambda\}$ is the holomorphic motion from the previous lemma. Let $\phi_\lambda : \mathbb{C} \smallsetminus K(f_*) \to \mathbb{C} \smallsetminus K(f_\lambda)$ be the conformal conjugacy between the quadratic polynomials f_* and f_λ on the complements of their Julia sets. By **??**, the map

$$\Psi_\lambda(z) = \begin{cases} \psi_\lambda(z), & z \in K(f_*), \\ \phi_\lambda(z), & z \in \mathbb{C} \smallsetminus K(f_*), \end{cases}$$

is a global qc conjugacy between f_* and f_λ conformal outside the Julia set.

Let $\nu_\lambda = (h_\lambda)_* \mu_\lambda$, where μ_λ is the conformal structure on $K(g_*)$ considered in the previous lemma. Since h_λ is confomal a.e. on the Julia set, we have:

$$\Psi^*_\lambda(\sigma|K(f_\lambda)) = h_* \circ (H_\lambda)^* \circ h^*_\lambda (\sigma|K(f_\lambda)) = h_* \circ (H_\lambda)^* (\sigma|K(g_\lambda)) = h_* \mu_\lambda = \nu_\lambda.$$

Since the push forward-map

$$h_* : \mu \mapsto \nu, \quad \nu = \left(\frac{h'}{\bar{h}'} \mu \right) \circ h^{-1}$$

is a complex isomorphism between the spaces of Beltrami differentials. the previous lemma implies that ν_λ holomorphically depends on $\lambda \in P$. By the holomorphic dependence of the solution of the Beltrami equation on parameters (ref) and **??**, $f_\lambda(0) = \chi(\lambda)$ holomorphically depends on λ as well. □

1.11. Discreteness of the fibers.

LEMMA 6.12. *For any* $c \in M$, *the fiber* $\chi^{-1}(c)$ *is finite.*

PROOF. Since $M(\mathbf{g})$ is compact, it is enough to show that the fibers are discrete. Assume that there exists some $c \in M$ with an infinite fiber $\chi^{-1}(c)$. Since $M(\mathbf{g})$ is compact, this fiber contains a sequence of distinct parameter values $\lambda_n \in \chi^{-1}(c)$ converging to some point $\lambda_\star \in \chi^{-1}(c)$ We will skip the subscript in all notations affiliated with the map $g_{\lambda_\star}$, i.e., $g_{\lambda_\star} \equiv g$, $U_{\lambda_\star} \equiv U$ etc.

Since χ is holomorphic on $\operatorname{int} M$, $\lambda_\star$ cannot belong to $\operatorname{int} M$ unless it belongs to a queer component U such that $\chi|U \equiv$ *const*. But in the latter case, we can replace $\lambda_\star$ by any boundary point of U. Thus we can always assume that $\lambda_\star \in \partial M$.

Since the quadratic-like family $g_\lambda : U_\lambda \to U'_\lambda$ is equipped, there exists an equivariant holomorphic motion $h_\lambda : A \to A_\lambda$ of the closed fundamental annulus $A_\lambda = \bar{U}'_\lambda \setminus U_\lambda$, i.e., $h_\lambda(gz) = g_\lambda(h_\lambda z)$ for $z \in \partial A$. Extend it by the λ-lemma **??** to a holomorphic motion $h_\lambda : \mathbb{C} \setminus U \to \mathbb{C} \setminus U_\lambda$ (keeping the same notation for the extension). We will construct a holomorphic family of hybrid deformations G_λ of g, $\lambda \in \Lambda$, naturally generated by this holomorphic motion.

To this end let us first pull back the standard conformal structure to $\mathbb{C} \setminus U$, $\mu_\lambda = h^*_\lambda(\sigma)$. Then extend μ_λ to a g-invariant conformal structure on $\mathbb{C} \setminus K(g)$ by pulling it back by iterates of g. Finaly extend it to $K(g)$ as a strandard structure. This gives us a holomorphic family of g-invariant conformal structures on $\mathbb{C}$. We will keep the same notation μ_λ for these structures. Solving the Beltrami equations, we obtain a holomorphic family of qc maps $H_\lambda : \mathbb{C} \to \mathbb{C}$ such that $\mu_\lambda = (H_\lambda)^*(\sigma)$ and $\bar{\partial} H(z) = 0$ a.e. on $K(g)$. Conjugating g by these maps, we obtain a desired hybrid deformation $G_\lambda = H_\lambda \circ g \circ H_\lambda^{-1}$, $\lambda \in \Lambda$.

On the other hand, for maps $g_{\lambda_n} \equiv g_n$, we can construct the Beltrami differentials $\mu_{\lambda_n} \equiv \mu_n$ in a different way. Indeed, since the map g_n is hybrid equivalent to g, the equivariant map $h_{\lambda_n} \equiv h_n$ uniquely extends to a hybrid conjugacy (Theorem **??**). Let us keep the same notation h_n for this conjugacy.

The above two constructions naturally agree: $(h_n)^*\sigma = \mu_n$. Indeed, it is true on $\mathbb{C} \setminus U$ by definition. It is then true on $U \setminus K(f)$, since the Beltrami differentials are pulled-back under conformal liftings (see Lemma 3.27). Finally, it is true on the filled Julia set $K(g)$ since h_n is conformal a.e. on it.

Thus the qc maps $H_n : \mathbb{C} \to \mathbb{C}$ and $h_n : \mathbb{C} \to \mathbb{C}$ satisfy the same Beltrami equation. They also coincide at two points, e.g., at the critical point and at the β-fixed point of g (in fact, by Corollary 3.34 they coincide on the whole Julia set of g). By uniqueness of the solution of the Beltrami equation, $H_n = h_n$. Hence $G_n = g_n$. Returning to the original notations, we have

$$G_{\lambda_n}(z) = g_{\lambda_n}(z). \tag{1.5}$$

Take an $\epsilon > 0$ such that both functions $G_\lambda(z)$ and $g_\lambda(z)$ are well-defined in the bidisk $\{(\lambda, z) \in \mathbb{C}^2 : \ |\lambda - \lambda_\star| < \epsilon, \ z \in V \equiv g^{-1}U\}$. For any $z \in V$, consider two holomorphic functions of λ:

$$\Phi_z : (\lambda) = G_\lambda(z) \quad \text{and} \quad \phi_z(\lambda) = g_\lambda(z), \quad |\lambda - \lambda_\star| < \epsilon.$$

By (1.5), they are equal at points λ_n converging to λ_*. Hence they are identically equal.

Thus for $|\lambda| < \epsilon$, two quadratic-like maps, G_λ and g_λ, coincide on V. But it is impossible since the Julia set of G_λ is always connected, while the Julia set of g_λ is disconnected for some λ arbitrary close to $\lambda_\star$ (recall that we assume that $\lambda_\star \in \partial M(\mathbf{g})$). ☐

COROLLARY 6.13. $\chi(\operatorname{int} M(\mathbf{g})) \subset \operatorname{int} M$.

Remark. Of course, it is not obvious only for queer components.

PROOF. Take a component P of $\operatorname{int} M$. We have proven that $\chi|P$ is a *non-constant* holomorphic function. Hence the image $\chi(P)$ is open. Since it is obviously contained in M, it must be contained in $\operatorname{int} M$. ☐

1.12. Bijectivity. What is left is to show that the map $\chi : M(\mathbf{g}) \to M$ is bijective. By §1.5, the winding number of the curve $\chi : \partial\Lambda \to \mathbb{C}$ around any point $c \in D_{r^2}$ is equial to 1. By the Topological Argument Principle (§4.1),

$$\sum_{a \in \chi^{-1} c} \operatorname{ind}_a(\chi) = w_c(\chi, \partial\Lambda) = 1, \quad c \in D_{r^2}. \tag{1.6}$$

It immediately follows that the map $\chi : \Lambda \to \mathbb{D}_{r^2}$ is *surjective* (for otherwise the sum in the left-hand side would vanish fo some $c \in D_{r^2}$).

Let us show that χ is injective on the interior of $M(\mathbf{g})$. Indeed, if $a_0 \in \operatorname{int} M$, then by Corollary 6.13 $c = \chi(a_0) \in \operatorname{int} M$, and by Lemma 6.7 $\chi^{-1}(c) \subset \operatorname{int} M$. But by §1.10, $\chi| \operatorname{int} M$ is holomorphic and hence $\operatorname{ind}_a(\chi) > 0$ for any $a \in \operatorname{int} M$. It follows that the sum in the left-hand side of (1.6) actually contains only one term, so that c has only one preimage, a_0.

Finaly, assume that there is a point $c \in \partial M$ with more than one preimage. By the Topological Argument Principle, χ has a non-zero index at one of those preimages, say, a_1. Take another preimage a_2. Both a_1 and a_2 belong to ∂M.

Take a point $a_2' \notin \partial M(\mathbf{g})$ near a_2, and let $c' = \chi(a_2')$. By Exercise 1.22, χ is locally surjective near a_1, so that c' has a preimage a_1' over there. This contradicts injectivity of χ on $\Lambda \smallsetminus \partial M(\mathbf{g})$.

This completes the proof of Theorem 6.1.

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