ON THE HYPERBOLICITY OF THE FEIGENBAUM FIXED POINT

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Abstract. We show the hyperbolicity of the Feigenbaum fixed point using the inflexibility of the Feigenbaum tower, the Manẽ-Sad-Sullivan λ-Lemma and the existence of parabolic domains (petals) for semi-attractive fixed points.

1. Introduction and statement of results

The renormalization theory in one-dimensional dynamics had its origin in the observation of universality in families of unimodal maps by Feigenbaum and Collet-Tresser. They conjectured that such universality could be explained by the existence of a hyperbolic fixed point (the so called Feigenbaum fixed point) of a operator called the Feigenbaum renormalization operator (defined in the "space of unimodal maps") and the characterization of its stable manifold. Lanford [Lan] proved the existence of such fixed point and Sullivan [Su], introducing quasiconformal methods in renormalization theory, proved the uniqueness of such fixed point. He also gave the characterization of the stable manifold (in Sullivan result, the "stable manifold" should be understood as the set of unimodal maps whose renormalizations converge to the fixed point). McMullen [McM96] proved that, in such "stable manifold", the convergence to the Feigenbaum fixed point is indeed exponentially fast. Finally, Lyubich [Lyu99] proved the Feigenbaum conjecture proving that the Feigenbaum fixed point is hyperbolic, with codimension one stable manifold. Furthermore, he proved that the "stable manifold"studied by Sullivan and McMullen, coincides with the stable manifold of this fixed point. The results cited above made deep use of methods of complex dynamics and represents a sample of the importance of such methods in the developments of one-dimensional dynamics in the last two decades. The main goal of this work is to provide a simpler and shorter proof to part of McMullen and Lyubich results in [McM96] and [Lyu99], in such way to make these results more accessible to a wider audience.

Let $g\colon U_0 \to V_0$ be a quadratic-like map. This means that g is a ramified holomorphic covering map of degree two, where U and V are simply connected domains, $U \Subset V$. We also assume that the filled-in Julia set of g, $K(g) := \cap_n g^{-n}V$, is connected. We say that g is renormalizable with period two if there exist simply

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connected subdomains U_1, V_1 so that $g^2\colon U_1 \to V_1$ is also a quadratic-like map with connected filled-in Julia set.

Two quadratic like maps $h\colon U_h \to V_h$ and $g\colon U_g \to V_g$, both with connected filled-in Julia set, defines the same quadratic-like germ if $K(h)$ coincides with $K(g)$ and h coincides with g in a neighborhood of $K(g)$. If g is renormalizable, then the renormalization of the germ defined by g, denoted $\mathcal{R}g$, is the unique quadratic-like germ defined by the normalization of any possible induced map $g^2\colon U_1 \to V_1$ which are quadratic-like maps with connected filled-in Julia set (normalize the germ using an affine conjugacy, setting the critical point at zero and the unique fixed point in $K(g_1)$ which does not cut $K(g_1)$ in two parts, the so-called β fixed point of g_1, to 1). The operator $\mathcal{R}$ is called the Feigenbaum renormalization operator. In the setting of quadratic-like germs which have real values in the real line, there exists an unique fixed point to the Feigenbaum renormalization operator (proved by D. Sullivan: see also [McM96]), denoted $f^\star$ (it is an open question if this is the unique fixed point in the set of all quadratic-like germs).

It is a consequence of the so-called a priori bounds [Lyu99] that we can choose a simply connected domain U, $K(f) \subset U$, so that, if $\mathcal{B}(U)$ denotes the Banach space of the complex analytic functions g, $Dg(0) = 0$, with a continuous extension to $\overline{U}$, provided with the sup norm, and $\mathcal{B}_{nor}(U)$ denotes the affine subspace of the functions g so that $g(1) = 1$, then the fixed point $f^\star$ has a complex analytic extension which belongs to $\mathcal{B}_{nor}(U)$ and there exists N so that the operator $\mathcal{R}^N$ can be represented as a compact operator defined in a small neighborhood of $f^\star$ in $\mathcal{B}_{nor}(U)$. More precisely, there exists a larger domain $\tilde{U} \Supset U$ and a complex analytic operator $\tilde{\mathcal{R}}\colon B_{\mathcal{B}_{nor}(U)}(f^\star, \epsilon) \to \mathcal{B}_{nor}(\tilde{U})$ so that, if i denotes the natural inclusion $i\colon \mathcal{B}(\tilde{U}) \to \mathcal{B}(U)$, then $\mathcal{R}^N = i \circ \tilde{\mathcal{R}}$, where the equality holds in the intersection of the domains of the operators. To simplify the notation, we will assume that $N = 1$ and identify $\mathcal{R}$ with its complex analytic extension in $\mathcal{B}_{nor}(U)$.

Two quadratic-like maps g_0 and g_1 are in the same hybrid class if there exists a quasiconformal conjugacy ϕ between them, in a neighborhood of their filled-in Julia sets, so that $\overline{\partial}\phi \equiv 0$ on $K(g_0)$. Note that quadratic-like maps in the hybrid class of $f^\star$ are infinitely renormalizable.

We will provide a new approach to the following result:

Theorem 1 (Exponential contraction:[McM96] and [Lyu99])**.** *There exists $\lambda < 1$ so that, for every quadratic-like map f which is in the hybrid class of $f^\star$, there exist $n_0 = n_0(f)$ and $C = C(f) > 0$ so that $\mathcal{R}^n f \in B_{\mathcal{B}(U)}(f^\star, \epsilon)$, for $n \geq n_0$, and $|\mathcal{R}^{n_0+n} f - f^\star|_{\mathcal{B}(U)} \leq C\lambda^n$.*

A major attractive of this new proof is that it is essentially infinitesimal and has a “dynamical flavor”: we will prove that the derivative of the renormalization operator is a contraction in the tangent space of the hybrid class (the contraction of the derivative of the renormalization operator on the hybrid class was proved by Lyubich [Lyu99], but his proof is not infinitesimal). Moreover, the method seems to be so general as the previous ones: it also applies to the classical renormalization horseshoe [Lyu99] and the Fibonacci renormalization operator [Sm02a], for instance.

We will also obtain, as a corollary of McMullen theory of towers [McM96], the local behavior of semi-attractive fixed points [H] and an easy application of the λ-lemma [MSS] that

Theorem 2 ([Lan][Lyu99]). *The Feigenbaum fixed point is hyperbolic.*

The reader will observe that we assume the Feigenbaum combinatorics just to simplify the notation: the argument in the proof of Theorem 2 works as well to prove the hyperbolicity of real periodic points of the renormalization horseshoe.

2. Preliminaries

2.1. **Parabolic domains for semi-attractive fixed points.** Consider a complex Banach space B, and let $F\colon A \subset B \to B$ be a complex analytic operator defined in an open set A. Suppose that $p \in A$ is a fixed point for F. We say that p is a **semi-attractive** fixed point for F if

- The value 1 is an eigenvalue for DF_p.
- There exists a Banach subspace E^s, with (complex) codimension one, which is invariant by the action of DF_p and furthermore the spectrum of DF_p, restricted to E^s, is contained in $\{z\colon\ |z| \leq r\}$, where $r < 1$.

The following result was proved by M. Hakim [H] for finite-dimensional complex Banach spaces ($\mathbb{C}^n$), but the proof can be carry out as well for a general complex Banach space:

Proposition 2.1 ([H]). *Consider a compact complex analytic operator F, defined in an open set of a complex Banach space B. Let p be a semi-attractive fixed point. Then one of the following statements holds:*

(1) ***Curve of fixed points:*** *There exists a complex analytic curve of fixed points which contains p.*
(2) ***Parabolic domains (Petals):*** *There exists $k \geq 1$ so that, for every $\epsilon > 0$ there exists a connected open set U, whose diameter is smaller than ϵ, which is forward invariant by the action of F and, moreover,*
$$F^n u \to_n p, \text{ for every } u \in U,$$
where the speed of this convergence is subexponential: for each $u \in U$, there exists $C = C(u)$ so that
$$\frac{1}{C}\frac{1}{n^{1/k}} \leq |F^n u - p| \leq C\frac{1}{n^{1/k}}.$$

An outline of Hakim's proof can be found in the Appendix.

3. Infinitesimal contraction on the horizontal space

Let $f\colon V_1 \to V_2$ be a quadratic-like map with connected Julia set and with an analytical extension to $\mathcal{B}_{nor}(U)$, with $K(f) \subset U$. The horizontal subspace (introduced by Lyubich[Lyu99]) of f, denoted E^h_f, is the subspace of the vectors $v \in \mathcal{B}(U)$ so that there exists a quasiconformal vector field in the Riemann sphere α satisfying $v = \alpha \circ f - Df \cdot \alpha$ in a neighborhood of $K(f)$, with $\overline{\partial}\alpha \equiv 0$ on $K(f)$ and $\alpha(0) = \alpha(1) = \alpha(\infty) = 0$. We will not use the following information here, but certainly it will clarify the spirit of our methods: in an appropriated setting, the hybrid class is a complex analytic manifold and the horizontal space is the tangent space of the hybrid class at f(see [Lyu99]).

Lemma 3.1 ([ALdM]). *Let f be a quadratic-like map with an extension to $\mathcal{B}_{nor}(U)$ and connected Julia set contained in U. Assume that f does not support invariant line fields in its filled-in Julia set. Let $V \Subset U$ be a domain with smooth boundary so* *that $K(f) \subset V$. Then there exist $C, \epsilon > 0$ so that, if $|f-g|_{\mathcal{B}(U)} \leq \epsilon$ and $g\colon g^{-1}V \to V$ is a quadratic-like map with connected Julia set, then, for every $v \in E^h_g$ there exists a $C|v|_{\mathcal{B}(U)}$-quasiconformal vector field α in $\overline{\mathbb{C}}$ so that $v = \alpha \circ g - Dg \cdot \alpha$ on V.*

With the aid of a compactness criterium to quasiconformal vector fields in $\overline{\mathbb{C}}$, we have:

Corollary 3.2 ([ALdM])**.** *Assume that $(f_n, v_n) \to_n (f_\infty, v_\infty)$ in $\mathcal{B}_{nor}(U) \times \mathcal{B}(U)$, where $f_i\colon f_i^{-1}V \to V$, $i \in \mathbb{N} \cup \{\infty\}$, are quadratic-like maps with connected filled-in Julia sets $K(f_i) \subset V \Subset U$. Furthermore, assume that $v_n \in E^h_{f_n}$, for $n \in \mathbb{N}$. If f does not support invariant line fields in $K(f)$, then $v_\infty \in E^h_f$. In particular E^h_f is closed.*

If $\mathcal{R}$ is the nth iteration of the Feigenbaum renormalization operator and f is close to $f^\star$ in $\mathcal{B}(U)$, denote by β_f the analytic continuation of the β-fixed point of the small Julia set associated with the nth renormalization of $f^\star$ [McM96]. The following result gives a description of the action of the derivative in a horizontal vector $v = \alpha \circ f - Df \cdot \alpha$ in terms of α:

Proposition 3.3. *Let V be a neighborhood of $K(f^\star)$. Replacing $\mathcal{R}$ by an iteration of it, if necessary, the following property holds: If $f \in \mathcal{B}_{nor}(U)$ is close enough to $f^\star$ and $v = \alpha \circ f - Df \cdot \alpha$ on V, where $v \in \mathcal{B}(U)$ and α is a quasiconformal vector field in the Riemann sphere, normalized by $\alpha(0) = \alpha(1) = \alpha(\infty) = 0$, then*

$$D\mathcal{R}_f \cdot v = r(\alpha) \circ \mathcal{R}f - D(\mathcal{R}f) \cdot r(\alpha), \tag{1}$$

on U, where

$$r(\alpha)(z) := \frac{1}{\beta_f}\alpha(\beta_f z) - \frac{1}{\beta_f}\alpha(\beta_f) \cdot z.$$

In particular, if f is renormalizable, then $D\mathcal{R}_f E^h_f \subset E^h_{\mathcal{R}f}$.

This result is consequence of a simply calculation and the complex bounds to $f^\star$. Note that, apart the normalization by a linear vector field, $r(\alpha)$ is just the pullback of the vector field α by a linear map. In particular, if α is a C-quasiconformal vector field, then $r(\alpha)$ is also a C-quasiconformal vector field: this will be a key point in the proof of the infinitesimal contraction of the renormalization operator in the horizontal subspace (Proposition 3.4).

Let $f^\star\colon V_1 \to V_2$ be a quadratic-like representation of the fixed point. The Feigenbaum tower is the indexed family of quadratic-like maps $f_i^\star\colon \beta_{f^\star}^{-i} V_1 \to \beta_{f^\star}^{-i} V_2$, $i \in \mathbb{N}$, defined by $f_i^\star(z) := \beta_{f^\star}^{-i} f^\star(\beta_{f^\star}^{i} \cdot z)$.

Proposition 3.1 ([McM96])**.** *The Feigenbaum tower does not support invariant line fields: this means that there is not a measurable line field which is invariant by all (or even an infinite number of) maps in the Feigenbaum tower.*

Proposition 3.2 ([Su] and [McM96])**.** *Let f be a quadratic-like map which admits a hybrid conjugacy ϕ with $f^\star$. Then $\phi_n(z) := \beta_{f^\star}^{-n} \cdot \phi(\beta_{\mathcal{R}^{n-1}f} \cdots \beta_f \cdot z)$ converges to identity uniformly on compact sets in the complex plane. In particular, there exists $n_0 = n_0(f)$ so that $\mathcal{R}^n f \in B_{\mathcal{B}_{nor}(U)}(f^\star, \epsilon)$, for $n > n_0$, and $\mathcal{R}^n f \to_n f^\star$ on $\mathcal{B}_{nor}(U)$.*

Theorem 1 says that this convergence is, indeed, exponentially fast. The following proposition has a straightforward proof:

Proposition 3.3. *Let $\mathcal{R} = i \circ \tilde{\mathcal{R}}$, where $\tilde{\mathcal{R}} \colon \mathcal{V} \subset \mathcal{B} \to \tilde{\mathcal{B}}$ is an operator defined in a neighborhood $\mathcal{V}$ of a Banach space $\mathcal{B}$, to another Banach space $\tilde{\mathcal{B}}$, and $i \colon \tilde{\mathcal{B}} \to \mathcal{B}$ is a compact linear transformation. Let $S \subset \mathcal{B} \times \mathcal{B}$ be a set with the following properties:*

1. ***Vector bundle structure:*** *If (f, v_1) and $(f, v_2) \in S$, then $(f, \alpha \cdot v_1 + v_2) \in S$, for every $\alpha \in \mathbb{C}$,*
2. ***Semicontinuity:*** *If $(f_n, v_n) \to (f, v)$ and $(f_n, v_n) \in S$, then $(f, v) \in S$,*
3. ***Invariance:*** *If $(f, v) \in S$ then $(\mathcal{R}f, D\mathcal{R}_f \cdot v) \in S$,*
4. ***Compactness:*** *$\{(\tilde{\mathcal{R}}f, D\tilde{\mathcal{R}}_f \cdot v) \colon \ (f, v) \in S, |v| \leq 1\}$ is a bounded set in $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$,*
5. ***Uniform continuity:*** *Denote $E_f := \{(f, v) \colon (f, v) \in S\}$. There exists $C > 0$ so that, for every f and $n \geq 0$, $|D\mathcal{R}^n_f|_{E_f} \leq C$,*
6. *If $(f, v) \in S$ then $|D\mathcal{R}^n_f \cdot v| \to_n 0$,*

Then there exist $\lambda < 1$ and $N \in \mathbb{N}$ so that $|D\mathcal{R}^N_f|_{E_f} \leq \lambda$, for every f so that $E_f \neq \varnothing$.

Proposition 3.3 is a generalization of the following fact about compact linear operators $T \colon \mathcal{B} \to \mathcal{B}$: if $T^n v \to 0$, for every $v \in \mathcal{B}$, then the spectral radius of T is strictly smaller than one.

Given ϵ, $K > 0$, and a domain $V \Subset U$ so that $K(f^\star) \subset V$, denote by $\mathcal{A}(\epsilon, K, V)$ the set of maps $f \in \mathcal{B}_{nor}(U)$ so that there exists a K- quasiconformal map ϕ in the complex plane so that $\phi(V) \subset \overline{U}$ and $\phi \circ f^\star = f \circ \phi$ on V; moreover, for $n \geq 0$, we have $|\mathcal{R}^n f - f^\star|_{\mathcal{B}(U)} \leq \epsilon$. Note that $\mathcal{A} := \mathcal{A}(\epsilon, K, V)$ is closed. By the topological convergence (Proposition 3.2) and Lemma 2.2 in [Lyu02], we can choose K and ϵ properly so that, replacing $\mathcal{R}$ by an iterate, if necessary, we can assume that $\mathcal{A}$ is invariant by the action of $\mathcal{R}$ and that for every f in the hybrid class of $f^\star$, there exists $N = N(f)$ so that $\mathcal{R}^N f \in \mathcal{A}$.

Proposition 3.4 (Infinitesimal contraction: cf. [Lyu99])**.** *There exist $\lambda < 1$ and $N > 0$ so that $|D\mathcal{R}^N_f|_{E^h_f} \leq \lambda$, for every $f \in \mathcal{A}(\epsilon, K, V)$.*

Proof. Consider the set $S := \{(f, v) \colon f \in \mathcal{A}, \ v \in E^h_f\}$. It is sufficient to verify the properties in the statement of Proposition 3.3. Since $\mathcal{A}$ is closed, property 2 follows of Corollary 3.2. Since $\mathcal{A}$ is invariant by $\mathcal{R}$, property 3 follows of Proposition 3.3. The compactness property is obvious, if ϵ is small enough. To prove the uniform continuity property, by Propositions 3.1 and 3.3, we have that, for $(f, v) \in S$ and $n \geq 1$, $D\mathcal{R}^n_f \cdot v = \alpha_n \circ \mathcal{R}^N f - D(\mathcal{R}^N f) \cdot \alpha_n$ on U, with

$$\alpha_n(z) := \frac{1}{\beta_{n-1} \ldots \beta_0} \alpha(\beta_{n-1} \ldots \beta_0 z) - \frac{1}{\beta_{n-1} \ldots \beta_0} \alpha(\beta_{n-1} \ldots \beta_0) z,$$

where $\beta_i = \beta_{\mathcal{R}^i f}$ and α_n are $K \cdot |v|_{\mathcal{B}(U)}$-quasiconformal vector fields. Note that K does not depends on $(f, v) \in S$ or $n \geq 1$. By the compactness of K- quasiconformal vector fields (recall that $\alpha_n(0) = \alpha_n(1) = \alpha_n(\infty) = 0$), we get $|D\mathcal{R}^n_f|_{E^h_f} \leq C$, for some $C > 0$. To prove assumption 6, note that $\overline{\partial} \alpha_n$ is an invariant Beltrami field to the finite tower

$$\mathcal{R}^n f, \frac{1}{\beta_{n-1}} \mathcal{R}^{n-1} f(\beta_{n-1} z), \ldots, \frac{1}{\beta_{n-1} \cdots \beta_0} f(\beta_{n-1} \cdots \beta_0 z).$$

But, by the topological convergence, these finite towers converges to the Feigenbaum tower. Hence, if a subsequence α_{n_k} converges to a quasiconformal vector field α_∞, then $\overline{\partial} \alpha_\infty$ is an invariant Beltrami field to the Feigenbaum tower (since $\overline{\partial} \alpha_{n_k}$

converges to $\overline{\partial}\alpha_\infty$ in the distributional sense), so, by Proposition 3.1, α_∞ is a conformal vector field in the Riemann sphere. Since α_∞ vanishes at three points, $\alpha_\infty \equiv 0$. Hence $\alpha_n \to 0$ uniformly on compact sets in the complex plane, so we get $\mathcal{R}^n_f \cdot v \to 0$ (Note that $|D(\mathcal{R}^n f)|$ is uniformly bounded, for $n \geq 1$). ☐

We are going to prove Theorem 1: Let f be a quadratic-like map in the hybrid class of $f^\star$. Then there exists a quasiconformal map $\phi\colon \mathbb{C} \to \mathbb{C}$ which is a conjugacy between them in a neighborhood of their Julia sets. Consider the following Beltrami path f_t between the two maps, induced by ϕ: if ϕ_t, $|t| \leq 1$, is the unique normalized quasiconformal map so that $\overline{\partial}/\partial\phi_t = t \cdot \overline{\partial}/\partial\phi$, define $f_t = \phi_t \circ f \circ \phi_t^{-1}$. By the topological convergence, there exists n_0 so that $\mathcal{R}^{n_0+n} f_t \in \mathcal{A}$, for $n \geq 0$, $|t| \leq 1$. An easy calculation shows that

$$\frac{d\mathcal{R}^{n_0+n} f_t}{dt}\Big|_{t=t_0} \in E^h_{\mathcal{R}^{n_0+n} f_{t_0}},$$

for $|t_0| \leq 1$. The infinitesimal contraction finishes the proof.

Remark 1. *The first step in the above proof of Theorem 1, to prove that* $\alpha_n \to 0$ *(in the proof of Proposition 3.4), must be compared with the proof of Lemma 9.12 in* [McM96]. *In C. McMullen argument, additional considerations should be done to arrive in exponential contraction; firstly it is proved that quasiconformal deformations (as the quasiconformal vector field* α *in the definition of the horizontal vectors) are* $C^{1+\beta}$*-conformal at the critical point (Lemma 9.12 in* [McM96] *and the deepness of the critical point have key roles in this proof), and then it is necessary to integrate this result. In M. Lyubich argument* [Lyu99], *firstly it is proved that the hybrid class is a complex analytic manifold and then the topological convergence is converted in exponential contraction via Schwartz's Lemma.*

4. Hyperbolicity of the Feigenbaum fixed point

We are going to prove Theorem 2. Firstly we will prove that

$$\sigma(D\mathcal{R}^2_{f^\star}) \cap \mathbb{S}^1 \subset \{1\}. \tag{2}$$

Indeed, if $D\mathcal{R}_{f^\star} \cdot v = \lambda v$, then the vector $\tilde{v}(z) := \overline{v(\overline{z})}$ is a solution to $D\mathcal{R}_{f^\star} \cdot \tilde{v} = \overline{\lambda}\tilde{v}$. So if $\lambda \in \mathbb{S}^1 \setminus \{-1, 1\}$ then *codim* $E^h > 1$, which is a contradiction (E^h has codimension one [Lyu99]. The same result can be proven in an easy way using the argument explained in section 12 on [Sm02a]). Indeed, we can prove, using the contraction on the horizontal direction and results on [Sm03], which uses only elementary methods, that $\sigma(D\mathcal{R}_{f^\star}) \cap \mathbb{S}^1 \subset \{1\}$, but the proof is more involving.

Furthermore $\sigma(D\mathcal{R}_{f^\star})$ is not contained in $\mathbb{D}$ (see Lyubich[Lyu99]. We can also use the results in [Sm03] to prove this claim). So either $f^\star$ is a hyperbolic fixed point (with a onedimensional expanding direction) or it is a semi-attractive fixed point, since by Proposition 3.4 the derivative of the renormalization operator at the fixed point is a contraction on the horizontal space, which has codimension one. Assume that $f^\star$ is semi-attractive and let's arrive in a contradiction. Indeed, by Proposition 2.1, one of the following statements holds:

Case i. There exists a connected open set of maps $\mathcal{U} \subset \mathcal{B}_{nor}(U)$, whose diameter can be taken small, which is forward invariant by the action of $\mathcal{R}^2$ and so that each map in $\mathcal{U}$ is attracted at a subexponential speed to the fixed point $f^\star$. Because the maps in $\mathcal{U}$ are very close to $f^\star$ and $\mathcal{U}$ is forward invariant, all the maps in $\mathcal{U}$ are infinitely renormalizable (this argument is easy: see Lemma 5.8 in [Lyu99]). So their filled-in Julia sets have empty interior and their periodic points are repelling, hence there are not bifurcations of periodic points in $\mathcal{U}$. Consider two maps g, $\tilde{g}$ in $\mathcal{U}$ which admit a complex analytic path $g\colon \mathbb{D} \to \mathcal{U}$ between them ($g_0 = g$ and, for some $|\lambda| < 1$, $\tilde{g} = g_\lambda$). Because $\mathbb{D}$ is simply connected and there are not bifurcations of periodic points in $\mathcal{U}$, each periodic point $p \in K(g_0)$ has an unique analytic continuation $h(p, \lambda)$, $\lambda \in \mathbb{D}$: this means that $h(p, 0) = p$ and $h(p, \lambda)$ is a periodic point of g_λ. So the function $h\colon Per(g_0) \times \mathbb{D} \to \mathbb{C}$ defines a holomorphic motion on $Per(g_0) = \{p\colon \exists n > 0 \ s.t.\ g_0^n(p) = p\}$ (note that $h(p, \lambda) \neq h(q, \lambda)$, if $p \neq q$, since there are not bifurcations of periodic points). Moreover, provided $\mathcal{U}$ is small enough, we can select a domain U_1 with a real analytic Jordan curve boundary so that, for every $\lambda \in \mathbb{D}$, $g_\lambda\colon g_\lambda^{-1} U_1 \to U_1$ is a quadratic-like representation. We can also easily define a holomorphic motion $h\colon U_1 \setminus g_0^{-1} U_0 \times \mathbb{D} \to \mathbb{C}$ so that $h(x, \lambda) \equiv x$, for $x \in \mathbb{C} \setminus U_1$ and $g_\lambda(h(x, \lambda)) = h(g_0(x), \lambda)$, for $x \in \partial g_0^{-1} U_1$. Since g_λ have connected filled-in Julia sets, we can extend the holomorphic motion to a holomorphic motion $h\colon \mathbb{C} \setminus K(g_0) \times \mathbb{D} \to \mathbb{C}$ so that $g_\lambda(h(x, \lambda)) = h(g_0(x), \lambda)$, for $x \in g_0^{-1} U_1 \setminus K(g_0)$. So we have defined a holomorphic motion h on the everywhere dense set $Per(g_0) \cup (\mathbb{C} \setminus K(g_0))$ which commutes with the dynamics. By the λ-lemma [MSS], this holomorphic motion extends to the whole Riemann sphere, so all maps g_λ are quasiconformaly conjugated. Since there is a piecewise complex analytic path between any two maps in $\mathcal{U}$, we conclude that all maps in $\mathcal{U}$ are in the same quasiconformal class. Note that the above construction does not give any upper bound for the quasiconformality of the conjugacy: the quasiconformality could be large when the Kobayashi distance between g and $\tilde{g}$ on $\mathcal{U}$ is large.

We claim that, provided $\mathcal{U}$ is small enough, it is possible to choose a quasiconformal conjugacy between any two maps in $\mathcal{U}$ so that the quasiconformality is uniformly bounded *outside* their filled-in Julia sets, using the argument in the proof of Lemma 2.3 in [Lyu02]: in a small neighborhood $\mathcal{V} \subset \mathcal{B}_{nor}(U)$ of $f^\star$, it is possible to find a domain U_1 so that $g\colon g^{-1} U_1 \to U_1$ is a quadratic-like restriction of g (but note that the Julia sets of these quadratic-like restrictions are not, in general, connected). This define the holomorphic moving fundamental annulus $U_1 \setminus g^{-1} U_1$. In particular, provided $\mathcal{U}$ is small enough, there exists $B > 0$ so that for every g_0 and g_1 which belong to $\mathcal{U}$, there exists a B-quasiconformal mapping h between $\mathbb{C} \setminus g_0^{-1} U_1$ and $\mathbb{C} \setminus g_1^{-1} U_1$ so that $h \equiv Id$ on $\mathbb{C} \setminus U_1$ and $g_1 \circ h = h \circ g_0$ on $\partial g_0^{-1} U_1$. Since the Julia sets of g_0 and g_1 are connected, we can extend h to a B-quasiconformal map

$$h\colon \mathbb{C} \setminus K(g_0) \to \mathbb{C} \setminus K(g_1)$$

which is a conjugacy on $g_0^{-1} U_1 \setminus K(g_0)$. Once we already know that g_0 and g_1 are in the same quasiconformal class, h has a quasiconformal extension h_{g_0, g_1} to $\mathbb{C}$ (this follows as in the proof of Lemma 1, in [DH, pg. 302]: if $\tilde{h}$ is a quasiconformal conjugacy between g_0 and g_1, then $\tilde{h}^{-1} \circ h$ commutes with g_0 outside $K(g_0)$, which implies that $\tilde{h}^{-1} \circ h$ extends to a homeomorphism in $\mathbb{C}$ which coincides with Id on $K(g_0)$. By the Rickmann removability theorem (see the statement in [DH]), this map is a quasiconformal homeomorphism, so h is a quasiconformal homeomorphism). This finishes the proof of the claim.

Since all renormalizations of these maps are very close to $f^\star$, they also satisfies the unbranched complex bounds condition (see, for instance, Lemma 9.3 and 9.4 in [Lyu97]). In particular there are not invariant line fields supported on their filled-in Julia sets [McM94], and hence the quasiconformality of the conjugacy $h_{g_0, g_1}\colon \mathbb{C} \to \mathbb{C}$

is uniformly bounded on the whole complex plane by B. But $f^\star$ is a boundary point of $\mathcal{U}$, so the compactness of B-quasiconformal maps (note that the conjugacies h_{g_0,g_1} satisfies $h_{g_0,g_1}(0) = 0$, $h_{g_0,g_1}(1) = 1$ and $h_{g_0,g_1}(\infty) = \infty$) and the non-existence of invariant line fields supported on the filled-in Julia set of $f^\star$ imply that all these maps are hybrid conjugated with $f^\star$. But this implies that the subexponential speed of convergence given by Proposition 2.1 is impossible, since by Theorem 1 the maps in the hybrid class of f converges to $f^\star$ exponentially fast.

Case ii. There exists a connected complex analytic curve of fixed points which contains f. We will apply essentially the same argument used in Case i: Note that in a similar way we can prove that all these fixed points of the operator $\mathcal{R}^2$ are polynomial-like maps which are infinitely renormalizable: in particular their filled-in Julia sets have empty interior and all their periodic points are repelling. So there are not bifurcations of periodic points in this curve of fixed points. Use the λ-lemma [MSS] to conclude that all these fixed points are quasiconformally conjugated (the argument is as in Case i). Since the fixed point $f^\star$ does not support invariant line fields in its filled-in Julia set, we conclude that all these fixed points are hybrid conjugated, which is impossible, since iterations of maps in the hybrid class of $f^\star$ converges to the fixed point $f^\star$.

So we concluded that $f^\star$ must be a hyperbolic fixed point with codimension one stable manifold.

Appendix: Outline of Hakim's proof

To convince the reader of the existence of parabolic petals for semi-attractive compact operators in Banach spaces, we will give an outline of Hakim's proof of the existence of parabolic domains: we do not claim any sort of originality for ourselves in the following exposition and we refer to the quite clear work [H] for details. We will use the notation introduced in Section 2.1. Consider a complex analytic operator T with a semi-attractive fixed point 0. Assume $DT_0 \cdot v = v$, $v \neq 0$. In the following lines, we will identify B with $\mathbb{C} \times E^s$ by the isomorphism $(x, y) \to x \cdot v + y$.

By the Stable Manifold Theorem for compact operators (see Mañé [M]), for $\delta > 0$ and $\epsilon > 0$ small the set

$$W^s_{\delta,\epsilon} = \{x\colon\ \exists\ C\ s.t.\ |T^n x| < \delta\ and\ |T^n x| \leq C(1-\epsilon)^n,\ for\ n \geq 0\}$$

is a codimension one complex analytic manifold. More precisely, there exists a holomorphic function $\psi\colon V \to \mathbb{C}$, where V is a neighborhood of 0 on E^s, with $D\psi(0) = 0$, so that

$$W^s_{\delta,\epsilon} = \{(\psi(y), y)\colon y \in V\}.$$

In particular, after the local biholomorphic changes of variables

$$\begin{aligned} X &= x + \psi(y) \\ Y &= y \end{aligned} \tag{3}$$

it is possible to represent T as $T\colon \mathbb{C} \times E^s \to \mathbb{C} \times E^s$, where $T(x, y) = (x', y')$, with

$$\begin{aligned} x' &= F(x, y) = a_1(y)x + O_y(x^2) \\ y' &= G(y) + xh(x, y) \end{aligned} \tag{4}$$

where G is a (compact) contraction around 0 and $a_1(0) = 1$. After the local biholomorphic change of variables

$$\begin{aligned} X &= v(y)x \\ Y &= y \end{aligned} \tag{5}$$

where

$$v(y) := \Pi_{i\geq 0}\ a_1(G^i(y)),$$

we can assume that $a_1 \equiv 1$.

Note that, for every n, T has the form

$$\begin{aligned} x' &= F(x,y) = x + \sum_{2\leq i\leq n} a_i(y)x^n + O_y(x^{n+1}) \\ y' &= G(y) + xh(x,y) \end{aligned} \tag{6}$$

where G is a (compact) contraction around 0. We claim that we can assume, after certain biholomorphic changes of variables, that a_2, a_3, $\cdots$, a_n do not depend on y. Indeed, assume by induction that T can put in the form

$$\begin{aligned} x' &= F(x,y) = x + \sum_{2\leq i\leq n} \tilde{a}_i x^n + \tilde{a}_{n+1}(y)x^{n+1} + O_y(x^{n+2}) \\ y' &= G(y) + xh(x,y) \end{aligned} \tag{7}$$

Then after the local change of variables

$$\begin{aligned} X &= x + v(y)x^{n+1} \\ Y &= y \end{aligned} \tag{8}$$

where $v(y) := \sum_{i\geq 0}(\tilde{a}_{n+1}(G^i(y)) - \tilde{a}_{n+1}(0))$, T will have the form

$$\begin{aligned} x' &= F(x,y) = x + \sum_{2\leq i\leq n} \tilde{a}_i x^n + \tilde{a}_{n+1}(0)x^{n+1} + \tilde{a}_{n+2}(y)x^{n+2} + O_y(x^{n+2}) \\ y' &= G(y) + xh(x,y). \end{aligned} \tag{9}$$

Now we are going to introduce the concept of multiplicity of the fixed point 0 for transformations on the form of Eq. (6). By the implicit function theorem, for each transformation in that form there exists a complex analytic curve $y\colon U \subset \mathbb{C} \to E^s$, with $0 \in U$, which is the unique solution for the equation

$$y(x) = G(y(x)) + xh(x, y(x)).$$

Consider the function $q\colon U \to \mathbb{C}$ defined by

$$q(x) := F(x, y(x)) - x.$$

The **multiplicity** of T at 0 is defined as the order of q at 0. Note that the multiplicity of T at 0 is finite if and only if 0 is an isolated fixed point and infinity if and only if $q(x)$ vanishes everywhere and $(x, y(x))$ is a complex analytic curve of fixed points for T (which contains all the fixed points in a neighborhood of 0). Moreover, if T has the form Eq. (7), with $\tilde{a}_2 = \cdots = \tilde{a}_{n-1} = 0$ and $\tilde{a}_n \neq 0$, then the multiplicity of T is exactly n.

Consider a transformation T as in Eq. (6) and biholomorphic change of variables $W(x,y) = (X,Y)$ of the type

$$\begin{aligned} X &= x + v(y)x^k \\ Y &= y \end{aligned} \tag{10}$$

where v is a holomorphic function and $k > 1$. Then $W^{-1} \circ T \circ W$ has also the form in Eq. (6). Moreover

Proposition 4.1. *The multiplicity of $W^{-1} \circ T \circ W$ at 0 is equal to the multiplicity of T at 0.*

Proof. (suggested by M. Lyubich) Assume that it is finite (otherwise the invariance is trivial): then 0 is an isolated fixed point. Consider the one- parameter family of change of variables W_λ defined by

$$\begin{aligned} X &= x + \lambda v(y) x^k \\ Y &= y \end{aligned} \tag{11}$$

Then $W_\lambda^{-1} \circ T \circ W_\lambda$ has the form

$$\begin{aligned} x' &= F_\lambda(x,y) = x + O_{y,\lambda}(x^2) \\ y' &= G_\lambda(y) + x h_\lambda(x,y) \end{aligned} \tag{12}$$

Note that we can choose δ_0 small enough so that for all $|\lambda| \leq 1$, 0 is the unique fixed point for $W_\lambda^{-1} \circ T \circ W_\lambda$ on $\{(x,y), |x| \leq \delta_0, |y| \leq \delta_0\}$. Moreover, by the implicit function theorem and the compactness of $\{\lambda \colon |\lambda| \leq 1\}$ there exists a holomorphic function $y_\lambda(x) = y(\lambda, x)$, defined on

$$\{\lambda \colon |\lambda| < 1 + \delta_1\} \times \{x \colon |x| < \delta_2\}$$

so that

$$y_\lambda(x) = G_\lambda(y) + x h_\lambda(x, y_\lambda(x))$$

Choosing δ_1, δ_2 small enough, for each λ the point 0 is the unique solution for the equation

$$q_\lambda(x) := F_\lambda(x, y_\lambda(x)) - x = 0$$

on $\{x \colon |x| \leq \delta_1\}$. By Rouché's Theorem, $ord_0\ q_\lambda$ does not depends on λ. □

Assume that T has the form of Eq. (6) and finite multiplicity n. After appropriated changes of variables, we can assume that $a_2, \ldots, a_{2n-1}$ does not depends on y. Since the multiplicity is invariant by the above changes of variables, we conclude that $a_2 = \cdots = a_{n-1} = 0$ and $a_n \neq 0$. Doing appropriated changes of variables in the form of Eq. (10) (indeed, in this case v does not depends on y) and replacing the coordenate x by θx, for some $\theta \neq 0$, if necessary, it is possible to put T in the form

$$\begin{aligned} x' &= x - \frac{1}{n-1} x^{n-1} + a x^{2(n-1)} + O_y(|x|^{2(n-1)+1}) \\ y' &= G(y) + x h(x,y). \end{aligned} \tag{13}$$

Under the above form, the set

$$P_{R,\rho} = \{(x,y) \colon |x^{n-1} - \frac{1}{2R}| < \frac{1}{2R} \text{ and } |y| < \rho\}$$

is a parabolic domain, provided R and ρ are small enough: here Hakim's proof is very similar to the one-dimensional situation: make the "change of variables"

$$\begin{aligned} X &= x^{n-1} \\ Y &= y \end{aligned} \tag{14}$$

and

$$X = 1/x \qquad (15)$$
$$Y = y$$

to put T in the form

$$x' = x + 1 + c\frac{1}{x} + O_y(\frac{1}{|x|^{1+1/(n-1)}}) \qquad (16)$$
$$y = G(y) + O_y(\frac{1}{|x|^{1/(n-1)}}).$$

and now the proof is easy.

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