STATISTICAL PROPERTIES OF UNIMODAL MAPS: SMOOTH FAMILIES WITH NEGATIVE SCHWARZIAN DERIVATIVE

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Abstract. We prove that there is a residual set of families of smooth or analytic unimodal maps with quadratic critical point and negative Schwarzian derivative such that almost every parameter is either regular or Collet-Eckmann with subexponential recurrence of the critical orbit, both cases allowing a detailed and robust statistical description of the dynamics. This proves a version of Palis conjecture in this setting.

1. Introduction

'The main strategy of the study of all mathematical models is, according to Poincaré, the consideration of each model as a point of a space of different but similar admissible systems'(V. Arnold in [Ar]). One of the main concerns of dynamical systems is to stablish properties valid for typical systems. Since the space of such systems is usually infinite dimension, there are of course many concepts of 'typical'. According [Ar] again, 'The most physical genericity notion is defined by Kolmogorov (1954), who suggested to call a property of dynamical systems exceptional, if it holds only on Lebesgue measure zero set of values of the parameters in every (topologically) generic family of systems, depending on sufficiently many parameters'.

In the last decade Palis [Pa] described a general program for (dissipative) dynamical systems in any dimension. He conjectured that a typical dynamical system has a finite number of atractors described by physical measures which are stochastically stable and whose basins have full Lebesgue measure (finitude and robustness of attractors). Typical was to be interpreted in the Kolmogorov sense: full measure in generic families. Our aim here is to give a proof of this conjecture for an important class of one-dimensional dynamical systems.

Here we consider unimodal maps, that is, continuous maps from an interval to itself which have a unique turning point. More specifically, we consider quasiquadratic unimodal maps, that is, we assume that the map is C^3 with negative Schwarzian derivative and that the critical point is non degenerate.

1.1. **The quadratic family.** The basic model for unimodal maps is the quadratic family, $f_a(x) = a - x^2$, where $-1/4 \leq a \leq 2$ is a parameter. Despite its simple appearance, the dynamics of those maps presents many remarkable phenomena. Restricting to the probabilistic point of view, its richness first became apparent with the work of Jakobson [J], where it was shown that for a positive measure of parameters the behavior is stochastic, more precisely, there is an absolutely continuous invariant measure (the physical measure) with positive Lyapunov exponent. On the other hand, it was later shown by Lyubich [L2] and Graczyk-Swiatek [GS] that regular parameters (with a periodic hyperbolic attractor) are (open and) dense. so at least two kinds of very distinct observable behavior are present on the quadratic family, and they alternate in a complicate way.

Besides regular and stochastic behavior, different behavior was shown to exist, including examples with bad statistics, like absence of a physical measure or a physical measure concentrated on a hyperbolic repeller. Those pathologies were shown to be non-observable in [L3] and [MN]. Finally in [L5] it was concluded the regular or stochastic dichotomy among quadratic maps.

Among stochastic maps, a specific class grabbed lots of attention in the 90's: Collet-Eckmann maps. They are characterized by a positive Lyapunov exponent for the critical value, and gradually they were shown to have 'best possible' near hyperbolic properties: exponential decay of correlations, validity of central limit and large deviations theorems, good spectral properties and zeta functions ([KN], [Y]). Let's call attention to the robustness of the statistical description, with a good understanding of stochastic perturbations: strong stochastic stability ([BV]), rates of convergence to equilibrium ([BBM]).

In [AM1] the regular or stochastic dichotomy was extended to show that almost every stochastic map is actually Collet-Eckmann and satisfy many strong assymptotics, in particular implying the validity of the above mentioned results.

The position of the quadratic family in the borderline of real and complex dynamics made its study a convergence point of many different techniques: most of the deeper results depend on this interaction. It gradually became clear however that studying the quadratic family allows to obtain results on more general unimodal maps.

1.2. **Universality.** Starting with the works of Milnor-Thurston, and also through the discoveries of Feigenbaum and Coullet-Tresser, the quadratic family was shown to be not only a prototype of unimodal maps, but to present universal features both combinatorially and geometrically. More recently, the result of density of hyperbolicity among unimodal maps was obtained in [K] exploiting the validity of this result for quadratic maps.

In [ALM], a general method was devised to transfer information from the quadratic family to real analytic families of unimodal maps. It is shown that the decomposition of spaces of analytic unimodal maps according to combinatorial behavior is essentially a codimension-one lamination.

Thinking of two analytic families as transversals to this lamination, one may try to compare the parameter space of both families via the holonomy map. A straightforward application of this method allowed to conclude that the bifurcation pattern of a general analytic family is locally the same as the quadratic family from the topological point of view (outside of countably many 'bad parameters').

The 'holonomy' method was then succesfully applied to extend the regular or stochastic dichotomy from the quadratic family to a general analytic family. The probabilistic point of view presents new difficulties however. First, the statistical properties of two topologically conjugate maps need not correspond by the (generally not absolutely continuous) conjugacy. Fortunately many properties are preserved, in particular the criterias used by Lyubich in his result.

The second difficulty is that the holonomy map is usually not be absolutely continuous, so typical combinatorics for the quadratic family may not be typical for other families: it is needed to show that the class of regular or stochastic maps is still typical after application of the holonomy map.

1.3. **Results and plan of the proof.** Let's call a k-parameter family good if almost every parameter is either regular or Collet-Eckmann (and some additional technical conditions). Our goal will be to prove that good families are generic. This question naturally makes sense in different spaces of unimodal maps (corresponding to different smoothness).

We start by describing how the holonomy method of [ALM] can be applyed to generalize the results of [AM1] to general analytic families. As a consequence we conclude that essentially all analytic families are good.

To get to the smooth setting (at least C^3, since we are assuming negative Schwarzian derivative), our strategy is different: we show a certain robustness of good families, which together with their denseness (from the analytic case) will yield genericity. Our main tool is one of the nice properties of Collet-Eckmann maps: persistence of the Collet-Eckmann condition under generic unfolding (a result of [T]). Using an abstract scheme we reduce the global result to this local one.

Let's mention that we have obtained the same results without the negative Schwarzian assumption, also allowing us to get to C^2 smoothness. The techniques are very different however, since we do not use anymore the holonomy method we use here (of global nature), which is replaced by a local holonomy analysis and use of a finite version of the 'infinitesimal perturbation' method of [ALM]. For analytic maps this also allowed us to obtain more refined assymptotic estimates which have interesting consequences [AM2] as pathological measure-theoretical behavior of the lamination.

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2. General definitions

2.1. **Notation.** Let $I = [-1, 1]$, B^k be the closed unitary ball in $\mathbb{R}^k$. Let $C^r(I)$ denote the space of C^r maps $f : I \to \mathbb{R}$.

By a unimodal map we will mean a smooth symmetric map $f : I \to I$ with a unique critical point at 0 such that $f(-1) = -1$, $Df(-1) \geq 1$. If f is C^3, we define the Schwarzian derivative as

$$Sf = \frac{D^3 f}{Df} - \frac{3}{2}\left(\frac{D^2 f}{Df}\right)$$

defined on $I \setminus \{0\}$.

For $a > 0$, let $\Omega_a \subset \mathbb{C}$ denote an a neighborhood I.

Let $\mathcal{A}_a$ denote the space of holomorphic maps on Ω_a which have a continuous extension to $\partial\Omega_a$, satisfying $\phi(z) = \phi(-z)$, $\phi(-1) = \phi(1) = -1$ and $\phi'(0) = 0$.

Notice that $\mathcal{A}_a$ is a closed affine subspace of the Banach space of bounded holomorphic maps of Ω_a. We endow it with the induced metric and affine structure.

We define $\mathcal{A}_a^{\mathbb{R}} \subset \mathcal{A}_a$ the space of maps which are real symmetric.

2.2. **More on unimodal maps.** A C^3 unimodal map such that $Sf < 0$ on $I \setminus \{0\}$ and such that its critical point is non-degenerate (that is, $D^2 f \neq 0$) will be called a S-unimodal map.

A k-parameter family of unimodal maps is a map $\Gamma : B^k \times I \to I$ such that for $p \in B^k$, $\gamma_p(x) = \Gamma(p, x)$ is a unimodal map.

A unimodal map f is called Collet-Eckmann (CE) if there exists constants $C > 0$, $\lambda > 1$ such that for every $n > 0$,

$$|Df^n(f(0))| > C\lambda^n.$$

This means that the map is strongly hyperbolic along the critical orbit. It is also useful to study the hyperbolicity o backward iterates of the critical point, so we say that f is backwards CE (BCE) if there exists $C > 0$, $\lambda > 1$ such that for any $n > 0$ and any x with $f^n(x) = 0$, we have

$$|Df^n(x)| > C\lambda^n.$$

By a result of Nowicki (see [MvS]), for S-unimodal maps CE implies BCE, so we will mostly discuss the Collet-Eckmann condition (except for the last section where we consider C^2 unimodal maps as well).

Very often it is useful to estimate how fast is the recurrence of the critical orbit. We will be mainly interested in two kinds of control: polynomial recurrence (P) if there exists $\alpha > 0$ such that

$$|f^n(0)| > n^{-\alpha}$$

for big enough n and subexponential recurrence (SE) if for all $\alpha > 0$,

$$|f^n(0)| > e^{-\alpha n}$$

for n big enough.

We will say that f is weakly regular (WR) if

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \ln(|Df(f^k(0))|) \chi_{(-\delta,\delta)}.$$

We will consider spaces of S-unimodal maps: we define $\mathcal{U}^r \subset C^r(I)$ the set of S-unimodal maps. Spaces of analytic unimodal maps are now easily defined: $\mathcal{U}_a = \mathcal{U}^3 \cap \mathcal{A}_a^{\mathbb{R}}$.

2.3. **The quadratic family.** The quadratic family is the most studied family of unimodal maps. It is usually defined by

$$q_t(x) = t - x^2,$$

so that for $-1/4 \leq t \leq 2$, there exists a unique symmetric interval $I_t = [-\beta_t, \beta_t]$ such that $p_t(I_t) = I_t$ and $p_t(-\beta_t) = -\beta_t$, so p_t can be seen as a unimodal map of I_t (which depends on t). It is also not difficult to see that $Sp_t(x) < 0$ if $x \neq 0$.

By an affine reparametrization of the parameter t and of each interval I_t, we obtain a canonical one-parameter family of S-unimodal maps in the interval I, which we denote p_t, $t \in B^1$, which will be called the quadratic family as well.

2.4. **Quasisymmetric maps.** Let $\gamma \geq 1$ be given. We say that a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is γ-quasisymmetric (γ-qs) if for all $h > 0$

$$\frac{1}{\gamma} \leq \frac{f(x+h) - f(x)}{f(x) - f(x-h)} \leq \gamma.$$

If $X \subset \mathbb{R}$ and $h : X \to \mathbb{R}$ has a γ-quasisymmetric extension to $\mathbb{R}$ we will also say that h is γ-qs.

3. Statement of the results

We would like to classify the typical behavior in generic families of unimodal maps. This classification should reveal refined information on the stochastic description of the dynamics of those typical parameter.

We will therefore consider a smooth enough family of unimodal maps Γ. The techniques of the present paper will need the fact that Γ is a family of S-unimodal maps. This includes two main restrictions: the negative Schwarzian and the quadratic critical point. The first one is serious, since this condition is not dense, but can be removed with more refined techniques (see [A]). The second one is no serious loss of generality, since quadratic critical point is certainly typical among unimodal maps.

We first consider the analytic case. In this case, the generic families we will consider will be families satisfying a non-degeneracy condition: an analytic family of S-unimodal maps is called standard if it contains a regular map. This condition is easy to verify, both in practice (if the family is non-degenerate this can be verified in finite time) as in theory (they are indeed generic for meaningful concepts). Such non-degenerate family will be called standard, and our main result is a dichotomy on the typical dynamical behavior in any such family.

Theorem A. *Let Γ be a standard k-parameter family of S-unimodal maps. Then for almost every $p \in B^k$, f_p is either regular or satisfy CE, P.*

Our second result about standard families is the robustness of a slightly weaker dichotomy under C^2 perturbations of the family.

Theorem B. *Let Γ be a standard k-parameter family of S-unimodal maps. Let Γ_n be a sequence of C^2 families such that $\Gamma^n \to \Gamma$ in the C^2 topology. Let X_n be the set of parameters $p \in B^k$ such that Γ_n is either regular or satisfy BCE, CE, SE and WR. Then $|X_n| \to 1$.*

As a consequence, we can use a Baire argument to conclude that a dichotomy is still valid among topologically generic (residual set) smooth families.

Smooth Dichotomy. *In topologically generic k-parameter C^r, $r = 3, 4, ..., \infty$ families of S-unimodal maps, almost every parameter is either regular or satisfy CE, SE and WR.*

It is good to recall that both types of behavior stablished on the dichothomy are indeed observable for open sets of families of unimodal maps ([J], [BC]).

3.1. **Ergodic meaning.** The importance of the above dichotomy is the fact that each of the two possibilities has very well defined stochastic properties. We quickly recall those (we assume tha maps are S-unimodal).

Regular maps have a periodic attractor whose basin is big both topologically (open and dense set) as in the measure-theoretical sense (full measure). Moreover the attractor and its basin are stable under C^1 perturbations. The dynamics of such maps can be described in deterministic terms.

Maps satisfying CE and SE have non-deterministic dynamics. They can however describe them through their stochastic properties, and it turns out that such maps have the main good properties usually found in hyperbolic maps. First, there is a physical measure, that is an invariant probability which describes assymptotic behavior of orbits: for almost every x and for every continuous $\phi : I \to \mathbb{R}$,

$$\lim \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)) = \int \phi d\mu.$$

This physical measure has a positive Lyapunov exponent and is indeed absolutely continuous and supported on a cycle of intervals, so the assymptotic behavior is non-deterministic. The convergence to the assymptotic stochastic model is exponential, see the results on decay of correlations ([KN], [Y]) and convergence to equilibrium. Those properties are beautifully related to a spectral gap of a transfer operator and to zeta functions, see [KN]. Notice finally that exponential decay of correlations is actually equivalent to the Collet-Eckmann condition (see [NS]).

While the dynamics is highly unstable under deterministic perturbations (nearby maps can be regular for instance), the stochastic description given by the physical measure μ is robust under stochastic perturbations: the perturbed system has a stationary measure which is close to μ in the sense of the L^1 distance between their densities. For studies of decay of correlations for the perturbed systems, see [BBM].

4. Analytic families

4.1. **Hybrid conjugacy and holonomy maps.** A hybrid conjugacy between two S-unimodal maps f and $\tilde{f}$ is a topologic conjugacy which, if there exists an attracting periodic orbit, is locally smooth in the basin of attraction of this periodic orbit.

Two S-unimodal maps $f, \tilde{f}$ are said to be hybrid equivalent if there exists a hybrid conjugacy between them. The set of all maps which are hybrid equivalent to some f is called hybrid class of f.

Theorem 4.1. *Let h be a hybrid conjugacy between two analytic S-unimodal maps f and $\tilde{f}$. Then h is quasisymmetric.*

It follows from a result of Guckenheimer (see [MvS]) that any S-unimodal map f is hybrid conjugate to a unique quadratic map $\chi(f)$. The map χ is called the straightening.

Lemma 4.2. *If f is an analytic S-unimodal map and $\chi(f)$ is either regular or CE and P then f is too.*

Proof. Since regularity is clearly preserved by hybrid conjugacy, we only have to analyze invariance of the conditions CE and P. By Theorem 4.1, we just have to prove invariance under quasisymmetric conjugacy.

By [NP] the Collet-Eckmann condition is invariant under quasisymmetric conjugacy (and even under topological conjugacy). Since quasisymmetric maps are Hölder, it is easy to see that P is also invariant. □

4.2. **Hybrid laminations.** It is natural to study the hybrid class of some map f. This is what is done in Theorem A of [ALM] in the analytic setting, where it is shown that in $\mathcal{U}_a$, every hybrid class is a codimension-one analytic submanifold. Moreover, different hybrid class fit together in some nice structure, called hybrid lamination.

If Γ is a one-parameter family of S-unimodal maps, it can be naturally thought of as an analytic curve in some $\mathcal{U}_a$.

A consequence of the nice structure of the hybrid lamination is the following

Lemma 4.3. *If Γ is not contained in an hybrid class then there is an open set of parameters, with countable complement, where Γ is transverse to the hybrid lamination.*

From Milnor-Thurston kneading theory, we know that any family that is not contained in an hybrid class must contain a regular map.

Standard families are strongly generic among analytic families *An analytic family of S-unimodal maps which is non standard must be contained in some hybrid class. In particular, standard families are dense in $\mathcal{U}^n$, $n = 3, ..., \infty$.*

Definine the map χ_Γ on B^1 defined by $\chi_\Gamma(t) = \chi(\gamma_t)$. In [ALM] the map χ_Γ is considered as the holonomy map from Γ to the quadratic family along the hybrid lamination in some $\mathcal{U}_a$. Using this interpretation, they obtain

Theorem 4.4. *Let Γ be a standard one-parameter family of unimodal maps. If Γ contains a non-regular map, then there is an open set $U \subset B^1$ with countable complement such that the straightening χ_Γ quasisymmetric in any compact interval $J \subset U$.*

4.3. **Dichotomy in the quadratic family.** The main result of [AM1] is that almost every parameter in the quadratic family is either regular or Collet-Eckmann with a polynomiar recurrence of the critical orbit. To obtain the same result for a standard family using Theorem 4.4, we will need a stronger estimate, since quasisymmetric maps are not in general absolutely continuous.

Let's say that a set $X \subset B^1$ has total qs-probability if the image of $B^1 \setminus X$ by any quasisymmetric map $h : B^1 \to B^1$ has 0-Lebesgue measure.

By a slight modification of the proofs in that [AM1] (see appendix), it is possible to obtain

Theorem 4.5. *The set of quadratic maps which are either regular or CE and P has total qs-probability.*

Remark 4.1. In [AM1] it is obtained a better result than polynomial recurrence, namely it is shown that the assymptotyc exponent of the recurrence

$$\limsup_{n\to\infty} \frac{-\ln(f^n(0))}{\ln(n)}$$

is exactly 1 for almost every non regular map. However, for a set of total qs-probability, we are only able to show that the assymptotic exponent is bounded.

4.4. **Proof of Theorem A.** Let Γ be a standard family. If all parameters are regular, there is nothing to prove, so assume that there is a non-regular parameter.

First assume Γ is one-parameter. By Theorems 4.5 and 4.4, for almost every $t \in B^1$, $\chi_\Gamma(t)$ is either regular or satisfies CE and P. By Lemma 4.2, this implies that γ_t is either regular or CE and P.

Assume now that Γ is a k-parameter family. Let $p \in B^k$ be a regular parameter. Let $L : B^1 \to B^k$ be an affine map such $p \in L(B^1)$. Let Γ^L be the one-parameter family defined by $\gamma^L_t = \gamma_{L(t)}$. Then Γ^L is a standard one-parameter family and for almost every t γ^L_t is either regular or CE and P. The result follows by application of Fubini's Theorem.

5. Robustness of the dichotomy

To obtain the robustness claimed on Theorem B our approach will be to exploit an important result of Tsujii, whose core is a strong generalization of Benedicks-Carleson result and techniques. This result establishes that the CE and SE conditions are infinitesimally persistent in one-parameter families unfolding generically: they are density points of CE and SE parameters. The connection with our robustness result, which has a global nature, is done using an abstract (and technical) scheme.

5.1. **Tsujii's theorem.** Let Γ be a C^2 k-parameter family of unimodal maps. Assume that p_0 is a parameter such that γ_{p_0} satisfies CE, BCE, SE and has a quadratic critical point. Tsujii's Theorem considers the case where Γ is a generic unfolding at p_0. For one-parameter families, generic unfolding means precisely

$$\sum_{k=0}^{\infty} \frac{\partial_t \gamma_p(\gamma^j_p(0))}{D\gamma^j_p(f^j(0))} \neq 0,$$

where ∂_t stands for the parameter derivative. This transversality condition will be called Tsujii transversality.

If Γ is a one-parameter family, we will say that (Γ, p_0) satisfies the Tsujii conditions if all above requirements are satisfied.

The following is an immediate consequence of the main theorem of Tsujii in [T].

Theorem 5.1. *Let Γ be a C^2 one-parameter family of unimodal maps. Assume (Γ, t_0) satisfies the Tsujii conditions. Then t_0 is a density point of parameters t for which (Γ, t) satisfies the Tsujii conditions and for which γ_t is WR.*

5.2. **A higher dimensional version.** To pass from one-parameter to k-parameters, we need the following easy proposition. Let's say that $p \in B^k$ is a density point of X along a line l through p if p is a density point of $l \cap X$ in l endowed with the linear Lebesgue measure.

Proposition 5.2. *If p is a density point of X along almost every line, then p is a density point of X in B^k.*

Proof. To compute $d(B_\epsilon(p), X)$, use polar coordinates, and integrate first along the rays, obtaining a density function $\rho_\epsilon(\theta)$, $\theta \in S^{k-1}$. To conclude, use Fatou's Lemma

$$\liminf_{\epsilon \to 0} \int \rho_\epsilon(\theta) d\theta \geq \int \liminf_{\epsilon \to 0} \rho_\epsilon(\theta) d\theta.$$

□

The k-parameter analog of the Tsujii transversality is the existence of a line along which the one-parameter transversality condition is satisfied. In other words, there exists a an affine map $L : B^1 \to B^k$ passing through p_0 such that the induced one-parameter family $\Gamma \circ L$ is Tsujii transverse at the parameter t_0 such that $L(t_0) = p_0$.

Notice that from linearity, if (Γ, p_0) is Tsujii transverse, then all lines passing through p_0 are Tsujii transverse except the lines parallel to a certain codimension-one space of $\mathbb{R}^k$.

Lemma 5.3. *Let Γ be a C^2 k-parameter family of unimodal maps. Assume (Γ, p_0) satisfies the Tsujii conditions. Then p_0 is a density point of parameters p for which (Γ, p) satisfies the Tsujii conditions and for which γ_p is WR.*

Proof. If Γ is Tsujii transverse at p_0 then it is Tsujii transverse along almost every line through p_0. Along such a line it is a density point of parameters satisfying the Tsujii conditions and which are WR. The result follows from proposition 5.2. □

5.2.1. *Tsujii transversality and hybrid lamination.* Let's take a closer look at the Tsujii transversality for an analytic Γ. Let $\gamma_p = f$.

Assuming

$$\sum_{k=0}^{\infty} \frac{1}{Df^k(f(0))} < \infty \tag{5.1}$$

(in particular if f is CE), let

$$\nu_f(v) = \sum_{k=0}^{\infty} v(f^k(0))/Df^k(f(0))$$

be a functional defined on continuous vectorfields v on the interval. It is easy to see (under the summability condition) that ν_f does not vanish on the set of symmetric polynomials.

Lemma 5.4. *The kernel of ν_f intersected with $\mathcal{A}_a^{\mathbb{R}}$ is the tangent space to the hybrid class of f.*

Proof. Since ν is non-trivial over symmetric polynomials the above intersection is a closed codimension-one subspace of $\mathcal{A}_a^{\mathbb{R}}$. It is enough to show that if v is tangent then $\nu_f(v) \neq 0$. It is remarked in [ALM] that

$$Df^n(f(0)) \sum_{k=0}^{n-1} v(f^k(0))/Df^k(f(0))$$

is precisely $\partial_t(\gamma_t^v)^{n+1}(0)$, and this sequence is bounded independent of n if v is tangent. Since the summability condition (5.1) $|Df^n(f(0))| \to \infty$, we have necessarily $\nu_f(v) = 0$. ☐

So Tsujii transversality can be interpreted for such a map (satisfying the summability condition (5.1)) as transversality of the family to the hybrid class of γ_p.

Since for maps with negative Schwarzian derivative CE implies the BCE, we can conclude from Theorem A, Lemma 4.3 and this discussion the following

Lemma 5.5. *If Γ is a standard k-parameter family then almost every parameter is regular or satisfies the Tsujii conditions.*

5.3. Generalities about density points.

5.3.1. *Density points in abstract.* Let $\mathcal{U}^2$ be the space of C^2 unimodal maps (without, naturally, the hypothesis of negative Schwarzian derivative), and K be the space of C^2 k-parameter families of unimodal maps.

Let $\mathcal{X} \subset K \times B^k$ be the set of (Γ, p) such that either γ_p is regular or satisfies the Tsujii conditions and WR. For $\Gamma \in K$, let $X_\Gamma = \{p \in B^k | (\Gamma, p) \in X\}$.

Let $Y \subset B^k$ be measurable with $|Y| > 0$. We define the density of $\mathcal{X}$ along Γ on Y as

$$d(\Gamma, Y) = \frac{|Y \cap X_\Gamma|}{|Y|}.$$

Instead of defining the classical infinitesimal density:

$$\liminf_{\epsilon \to 0} d(\Gamma, B_\epsilon(p))$$

we will need to consider the stability of the density with respect to perturbations of Γ. With this in mind we introduce two parameters. Let

$$D^-(\Gamma^0, p) = \liminf_{\substack{\Gamma \to \Gamma^0 \\ \gamma_p = \gamma_p^0}} \liminf_{\epsilon \to 0} d(\Gamma, B_\epsilon(p)),$$

$$D^+(\Gamma^0, p) = \liminf_{\epsilon \to 0} \liminf_{\Gamma \to \Gamma^0} d(\Gamma, B_\epsilon(p)).$$

Tsujii's result gives a direct way to estimate D^-

Lemma 5.6. *Let* Γ *be a standard family. Then for almost every* $p \in B^k$*,* $D^-(\Gamma, p) = 1$*.*

However, for computations of stability D^+ is more relevant. We proceed to discuss the effect of the interchange of limits in the definition of D^- and D^+.

5.3.2. *Approximation condition.* Let's say that (Γ^0, p_0) is well approximable if for any $\Gamma^n \to \Gamma^0$, $\epsilon_n \to 0$ and any neighborhood $\mathcal{V}$ of Γ^0, there exists sequences k_n and l_n such that for any sequence $m_n \geq l_n$ we can find a family $\tilde{\Gamma} \in \mathcal{V}$ such that

$$\lim_{n\to\infty} \frac{|\{p \in B_{\epsilon_{k_n}}(p_0) | \tilde{\gamma}_p \neq \gamma_p^{m_n}\}|}{|B_{\epsilon_{k_n}}|} = 0$$

and furthermore $\tilde{\gamma}_{p_0} = \gamma^0_{p_0}$) (this actually follows from the last property).

Lemma 5.7. *For any* $\Gamma \in \mathcal{K}$ *and* $p \in B^k$*,* (Γ, p) *is well approximable.*

Proof. We use the following easy interpolation result, which is a consequence of existence of C^∞ partitions of unity (this interpolation obviously fails for analytic maps):

Let $U \subset B^k$ *be open. For each compact* $K \subset U$*, for all* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that if* Γ' *is* δ*-close to* Γ *then there exists* Γ'' ϵ*-close to* Γ *coinciding with* Γ *out of* U *and coinciding with* Γ' *in* K*.*

We can now obtain $\tilde{\Gamma}$ in the following way. Consider k_n such that $\epsilon_{k_{n+1}}/\epsilon_{k_n} \to 0$, and let $U_n = \operatorname{int}(B_{\epsilon_{k_n}} \setminus B_{\epsilon_{k_{n+1}}})$, $K_n \subset U_n$ compact such that $|K_n|/|U_n| \to 1$.

If we now choose l_n growing fast enough, for $m_n \geq l_n$ then Γ_{m_n} is very close to Γ and the above interpolation result shows that we can interpolate Γ and Γ_{m_n} inside each U_n, obtaining a family $\tilde{\Gamma}$, ϵ-close to Γ, such that for each n, $\tilde{\Gamma}$ and Γ_{m_n} coincide in K_n. □

Lemma 5.8. *In this setting,*

$$D^+(\Gamma^0, p_0) \geq D^-(\Gamma^0, p_0).$$

Proof. Take sequences Γ^n and ϵ_j satisfying

$$\lim_{j\to\infty} \liminf_{n\to\infty} d(\Gamma^n, B_{\epsilon_j}(p_0)) = D + (\Gamma^0, p_0).$$

and fix a neighborhood $\mathcal{V}$ of Γ^0. Let k_n, l_n be the sequences from the approximation condition. Let $m_n > l_n$ be a sequence such that

$$\lim_{n\to\infty} d(\Gamma^{m_n}, B_{\epsilon_{k_n}}(p_0)) = D + (\Gamma^0, p_0).$$

and let $\tilde{\Gamma}$ be as in the approximation condition. Then it is clear that

$$\lim_{n\to\infty} d(\tilde{\Gamma}, B_{\epsilon_{k_n}}(p_0)) = D^+(\Gamma^0, p_0).$$

Since we can get $\tilde{\Gamma}$ to approximate Γ^0 by shrinking $\mathcal{V}$, we get $D^+ \geq D^-$. ☐

5.3.3. *Proof of Theorem B.* Let Γ be a standard analytic family. Then almost every parameter satisfy $D^-(\Gamma, p) = 1$. In particular, for almost every p we have $D^+(\Gamma, p) = 1$.

Fix $\epsilon > 0$. Let $p \in B^k$ be such that $D^+(\Gamma, p) = 1$. By definition of D^+ there exists a sequence of balls $U^n(p)$ centered on p and neighborhoods $\mathcal{V}^n(p)$ such that if $\tilde{\Gamma} \in \mathcal{V}^n(p)$ then

$$d(\tilde{\Gamma}, U^n(p)) > 1 - \epsilon/2.$$

By Vitali's Lemma, there exists finite sequences p_k, n_k such that $U^{n_k}(p_k)$ are disjoint and $|\cup U^n(k)| > 1 - \epsilon/2$. Let $\mathcal{V} = \cap \mathcal{V}^{n_k}(p_k)$. Then if $\Gamma \in \mathcal{V}$, $d(\tilde{\Gamma}, B^k) > 1 - \epsilon$.

6. Appendix

Our aim here is to sketch a proof of Theorem 4.5. This proof is very similar to the one of the main results of [AM1], a full proof of this result is given in [AM3]. We think however, that it is worthwhile to present in a concise way the main differences between both results.

In what follows, we will use the notation and concepts of [AM1].

6.1. **Simple maps.** The starting point of [AM1] is the following result due to Lyubich: almost every real quadratic map is either regular or simple (this implies the regular or stochastic dichotomy, since simple maps are stochastic).

In [ALM], it is remarked that the proof of Lyubich actually shows that the set of regular or simple maps has total qs-probability. We can use this as the starting point for our purposes: it is enough to show that CE and P have total qs-probability among simple maps. We will prove this by describing (with total qs-probability) the eventual behavior of the principal nest.

Let's say that a set Y has total k-qs probability in a set X if $Y \subset X$ and for any $h : \mathbb{R} \to \mathbb{R}$ which is k-qs we have $|h(Y \setminus X)| = 0$. If X is the set of simple maps, we will say for shork that Y has total k-qs probability.

We will need the following version of the Borel-Cantelli Lemma

Lemma 6.1. *Let X be a set of simple maps with total k-qs probability. Let Q_n be the set of all simple maps f with some (bad) property related to the n-th stage of the principal nest of f. And let $Y \subset X$ be the set of maps belonging to (at most) finitely many Q_n. Let $q_n(f)$ be either $p_k(Q_n \cap J_n[f] | J_n[f])$ or $p_k(Q_n \cap J_n^{\tau_n}[f] | J_n^{\tau_n}[f])$. If for any $f \in X$, $\sum_{n=1}^{\infty} q_n(f) < \infty$, then Y has total k-qs probability.*

6.2. **Initial estimates.** From now on, we fix some $\gamma > k$

6.2.1. *Decay of geometry.* Following [AM1], we first estimate the distribution of the length $|\underline{d}^{(n)}(x)|$ of branches of the landing map to I_{n+1}. As a first application, we obtain an estimate for s_n.

Lemma 6.2. *There exists $0 < a < 1 < b$ such that with total k-qs probability, for all n sufficiently big we have for all $k > 0$,*

1. $$p_\gamma(|\underline{d}^{(n)}(x)| < k | x \in I_n) < k c_n^a,$$
2. $$p_\gamma(|\underline{d}^{(n)}(x)| > k | x \in I_n) < e^{-k c_n^b},$$
3. $$p_\gamma(|\underline{d}^{(n)}(x)| < k | x \in I_n^{\tau_n}) < k c_n^a,$$
4. $$p_\gamma(|\underline{d}^{(n)}(x)| > k | x \in I_n^{\tau_n}) < e^{-k c_n^b}.$$

Using PhPa1 we obtain

Lemma 6.3. *There exists $0 < a < 1 < b$, such that with total k-qs probability, for all n sufficiently big we have*

$$c_n^{-a} < s_n < c_n^{-b}.$$

Which implies that with total k-qs probability c_n decays at least torrentially.

6.2.2. *Derivatives.* We then proceed to estimate derivatives of branches of the return map. For this we must initially exclude the possibility of a 'too recurrent' or 'too low' return. This is easy using PhPa2.

Lemma 6.4. *There exists $b > 1$ such that with total k-qs probability, for all n sufficiently big we have*

$$d(R_n(0), \partial I_n \cup \{0\}) > n^{-b} |I_n|.$$

The other tool is to estimate the distortion, which is done using PhPa1 to obtain

Lemma 6.5. *There exists $b > 1$ such that with total k-qs probability, for all n sufficiently big, for all $\underline{d} \in \Omega$ we have*

$$\mathrm{dist}(R_n^{\underline{d}}) < n^b.$$

Since return maps are torrentially expansive on average (from the decay of geometry), we get

Lemma 6.6. *With total k-qs probability, for all n sufficiently big, for all $j \neq 0$ and any $x \in I_n^j$, $|R_n'(x)| > 1$.*

The above estimates allow a direct estimate of derivatives under iterates of f.

Lemma 6.7. *With total k-qs probability, if n is sufficiently big and if $x \in \cup_{j\neq 0} I_n^j$ and $R_n|_{I_n^j} = f^r$, then for $1 \leq k \leq r$, $|(Df^k(x))| > |x| c_{n-1}^3$.*

6.3. **Distribution of return times.** We proceed to estimate the distribution of return times. The main tool is a Large Deviations estimate.

Lemma 6.8. *There exists $b > 1$, such that with total k-qs probability, such that for all n sufficiently big we have, for $k > 1$,*

$$p_\gamma(r_n(x) > k c_{n-1}^{-b} | x \in I_n) < e^{-k}.$$

In the course of this estimate it is also stablished

Lemma 6.9. *There exists $b > 1$, such that with total k-qs probability for all n sufficiently big we have, $v_n < c_{n-1}^{-b}$.*

This has many consequences (for instance, c_n decays torrentially). We summarize below the more important ones.

Lemma 6.10. *There exist $0 < a < 1 < b$, such that with total k-qs probability, for all n sufficiently big we have*

1. $p_\gamma(l_n(x) < c_n^{-s} | x \in I_n) < c_n^{a-s}$, *with* $s > 0$,
2. $p_\gamma(l_n(x) < c_n^{-s} | x \in I_n^{\tau_n}) < c_n^{a-s}$, *with* $s > 0$,
3. $p_\gamma(l_n(x) > c_n^{-s} | x \in I_n) < e^{-c_n^{b-s}}$, *with* $s > b$,
4. $p_\gamma(l_n(x) > c_n^{-s} | x \in I_n^{\tau_n}) < e^{-c_n^{b-s}}$, *with* $s > b$,
5. $p_\gamma(r_n(x) < c_{n-1}^{-s} | x \in I_n) < c_{n-1}^{a-s}$, *with* $s > 0$,
6. $p_\gamma(r_n(x) > c_n^{-s} | x \in I_n) < e^{-c_{(n-1)}^{b-s}}$ *with* $s > b$.

6.4. **Dealing with hyperbolicity.** In this section we show by an inductive proccess that the great majority of branches are reasonably hyperbolic. In order to do that, in the following subsection, we define some classes of branches with 'good' distribution of times and which are not too close to the critical point. The definition of 'good' distribution of times has an inductive component: they are composition of many 'good' branches of the previous level. The fact that most branches are good is related to the validity of some kind of Law of Large Numbers estimate.

We define as in [AM1] the sequences $\rho_n = (n+1)/n$ and $\mathrm{tr}_n = (2n+3)/(2n+1)$, so that $\rho_n > \mathrm{tr}_n > \rho_{n+1}$ and $\lim \rho_n = 1$. We define the sequence $\gamma_n = \gamma\rho_n$ and an intermediate sequence $\tilde{\gamma}_n = \gamma \mathrm{tr}_n$.

6.4.1. *Some kinds of branches and landings.* Standard landings

We define the set of standard landings at time n, $LS(n) \subset \Omega$ as the set of all $\underline{d} = (j_1, ..., j_m)$ satisfying the following.

LS1: (m not too small or large) $c_n^{-a/2} < m < c_n^{-2b}$,

LS2: (No very large times) $r_n(j_i) < c_{n-1}^{-3b}$ for all i.

LS3: (Short times are sparse in not too small initial segments) For $c_{n-1}^{-2b} \leq k \leq m$

$$\#\{r_n(j_i) < c_{n-1}^{-a/2}\} \cap \{1, ..., k\} < (6 \cdot 2^n) c_{n-1}^{a/2} k.$$

We also define the set of fast landings at time n, $LF(n) \subset \Omega$ by the following condition.

LF: (m small) $m < c_n^{-a/2}$.

The following lemma is analogous to lemma 10.1 of [AM1].

Lemma 6.11. *With total probability, for all n sufficiently big,*

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n) | x \in I_n) < c_n^{a/3}/2,$$

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin (LS(n) \cup F(n)) | x \in I_n) < c_n^{2n}/2,$$

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n) | x \in I_n^{\tau_n}) < c_n^{a/3}/2,$$

and

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin (LS(n) \cup F(n)) | x \in I_n^{\tau_n}) < c_n^{2n}/2.$$

6.4.2. *Very good returns , bad returns and excellent landings.* Define the set of very good returns, $VG(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0, n \in \mathbb{N}$ and the set of bad returns, $B(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0, n \in \mathbb{N}$, $n \geq n_0$ by induction as follows. We let $VG(n_0, n_0) = \mathbb{Z} \setminus \{0\}$,$B(n_0, n_0) = \emptyset$ and supposing $VG(n_0, n)$ and $B(n_0, n)$ defined, define the set of excellent landings $LE(n_0, n) \subset LS(n)$ satisfying the following extra assumptions.

LE1: (Not very good moments are sparse in not too small initial segments) For all $c_{n-1}^{-2b} < k \leq m$

$$\#\{i | j_i \notin VG(n_0, n)\} \cap \{1, ..., k\} < 6 \cdot 2^n c_{n-1}^{a/3} k,$$

LE2: (Bad moments are sparse in not too small initial segments) For all $c_n^{-1/n} < k \le m$

$$\#\{i | j_i \notin B(n_0, n)\} \cap \{1, ..., k\} < 6 \cdot 2^n c_{n-1}^n k,$$

We define $VG(n_0, n+1)$ as the set of j such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ with $\underline{d} \in LE(n_0, n)$ and the extra condition.

VG: (distant from 0) The distance of I_{n+1}^j to 0 is bigger than $c_n^{n^2} |I_{n+1}|$.

And we define $B(n_0, n+1)$ as the set of $j \notin VG(n_0, n+1)$ such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ with $\underline{d} \notin LF(n)$.

Lemma 6.12. *With total probability, for all n_0 sufficiently big,*

$$p_{\gamma_n}(j^{(n)}(x) \notin VG(n_0, n) | x \in I_n) < c_{n-1}^{2a/5}.$$

$$p_{\gamma_n}(j^{(n)}(x) \in B(n_0, n) | x \in I_n) < c_{n-1}^{2n}.$$

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0, n) | x \in I_n) < 2 c_n^{2a/5}/3,$$

and

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0, n) | x \in I_n^{\tau_n}) < 2 c_n^{2a/5}/3.$$

Using PhPa2 we get

Lemma 6.13. *With total probability, for all n_0 big enough, for all n big enough, $\tau_n \in VG(n_0, n)$.*

Lemma 6.14. *With total probability, for all n_0 big enough and for all $n \ge n_0$, if $j \in VG(n_0, n+1)$ then*

$$m < r_{n+1}(j) < m c_{n-1}^{-2b},$$

where as usual, m is such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ and $\underline{d} = (j_1, ..., j_m)$.

The next lemma is a version of Lemma 10.6 of [AM1] , but its proof changes a little bit , and so will be explicitely given . In the proof we will use the notions of fast and bad branches that we introduced above.

Let $j \in VG(n_0, n+1)$. We can write $R_{n+1}|_{I_{n+1}^j} = f^{r_{n+1}(j)}$, that is, a big iterate of f. One may consider which proportion of those iterates belong to very good branches of the previous level. More generally, we can truncate the return R_{n+1}, that is, we may consider $k < r_{n+1}(j)$ and ask which proportion of iterates up to k belong to very good branches.

Lemma 6.15. *With total probability, for all n_0 big enough and for all $n \ge n_0$, the following holds.*

Let $j \in VG(n_0, n+1)$, as usual let $R_n(I^j_{n+1}) = I^{\underline{d}}_n$ and $\underline{d} = (j_1, ..., j_m)$. Let m_k be biggest possible with

$$v_n + \sum_{j=1}^{m_k} r_n(j_i) \leq k$$

(the amount of full returns to level n before time k) and let

$$\beta_k = \sum_{\substack{1\leq i\leq m_k,\\ j_i\in VG(n_0,n)}} r_n(j_i).$$

(the total time spent in full returns to level n which are very good before time k) Then $1 - \beta_k/k < c_{n-1}^{a/5}$ if $k > c_n^{-2/n}$.

Proof. Let's estimate first the time i_k which is not spent on full returns:

$$i_k = k - \sum_{j=1}^{m_k} r_n(j_i).$$

This corresponds exactly to v_n plus some incomplete part of the return $j_{m_{k+1}}$. This part can be bounded by $c_{n-1}^{-b} + c_{n-1}^{-3b}$ (use the estimate of v_n and LS2 to estimate the incomplete part).

Using LS2 we conclude now that

$$m_k > (k - c_{n-1}^{-b})c_{n-1}^{3b} > c_n^{-1/n}$$

so m_k is not too small.

Let's now estimate the contribution h_k from bad full returns j_i . The number of such returns must be less than $c_{n-1}^{n/2} m_k$, so their total time is at most $c_{n-1}^{(n/2)-3b} m_k < m_k$.

The non very good full returns on the other hand can be estimated by LE1 (given the estimate on m_k), they are at most $c_{n-1}^{a/4} m_k$. So we can estimate the total time l_k of non very good full returns with time less then $c_{n-1}^{-a/2}$ by

$$c_{n-1}^{a/4} c_{n-1}^{-a/2} m_k,$$

while β_k can be estimated from below by

$$(1 - c_{n-1}^{a/4}) c_{n-1}^{-a/2} m_k.$$

It is easy to see then that $i_k/\beta_k \ll c_{n-1}^{a/5}$, $h_k/\beta_k \ll c_{n-1}^{a/5}$. We also have

$$l_k/\beta_k < 2c_{n-1}^{a/4}.$$

So $(i_k + h_k + l_k)/\beta_k$ is less then $c_{n-1}^{a/5}$. Since $i_k + h_k + l_k + \beta_k = k$ we have $1 - \beta_k/k < (i_k + h_k + l_k)/\beta_k$. □

6.4.3. *Cool landings.* Let's define the set of cool landings $LC(n_0, n) \subset \Omega$, $n_0, n \in \mathbb{N}$, $n \geq n_0$ as the set of all $\underline{d} = (j_1, ..., j_m)$ in $LE(n_0, n)$ satisfying.

LC1: (Starts very good) $j_i \in VG(n_0, n)$, $1 \leq i \leq c_{n-1}^{-a/3}$.
LC2: (Short times are sparse in not too small initial segments) For $c_{n-1}^{-a/2} \leq k \leq m$

$$\#\{r_n(j_i) < c_{n-1}^{-a/2}\} \cap \{1, ..., k\} < (6 \cdot 2^n) c_{n-1}^{a/3} k,$$

LC3: (Not very good moments are sparse in not too small initial segments) For all $c_{n-1}^{-a/4} < k \leq m$

$$\#\{i | j_i \notin VG(n_0, n)\} \cap \{1, ..., k\} < (6 \cdot 2^n) c_{n-1}^{a/5} k,$$

LC4: (Bad times are sparse in not too small initial segments) For $c_{n-1}^{-n/3} \leq k \leq m$

$$\#\{i | j_i \in B(n_0, n)\} \cap \{1, ..., k\} < (6 \cdot 2^n) c_{n-1}^{n/6} k,$$

LC5: (Starts with no bad times) $j_i \notin B(n_0, n)$, $1 \leq i \leq c_{n-1}^{-n/2}$.

Lemma 6.16. *With total probability, for all n_0 sufficiently big and all $n \geq n_0$,*

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LC(n_0, n) | x \in I_n) < c_{n-1}^{a/10}$$

and for all n big enough

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LC(n_0, n) | x \in I_n^{\tau_n}) < c_{n-1}^{a/10}.$$

Lemma 6.17. *With total probability, for all n_0 big enough, for all n big enough we have $R_n(0) \in LC(n_0, n)$.*

6.4.4. *Hyperbolicity.* Preliminaries

For $j \neq 0$, we define

$$\lambda_n^{(j)} = \inf_{x \in I_n^j} \frac{\ln(|R_n'(x)|)}{r_n(j)}.$$

And $\lambda_n = \inf_{j \neq 0} \lambda_n^{(j)}$. As a consequence of the exponential estimate on distortion, together with hyperbolicity of f in the complement of I_n^0 we immediately have the following.

Lemma 6.18. *With total probability, for all n sufficiently big, $\lambda_n > 0$.*

6.4.5. *Good branches.* We define the set of good returns $G(n_0,n) \subset \mathbb{Z} \setminus \{0\}$, $n_0, n \in \mathbb{N}$, $n \geq n_0$ as the set of all j such that

G1: (hyperbolic return)

$$\lambda_n^{(j)} \geq \lambda_{n_0} \frac{1+2^{n_0-n}}{2},$$

G2: (hyperbolicity in partial return) for $c_{n-1}^{-3/(n-1)} \leq k \leq r_n(j)$ we have

$$\inf_{I_n^j} \frac{\ln(|Df^k|)}{k} \geq \lambda_{n_0} \frac{1+2^{n_0-n+1/2}}{2} - c_n^{-4/(n-1)}.$$

Notice that since c_n decreases torrentially, for n sufficiently big G2 implies for $c_{n-1}^{-3/(n-1)} \leq k \leq r_n(j)$ we have

$$\inf_{I_n^j} \frac{\ln(|Df^k|)}{k} \geq \lambda_{n_0} \frac{1+2^{n_0-n}}{2}.$$

Lemma 6.19. *With total probability, for n_0 big enough, for all $n > n_0$, $VG(n_0,n) \subset G(n_0,n)$.*

6.4.6. *Hyperbolicity in cool landings.*

Lemma 6.20. *With total probability, if n_0 is sufficiently big, for all n sufficiently big, if $\underline{d} \in LC(n_0, n+1)$ then for all $c_{n-1}^{-4/(n-1)} < k \leq l_n(\underline{d})$,*

$$\inf_{C_n^{\underline{d}}} \frac{\ln(|Df^k|)}{k} \geq \frac{\lambda_{n_0}}{2}.$$

With these results we can adapt the proofs of Theorems A and B of [AM1] and obtain Theorem 4.5.

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