PHASE-PARAMETER RELATION AND SHARP STATISTICAL PROPERTIES FOR GENERAL FAMILIES OF UNIMODAL MAPS

ARTUR AVILA AND CARLOS GUSTAVO MOREIRA

Abstract. We obtain precise estimates relating the phase space and the parameter space of analytic families of unimodal maps, which generalize the case of the quadratic family obtained in [AM1]. This result implies a statistical description of the dynamics of typical analytic *quasiquadratic maps* which is much sharper than what was previously known: as an example, we can conclude that the recurrence of the critical point is polynomial *with exponent one*. To complete the picture, we show that typical analytic non-regular unimodal maps admit a quasiquadratic renormalization, so that the previous result applies also without the quasiquadratic assumption. Those ideas lead to a proof of a theorem of Shishikura: the set of non-renormalizable parameters in the boundary of the Mandelbrot set has zero Lebesgue measure. Further applications of those results can be found in [AM3].

Contents

1. Introduction

A *unimodal map* is a smooth (at least C^2) map $f : I \to I$, where $I \subset \mathbb{R}$ is an interval, which has one unique critical point $c \in \operatorname{int} I$ which is a maximum. Let us say that f is *regular* if it has a quadratic critical point, is hyperbolic and its critical point is not periodic or preperiodic. By a result of Kozlovski [K2], the set of regular maps coincides with the set of structurally stable unimodal maps, and

Date: December 19, 2004.
Partially supported by Faperj and CNPq, Brazil.

it follows that the set of regular maps is open and dense in all smooth (and even analytic) topologies.

A central problem in dynamical systems is to give a good statistical description of "typical systems", for some reasonable measure-theoretical notion of typical: Ideally (according to the Palis Conjecture [P]) the set of systems with a good statistical description should correspond to a Lebesgue full measure set in (a large set of) parametrized families.

In order to establish such a picture, one should understand well how the dynamics varies with the parameter. In one-dimensional dynamics, a basic approach has been to investigate thoroughly the dynamics of an individual map and show that some of its properties are reflected on the nearby parameter space. As described by Adrien Douady: "You first plow in the dynamical plane and then harvest in the parameter plane". More specifically, in the situations we will discuss one uses the dynamics to define tilings of both the phase and parameter spaces. Estimates on the geometry of the tilings are done first in the phase space and later one tries to transfer them to the parameter space.[1]

The most studied family of unimodal maps is the quadratic family $p_\lambda = \lambda - x^2$, $-1/4 \leq \lambda \leq 2$. The analysis of the relation between phase space and parameter space in this case has been efficiently carried out using complex methods. Estimates of this type are central to the proof of Yoccoz [H] of the local connectivity of the Mandelbrot set for finitely renormalizable parameters. In [L3], Lyubich used holomorphic motions[2] to relate phase and parameter in a rather robust way (in the finitely renormalizable case). This was used in a probabilistic argument to show that almost every non-regular finitely renormalizable p_λ satisfies the Martens-Nowicki criterion [MN] for the existence of an absolutely continuous invariant measure. Later in [L5] Lyubich also established that infinitely renormalizable parameters have zero Lebesgue measure, thus concluding that almost every non-regular p_λ is stochastic (admits an absolutely continuous invariant measure).

In [AM1], the analysis of [L3] was pushed further. It was shown that for a typical non-regular quadratic map p_{λ_0}, the phase space of p_{λ_0} near the critical point 0 and the parameter space near λ_0 are related by some metric rules called the *Phase-Parameter relation*. Informally, those rules state that a phase-parameter map corresponding to certain tilings (arising naturally in the consideration of first return maps to some small intervals around the critical point) is quasisymmetric with good constants. The importance of this particular set of rules is that it can be efficiently used in probabilistic arguments (in [AM1] those rules are used to establish the Collet-Eckmann condition and polynomial recurrence for the critical orbit for almost every non-regular parameter, we will come back to those properties later).

The proof of [L3], and hence of [AM1], was tied to the combinatorial theory of the Mandelbrot set, so it can only work for quadratic maps (or, at most, full unfolded families of quadratic-like maps, see [L3]).

Let us say that an analytic family of unimodal maps (depending on finitely many parameters) is *non-trivial* if regular parameters are dense. Such a definition has the advantage of brevity, but at first it may seem to be too strong. However, it actually

[1]In order to have this picture it is of course convenient that the phase and parameter space have the same dimension.

[2]Here it is crucial that the parameter space has dimension one.

can be shown that it corresponds to two natural non-degeneracy conditions: there is no open subset of parameters where either (1) there is a persistently degenerate periodic orbit or (2) the family is contained in a leaf of a certain codimension-one lamination (the "hybrid lamination" of [ALM]). In particular non-trivial analytic families are dense in any topology. The first main result of this paper is the following (see §7 for the precise definition of the Phase-Parameter relation):

Theorem A. *Let f_λ be a one-parameter non-trivial analytic family of unimodal maps. Then f_λ satisfies the Phase-Parameter relation at almost every non-regular parameter.*

As previously discussed, the Phase-Parameter relation has many remarkable consequences for the study of the dynamical behavior of typical parameters. Our second main result is an application of the Phase-Parameter relation:

Theorem B. *Let f_λ be a non-trivial analytic family of unimodal maps.*[3] *Then almost every parameter is either regular or has a renormalization which is topologically conjugate to a quadratic polynomial.*

This result allows one to reduce the study of typical unimodal maps to the special case of unimodal maps which are quasiquadratic (i.e., persistently topologically conjugate to a quadratic polynomial).

1.1. **Statistical properties of typical unimodal maps.** Typical quasiquadratic maps had been previously studied in [ALM], [AM2], which extend several results ([L3], [MN], [L5] and [AM1]) first obtained for the quadratic family. In particular it was concluded that the dynamics of typical quasiquadratic maps have an excellent statistical description (in terms of physical measures, decay of correlations and stochastic stability), thus answering the Palis Conjecture (see [AM2] for details) in the unimodal quasiquadratic case.

For regular maps, the good statistical description comes for free. For a non-regular map f, it is related to essentially two properties regarding its critical point c: the Collet-Eckmann condition[4] and subexponential recurrence[5].

Thus, [AM2] achieves the good statistical description via a dichotomy: typical quasiquadratic maps are either regular or Collet-Eckmann and subexponentially recurrent. This is done in both the analytic case and the smooth case (C^k, $k = 3, ..., \infty$). For typical non-regular analytic quasiquadratic maps, it is proved even more, that the critical point is polynomially recurrent[6].

Our Theorem B allows us to immediately obtain the analytic case in our more general setting (see Theorem 10.1 for a more precise statement):

Corollary C. *Let f_λ be a non-trivial analytic family of unimodal maps. Then almost every non-regular parameter is Collet-Eckmann and its critical point is polynomially recurrent.*

This allows us not only to generalize the smooth case of [AM2] besides quasiquadratic maps, but to reduce the differentiability requirements, including the C^2 case in the description (see Theorem 10.3 for a more precise statement):

[3]Notice that here we do not assume that the parameter space is one-dimensional.

[4]A unimodal map f is Collet-Eckmann if $|Df^n(f(c))| > C\lambda^n$ for some constants $C > 0$ and $\lambda > 1$.

[5]That is, for every $\alpha > 0$, $|f^n(c) - c| > e^{-\alpha n}$ for n sufficiently big.

[6]That is, there exists $\gamma > 0$ such that $|f^n(c) - c| > n^{-\gamma}$ for every n sufficiently big.

Corollary D. *In generic smooth (C^k, $k = 2, ..., \infty$) families of unimodal maps, almost every parameter is regular, or has a renormalization which is conjugate to a quadratic map, is Collet-Eckmann and its critical point is subexponentially recurrent.*

Remark 1.1. The dichotomies in Corollaries C and D imply that the dynamics of typical non-regular unimodal maps have the same excellent statistical description of the quasiquadratic case studied by [AM2] (some of the statistical properties one obtains are discussed in Remark 10.1). In particular, our Corollaries C and D give an answer to the Palis Conjecture in the general unimodal case.

1.2. **Sharpness.** The Phase-Parameter relation allows one to obtain very precise estimates on the dynamics of typical parameters. For instance, the statistical analysis of [AM1] could compute the exact exponent of the polynomial recurrence[7] in the case of the quadratic family. The method used in [AM2] to extend results from the quadratic family to other non-trivial families of quasiquadratic maps (based on comparison of the respective parameter spaces) introduces unavoidable distortion and can not be used to estimate the exponent of the recurrence even in the quasiquadratic case. Our Theorem A implies that the same sharp estimates obtained for the quadratic family remain valid for general analytic maps.

Corollary E. *Let f_λ be a non-trivial analytic family of unimodal maps. Then almost every parameter is either regular or has a polynomially recurrent critical point with exponent 1.*

We call the attention of the reader to [AM3] where much more refined statistical applications of Theorem A are obtained. Those results are inaccessible by the methods of [AM2], and indeed are used to show the limitations of estimates based on comparison of parameter spaces of different families with respect to direct Phase-Parameter estimates.

1.3. **Complex parameters.** A very natural question raised by the description of typical parameters in the real quadratic family is if the results generalize to complex parameters. Actually it can be hoped for

In this direction, let us remark that the argument of the proof of Theorem B can be also applied in the complex setting. In this setting, it gives a proof of the following result of Shishikura (unpublished, a sketch can be found as Theorem 4 in [Sh]):

Theorem F. *The set of non-hyperbolic, non-infinitely renormalizable complex quadratic parameters has zero Lebesgue measure.*

We discuss this application in Appendix B.

1.4. **Outline of the proof of Theorem A.** The proof of Theorem A can be divided in four parts. The crucial step of this paper is step (2) below, which allows us to integrate the work of [AM1] and [ALM].

(1) Following [L3] and the Appendix of [AM1], we describe a complex analogous of the Phase-Parameter relation for certain families of complex return type maps, which model complex extensions of the return maps $R_n : I_n \to I_n$ to the principal

[7] The exponent of the polynomial recurrence of the critical point c of a unimodal map f is the infimum of all $\gamma > 0$ such that, for n sufficiently big, $|f^n(c) - c| > n^{-\gamma}$.

nest of a unimodal map f. This study is restricted to the class of so called full families.
(2) We show that through any given analytic unimodal map f which is at most finitely renormalizable with a recurrent critical point, there exists an analytic family $\tilde{f}_\lambda$ (constructed explicitly) which gives rise (after a generalized renormalization procedure) to a full family of complex return type maps. Using step (1), we conclude that the Phase-Parameter relation is valid at f for this special family $\tilde{f}_\lambda$. By construction, this family is tangent to a certain special infinitesimal perturbation considered in [ALM], where this perturbation had been shown to be transverse to the topological class of f (which is a codimension-one analytic submanifold).
(3) We show that if the Phase-Parameter relation is valid for one transverse family at f, then it is valid for all transverse families at f. This step is heavily based on the results of [ALM]: in order to compare the parameter space of both families, one uses the local holonomy of the lamination associated to the partition of spaces of unimodal maps into topological classes.
(4) Using a simple generalization of [ALM] we conclude that a non-trivial family of unimodal maps is transverse to the topological class of almost every non-regular parameter, and that typical parameters are finitely renormalizable with a recurrent critical point. This concludes the proof of Theorem A.

1.5. **Structure of the paper.** In §2 we give some basic background on quasiconformal maps and holomorphic motions. In §3, we discuss the dynamics of families of complex return-type maps (this is based on [L3]) and obtain some Phase-Parameter estimates in this context (following the sketch of the Appendix of [AM1]). In §4 we present the results of Lyubich in [L2] and [L3] in the generality needed for our applications. In §5 we present the basic theory of unimodal maps, and in §5.6 we introduce the results of [ALM] on the lamination structure of topological classes of unimodal maps and state some straightforward generalizations (some details are given in Appendix A). In §6 we construct a special analytic family of unimodal maps which induces a full family of return type maps and in §7 we state and prove the Phase-Parameter relation for the special family. In §8 and §9 we prove Theorems A and B, and in §10 we show the relation to the corollaries. In Appendix B we give a proof of Theorem F.

Acknowledgements: Most of the results of this paper were announced in [Av2], and, together with [AM1], [AM2] and [ALM], formed the thesis of the first author. The first author would like to thank Welington de Melo and Mikhail Lyubich who collaborated in those works, and to Viviane Baladi and Jean-Christophe Yoccoz for useful conversations.

2. Preliminaries

2.1. **General notation.** Let Ω be the set of finite sequences (possibly empty) of non-zero integers $\underline{d} = (j_1, ..., j_m)$.

A *Jordan curve* T is a subset of $\mathbb{C}$ homeomorphic to a circle. A *Jordan disk* is a bounded open subset U of $\mathbb{C}$ such that ∂U is a Jordan curve.

We let $\mathbb{D}_r(w) = \{z \in \mathbb{C} | |z-w| < r\}$. Let $\mathbb{D}_r = \mathbb{D}_r(0)$, and $\mathbb{D} = \mathbb{D}_1$. If $r > 1$, let $A_r = \{z \in \mathbb{C} | 1 < |z| < r\}$. An *annulus* A is a subset of $\mathbb{C}$ such that there exists a conformal map from A to some A_r. In this case, r is uniquely defined and we denote the *modulus* of A as $\mathrm{mod}(A) = \ln(r)$.

2.2. **Graphs and sections.** Let us fix a complex Banach space $\mathbb{E}$. If $\Lambda \subset \mathbb{E}$, a *graph* of a continuous map $\phi : \Lambda \to \mathbb{C}$ is the set of all $(z, \phi(z)) \in \mathbb{E} \oplus \mathbb{C}$, $z \in \Lambda$.

Let $\mathbf{0} : \mathbb{E} \to \mathbb{E} \oplus \mathbb{C}$ be defined by $\mathbf{0}(z) = (z, 0)$.

Let $\pi_1 : \mathbb{E} \oplus \mathbb{C} \to \mathbb{E}$, $\pi_2 : \mathbb{E} \oplus \mathbb{C} \to \mathbb{C}$ be the coordinate projections. Given a set $\mathcal{X} \subset \mathbb{E} \oplus \mathbb{C}$ we denote its fibers $X[z] = \pi_2(\mathcal{X} \cap \pi_1^{-1}(z))$.

A *fiberwise map* $\mathcal{F} : \mathcal{X} \to \mathbb{E} \oplus \mathbb{C}$ is a map such that $\pi_1 \circ \mathcal{F} = \pi_1$. We denote its fibers $F[z] : X[z] \to \mathbb{C}$, so that $\mathcal{F}(z, w) = (z, F[z](w))$.

Let $B_r(\mathbb{E})$ be the ball of radius r around 0.

2.3. **Quasiconformal and quasisymmetric maps.** Let $U \subset \mathbb{C}$ be a domain. A map $h : U \to \mathbb{C}$ is K-*quasiconformal* (K-qc) if it is a homeomorphism onto its image and for any annulus $A \subset U$, $\operatorname{mod}(A)/K \leq \operatorname{mod}(h(A)) \leq K \operatorname{mod}(A)$. The minimum such K is called the *dilatation* $\operatorname{Dil}(h)$ of h.

A homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is said to be γ-quasisymmetric if it has a real-symmetric extension $h : \mathbb{C} \to \mathbb{C}$ which is quasiconformal with dilatation bounded by γ. If $X \subset \mathbb{R}$, we will also say that $h : X \to \mathbb{R}$ is γ-qs if it has a γ-qs extension.

A quasiconformal vector field α of $\mathbb{C}$ is a continuous vector field with locally integrable distributional derivatives $\overline{\partial}\alpha$ and $\partial\alpha$ in L^1 and $\overline{\partial}\alpha \in L^\infty$.

2.4. **Holomorphic motions.** Let Λ be a connected open set of a Banach space $\mathbb{E}$. A *holomorphic motion* h over Λ is a family of holomorphic maps defined on Λ whose graphs (called *leaves* of h) do not intersect. The *support* of h is the set $\mathcal{X} \subset \mathbb{C}^2$ which is the union of the leaves of h.

The *transition* (or holonomy) maps $h[z, w] : X[z] \to X[w]$, $z, w \in \Lambda$, are defined by $h[z, w](x) = y$ if (z, x) and (w, y) belong to the same leaf.

Given a holomorphic motion h over a domain Λ, a holomorphic motion h' over a domain $\Lambda' \subset \Lambda$ whose leaves are contained in leaves of h is called a *restriction* of h.[8] If h is a restriction of h' we also say that h' is an *extension* of h.

Let $K : [0, 1) \to \mathbb{R}$ be defined by $K(r) = (1 + \rho)/(1 - \rho)$ where $0 \leq \rho = \rho(r) < 1$ is such that the hyperbolic distance in $\mathbb{D}$ between 0 and ρ is r.

λ-Lemma ([MSS], [BR]) *Let h be a holomorphic motion over a hyperbolic domain $\Lambda \subset \mathbb{C}$ and let $z, w \in \Lambda$. Then $h[z, w]$ extends to a quasiconformal map of $\mathbb{C}$ with dilatation bounded by $K(r)$, where r is the hyperbolic distance between z and w in Λ.*

In the general case (Λ not one-dimensional), the same estimate holds with the Kobayashi distance instead of the hyperbolic distance. In particular, if h is a holomorphic motion over $B_r(\mathbb{E})$, and if $z, w \in B_{r/2}(\mathbb{E})$ then $h[z, w] = 1 + O(\|z - w\|)$.

If $h = h_U$ is a holomorphic motion of an open set, we define $\operatorname{Dil}(h)$ as the supremum of the dilatations of the maps $h[z, w]$.

A *completion* of a holomorphic motion means an extension of h to the whole complex plane: $X[z] = \mathbb{C}$ for all $z \in \Lambda$. The problem of existence of completions is considerably different in one-dimension or higher:

Extension Lemma ([Sl]) *Any holomorphic motion over a simply connected domain $\Lambda \subset \mathbb{C}$ can be completed.*

[8]Notice that the notion of restriction allows shrinking of both the parameter and the moving set.

Canonical Extension Lemma ([BR]) *Let h be a holomorphic motion over $B_r(\mathbb{E})$. Then the restriction of h to $B_{r/3}(\mathbb{E})$ can be completed in a canonical way.*

2.4.1. *Symmetry.* Let us assume that $\mathbb{E}$ is the complexification of a real-symmetric space $\mathbb{E}^{\mathbb{R}}$, that is, there is a anti-linear isometric involution conj fixing $\mathbb{E}^{\mathbb{R}}$. Let us use conj to denote also the map $(z, w) \to (\operatorname{conj} z, \overline{w})$ in $\mathbb{E} \oplus \mathbb{C}$.

A set $X \subset \mathbb{E}, \mathbb{E} \oplus \mathbb{C}$ is called real-symmetric if $\operatorname{conj}(X) = X$. Let $\Lambda \subset \mathbb{E}$ be a real-symmetric domain. A holomorphic motion h over Λ is called real-symmetric if the image of any leaf by conj is also a leaf.

The systems we are interested on are real, so they naturally possess symmetry. In many cases, we will consider a real-symmetric holomorphic motion associated to the system, which will need to be completed using the Extension Lemma (in one-dimension) or the Canonical Extension Lemma (in higher dimensions).

The Canonical Extension Lemma implies that a real-symmetric holomorphic motion over B_r extends to a real-symmetric holomorphic motion over $B_{r/3}$ (see Remark 2.2 of [ALM]). On the other hand, the Extension Lemma adds ambiguity on the procedure, since the extension is not unique. In particular, this could lead to loss of symmetry. In order to avoid this problem, we will choose a little bit more carefully our extensions. The relevant result is then the following:

Real Extension Lemma. *Any real-symmetric holomorphic motion over a simply connected domain $\Lambda \subset \mathbb{C}$ can be completed to a real-symmetric holomorphic motion.*

This version of the Extension Lemma can be proved in the same way as the non-symmetric one[9].

So we can adopt the following convention:

Symmetry assumption. *Extensions of real-symmetric motions will always be taken real-symmetric.*

2.4.2. *Notation conventions.* We will use the following conventions. Instead of talking about the sets $X[z]$, fixing some $z \in \Lambda$, we will say that h is the motion of X over Λ, where X is to be thought of as a set which depends on the point $z \in \Lambda$. In other words, we usually drop the brackets from the notation.

We will also use the following notation for restrictions of holomorphic motions: if $Y \subset X$, we denote $\mathcal{Y} \subset \mathcal{X}$ as the union of leaves through Y.

2.5. **Codimension-one laminations.** Let $\mathbb{F}$ be a Banach space. A codimension-one *holomorphic lamination* $\mathcal{L}$ on an open subset $\mathcal{W} \subset \mathbb{F}$ is a family of disjoint codimension-one Banach submanifolds of $\mathbb{F}$, called the *leaves* of the lamination such that for any point $p \in \mathcal{W}$, there exists a holomorphic local chart $\Phi : \tilde{\mathcal{W}} \to \mathcal{V} \oplus \mathbb{C}$, where $\tilde{\mathcal{W}} \subset \mathcal{W}$ is a neighborhood of p and $\mathcal{V}$ is an open set in some complex Banach space $\mathbb{E}$, such that for any leaf L and any connected component L_0 of $L \cap \mathcal{W}$, the image $\Phi(L_0)$ is a graph of a holomorphic function $\mathcal{V} \to \mathbb{C}$.

[9] This is particularly easy to check in Douady's [D] proof of the Extension Lemma (similar considerations can be applied also to Astala-Martin's proof [AsM]). Indeed, there exists only one step which could lead to loss of symmetry, and thus needs to be looked more carefully in order to obtain the Real Extension Lemma: in Proposition 1 we should make sure that the (not uniquely defined) diffeomorphism F is chosen real-symmetric (the proof that this is possible is the same).

It is clear that the local theory of codimension-one laminations is the theory of holomorphic motions. For instance, the λ-Lemma implies that holonomy maps of codimension-one laminations have quasiconformal extensions, and gives bounds on the dilatation of those extensions.

3. Complex dynamics

In this section we introduce some basic language necessary to describe precisely the constructions of [L3]. Although this language may seem at first technical and heavy, it will allow us to give formal and concise proofs of the results we need (which are extensions of the results of [L3]). We warn the reader that the notation is different from [L3].

Through this section, we will deal exclusively with one-dimensional holomorphic motions over some Jordan domain $\Lambda \subset \mathbb{C}$.

3.1. ***R*-maps and *L*-maps.** Let U be a Jordan disk and U^j, $j \in \mathbb{Z}$ be a family of Jordan disks with disjoint closures such that $\overline{U^j} \subset U$ for every $j \in \mathbb{Z}$. We assume further that $0 \in U^0$. A holomorphic map $R : \cup U^j \to U$ is called a R-map (return type map) if for $j \neq 0$, $R|U^j$ extends to a homeomorphism $R : \overline{U^j} \to \overline{U}$ and $R|U^0$ extends to a double covering map $R : \overline{U^0} \to \overline{U}$ ramified at 0.

For $\underline{d} \in \Omega$, $\underline{d} = (j_1, ..., j_m)$, we define $U^{\underline{d}} = \{z \in U |\, R^{k-1}(z) \in U^{j_k}, 1 \le k \le m\}$ and we let $R^{\underline{d}} = R^m|U^{\underline{d}}$. Let $W^{\underline{d}} = (R^{\underline{d}})^{-1}(U^0)$.

Given an R-map R we define an *L-map* (landing type map) $L(R) : \cup W^{\underline{d}} \to U^0$, by setting $L(R)|W^{\underline{d}} = R^{\underline{d}}$ (thus $L(R)$ is the first landing map to U^0 under the dynamics of R). We will say that $L(R)$ is the landing map associated with (or induced from) R.

3.1.1. *Renormalization.* Given an R-map R such that $R(0) \in \cup W^{\underline{d}}$ we can define the (generalized in the sense of Lyubich) *renormalization* $N(R)$ as the first return map to U^0 under the dynamics of R. It follows that $N(R) = L(R) \circ R|U^0$ where defined in U^0, and that $N(R)$ is also an R-map.

3.2. **Tubes and tube maps.** A *proper motion* of a set X over Λ is a holomorphic motion of X over Λ such that for any $z \in \Lambda$, the map $\mathbf{h}[z] : \Lambda \times X[z] \to \mathcal{X}$ defined by $\mathbf{h}[z](w, x) = (w, h[z, w](x))$ has an extension to $\overline{\Lambda} \times X[z]$ which is a homeomorphism.

An *equipped tube* h_T is a holomorphic motion of a Jordan curve T. Its support is called a *tube*. We say that an equipped tube is *proper* if it is a proper motion. Its support is called a *proper tube*. The *filling* of a tube $\mathcal{T}$ is the set $\mathcal{U} \subset \Lambda \times \mathbb{C}$ such that $U[z]$ is the bounded component of $\mathbb{C} \setminus T[z]$, $z \in \Lambda$.

A *special motion* is a holomorphic motion $h = h_{X \cup T}$ such that $\mathcal{X}$ is contained in the filling $\mathcal{U}$ of $\mathcal{T}$, $h|T$ is an equipped proper tube and the closure of any leaf through X does not intersect the closure of $\mathcal{T}$.

If $\mathcal{T}$ is a tube over Λ, and $\mathcal{U}$ is its filling, a fiberwise holomorphic map $\mathcal{F} : \mathcal{U} \to \mathbb{C}^2$ is called a *tube map* if it admits a continuous extension to $\overline{\mathcal{U}}$.

3.2.1. *Tube pullback.* Let $\mathcal{F} : \mathcal{V} \to \mathbb{C}^2$ be a tube map such that $\mathcal{F}(\partial \mathcal{V}) = \partial \mathcal{U}$, where $\mathcal{U}$ is the filling of a tube over Λ and let h be a holomorphic motion supported on $\overline{\mathcal{U}} \cap \pi_1^{-1}(\Lambda)$. Let Γ be a (parameter) open set such that $\overline{\Gamma} \subset \Lambda$ and W be a (phase) open set which moves holomorphically by h over Λ and such that $\overline{W} \subset U$. Assume that $\overline{W}$ contains critical values of $\mathcal{F}|(\mathcal{V} \cap \pi_1^{-1}(\overline{\Gamma}))$, that is, if $\lambda \in \overline{\Gamma}$, $z \in V[\lambda]$ and $DF[\lambda](z) = 0$ then $F[\lambda](z) \in \overline{W[\lambda]}$.

Let us consider a leaf of h through $z \in U \setminus \overline{W}$, and let us denote by $\mathcal{E}(z)$ its preimage by $\mathcal{F}$ intersected with $\pi_1^{-1}(\Gamma)$. Each connected component of $\mathcal{E}(z)$ is a graph over Γ, moreover, $\overline{\mathcal{E}}(z) \subset \mathcal{U}$. So the set of connected components of $\mathcal{E}(z)$, $z \in U \setminus \overline{W}$ is a holomorphic motion over Γ. We define a new holomorphic motion over Γ, called *a lift of h by $(\mathcal{F}, \Gamma, W)$*, as an extension to the closure of V of the holomorphic motion whose leaves are the connected components of $\mathcal{E}(z)$, $z \in U \setminus \overline{W}$ (the lift is not uniquely defined). It is clear that this holomorphic motion is a special motion of V over Γ and its dilatation over $\mathcal{F}^{-1}(U \setminus \overline{W})$ is bounded by $K(r)$ where r is the hyperbolic diameter of Γ in Λ.

3.2.2. *Diagonal and Phase-Parameter holonomy maps.* Let h be an equipped proper tube supported on $\mathcal{T}$. A *diagonal* of $\mathcal{T}$ is a holomorphic section $\Phi : \Lambda \to \mathbb{C}^2$ (so that $\pi_1 \circ \Phi = \mathrm{id}$), admitting a continuous extension to Λ, and such that $\Phi(\Lambda)$ is contained in the filling of $\mathcal{T}$ and for $\lambda \in \Lambda$, $h[\lambda] \circ \Phi|\partial\Lambda$ has degree one onto $T[\lambda]$.

Let $h = h_{X \cup T}$ be a special motion and let Φ be a diagonal of $h|T$. It is a consequence of the Argument Principle (see [L3]) that any leaf of $h|X$ intersects $\Phi(\Lambda)$ at a unique point (with multiplicity one). From this we can define a map $\chi[\lambda] : X[\lambda] \to \Lambda$ such that $\chi[\lambda](z) = w$ if (λ, z) and $\Phi(w)$ belong to the same leaf of h. It is clear that each $\chi[\lambda]$ is a homeomorphism onto its image. Moreover, if $U \subset X$ is open, $\chi[\lambda]|U[\lambda]$ is locally quasiconformal, and if $\mathrm{Dil}(h|U) < \infty$ then $\chi[\lambda]|U[\lambda]$ is globally quasiconformal with dilatation bounded by $\mathrm{Dil}(h|U)$.

We will say that χ is the *holonomy family* associated to the pair (h, Φ).

Remark 3.1. Let $h_{U \cup T}$ be a special motion, Φ a diagonal, and let χ be the holonomy family associated to (h, Φ). Let X be compactly contained in U. Then the λ-lemma implies that for every $\lambda \in X$, $\chi[\lambda]|X[\lambda]$ extends to a qc map of the whole plane[10].

If $\chi(X)$ has small hyperbolic diameter in $\chi(U)$ then one can say more: this qc extension has dilatation close to 1. Indeed, in this case there is a Jordan domain $X \subset U' \subset U$ such that $\chi(U')$ has small hyperbolic diameter in $\chi(U)$ and $\chi(X)$ has small hyperbolic diameter in $\chi(U')$. Using the λ-lemma, one sees that for $\lambda \in U'$, $\chi[\lambda]|U'[\lambda]$ has dilatation close to 1, and we may apply the previous argument. (This does not work if we only know that X has small hyperbolic diameter in U.)

3.3. **Families of R-maps.** An *R-family* is a pair $(\mathcal{R}, h)$, where $\mathcal{R}$ is a holomorphic map $\mathcal{R} : \cup\mathcal{U}^j \to \mathcal{U}$ such that the fibers $R[\lambda]$ of $\mathcal{R}$ are R-maps, for every j, $\mathcal{R}|\mathcal{U}^j$ is a tube map, and $h = h_{\overline{U}}$ is a holomorphic motion such that $h|(\partial U \cup \cup_j \partial U_j)$ is special. If additionally $\mathcal{R} \circ \mathbf{0}$ is a diagonal to h, we say that the $\mathcal{R}$ is *full*.

3.3.1. *From R-families to L-families.* Given an R-family $\mathcal{R}$ with motion $h = h_{\overline{U}}$ we induce a family of L-maps as follows. If $\underline{d} \in \Omega$, $\underline{d} = (j_1, ..., j_m)$, we let $\mathcal{U}^{\underline{d}} = \{(\lambda, z) \in \mathcal{U} | R^{k-1}[\lambda](z) \in U^{j_k}[\lambda]\}$ and define $\mathcal{R}^{\underline{d}} = \mathcal{R}^m|\mathcal{U}^{\underline{d}}$. Let $\mathcal{W}^{\underline{d}} = (\mathcal{R}^{\underline{d}})^{-1}(\mathcal{U}^0)$. We define $L(\mathcal{R}) : \cup\mathcal{W}^{\underline{d}} \to \mathcal{U}^0$ by $L(\mathcal{R})|\mathcal{W}^{\underline{d}} = \mathcal{R}^{\underline{d}}$. Notice that the L-maps which are associated with the fibers of $\mathcal{R}$ coincide with the fibers of $L(\mathcal{R})$.

We define a holomorphic motion $L(h)$ in the following way. The leaf through $z \in \partial U$ is the leaf of h through z. If there is a smallest $U^{\underline{d}}$ such that $z \in U^{\underline{d}}$, we

[10] Here we use that the restriction of a quasiconformal map χ to a compact subset of its domain always admits a global qc extension (the bounds on the dilatation of the global extension depending on the original bounds and on the hyperbolic diameter of X in U).

let the leaf through z be the preimage by $\mathcal{R}^{\underline{d}}$ of the leaf of h through $\mathcal{R}^{\underline{d}}(z)$. We finally extend it to $\overline{U}$ using the Extension Lemma.

The *L-family* associated to $(\mathcal{R},h)$ is the pair $(L(\mathcal{R}),L(h))$.

3.3.2. *Parameter partition and family renormalization.* Let $(\mathcal{R},h)$ be a full R-family. Since $L(h)|(U\cup\cup_j\overline{U^j})$ is special, we can consider the holonomy family of the pair $(L(h)|(U\cup\cup_j\overline{U^j}),\mathcal{R}(\mathbf{0}))$, which we denote by χ. We use χ to partition Λ: we will denote $\Lambda^{\underline{d}}=\chi(U^{\underline{d}})$ and $\Gamma^{\underline{d}}=\chi(W^{\underline{d}})$.

The $\underline{d}$-renormalization of $(\mathcal{R},h)$ is the R-family $(N^{\underline{d}}(\mathcal{R}),N^{\underline{d}}(h))$ over $\Gamma^{\underline{d}}$ defined as follows. We take $N^{\underline{d}}(h)$ as the lift of $L(h)$ by $(\mathcal{R}|\mathcal{U}^0,\Gamma^{\underline{d}},W^{\underline{d}})$ where defined. It is clear that $(N^{\underline{d}}(\mathcal{R}),N^{\underline{d}}(h))$ is full, and its fibers are renormalizations of the fibers of $(\mathcal{R},h)$. Moreover, $N^{\underline{d}}(h)$ is a special motion.

3.3.3. *Chains.* An *R-chain* over λ_0 is a sequence of full R-families $(\mathcal{R}_i,h_i)$, over domains Λ_i, $i\geq 1$, such that $\lambda_0\in\cap\Lambda_i$ and which are related by renormalization: $\mathcal{R}_{i+1}=N^{\underline{d}_i}(\mathcal{R}_i)$, $h_{i+1}=N^{\underline{d}_i}(h_i)$ for some sequence $\underline{d}_i$. We will say that a level $\mathcal{R}_i$ of the chain is central if $|\underline{d}_i|=0$.

3.3.4. *Gape motion.* In the situations we shall face, the central puzzle piece U_i^0 degenerates to a figure eight when λ goes to the boundary of $\partial\Lambda_i$. This will force us to consider a technical modification of the holomorphic motion h_i as follows.

For $i>1$, let $G(h_{i-1})$ be a holomorphic motion of U_{i-1} over $\Lambda_{i-1}^{\underline{d}_{i-1}}$ which coincides with $L(h_{i-1})$ on $U_{i-1}\setminus\overline{U_{i-1}^0}$, and coincides with the lift of $L(h)$ by $(\mathcal{R}|\mathcal{U}_{i-1}^0,\Lambda_{i-1}^{\underline{d}_{i-1}},U_{i-1}^{\underline{d}_{i-1}})$ on U_{i-1}^0.

Notice that for $i>1$, the motion h_i (and hence $L(h_i)$) is special, since it is obtained by renormalization. So for $i>2$, the motion $G(h_{i-1})$ is special. Moreover, it is easy to see that $(\mathcal{R}_{i-1}^{|\underline{d}_{i-1}|+1}\circ\mathbf{0})|\Lambda^{\underline{d}_{i-1}}$ (which extends $(\mathcal{R}_i\circ\mathbf{0})|\Lambda_{i+1})$ is a diagonal to $G(h_{i-1})$.

3.3.5. *Real chains.* A fiberwise map $\mathcal{F}:\mathcal{X}\to\mathbb{C}^2$ is real-symmetric if $\mathcal{X}$ is real-symmetric and $\mathcal{F}\circ\operatorname{conj}=\operatorname{conj}\circ\mathcal{F}$. We will say that a chain $\{\mathcal{R}_i\}$ over a parameter $\lambda\in\mathbb{R}$ is real-symmetric if each $\mathcal{R}_i$ and each underlying holomorphic motion h_i is real-symmetric.

Because of the Symmetry assumption, a chain $\{\mathcal{R}_i\}$ over a parameter $\lambda\in\mathbb{R}$ is real-symmetric provided the first step data $\mathcal{R}_1$ and h_1 is real-symmetric. In this case, all objects related to the chain are real-symmetric.

Remark 3.2. If $\mathcal{R}_1$ is real-symmetric then h_1 can always be modified to be real-symmetric. Indeed if $\mathcal{R}_1$ is real-symmetric then $\partial\mathcal{U}_1\cup\cup\partial\mathcal{U}_1^j$ is a real-symmetric set and it is enough to check that $h_1|(\partial U_1\cup\cup\partial U_1^j)$ is already real-symmetric. To see this, first notice that if $X[\lambda]$ moves holomorphically and X has empty interior then the holomorphic motion of X is unique[11]. This implies that if $X[\lambda]$ is real symmetric with empty interior then any motion of $X[\lambda]$ is also real-symmetric.

[11]In this case $\mathbb{C}\setminus X$ also moves holomorphically by some motion $h_{\mathbb{C}\setminus X}$ obtained from the Extension Lemma, and the motion of X can be seen as coming from the extension of $h_{\mathbb{C}\setminus X}$ to the closure $\overline{\mathbb{C}\setminus X}=\mathbb{C}$, and this extension is unique.

3.4. **Complex Phase-Parameter estimates.** We shall now show how estimates on the geometry of parapuzzle pieces yield automatically estimates on the regularity of holonomy maps. We shall need four specific statements, contained in two lemmas.

Lemma 3.1. *Let us consider an R-chain $(\mathcal{R}_i, h_i)$ over λ_0, and let τ_i be such $R_i[\lambda_0](0) \in U_i^{\tau_i}[\lambda_0]$. For $i > 1$, let χ_i be the holonomy family associated to $(L(h_i), \mathcal{R}_i \circ \mathbf{0})$. For every $\gamma > 1$ there exists $K > 0$ such that if* $\operatorname{mod}(\Lambda_{i-1} \setminus \overline{\Lambda_i}) > K$ *and* $\operatorname{mod}(U_{i-1}[\lambda_0] \setminus \overline{U_i[\lambda_0]}) > K$ *then*
[CPhPh1] *For every $\lambda \in \Lambda_i^{\tau_i}[\lambda_0]$, $L(h_i)[\lambda_0, \lambda]|U_i[\lambda_0]$ has a γ-quasiconformal extension to the whole complex plane,*
[CPhPa1] *$\chi_i[\lambda_0]|U_i^{\tau_i}[\lambda_0]$ has a γ-quasiconformal extension to the whole complex plane.*

Moreover, if the R-chain $(\mathcal{R}_i, h_i)$ is real then the claimed extensions can be taken real as well.

Proof. Both items (1) and (2) follow easily from the λ-lemma (see also Remark 3.1) if we can establish that $\operatorname{mod}(\Lambda_i \setminus \overline{\Lambda_i^{\tau_i}})$ is big.

The hypothesis on $\operatorname{mod}(U_{i-1}[\lambda_0] \setminus \overline{U_i[\lambda_0]})$ implies that $\operatorname{mod}(U_i[\lambda_0] \setminus \overline{U_i^{\tau_i}[\lambda_0]})$ and $\operatorname{mod}(U_i[\lambda_0] \setminus \overline{U_i^0[\lambda_0]})$ are bigger than $K/2$. If K is big, this implies that there is an annulus of big modulus contained in $U_i[\lambda_0] \setminus \overline{U_i^0[\lambda_0]}$ and going around $U_i^{\tau_i}[\lambda_0]$. Using again that K is big and the hypothesis on $\operatorname{mod}(\Lambda_{i-1} \setminus \overline{\Lambda_i})$, we see that the dilatation of $h_i|(U_i \setminus \overline{U_i^0})$ is small (λ-lemma). We conclude that $\operatorname{mod}(\Lambda_i \setminus \overline{\Lambda_i^{\tau_i}})$ is big as required. □

Lemma 3.2. *Let us consider an R-chain $(\mathcal{R}_i, h_i)$ over λ_0. For $i > 2$, let $\tilde{\chi}_i$ be the holonomy family associated to $(G(h_{i-1}), \mathcal{R}_{i-1}^{|\underline{d}_{i-1}|+1} \circ \mathbf{0})$ For every $\gamma > 1$ there exists $K > 0$ such that if* $\operatorname{mod}(\Lambda_{i-2} \setminus \overline{\Lambda_{i-1}}) > K$ *and* $\operatorname{mod}(U_{i-1}[\lambda_0] \setminus \overline{U_{i-1}[\lambda_0]}) > K$ *then*
[CPhPh2] *For every $\lambda \in \Lambda_i$, $G(h_{i-1})[\lambda_0, \lambda]|U_{i-1}[\lambda_0]$ has a γ-quasiconformal extension to the whole complex plane,*
[CPhPa2] *$\tilde{\chi}_i[\lambda_0]|U_i[\lambda_0]$ has a γ-quasiconformal extension to the whole complex plane.*

Moreover, if the R-chain $(\mathcal{R}_i, h_i)$ is real then the claimed extensions can be taken real as well.

Proof. Both items (1) and (2) follow easily from the λ-lemma (see also Remark 3.1) if we can establish that $\operatorname{mod}(\Lambda_{i-1}^{\underline{d}_{i-1}} \setminus \overline{\Lambda_i})$ is big.

The hypothesis on $\operatorname{mod}(\Lambda_{i-2} \setminus \overline{\Lambda_{i-1}}$ implies that the dilatation of $L(h_{i-1})|(U_{i-1} \setminus \overline{U_i})$ is less than 2 (provided K is sufficiently big). Notice that $\Lambda_{i-1}^{\underline{d}_{i-1}} \setminus \overline{\Lambda_i} = \chi_{i-1}[\lambda_0](U_{i-1}^{\underline{d}_{i-1}}[\lambda_0] \setminus \overline{W_{i-1}^{\underline{d}_{i-1}}[\lambda_0]})$, where χ_{i-1} is the holonomy family associated to $(L(h_{i-1}), \mathcal{R}_{i-1} \circ \mathbf{0})$. The hypothesis on $\operatorname{mod}(U_{i-1}[\lambda_0] \setminus \overline{U_{i-1}^0[\lambda_0]})$ (which equals $\operatorname{mod}(U_{i-1}^{\underline{d}_{i-1}}[\lambda_0] \setminus \overline{W_{i-1}^{\underline{d}_{i-1}}[\lambda_0]})$) then implies that $\operatorname{mod}(\Lambda_{i-1}^{\underline{d}_{i-1}} \setminus \overline{\Lambda_i})$ is big (at least $K/2$) as required. □

4. Puzzle and parapuzzle geometry

In this section we will recall an important part of Lyubich's theory of the quadratic family (regarding linear growth of moduli of certain phase and parameter annuli), and will discuss the validity of those results in the context of more general R-chains.

4.1. **Puzzle estimates.** The following result is contained on (the proof of) Theorem II of [L2]:

Theorem 4.1. *For every $C > 0$, there exists $C' > 0$ with the following property. Let R_i be a sequence of R-maps such that $R_{i+1} = N(R_i)$ and let $n_k - 1$ be the sequence of non-central levels, so that $R_{n_k-1}(0) \notin U^0_{n_k-1}$. If $\operatorname{mod}(U_1 \setminus \overline{U^0_1}) > C'$ then $\operatorname{mod}(U_{n_k} \setminus \overline{U^0_{n_k}}) > C$.*

(In Lyubich's notation, R-maps are called generalized quadratic maps.)

The following result is Theorem III of [L2]:

Theorem 4.2. *For every $C' > 0$, there exists $C'' > 0$ with the following property. Let R_i be a sequence of R-maps such that $R_{i+1} = N(R_i)$ and let $n_k - 1$ be the sequence of non-central levels. If $\operatorname{mod}(U_1 \setminus \overline{U^0_1}) > C'$ then $\operatorname{mod}(U_{n_k} \setminus \overline{U^0_{n_k}}) > C''k$.*

4.2. **Parapuzzle estimates.**

4.2.1. *The quadratic family.* Let $p_c(z) = z^2 + c$ be the quadratic family. The following result is contained in Lemma 3.6 of [L3]:

Theorem 4.3. *Let us fix a non-renormalizable quadratic polynomial p_{c_0} with a recurrent critical point and no neutral periodic orbits. Then there exists a full R-family $\mathcal{R}_1$ over some $c_0 \in \Lambda_1$ such that if $c \in \Lambda_1$ then $R[c] : \cup U^j_1[c] \to U_1[c]$ is the first return map under iteration by p_c.*

The following is Theorem A of [L3]:

Theorem 4.4. *In the setting of Theorem 4.3, let $\mathcal{R}_i$ be the R-chain over c_0 with first step $\mathcal{R}_1$. If $n_k - 1$ denotes the k-th non-central return, then $\operatorname{mod}(\Lambda_{n_k} \setminus \overline{\Lambda_{n_k+1}}) > Tk$, for some constant $T > 0$.*

Remark 4.1. In Lyubich's notation he lets $\Delta^i = \Lambda_{n_i}$ and $\Pi^i = \Lambda^0_{n_i}$. He states that both $\operatorname{mod}(\Delta^i \setminus \overline{\Delta^{i+1}})$ and $\operatorname{mod}(\Delta^i \setminus \overline{\Pi^i})$ grow linearly. His statement implies ours after one notices that if $n_i + 1 = n_{i+1}$ then $\Delta^{i+1} = \Lambda_{n_i+1}$, otherwise $\Pi^i = \Lambda_{n_i+1}$.

Those two results are proved in a slightly more general setting then we state here: they are valid for so-called full unfolded families of quadratic-like maps. This version allows one to state results also for finitely renormalizable quadratic polynomials (via renormalization).

4.2.2. *General case.* The following more general theorem can be proved using the ideas of Theorem A of [L3] but it is a little bit tedious to check the details (it is necessary to get deep into the construction of [L2]).

Theorem 4.5. *For every $K > 1$, $T > 0$, there exists $T' > 0$ with the following property. Let $(\mathcal{R}_i, h_i)$ be a R-chain over λ_0 and let $n_k - 1$ be the sequence of non-central levels. If $\operatorname{Dil}(h_1|(U_1 \setminus \overline{U^0_1})) < K$ and $\operatorname{mod}(U_1[\lambda] \setminus \overline{U^0_1[\lambda]}) > T$ then $\operatorname{mod}(\Lambda_{n_k} \setminus \overline{\Lambda_{n_k+1}}) > T'k$.*

Since we do not need the full strength of the previous theorem, we will state and prove a weaker estimate using a simple inductive argument.

Theorem 4.6. *For every $K > 1$, there exists constants $T' > 0$, $T'' > 0$ with the following properties. Let $(\mathcal{R}_i, h_i)$ be a R-chain over λ_0 and let $n_k - 1$ be the sequence of non-central levels. If $\operatorname{Dil}(h_1|(U_1 \setminus \overline{U^0_1}) < K$ and $\operatorname{mod}(U_1[\lambda_0] \setminus \overline{U^0_1[\lambda_0]}) > T'$ then $\operatorname{mod}(\Lambda_{n_k} \setminus \overline{\Lambda_{n_k+1}}) > T''k$.*

Proof. Let $\nu_i = \mathrm{mod}(\Lambda_i \setminus \overline{\Lambda_{i+1}})$, $\mu_i = \mathrm{mod}(U_i[\lambda_0] \setminus \overline{U_{i+1}[\lambda_0]})$, $k_i = \mathrm{Dil}(h_i|U_i[\lambda_0] \setminus \overline{U_{i+1}[\lambda_0]})$. For $i > 1$, denote by χ_i^0 the holonomy family associated to $(h_i, \mathcal{R}_i \circ \mathbf{0})$.

Notice that if $\nu_i > C_0 = 1000$ then $k_{i+1} \leq 2$. Moreover, for $i > 1$, and in particular for $i = n_k$, we have $\nu_i = \mathrm{mod}(\chi_i^0(U_i[\lambda_0] \setminus \overline{U_{i+1}[\lambda_0]})) \geq \mu_i/k_i$.

Let $T = (4+2C_0)(2+K)$ and let $T' > T$ be so big that if $\mu_1 > T'$ then $\mu_{n_k} > T$, $k \geq 1$. Let also T'' be such that if $\mu_1 > T'$ then $\mu_{n_k} > kT''(2+K)$.

Let us assume that for some m, we have $\mu_m > T$ and $k_m \leq (2+K)$, and let $m' \geq m$ be the next non-central return.

For $\lambda \in \Lambda_{m'+1}$, we have $R_m^{m'-m+1}(0) \in W_m^{\underline{d}}$ for some $\underline{d}$. Let Υ be the component of $R_m(0)$ of $(R_m^{m'-m}|U_{m'})^{-1}(U_m^{\underline{d}})$ and Υ' be the component of $R_m(0)$ of $(R_m^{m'-m}|U_{m'})^{-1}(W_m^{\underline{d}})$.

Let $H_m = L(h_m)$ outside of $U_m^{\underline{d}}$ and let the leaves of $H_m|U_m^{\underline{d}}$ be the preimages by $R_m^{\underline{d}}$ of the leaves of h_m. If $m = m'$, let $H = H_m$. Otherwise, notice that if $\lambda \in \Lambda_{m'-1}$, then $R_m^{m'-m}|U_{m'-1}^0[\lambda]$ is a $2^{m'-m}$ branched covering map over $U_m[\lambda]$, and for $\lambda \in \Lambda_{m'}$, $R_m^{m'-m}|U_{m'}[\lambda] \setminus \overline{U_{m'}^0[\lambda]}$ is unbranched. Let H be the lift of H_m by $(\mathcal{R}_m^{m'-m}|U_{m'}, U_{m'}^0, \Lambda_{m'})$. So in both cases, H is a holomorphic motion over $\Lambda_{m'}$.

With this definition, Υ and Υ' (which are apriori defined over $\Lambda_{m'+1}$) move holomorphically with H (over $\Lambda_{m'}$).

Let χ be the holonomy family of the pair $(\mathcal{R}_{m'} \circ \mathbf{0}, H|\partial U_{m'} \cup \overline{\Upsilon})$. It is clear that $\mathrm{Dil}(\chi|\Upsilon)$ is bounded by k_m. In particular, we can estimate $\nu_{m'} \geq \mathrm{mod}(\Upsilon[\lambda_0] \setminus \overline{\Upsilon'[\lambda_0]})/k_m = \mu_m/k_m \geq \mu_m/(2+K) > C_0$. With $m = 1$, we have $k_1 \leq K \leq 2+K$ by hypothesis and $m' = n_1 - 1$, so $\nu_{n_1-1} \geq \mu_1/(2+K) \geq T/(2+K) \geq C_0$ and $k_{n_1} \leq 2 \leq 2+K$. With $m = n_k$, we have that $m' = n_{k+1} - 1$ and $\nu_{n_{k+1}-1} \geq \mu_{n_k}/(2+K) \geq T/(2+K) \geq C_0$ and $k_{n_{k+1}} \leq 2 \leq 2+K$, provided $k_{n_k} \leq 2+K$. By induction, we have $k_{n_k} \leq 2+K$ for every k, so $\nu_{n_k} \geq \mu_{n_k}/(2+K) > T''k$. □

This simple inductive argument does not seem to work easily to get the full Theorem 4.5[12].

5. Unimodal maps

We refer to the book of de Melo & van Strien [MS] for the general background in one-dimensional dynamics.

We will say that a smooth (at least C^2) map $f : I \to I$ of the interval $I = [-1, 1]$ is *unimodal* if $f(-1) = -1$, $f(x) = f(-x)$ and 0 is the only critical point of f and is non-degenerate, so that $D^2 f(0) \neq 0$.

Remark 5.1. The introduction of normalization and symmetry assumptions was made in order to avoid cumbersome notations: all results and proofs generalize to the non-symmetric case. See also Appendix C of [ALM].

Remark 5.2. The assumption that the critical point is non-degenerate is made just for convenience: typical unimodal maps certainly have non-degenerate critical point. If one is not willing to make this assumption already in the definition, one should add the non-degeneracy condition to the Kupka-Smale definition below. In this case it would still hold that in non-trivial analytic families the set of parameters

[12] However, we will see that this is enough to yield the full power of Theorem B of [L3] (almost every non-regular finitely renormalizable quadratic map is stochastic), through the arguments of this paper. This approach only uses geometric estimates of puzzle pieces for *real maps*, and may be useful for generalizations beyond unimodal maps with a quadratic critical point.

with a degenerate critical point have zero Lebesgue measure (and is contained in a countable number of analytic subvarieties with codimension at least 1), see Lemma 9.6.

The theory of unimodal maps with fixed non-quadratic criticality is considerably different and less complete than the typical case, and the proofs of this work do not apply.

Let $\mathbb{U}^k$, $k \geq 2$ be the space of C^k unimodal maps. We endow $\mathbb{U}^k$ with the C^k topology.

Basic examples of unimodal maps are given by quadratic maps

$$q_\tau : I \to I, \quad q_\tau(x) = \tau - 1 - \tau x^2, \tag{5.1}$$

where $\tau \in [1/2, 2]$ is a real parameter.

A map $f \in \mathbb{U}^2$ is said to be Kupka-Smale if all periodic orbits are hyperbolic. It is said to be hyperbolic if it is Kupka-Smale and the critical point is attracted to a periodic attractor. It is said to be regular if it is hyperbolic and its critical point is not periodic or preperiodic. It is well known that regular maps are structurally stable.

A k-parameter C^r (or analytic) family of unimodal maps is a C^r (or analytic) map $F : \overline{\Lambda} \times I \to I$ such that $f_\lambda \in \mathbb{U}^2$, where $f_\lambda(x) = F(\lambda, x)$ where $\Lambda \subset \mathbb{R}^k$ is a bounded open connected domain with smooth (C^∞) boundary. We denote $\mathbb{UF}^r(\Lambda)$ the space of C^r families of unimodal maps, endowed with the C^r topology. Notice that $\mathbb{UF}^r(\Lambda)$ is a separable Baire space.

We will not introduce a topology in the space of analytic families of unimodal maps.

5.1. **Combinatorics and hyperbolicity.**

Let $f \in \mathbb{U}^2$. A symmetric interval $T \subset I$ is said to be nice if the iterates of ∂T never return to $\operatorname{int} T$. A nice interval $T \neq I$ is said to be a restrictive (or periodic) interval of period m for f if $f^m(T) \subset T$ and m is minimal with this property. In this case, the map $A \circ f^m \circ A^{-1} : I \to I$ is again unimodal for some affine map $A : T \to I$: this map is usually called a renormalization of f if $m > 1$ or a unimodal restriction if $m = 1$.

If $T \subset I$ is a nice interval, the domain of the first return map R_T to T consists of a (at most) countable union of intervals which we denote T^j. We reserve the index 0 for the component of 0: $0 \in T^0$, if 0 returns to T. From the nice condition, $R_T|T^j$ is a diffeomorphism if $0 \notin T^j$, and is an even map if $0 \in T^j$. We call T^0 the central domain of R_T. The return R_T is said to be central if $R_T(0) \in T^0$.

Under the Kupka-Smale condition, the dynamics outside a nice interval is hyperbolic, and in particular persistent:

Lemma 5.1. *Let $f \in \mathbb{U}^2$ and let $T \subset I$ be a symmetric interval. If all periodic orbits contained if $I \setminus \operatorname{int} T$ are hyperbolic (in particular if f is Kupka-Smale), then*

(1) The set of points $X \subset I$ which never enter $\operatorname{int} T$ splits in two forward invariant sets: an open set U attracted by a finite number of periodic orbits and a closed set K such that $f|K$ is uniformly expanding: $|Df^n(x)| > C\lambda^n$, for $x \in K$ and for some constants $C > 0$, $\lambda > 1$. Moreover, preperiodic points are dense in K.

(2) There exists a neighborhood $\mathcal{V} \subset \mathbb{U}^2$ of f and a continuous family of homeomorphisms $H[g] : I \to I$, $g \in \mathcal{V}$ such that $g \circ H[g]|I \setminus T = H[g] \circ f$, and $H[f] = \mathrm{id}$.

Proof. The first item is a consequence of Mañé's Theorem (see [MS], Theorem 5.1 and Corollary 1, page 248). Since hyperbolic sets are persistent, the second item follows. □

The following well known result shows that nice intervals allow one to study arbitrarily small neighborhoods of 0.

Lemma 5.2. *Let $f \in \mathbb{U}^2$ be Kupka-Smale. If f is not hyperbolic and the critical orbit is infinite, then for every $\epsilon > 0$, there exists a nice interval $[-p, p] \subset (-\epsilon, \epsilon)$ with p preperiodic.*

Proof. Let T be the intersection of all nice intervals containing 0 whose boundary is preperiodic. If $T \neq \{0\}$, then the domain of R_T is either T or empty. In the first case, $R_T : T \to T$ has no fixed point in $\operatorname{int} T$ and it follows that $R_T^m(\operatorname{int} T)$ converge to a periodic attractor in ∂T. Otherwise, by Lemma 5.1, $\operatorname{int} f(T)$ must be contained in the basin of a periodic attractor, so f is either hyperbolic or the critical point is preperiodic. □

The following is an easy consequence of Lemma 5.1.

Lemma 5.3. *Let f_λ, $\lambda \in (-\epsilon, \epsilon)$ be a C^2 family of unimodal maps, and let T be a nice interval with preperiodic boundary for $f = f_0$. Assume that there exists an interval $0 \in J$ and a family $T[\lambda]$ of intervals with preperiodic boundary, such that $T[0] = T$ and for $\lambda \in J$, all non-hyperbolic periodic orbits of f_λ intersect $\operatorname{int} T[\lambda]$. Then there exists a continuous family of homeomorphisms $H[\lambda] : I \to I$, $\lambda \in J$ such that $H[\lambda](T) = T[\lambda]$ and $f_\lambda \circ H[\lambda]|(I \setminus T) = H[\lambda] \circ f$ and $H[0] = \mathrm{id}$.*

5.1.1. *Principal nest.* We say that f is infinitely renormalizable if there exists arbitrarily small restrictive intervals $T \subset I$. Otherwise we say that f is finitely renormalizable.

Let $\mathcal{F} \subset \mathbb{U}^2$ be the class of Kupka-Smale finitely renormalizable maps whose critical point is recurrent, but not periodic. If $f \in \mathcal{F}$, the first return map $f^m : T \to T$ to its smallest restrictive interval has an orientation reversing fixed point which we call p. Let $I_1 = [-p, p]$. Define a nested sequence of intervals I_i as follows. Assuming I_i defined, let R_i be the first return map to I_i and let I_{i+1} be the central domain I_i^0 of R_i.

The sequence I_i is called the *principal nest* of f. A level of the principal nest is called central if R_i is a central return. We say that a map $f \in \mathcal{F}$ is simple if there are only finitely many non-central levels in the principal nest.

5.2. **Negative Schwarzian derivative.** The Schwarzian derivative of a C^3 map $f : I \to I$ is defined by

$$Sf = \frac{D^3 f}{Df} - \frac{3}{2}\left(\frac{D^2 f}{Df}\right)^2$$

in the complement of the critical points of f. If Sf and Sg are simultaneously positive (or negative) then $S(g \circ f)$ is positive (or negative).

If f is a unimodal map the condition of negative Schwarzian derivative is very useful and can be exploited in several ways. One of the most used tools is the Koebe Principle:

Lemma 5.4 (Koebe Principle, see [MS], page 258)**.** *Let $f : T \to \mathbb{R}$ be a diffeomorphism with non-negative Schwarzian derivative. Then for every K_0, there exists a* *constant k_0 such that if $T' \subset T$ and both components L and R of $T \setminus T'$ are bigger than $K|T'|$ for some constant $K > K_0$ then the distortion of $f|T'$ is bounded by k_0. In particular, we have $\min\{|f(L)|, |f(R)|\} \geq \hat{k}_0 K|f(T')|$, for some $\hat{k}_0$ depending only on K_0. Moreover, $k_0 \to 1$ as $K_0 \to \infty$.*

Quadratic maps have negative Schwarzian derivative. Moreover, one can often reduce to this situation as is shown by the following well known estimate:

Lemma 5.5. *If $f \in \mathbb{U}^3$ is infinitely renormalizable, then if $T \subset I$ is a small enough periodic nice interval, the first return map to T has negative Schwarzian derivative.*

Recently, Kozlovski showed that the assumption of negative Schwarzian can be often removed. The next result follows from Lemma 5.1 and [GSS] (which is based on the work of Kozlovski [K1]).

Lemma 5.6. *Let $f \in \mathcal{F} \cap \mathbb{U}^3$. There exists $i > 0$, an analytic diffeomorphism $s : I \to I$ and a neighborhood $\mathcal{V} \subset \mathbb{U}^3$ of f, such that there exists a continuation $I_i[g]$, $g \in \mathcal{V}$ of I_i ($H[g](I_i) = I_i[g]$ in the notation of Lemma 5.1) such that the first return map to $s(I_i[g])$ by $s \circ g \circ s^{-1} : I \to I$ has negative Schwarzian derivative.*

5.3. **Decay of geometry.** The following result is due to Lyubich [L1] in the case of negative Schwarzian derivative and holds in general due to the work of Kozlovski:

Lemma 5.7. *Let $f \in \mathcal{F}$ be at least C^3, and let $n_k - 1$ denote the sequence of non-central levels in the principal nest of f. Then $|I_{n_k+1}|/|I_{n_k}| < C\lambda^k$ for some constants $C > 0$, $\lambda < 1$.*

5.4. **Quasiquadratic maps.** A map $f \in \mathbb{U}^3$ is *quasiquadratic* if any nearby map $g \in \mathbb{U}^3$ is topologically conjugate to some quadratic map. By the theory of Milnor-Thurston and Guckenheimer [MS], a map $f \in \mathbb{U}^3$ with negative Schwarzian derivative and $D^2 f(-1) < 0$ is quasiquadratic, so quadratic maps are quasiquadratic. The following results give sufficient conditions for a unimodal map to be quasiquadratic:

Theorem 5.8 (see Lemma 2.13 of [ALM]). *Let $f \in \mathbb{U}^3$ be a Kupka-Smale unimodal map which is topologically conjugate to a quadratic map. Then f is quasiquadratic.*

Theorem 5.9 (see Remark 2.6 of [ALM]). *Let $f \in \mathbb{U}^3$. If f is not conjugate to a quadratic polynomial then there exists a (not necessarily hyperbolic) periodic orbit which attracts an open set. In particular, if all periodic orbits of f are repelling then f is quasiquadratic.*

Remark 5.3. Theorem 5.8 is the reason that the quasiquadratic condition considers only C^3 maps and the C^3 topology (otherwise it would not be possible to guarantee that even quadratic maps are quasiquadratic).

5.5. **Spaces of analytic unimodal maps.** Let $a > 0$, and let $\Omega_a \subset \mathbb{C}$ be the set of points at distance at most a of I. Let $\mathcal{E}_a$ be the complex Banach space of holomorphic maps $v : \Omega_a \to \mathbb{C}$ continuous up to the boundary which are 0-symmetric (that is, $v(z) = v(-z)$) and such that $v(-1) = v(1) = 0$, endowed with the sup-norm $\|v\|_a = \|v\|_\infty$. It contains the real Banach space $\mathcal{E}_a^{\mathbb{R}}$ of "real maps" v, i.e, holomorphic maps symmetric with respect to the real line: $v(\overline{z}) = \overline{v(z)}$.

Let us consider the constant function $-1 \in \Omega_a$. The complex affine subspace $-1 + \mathcal{E}_a$ will be denoted as $\mathcal{A}_a$.

Let $\mathbb{U}_a = \mathbb{U}^2 \cap \mathcal{A}_a$. It is clear that any analytic unimodal map belongs to some $\mathbb{U}_a$. Note that $\mathbb{U}_a$ is the union of an open set in the affine subspace $\mathcal{A}_a^{\mathbb{R}} = -1 + \mathcal{E}_a^{\mathbb{R}}$ and a codimension-one space of unimodal maps satisfying $f(0) = 1$.

5.6. **Hybrid lamination.** One of the main results of [ALM] is to describe the structure of the partition in topological classes of spaces of analytic unimodal maps. In that paper, they consider only the quasiquadratic case, but their proof works for the general case (due to the results of Kozlovski) and gives the following:

Theorem 5.10 (Theorem A of [ALM])**.** *Let $f \in \mathbb{U}_a$ be a Kupka-Smale map. There exists a neighborhood $\mathcal{V} \subset \mathcal{A}_a$ of f endowed with a codimension-one holomorphic lamination $\mathcal{L}$ (also called hybrid lamination) with the following properties:*

(1) the lamination is real-symmetric;

(2) if $g \in \mathcal{V} \cap \mathcal{A}_a^{\mathbb{R}}$ is non-regular, then the intersection of the leaf through g with $\mathcal{A}_a^{\mathbb{R}}$ coincides with the intersection of the topological conjugacy class of g with $\mathcal{V}$;

(3) Each $g \in \mathcal{V} \cap \mathcal{A}_a^{\mathbb{R}}$ belongs to some leaf of $\mathcal{L}$.

(For the definition of the leaves of $\mathcal{L}$ in the regular case, see Appendix A.)

Theorem 5.11. *In the setting of Theorem 5.10, if $g_1, g_2 \in \mathcal{V}$ are in the same leaf of $\mathcal{L}$ and $\gamma_1(\lambda)$, $\gamma_2(\lambda)$ are real analytic paths in $\mathcal{V} \cap \mathcal{A}_a^{\mathbb{R}}$, transverse to the leaves of $\mathcal{V}$ and such that $\gamma_1(\lambda_1) = g_1$, $\gamma_2(\lambda_2) = g_2$, then the local holonomy map $\psi : (\lambda_1 - \epsilon, \lambda_1 + \epsilon) \to (\lambda_2 - \epsilon', \lambda_2 + \epsilon')$ is quasisymmetric. Moreover, for δ sufficiently small, $\psi|(\lambda_1 - \delta, \lambda_1 + \delta)$ is $1 + O(\|g_1 - g_2\|_a)$-qs.*

Proof. This estimate is just the λ-Lemma in the context of codimension-one complex laminations. □

Moreover, each non-regular topological class is like a Teichmuller space:

Theorem 5.12. *In the setting of Theorem 5.10, if $g_1, g_2 \in \mathcal{V} \cap \mathbb{U}_a$ belong to the same leaf of $\mathcal{L}$, then there exists a $1 + O(\|g_1 - g_2\|_a)$-qs map $h : I \to I$ such that $g_2 \circ h = h \circ g_1$.*

Proof. This follows from Proposition 8.9 of [ALM] and the λ-Lemma. □

The tangent space to topological classes has a nice characterization:

Theorem 5.13 (Theorem 8.10 of [ALM])**.** *If $f \in \mathbb{U}_a$ is a non-regular Kupka-Smale map then the tangent space to the topological class of f is given by the set of vector fields $v \in \mathcal{E}_a$ which do not admit a representation $v = \alpha \circ f - \alpha Df$ on the critical orbit with α a qc vector field of $\mathbb{C}$.*

5.7. **Analytic families.** Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be an analytic family of unimodal maps. Then for $a > 0$ sufficiently small, $\lambda \mapsto f_\lambda$ is an analytic map from Λ to $\mathbb{U}_a$. We say that f_λ is *non-trivial* if the set of regular parameters is dense.

If $\lambda_0 \in \Lambda$ is a Kupka-Smale parameter, transversality to the topological class of λ_0 has the obvious meaning (using Theorem 5.10). We remark that this definition does not depend on the choice of $\mathbb{U}_a$.

Remark 5.4. Let B_i be an enumeration of all open balls contained in Λ of rational radius and center. The condition of non-triviality of a family $\{f_\lambda\}$, $\lambda \in \Lambda$ is an intersection of a countable number of conditions (existence of a regular parameter $\lambda \in B_i$). Each of those conditions is open in $\mathbb{UF}^2(\Lambda)$. The set of non-trivial analytic families is also dense in the $\mathbb{UF}^\infty(\Lambda)$ (this would still hold natural topology of analytic families in Λ, which we did not introduce), due to Theorem 5.10.

We should remark that for an analytic family of quasiquadratic maps, non-triviality is equivalent to existence of *one* regular parameter (since all non-regular topological classes are analytic submanifolds in the quasiquadratic case). In particular, non-triviality is a C^3 open condition in the quasiquadratic case.

6. Construction of the special family

6.1. **Puzzle maps.** Let $f \in \mathbb{U}_a$ be a finitely renormalizable unimodal map with a recurrent critical point. Let us consider some nice interval A^0 and let $\{A^j\}$ be the connected components of the domain of the first landing map from I to A^0. We call the family $\{A^j\}$ the real puzzle for f associated to A^0. The basic object used in [ALM] to analyze the dynamics of unimodal maps can be viewed as a complexification of such real puzzles, which are called simply a puzzle.

The definition of puzzle in [ALM] is too general and technical for our purposes. In this paper, we will simply describe how to construct a puzzle for f (or rather a geometric puzzle, in the language of [ALM]). Instead of giving the precise definitions of a puzzle, we will just obtain the properties that are needed for our results.

Let us fix some advanced level $\mathbf{n}$ of the principal nest of f and assume that $|I_\mathbf{n}|/|I_{\mathbf{n}-1}|$ is very small. Let us fix the following notation: let $A^0 = I_\mathbf{n}$ and let $\{A^j\}$ be the real puzzle associated to A^0. We let A^1 be such that $f(0) \in A^1$.

Given $0 < \theta \leq \pi/2$, and $A \subset \mathbb{R}$, let $D_\theta(A)$ be the intersection of two round disks D_1 and D_2 where $D_1 \cap \mathbb{R} = A$, ∂D_1 intersects $\mathbb{R}$ making an angle θ, and D_2 is the image of D_1 by symmetry about $\mathbb{R}$. The complexification of the real puzzle $\{A^j\}$ should be imagined as $\{D_\theta(A^j)\}$ for a suitable value of θ. Of course, since the system is non-linear, the definition can not be so simple. Nevertheless, the condition $|I_\mathbf{n}|/|I_{\mathbf{n}-1}|$ small allows one to bound the nonlinearity of the first landing map to $I_\mathbf{n}$ and we can obtain (see [ALM], Lemma 5.5):

Lemma 6.1. *Let $0 < \phi < \psi < \gamma < \pi/2$ be fixed. For arbitrarily big $k > 0$, if $|I_\mathbf{n}|/|I_{\mathbf{n}-1}|$ is small enough, there exists a sequence V^j of open Jordan disks such that $D_\phi(A^j) \subset V^j \subset D_\psi(A^j)$ and $V^0 = D_{(\phi+\psi)/2}(A^0)$ with the following properties:*

(1) If $j \neq 0$ and $f(A^j) \subset A^k$ then $f : V^j \to V^k$ is a diffeomorphism;

(2) If $f(A^0) \cap A^j \neq \emptyset$, then $\operatorname{mod} f(V^0) \setminus \overline{D_\gamma(A^j)} > k$.

6.2. **A special Banach space of perturbations.** Let $A^1 = [l, r]$ with $l < r$, and let $N = [-l, l]$. Domains V^j which do not intersect A^1 or N will play no role in the construction to follow. Let V be the union of all V^j such that $A^j \subset N \cup A^1$.

One of the main problems of [ALM] is to obtain a direction v (or infinitesimal perturbation) which is transverse to the topological class of f. The idea is to consider a perturbation which does not affect much f in N, but causes a bump near the critical value, localized in A^1. There are several difficulties related to this scheme, the first of which is that such a bump can only be reasonably controlled up to its first derivative. Another difficulty is that we want an analytic perturbation, so it cannot vanish in N and be a bump at A^1. The solution involves the consideration of certain Banach spaces of smooth (C^1) functions in $N \cup A^1$ which are analytic in $\operatorname{int} N \cup \operatorname{int} A^1$, which allows one to construct perturbations that, while badly behaved in the real line (can be only controlled up to the first derivative), are well behaved with respect to the complex puzzle structure.

While the proof in [ALM] involves two steps, construction of a transverse smooth vector field and approximation of this vector field by polynomials, which need two different Banach spaces, we will realize the same construction with just one Banach space. This is important to estimate the asymmetric roles of perturbations concentrated in N and A^1. The proof of our main perturbation estimate (Lemma 6.4) is a mixture of two estimates, Lemma 7.4 (for perturbations localized in A^1) and Lemma 7.9 (for perturbations supported on $N \cup A^1$) of [ALM].

Let $Z = D_\gamma(A^1) \cup D_\gamma(N)$, and let Υ be the space of all vector fields v holomorphic on Z and whose derivative admits a continuous extension to $\overline{Z}$, which vanish up to the first derivative in ∂A^1 and its forward iterates (this is a finite set) and such that $v|D_\gamma(N)$ is a symmetric (odd) vector field. We use the norm $\|v\| = \sup_{z \in Z} |Dv|$.

Let $\Upsilon = \Upsilon_1 \oplus \Upsilon_2$, where $v \in \Upsilon_1$ if $v|D_\gamma(N) = 0$ and $v \in \Upsilon_2$ if $v|D_\gamma(A^1) = 0$.

Let $f_v = f \circ (\mathrm{id} + v)$. The reader should think of vector fields $v \in \Upsilon$ as perturbations of f acting by $v \to f_v$. One of the main advantages of the definition of Υ is that, for small $v \in \Upsilon$, "the puzzle persists", that is, there exists a continuation V^v of the set V inside Z, whose connected components behave, under iteration by f_v, in the same way that the connected components of V behaved under iteration by f.

To make this more precise, let us say that $v \in \Upsilon$ is admissible if there exists a holomorphic motion h^v over $\mathbb{D}$, which is real-symmetric if v is real-symmetric, and is defined by the family of transition maps $h^v[0, \lambda] \equiv h^v_\lambda : \mathbb{C} \to \mathbb{C}$, $\lambda \in \mathbb{D}$ such that:

(1) $h^v_\lambda|\mathbb{C} \setminus Z = \mathrm{id}$, $h^v_\lambda|\partial f(V^0) = \mathrm{id}$;
(2) $f_{\lambda v} \circ h^v_\lambda | V \setminus V^0 = h_\lambda \circ f$, $f_{\lambda v} \circ h^v_\lambda|\partial V^0 = f$.

The holomorphic motion h^v will be said to be *compatible* with v.

The following is a restatement of Lemma 7.9 of [ALM].

Lemma 6.2. *There exists $\epsilon > 0$ such that if v belongs to $\{v \in \Upsilon | \|v\| < \epsilon\}$ then v is admissible.*

We also need the following simple estimate (see the proof of Lemma 7.4 of [ALM]):

Lemma 6.3. *Let $0 < \theta < \gamma < \pi/2$. There exists $\epsilon' > 0$ such that if A is an interval and v is holomorphic on $D_\gamma(A)$ whose derivative extends continuously to $\overline{D_\gamma(A)}$ satisfying $|Dv| < \epsilon'$ then $\mathrm{id} + v : D_\gamma(A) \to \mathbb{C}$ is a diffeomorphism and $D_\theta(A) \subset (\mathrm{id} + v)(D_\gamma(A))$.*

Now we can prove:

Lemma 6.4. *There exists constants $\epsilon_1 > 0$, $\epsilon_2 > 0$, where ϵ_1 depends only on ψ and γ such that if $v_1 \in \Upsilon_1$, $\|v_1\| < \epsilon_1$ and $v_2 \in \Upsilon_2$, $\|v_2\| < \epsilon_2$ then $v = v_1 + v_2$ is admissible.*

Proof. Let n_1 be such that $f^{n_1}(V^1) = V^0$ and let $\theta = (\psi + \gamma)/2$.

Let $v \in \Upsilon$ with $\|v\| < \epsilon$. By Lemma 6.2, there exists a holomorphic motion h^v compatible with v.

We claim that if $0 < \epsilon_2 < \epsilon$ is small enough and $\|v\| < \epsilon_2$ then for $\lambda \in \mathbb{D}$, $h^v_\lambda(V^1) \subset D_\theta(A^1)$. Indeed, if this is not the case, there would be a sequence $z_k \in \partial D_\theta(A^1)$, $v_k \in \Upsilon$, $v_k \to 0$, such that $f^{n_1+1}_{v_k}(z_k) \in f(V^0)$. It clearly follows that $z_k \to \partial A^1 = \{l, r\}$, let us say that $z_k \to l$. It is clear that

$$f^{n_1+1}_{v_k}(z_k) = f^{n_1+1}_{v_k}(l) + Df^{n_1+1}_{v_k}(l) z_k + o(z_k) = f^{n_1+1}(l) + Df^{n_1+1}(l) z_k + o(z_k).$$

In particular, the sequence $f^{n_1+1}_{v_k}(z_k)$ converges to $f^{n_1+1}(l)$ along a direction which makes angle θ with the real line (since $Df^{n_1+1}(l) \in \mathbb{R} \setminus \{0\}$), so $f^{n_1+1}_{v_k}(z_k) \notin f(V^0)$ for k big, which is a contradiction.

Let ϵ_1 be as in Lemma 6.3. If $v = v_1 + v_2$, with $v_i \in \Upsilon_i$ and $\|v_i\| < \epsilon_i$, let $h^v_\lambda : \mathbb{C} \setminus (D_\gamma(A^1) \setminus V^1)$ be given by $h^v_\lambda|(\mathbb{C} \setminus D_\gamma(A^1)) = h^{v_2}_\lambda$ and $h^v_\lambda|V^1 = ((\mathrm{id} + \lambda_{v_1})|D_\gamma(A^1))^{-1} \circ h^{v_2}_\lambda$. Any extension of h^v_λ to $\mathbb{C}$ is clearly compatible with v. $\square$

We will also need the following easy lemma:

Lemma 6.5. *If $|I_{\mathbf{n}}|/|I_{\mathbf{n-1}}|$ is sufficiently small, then for $w = w_1 + w_2$ with $w_i \in \Upsilon_i$, $\|w_i\| < \epsilon_i$, and for $\lambda \in \mathbb{D}$, then $(f_{\lambda w}|h^w_\lambda(V^0))^{-1}(D_\gamma(V^1)) \subset \mathbb{D}_{\rho|A^0|}(0)$, where $\rho \to 0$ as $|I_{\mathbf{n}}|/|I_{\mathbf{n-1}}| \to 0$.*

Proof. Let $U = h_\lambda(V^0)$ and $U^0 = (f_{\lambda w}|W)^{-1}(D_\gamma(A^1))$. Notice that $f_{\lambda w}(0) = f(0) \in D_\gamma(V^1)$. Thus, $f_{\lambda w}|(U \setminus \overline{U^0})$ is a double covering of $f(U_0) \setminus \overline{D_\gamma(A^1)}$. By Lemma 6.1, if $|I_{\mathbf{n}}|/|I_{\mathbf{n-1}}|$ is small then $\mathrm{mod}(f(U_0) \setminus \overline{D_\gamma(A^1)})$ is large, and so $\mathrm{mod}(U \setminus \overline{U^0})$ is also big. Since the derivative of $\mathrm{id} + \lambda w$ is smaller than $\max\{1+\epsilon_1, 1+\epsilon_2\}$, we see that the diameter of U is at most $2|A^0|$, so the diameter of U^0 can be bounded by $\rho|A^0|/2$ with small ρ as required. $\square$

6.3. **Analytic vector fields.** We will be specially concerned with special types of w which generate analytic families of unimodal maps. The following lemma is obvious:

Lemma 6.6. *If $w \in \Upsilon$ is real-symmetric and has an analytic extension $w : I \to I$ of C^1 of norm less than one, such that $w(-1) = w(1) = 0$, then $f_{\lambda w}$, $\lambda \in (-1, 1)$ is an analytic family of unimodal maps, and $I_{\mathbf{n}}$ is a nice interval with preperiodic boundary for each $f_{\lambda w}$.*

The following is a consequence of the Mergelyan Polynomial Approximation theorem:

Lemma 6.7. *Let $w \in \Upsilon$. Then there exists a sequence $w_m \in \Upsilon$ such that the C^1 norm of $w_m|I$ is less than $\|w\|$, $w_m(-1) = w_m(1) = 0$ and $w_m \to w$ in Υ. If w is real-symmetric then we can also choose w_m real-symmetric.*

Lemma 6.8. *Let $w \in \Upsilon$ be as in Lemma 6.6. If w is admissible, then the domain of the first return map to $I_{\mathbf{n}}$ under iteration by $f_{\lambda w}$ is $((\mathrm{id} + \lambda w)|h^w_\lambda(V^0))^{-1}(V) \cap \mathbb{R}$.*

Proof. By construction, all components of $((\mathrm{id} + \lambda w)|h^w_\lambda(V^0))^{-1}(V) \cap \mathbb{R}$ are components of the first return map to $I_{\mathbf{n}}$, so we just have to check that all components are of this form. Notice that each $x \in V \cap (f(-l), l)$ has two preimages by f in $V \cap ((-l, l) \setminus I_{\mathbf{n}})$. It follows that each $x \in h^w_\lambda(V) \cap (f(-l), l)$ has two preimages by f^w_λ in $h^w_\lambda(V) \cap ((-l, l) \setminus I_{\mathbf{n}})$. Let now T be a component of the first return map to $I_{\mathbf{n}}$ under iteration by $f_{\lambda w}$. If T is the central component, then T must be the preimage of A_1. Otherwise, all iterates of T up to the return are contained in $(f(-l), l)$. Since $\mathrm{int}\, I_{\mathbf{n}} \subset h^w_\lambda(V)$, we conclude that all iterates of T up to the return belong to $h^w_\lambda(V)$. $\square$

6.4. **A special perturbation.** Let us consider an affine map $Q : A^1 \to I$, and let

$$\tilde{v}_n(z) = (1 - z^2)(1 - e^{-2n}) + \frac{2}{n}(e^{-n(1+z)} + e^{-n(1-z)} - e^{-2n} - 1),$$

and let $v_n \in \Upsilon_1$ be such that $v_n|D_\gamma(A^1) = Q^* \tilde{v}_n \epsilon_1/8$. Notice that $\|v_n\| < \epsilon_1$.

6.4.1. *Infinitesimal transversality.* The importance of the sequence v_m in [ALM] is that it is eventually transverse to the topologically class of f.

Let us say that w is *formally transverse at* f if there is no quasiconformal vector field α of $\mathbb{C}$, such that for $z \in \operatorname{orb}_f(0)$, $w(z) = f^*\alpha(z) - \alpha(z)$. (This definition is motivated by Theorem 5.13, see also Lemma 7.3.)

The following summarizes Lemmas 7.6, 7.7 and 7.8 of [ALM].

Lemma 6.9. *Let v_m be defined as above. If $|I_{\mathbf{n}}|/|I_{\mathbf{n}-1}|$ is sufficiently small, then for m sufficiently big, v_m is formally transverse at f.*

The following is due to (a version of) the so called Key estimate of [ALM] (more precisely we use Corollary 7.14 of [ALM]):

Lemma 6.10. *The set of vector fields $w \in \Upsilon$ which are not formally transverse at f is a closed subspace of Υ.*

Remark 6.1. In particular, if m is sufficiently big and w is close to v_m then w is formally transverse at f.

6.4.2. *Macroscopic transversality.* The following result can be interpreted as the macroscopic counterpart to the infinitesimal transversality of v_m.

Let $r > 0$ be minimal with $f^{r+1}(0) \in V^1$.

Lemma 6.11. *There exists a constant $\tau_0 > 0$ depending only on ϵ_1 and ϕ, such that if $|I_{\mathbf{n}}|/|I_{\mathbf{n}-1}|$ is sufficiently small the following holds. Let v_m be defined as above and let $r > 0$ be minimal with $f^{r+1}(0) \in V^1$. Then for m sufficiently big, there exists a domain $\hat{\Theta} \subset \mathbb{D}$ such that the map $\theta : \hat{\Theta} \to \mathbb{C}$ given by $\theta(\lambda) = f^r_{\lambda v_m}(0)$ is a diffeomorphism onto $\mathbb{D}_{\tau_0|I_{\mathbf{n}}|}$.*

Proof. Since $\|v_m\| < \epsilon_1$, there exists a holomorphic motion h^{v_m} which is compatible with v_m.

Let $\Psi : \mathbb{D} \to \mathbb{C}$, $\Psi(\lambda) = (\mathrm{id} + \lambda v_m)(f(0))$. It is clearly a diffeomorphism over a round disk D_m centered on 0. Let d_m be the hyperbolic distance between $f(0)$ and ∂D_m in $D_{\pi/2}(A^1)$. It is easy to estimate directly d_m from below in terms of ϵ_1 and m. In particular, for m big, $d_m > \tilde{\tau} > 0$ where $\tilde{\tau}$ depends only on ϵ_1, not on the position of $f(0)$ in A^1.[13]

Now let Q be the connected component of $f(0)$ on $f^{-(r-1)}(V^0)$, so that $f^{r-1} : Q \to V^0$ is a diffeomorphism. The hyperbolic distance between $\partial D \cap Q$ and $f(0)$ in Q is bounded from below by $\tilde{\tau}$ by the Schwarz Lemma (if $\partial D \cap Q = \emptyset$, we let this distance be ∞). It follows that $f^{r-1}(Q \cap D)$ contains a $\tilde{\tau}$ hyperbolic neighborhood of $f^r(0)$ on V^0. Now, if $|I_{\mathbf{n}}|/|I_{\mathbf{n}-1}|$ is very small, then $|I_{\mathbf{n}+1}|/|I_{\mathbf{n}}|$ is also very small, so $f^r(0)$ (which is contained in $I_{\mathbf{n}+1}$) is $\tilde{\tau}/2$ close to 0 in the hyperbolic metric of $V^0 \supset D_\phi(A^0)$.

As a consequence, $f^{r-1}(Q \cap D)$ contains a $\tilde{\tau}/2$ hyperbolic neighborhood of 0 in V^0, and since $V^0 \supset D_\phi(A^0)$, it must contain $\mathbb{D}_{\tau|A^0|}$, where τ depends on ϵ_1 and ϕ. $\square$

[13] To see this, notice that $D\Psi(0) = v_m(f(0))$, and the norm of $v_m(f(0))$ in the hyperbolic metric of $D_{\pi/2}(A^1)$ at $f(0)$ is at least $\epsilon_1/10$ for m big. Let $P : D_{\pi/2}(A^1) \to \mathbb{D}$ be a Moebius transformation taking $f(0)$ to 0. The the norm of $D(P \circ \Psi)(0)$ in the hyperbolic metric of $\mathbb{D}$ at 0 is at least $\epsilon_1/10$, so the Euclidean norm of $D(P \circ \Psi)(0)$ is at least $\epsilon_1/10$. By the Koebe 1/4 Theorem, $P(D_m)$ contains a round disk of radius $\epsilon_1/40$ around 0, thus the hyperbolic distance from $\partial P(D_m)$ to 0 in $\mathbb{D}$ is at least $\epsilon_1/40$.

6.4.3. *Construction of a full R-family.* Let τ_0 be the constant of Lemma 6.11 and let $|I_{\mathbf{n}}|/|I_{\mathbf{n}-1}|$ be such that Lemma 6.5 holds with $\rho < \tau_0/4$.

Let m be big and let us fix $v = v_m$ such that Lemmas 6.11 and 6.9 are valid, and let $\hat{\Theta}$ be the domain of Lemma 6.11.

Let $w = w_1 + w_2$ with $w_i \in \Upsilon_i$, $\|w_i\| < \epsilon_i$.

Let $U[0] = V^0$ and let the family $\{U^j[0]\}$ denote the connected components of $(f|V^0)^{-1}(\cup V^j)$, letting $0 \in U^0[0]$.

Let us consider a holomorphic motion $\tilde{H}$ over $\mathbb{D}$ given by the transition maps $\tilde{H}[0,\lambda] = \tilde{H}_\lambda : \mathbb{C} \to \mathbb{C}$ such that:

$$\tilde{H}_\lambda|\mathbb{C} \setminus U[0] = h_{\lambda w}$$

$$f_{\lambda w} \circ \tilde{H}_\lambda|U[0] \setminus U^0[0] = h_{\lambda w} \circ f.$$

Let $U[\lambda] = \tilde{H}_\lambda(U[0])$, $U^j[\lambda] = \tilde{H}_\lambda(U^j[0])$.

Let $R[\lambda]$ be the first return map from $U^j[\lambda]$ to U_0. It is clear that $(R[\lambda], \tilde{H}_\lambda)$ has a structure of a (non-full) R-family over $\mathbb{D}$. Let us consider the landing family $(L[\lambda], H_\lambda)$ associated to $(R[\lambda], \tilde{H}_\lambda)$.

Let $W^{\underline{d}}[0]$ be the domain of $L[0]$ containing $R[0](0)$. Notice that $L[\lambda]|W^{\underline{d}}[\lambda]$ extends to a diffeomorphism $R^{\underline{d}}[\lambda]$ onto $U[\lambda]$. For $\tau < \tau_0$, let $\Delta_\tau[\lambda]$ be the preimage of $\mathbb{D}_{\tau|A^0|}(0)$ by this diffeomorphism.

If $w = v$ then $R^{\underline{d}}[\lambda] = R^{\underline{d}}[0]$ for all λ, since v is supported on $D_\gamma(A^1)$.

In particular, $R^{\underline{d}}[\lambda] = R^{\underline{d}}[0]$ and $\Delta_\tau[\lambda] = \Delta_\tau[0]$ for all λ. So $\lambda \mapsto R[\lambda](0)$ is a map which restricts (in some domain $0 \in \mathbb{O}^v$) to a diffeomorphism onto $\Delta_\tau[0]$. It follows that taking $\tau = \tau_0/2$, for any w close to v there exists a domain $0 \in \mathbb{O}^w$ where $\lambda \mapsto R[\lambda](0)$ is a diffeomorphism onto $\Delta_\tau[0]$ (of course, $\mathbb{O}^w$ depends on w).

But for $w \in \Upsilon$ close to v and for all $\lambda \in \mathbb{D}$, $U^0[\lambda]$ is contained in $\mathbb{D}_{\rho|A^0|}$, so $W^{\underline{d}}[\lambda]$ is contained in $\Delta_{\tau_0/2}[0]$ with space. By the argument principle, letting Θ be the connected component of 0 of the set of $\lambda \in \hat{\Theta}$ with $R[\lambda](0) \in W^{\underline{d}}[\lambda]$, the map $S : \overline{\Theta} \to \overline{W^{\underline{d}}[0]}$ such that $S(\lambda) = H_\lambda^{-1}(R[\lambda](0))$ is a homeomorphism. We also have that the diameter of Θ is very small if ρ is small (in particular if $|I_{\mathbf{n}}|/|I_{\mathbf{n}-1}|$ is small).

Let $U_1[0] = U^0[0]$ and let $\{U_1^j[0]\}$ be the connected components of the preimage by $R[0]|U^0[0]$ of $\cup W^{\underline{d}}[0]$, and let $0 \in U_1^0$.

Let h be a holomorphic motion over Θ given by transition maps $h[0,\lambda] = h_\lambda : \mathbb{C} \to \mathbb{C}$ such that

$$h_\lambda|\mathbb{C} \setminus U_1 = H_\lambda,$$

$$R[\lambda] \circ h_\lambda|U_1 \setminus U_1^0 = h_\lambda \circ R[0].$$

Let $U_1[\lambda] = h_\lambda(U_1[0])$ and $U_1^j[\lambda] = h_\lambda(U_1^j[0])$.

Our construction shows clearly that the first return map $R_1[\lambda]$ from $\cup U_1^j[\lambda]$ to $U_1[\lambda]$ is an R-map for $\lambda \in \Theta$, so $(R_1[\lambda], h_\lambda)$ is an R-family, and our choice of Θ implies that $R_1[\lambda]$ is a *full* R-family.

Let us summarize the properties we obtained in this construction:

Lemma 6.12. *If $|I_{\mathbf{n}}|/|I_{\mathbf{n}-1}|$ is small enough, there exists a real-symmetric vector field $v \in \Upsilon$ and a neighborhood $v \in \mathcal{V} \subset \Upsilon$ such that for any $w \in \mathcal{V}$ real-symmetric, there exists a domain $0 \in \Theta \subset \mathbb{D}$, a family of R-maps $R_1[\lambda] : U_1^j[\lambda] \to U_1[\lambda]$, $\lambda \in \Theta$, and a real-symmetric holomorphic motion h over Θ such that:*

(1) For $\lambda \in \Theta \cap \mathbb{R}$, $U_1[\lambda] \cap \mathbb{R} = I_{\mathbf{n}+1}$;

(2) $R_1[\lambda]$ *is the first return map from* $\cup U_1^j[\lambda]$ *to* $U_1[\lambda]$ *under iteration by* $f_{\lambda w}$*;*

(3) $(R_1[\lambda], h)$ *form a full real-symmetric R-family.*

And moreover, if w *is as in Lemma 6.6 and* $\lambda \in \Theta \cap \mathbb{R}$ *then:*

(4) $I_{\mathbf{n}+1}[\lambda] \equiv U_1[\lambda] \cap \mathbb{R}$ *is the component of* 0 *of the first return map to* $I_{\mathbf{n}}$ *under iteration by* $f_{\lambda w}$*;*

(5) $I^j_{\mathbf{n}+1}[\lambda] \equiv U_1^j[\lambda] \cap \mathbb{R}$ *are the domains of the first return map to* $I_{\mathbf{n}+1}[\lambda]$ *under iteration of the real analytic extension* $f_{\lambda w} : I \to I$*.*

The construction of the R-family gives us also a good control of its geometry.

Lemma 6.13. *In the setting of Lemma 6.12,* $\mathrm{Dil}(h|\mathbb{C} \setminus \overline{U_1^0}) < 1+\epsilon$*, and* $\mathrm{mod}(U_1[0] \setminus \overline{U_1^0[0]}) > C$*, where* $\epsilon \to 0$ *and* $C \to \infty$ *when* $|I_{\mathbf{n}}|/|I_{\mathbf{n}-1}| \to 0$*.*

Proof. Indeed, $\mathrm{Dil}(h|\mathbb{C} \setminus \overline{U_1^0}) < 1+\epsilon$ is bounded by the hyperbolic diameter of Θ in $\mathbb{D}$, which is small if $|I_{\mathbf{n}}|/|I_{\mathbf{n}-1}| \to 0$ is big. On the other hand, $\mathrm{mod}(U_1[0] \setminus \overline{U_1^0[0]}) \geq \mathrm{mod}(U[0] \setminus \overline{U^0[0]})/2 \geq \mathrm{mod}(f(V^0) \setminus \overline{V^1})/4 > k/4$, which is big if $I_{\mathbf{n}} \setminus I_{\mathbf{n}-1}$ is small by Lemma 6.1. □

6.5. **Remarks on the infinitesimal transversality of the special perturbation.** We would like to point out that the "macroscopic transversality" of v_m is very much related to its infinitesimal transversality. The (formalizable) argument relating both properties is as follows (notice that this argument is different from the one given in [ALM], which emphasizes estimates at the infinitesimal level):

(1) v_m can be C^1 extended to I as an odd vector field which vanishes on $[r,1]$, $[-1,-r]$ and $[-l,l]$. This vector field is not C^2 but its C^1 norm is small (ϵ_1).

(2) (Macroscopic transversality implies a C^1 connecting lemma) Notice that the interval $(f^r_{-v_m}(0), f^r_{v_m}(0))$ contains the interval $I_{\mathbf{n}+1}$ (with lots of space). We conclude that the family $f_{\lambda v_m}$, $\lambda \in (-1,1)$ must go through a parameter λ where $f^r_{\lambda v_m}(0) = 0$, and so changes the combinatorics of f.

(3) Using the Key Estimate of [ALM], we see that, if v_m is not formally transverse at f, then it is actually tangent to the topological class of f in the following sense. There exists a (real-symmetric) holomorphic motion h over $\mathbb{D}$ whose transition maps $h[0,\lambda] \equiv h_\lambda : \mathbb{C} \to \mathbb{C}$ are such that $f_\lambda = h_\lambda \circ f \circ h_\lambda^{-1}$ is a family of so called "puzzle maps" (which behave as unimodal maps) such that

$$\frac{d}{d\lambda} f_\lambda \bigg|_{\lambda=0} = \frac{d}{d\lambda} f_{\lambda v_m} \bigg|_{\lambda=0} = Df \cdot v_m$$

(the maps h_λ are characterized by $\overline{\partial} h_\lambda / \partial h_\lambda = \lambda \partial \alpha$ for a specially chosen quasiconformal vector field α satisfying $v_m = f^*\alpha - \alpha$ on the critical orbit). This family can be considered the Beltrami path through f in the direction of $Df \cdot v_m$.

(4) The family f_λ is tangent to $f_{\lambda v_m}$ at $\lambda = 0$ and both families have big extensions (to $\mathbb{D}$). In particular, they must be close together in a slightly smaller disk, where we can detect the change of combinatorics: there is a parameter $\lambda \in \mathbb{D}$ such that $f^r_\lambda(0) = 0$[14].

(5) In particular, the family f_λ must change combinatorics, but this is a contradiction, since it consists of maps topologically conjugate to f. So we conclude

[14]More precisely, we use that the holomorphic map $\lambda \mapsto f^r_\lambda(0)$ has the same derivative at 0 as the almost linear map $\lambda \mapsto f^r_{\lambda v_m}(0)$, and a simple estimate shows that there exists a parameter $\lambda \in \mathbb{D}$ such that $f^r_{\lambda v_m}(0) = 0$.

that v_m is formally transverse at f. Notice that our argument is that a "reasonably efficient"[15] tangent path to v_m closes macroscopically the critical orbit.

(6) (Infinitesimal analytic connecting lemma) Although v_m is only C^1 in the interval, we can approximate it in the topology of Υ by polynomials w which will be still formally transverse to f. Those vector fields w are transversal to the topological class of f: they close "infinitesimally" the critical orbit.

7. The Phase-Parameter relation

7.1. **Phase-Parameter relation for the special family.** Let $f \in \mathcal{F}$ and let $R_i : \cup I_i^j \to I_i$ be the first return map. For $\underline{d} \in \Omega$, $\underline{d} = (j_1, ..., j_m)$, let $I_i^{\underline{d}} = \{x \in I_i | \, R_i^{k-1}(x) \in I_i^{j_{i+1}}, 1 \leq k \leq m\}$, and let $R_i^{\underline{d}} = R_i^m | I_i^{\underline{d}}$. Let $C_i^{\underline{d}} = (R_i^{\underline{d}})^{-1}(I_i^0)$. The map $L_i : \cup C_i^{\underline{d}} \to I_i^0$ is the first landing map from I_i to I_{i+1}.

Definition 7.1. Let us say that a family f_λ of unimodal maps satisfies the Topological Phase-Parameter relation at a parameter λ_0 if $f = f_{\lambda_0} \in \mathcal{F}$, and there exists $i_0 > 0$ and a sequence of nested intervals J_i, $i \geq i_0$ such that:

1. J_i is the maximal interval containing λ_0 such that for all $\lambda \in J_i$ there exists a homeomorphism $H_i[\lambda] : I \to I$ such that $f_\lambda \circ H_i[\lambda] | (I \setminus I_{i+1}) = H_i[\lambda] \circ f$.
2. There exists a homeomorphism $\Xi_i : I_i \to J_i$ such that $\Xi_i(C_i^{\underline{d}})$ (respectively, $\Xi_i(I_i^{\underline{d}})$) is the set of λ such that the first return of 0 to $H_i[\lambda](I_i)$ under iteration by f_λ belongs to $H_i[\lambda](C_i^{\underline{d}})$ (respectively, $H_i[\lambda](I_i^{\underline{d}})$).

Definition 7.2. Let f_λ be a family of unimodal maps. We say that f_λ has Decay of Parameter Geometry at λ_0 if $f = f_{\lambda_0} \in \mathcal{F}$, it satisfies the Topological Phase-Parameter relation at λ_0 and $|J_{n_k+1}|/|J_{n_k}| < C\lambda^k$ for some constants $C > 0$, $\lambda < 1$, where $n_k - 1$ counts the non-central levels of the principal nest of f.

Theorem 7.1. *Let $f \in \mathcal{F}$ be analytic. There exists a polynomial vector field w such that the family $f_{\lambda w} = f \circ (\mathrm{id} + \lambda w)$, $\lambda \in (-\epsilon, \epsilon)$ is an analytic family of unimodal maps which satisfies the Topological Phase-Parameter relation and has Decay of Parameter Geometry at 0.*

Proof. Let w and $\mathbf{n}$ be as in Lemma 6.12. Denote by $(\mathcal{R}_1, h_1)$ the R-family of that lemma. Since $f \in \mathcal{F}$, the critical point is recurrent and we can clearly construct a R-chain $(\mathcal{R}_i, h_i)$ over $\lambda = 0$. It is clear that the real trace of $R_i[0] : \cup U_i^j[0] \to U_i[0]$ is the first return map to $I_{\mathbf{n}+i}$. Let $J_{\mathbf{n}+i} = \Lambda_i \cap \mathbb{R}$, let $\Xi_{\mathbf{n}+i} = \chi_i[0] | I_{\mathbf{n}+i}$. It is clear that $|J_{n_k+1}|/|J_{n_k}|$ decays exponentially by Lemma 6.13 and Theorem 4.6, where $n_k - 1$ counts the non-central levels of the principal nest of f. In particular, $|J_n| \to 0$.

In order to conclude the result, we just have to show the existence of the continuous family of homeomorphisms $H_i[\lambda]$, for i sufficiently big. Notice that if $\lambda \in J_{\mathbf{n}+i}$, if $p \in I_{\mathbf{n}+i}[\lambda]$ is a periodic orbit for f_λ which never enters $I^0_{\mathbf{n}+i}[\lambda]$ then p is hyperbolic by the Schwarz Lemma. So, if $\lambda \in J_{\mathbf{n}+i}$, the only non-hyperbolic periodic orbits for f_λ must be entirely contained in $I \setminus I_{\mathbf{n}+1}$. But since $f | I \setminus I_{\mathbf{n}+1}$ is hyperbolic, there exists $\epsilon > 0$ such that if $\lambda \in (-\epsilon, \epsilon)$, all periodic orbits in $I \setminus I_{\mathbf{n}+1}[\lambda]$ of f_λ are hyperbolic (by Lemma 5.1). In particular, if i is sufficiently big, $J_i \subset (-\epsilon, \epsilon)$, and all periodic orbits of f_λ in $I \setminus I_{i+1}[\lambda]$ are hyperbolic. The result follows by Lemma 5.3. □

[15] In the sense of admitting a controlled extension to a big domain, as the Beltrami path we constructed.

Let K_i be the closure of the union of all $\partial C_i^{\underline{d}}$ and $\partial I_i^{\underline{d}}$. Notice that H_i and Ξ_i are only uniquely defined in K_i. Condition (2) of the Topological Phase-Parameter relation can be equivalently formulated as the existence of a homeomorphism $\Xi_i : I_i \to J_i$ such that the first return of the critical point (under iteration by f_λ) to $H_i[\lambda](I_i)$ belongs to $H_i[\lambda](K_i)$ if and only if $\lambda \in \Xi_i(K_i)$.

Let us now estimate the metric properties of $H_i|K_i$ and $\Xi_i|K_i$. In order to do so, we will need to consider convenient restrictions of those maps.

Let $\tilde{I}_{i+2} = (R_i|I_i^0)^{-1}(I_i^{\underline{d}})$, where $\underline{d}$ is such that $(R_i|I_i^0)^{-1}(C_i^{\underline{d}}) = I_{i+2}$.

Let τ_i be such that $R_i(0) \in I_i^{\tau_i}$.

Let $\tilde{K}_i = \overline{(\cup_j \partial I_i^j \cup \partial I_i) \setminus \operatorname{int} \tilde{I}_{i+1}}$.

Let $J_i^j = \Xi_i(I_i^j)$.

Definition 7.3. Let f_λ be a family of unimodal maps. We say that f_λ satisfies the Phase-Parameter relation at λ_0 if $f = f_{\lambda_0}$ is simple, f_λ satisfies the Topological Phase-Parameter relation at λ_0 and for every $\gamma > 1$, there exists $i_0 > 0$ such that for $i > i_0$ we have:

PhPa1: $\Xi_i|(K_i \cap I_i^{\tau_i})$ is γ-qs,
PhPa2: $\Xi_i|\tilde{K}_i$ is γ-qs,
PhPh1: $H_i[\lambda]|K_i$ is γ-qs if $\lambda \in J_i^{\tau_i}$,
PhPh2: the map $H_i[\lambda]|\tilde{K}_i$ is γ-qs if $\lambda \in J_i$.

Theorem 7.2. *In the same setting of the previous theorem, if f is simple, the family $f_{\lambda w}$ satisfies the Phase-Parameter relation at* 0.

Proof. Let $(\mathcal{R}_i, h_i)$ be the R-chain of the proof of Theorem 7.1. By Theorems 4.2 and 4.6, $\operatorname{mod}(U_i[0] \setminus \overline{U_i^0[0]}) \to \infty$ and $\operatorname{mod}(\Lambda_i \setminus \overline{\Lambda_{i+1}}) \to \infty$ (notice that since f is simple, all deep enough levels $\mathcal{R}_i$ are non-central). This implies that, for any fixed $\gamma > 1$, there exists $i_0 > 0$ such that for $i > i_0$ the hypothesis of Lemmas 3.1 and 3.2 are fulfilled and hence their conclusions (CPhPa1, CPhPh1, CPhPa2, and CPhPh2) apply. Those immediately imply the four conditions (PhPa1, PhPh1, PhPa2, and PhPh2) of the Phase-Parameter relation by restriction to the real line. $\square$

7.2. **Phase-parameter relation for transverse families.**

Let $f_{\lambda w}$ be the special family constructed before.

Lemma 7.3. *The family $f_{\lambda w}$ is transverse to the topological class of f at $\lambda = 0$.*

Proof. Indeed, if $f_{\lambda w}$ is not transverse then by Theorem 5.13, there exists a qc vector field $\alpha : \mathbb{C} \to \mathbb{C}$ such that

$$wDf = \frac{d}{d\lambda} f_{\lambda w}\bigg|_{w=0} = \alpha \circ f - \alpha Df$$

on $\operatorname{orb}_f(0)$. Dividing by Df we get $w = f^*\alpha - \alpha$ on $\operatorname{orb}_f(0)$. But this contradicts Remark 6.1. $\square$

We will now show how to use the lamination of [ALM] to transfer the Phase-Parameter relation from the transversal family $f_{\lambda w}$ to any transversal family f_λ.

The basic idea is contained in the following diagram:

$$\begin{array}{ccc} \text{Phase of } f_{\lambda w} & \underset{\text{qs conjugacy}}{\overset{\text{Theorem 5.12}}{\longrightarrow}} & \text{Phase of } f_\lambda \\ \text{Phase-Parameter for } f_{\lambda w}\big\downarrow & & \big\downarrow\text{Phase-Parameter for } f_\lambda \\ \text{Parameter of } f_{\lambda w} & \underset{\text{holonomy map of } \mathcal{L}}{\overset{\text{Theorem 5.11}}{\longrightarrow}} & \text{Parameter of } f_\lambda \end{array}$$

(notice that the estimates for all arrows are all ultimately based on the λ-Lemma).

Theorem 7.4. *Let $f \in \mathcal{F}$, and let f_λ be a one-parameter analytic family of unimodal maps through f such that $f_{\lambda_0} = f$ and f_λ is transverse to the topological class of f at $\lambda = \lambda_0$. Then the Topological Phase-Parameter relation and Decay of Parameter Geometry holds for the family f_λ at λ_0. Moreover, if f is simple, then the Phase-Parameter relation also holds.*

Proof. Using Theorems 7.1, 7.2 and Lemma 7.3 consider the family $f_{\lambda w}$ through f, which is transverse to the hybrid class of f and which satisfies the Topological Phase-Parameter relation and Decay of Parameter Geometry (and the Phase-Parameter relation if f is simple). Fix a such that both $f_{\lambda w}$ and f_λ are analytic paths in $\mathbb{U}_a$. Let $\mathcal{L}$ be the lamination from Theorem 5.10. Since both f_λ and $f_{\lambda w}$ are transverse to the topological class of f (at λ_0 and 0), we can consider the local holonomy map of the lamination $\mathcal{L}$, $\psi : (-\epsilon, \epsilon) \to (\lambda_0 - \epsilon', \lambda_0 + \epsilon')$.

Let $\tilde{\Xi}_i : I_i \to \tilde{J}_i$ be the phase-parameter map for the family $f_{\lambda w}$, and let $\tilde{H}_i[\lambda]$ be the phase-phase map. We obtain the phase-parameter map for f_λ as a composition $\Xi_i = \psi \circ \tilde{\Xi}_i$. Since $|\tilde{J}_i| \to 0$,

$$\lim_{i\to\infty} \sup_{\lambda \in \tilde{J}_i} \|f_{\lambda w} - f_{\psi(\lambda)}\|_a = 0.$$

In particular, by Theorem 5.11, $\psi|\tilde{J}_i$ is γ_i-qs with $\lim \gamma_i = 1$.

Since for each $\lambda \in J_i = \psi(\tilde{J}_i)$, f_λ is qs conjugate to $f_{\psi^{-1}(\lambda)w}$, we see that if $\lambda \in J_i$ then there are no non-hyperbolic periodic orbits for f_λ in the complement of the continuation of I_{i+1}. Using Lemma 5.1 we conclude as in Theorem 7.1 the existence of a continuous family $H_i[\lambda]$ of phase-phase maps for the family f_λ. It follows that the Topological Phase-Parameter relation holds for f_λ at λ_0.

Since ψ is quasisymmetric, it is Hölder and the Decay of Parameter Geometry also follows from Theorem 7.1. If f is simple, estimates PhPa1 and PhPa2 follow from Theorem 7.2.

Let $h_\lambda : I \to I$ be a quasisymmetric conjugacy between $f_{\lambda w}$ and $f_{\psi(\lambda)}$ which is $1 + O(\|f_{\lambda w} - f_{\psi(\lambda)}\|_a)$-qs. This family might not be continuous, but $H_i[\psi(\lambda)]|K_i = h_\lambda \circ \tilde{H}_i[\lambda]$, which is enough for our purposes. In particular, if f is simple, PhPh1 and PhPh2 follow from Theorem 7.2. $\square$

Remark 7.1. Notice that even if we are only interested in the phase-parameter relation for individual families, this proof needs the knowledge of the behavior of topological conjugacy classes of unimodal maps in infinite dimensional spaces. For the case of the quadratic family, this is not needed: the argument of [L3] is based on the combinatorial theory of the Mandelbrot set (Douady-Hubbard, Yoccoz), which allows to show directly that the real quadratic family gives rise to full unfolded complex return type families. In particular, our proof also gives a somewhat different approach to the phase-parameter relation on the quadratic family itself.

8. Proof of Theorem A

Let f_λ be a one-parameter non-trivial analytic family of unimodal maps. In view of Theorem 7.4, to conclude Theorem A it is enough to show that

(1) Almost every non-regular parameter belongs to $\mathcal{F}$, that is, it is Kupka-Smale, has a recurrent critical point and is not infinitely renormalizable,
(2) Almost every parameter in $\mathcal{F}$ is simple,
(3) f_λ is transverse to the topological class of almost every parameter.

We will take care of these issues separately below: item (1) will follow from Lemmas 8.1, 8.4, and 8.5, item (2) from Lemma 8.6 and item (3) from Lemma 8.3.

8.1. **Transversality.**

Lemma 8.1. *Let f_λ be a non-trivial analytic family of unimodal maps. Then at most countably many parameters are not Kupka-Smale or have a periodic or preperiodic critical point.*

Proof. Indeed, the set of parameters which are not Kupka-Smale correspond to solutions of countably many analytic equations of the type $f^n_\lambda(p) = p$, $Df^{2n}_\lambda(p) = 1$, $n > 0$. Similarly, the set of parameters with periodic or preperiodic critical point corresponds to countably many equations of the type $f^m_\lambda(0) = f^n_\lambda(0)$, $0 \leq m < n$. So the set of parameters which are not Kupka-Smale is either countable or contains intervals. Since regular parameters are dense, the first possibility holds. □

The following result is due to Douady, see Lemma 9.1 of [ALM]:

Lemma 8.2. *Let $\mathcal{L}$ be a codimension-one complex lamination on an open set $\mathcal{V}$ of some Banach space, and let γ be an analytic path in $\mathcal{V}$. If γ is not contained in a leaf of $\mathcal{L}$, then the set of parameters where γ is not transverse to the leaves of $\mathcal{L}$ consists of isolated points.*

This result immediately implies:

Lemma 8.3. *Let f_λ be a non-trivial analytic family of unimodal maps. Then the set of non-regular Kupka-Smale parameters λ_0 such that f_λ is not transverse to the topological class of f_{λ_0} at λ_0 is countable.*

8.2. **Non-recurrent parameters.**

The following result is due to Duncan Sands [S], but we will provide a quick proof based on holomorphic motions and Lemma 8.2.

Lemma 8.4. *Let f_λ be a non-trivial analytic family of unimodal maps. Then almost every parameter is regular or has a recurrent critical point.*

Proof. If this is not the case, there would exist $\epsilon > 0$ and a set X of parameters λ of positive measure such that for $\lambda \in X$,
(1) $\inf_{m \geq 1} |f^m_\lambda(0)| > \epsilon$ (by hypothesis),
(2) f_λ is non-regular, Kupka-Smale and the critical orbit is infinite (Lemma 8.1).

Let us fix a density point $\lambda_0 \in X$ of X. Using Lemma 5.2, consider a nice interval $T = T[\lambda_0] = [-p, p] \subset (-\epsilon, \epsilon)$ for f_{λ_0}, with p preperiodic. Let $T[\lambda]$, $\lambda - \lambda_0 \in (-\delta, \delta)$, $\delta > 0$ small denote the continuation of T. Let $K[\lambda]$, $\lambda - \lambda_0 \in (-\delta, \delta)$ denote the set of points in $I \setminus T[\lambda]$ which never enter $T[\lambda]$ and do not belong to the basin of hyperbolic attractors.

Since $K = K[\lambda_0]$ is an expanding set by Lemma 5.1, it persists in a complex neighborhood of λ_0: there exists a family of homeomorphisms $h_\lambda : K \to \mathbb{C}$, $\lambda \in \mathbb{D}_{\delta'}(\lambda_0)$, $\delta' < \delta$ depending continuously on λ, such that $h_{\lambda_0} = \mathrm{id}$ and $f_\lambda \circ h_\lambda = h_\lambda \circ f_{\lambda_0}$. It is easy to see (using Lemma 5.1) that for $\lambda \in \mathbb{R}$, $h_\lambda(K) = K[\lambda]$. For each preperiodic orbit p of f in K, it is clear that $\lambda \mapsto h_\lambda(p)$ is holomorphic in $\mathbb{D}_{\delta'}(\lambda_0)$. Since preperiodic orbits are dense in K, it follows that $h[\lambda_0, \lambda] \equiv h_\lambda$ are actually transition maps of a holomorphic motion h over $\mathbb{D}_{\delta'}(\lambda_0)$.

Since f_λ is non-trivial, $f_\lambda(0)$ does not belong to $K[\lambda]$ for a dense set of $\lambda \in (-\delta, \delta)$, so by Lemma 8.2, the path $\lambda \mapsto (\lambda, f_\lambda(0))$ is transverse to the leaves of h outside of countably many parameters λ. So there exist parameters $\lambda \in X$ arbitrarily close to λ_0 which are density points of X and transversality points of the above path. In order to avoid cumbersome notation, let us assume that λ_0 is itself a transversality point.

It follows that there exists a real-symmetric quasiconformal map χ (phase-parameter holonomy map[16]) taking a neighborhood V of $f_{\lambda_0}(0)$ to a neighborhood of λ_0, and taking points in $K \cap V$ to parameters $\lambda \in \chi(V)$ with $f_\lambda(0) \in K[\lambda]$. In particular, $\chi(K \cap V) \supset X \cap \chi(V)$.

Since K is an expanding set, it follows that there exists $\rho > 0$ such that in every r neighborhood of $f_{\lambda_0}(0)$ there exists an interval of size at least ρr disjoint from K. Since $\chi | V \cap \mathbb{R}$ is quasisymmetric, this property is preserved: there exists $\rho' > 0$ such that in every r neighborhood of λ_0 there exists an interval of size at least $\rho' r$ not intersecting X. This contradicts the hypothesis that λ_0 is a density point of X[17]. □

8.3. **Infinitely renormalizable maps.**

Lemma 8.5. *Let f_λ be a non-trivial analytic family of unimodal maps. Then the set of infinitely renormalizable parameters has Lebesgue measure zero.*

Proof. Let X be the set of parameters λ such that f_λ is infinitely renormalizable, and let $\lambda_0 \in X$ be a density point of X. By Lemma 5.5, there exists a nice interval $T[\lambda]$, $|\lambda - \lambda_0| < \delta$, which is periodic (of period, say, m) such that $f^m | T[\lambda]$ has negative Schwarzian derivative. In particular, if $A_\lambda : T[\lambda] \to I$ is affine, $g_\lambda = A_\lambda \circ f_\lambda^m \circ A_\lambda^{-1}$, $|\lambda - \lambda_0| < \delta'$ is an analytic family of quasiquadratic maps, which is non-trivial (because f_λ is). By Theorem B of [ALM], for almost every λ, g_λ is not infinitely renormalizable. It is clear that if $\lambda \in X$ and $|\lambda - \lambda_0| < \delta'$ then g_λ is infinitely renormalizable, so λ_0 is not a density point of X, contradiction. □

8.4. **Simple maps.** The following argument is adapted from the corresponding result of Lyubich for the quadratic family [L3].

[16]More precisely, χ is obtained by applying first the local holonomy map between the two transverse holomorphic curves $\{\lambda_0\} \times V$ (V a small neighborhood of $f_{\lambda_0}(0)$) and $\{(\lambda, f_\lambda(0)) | \lambda \in \mathbb{D}_{\delta'}(\lambda_0)\}$ followed by the projection in the first coordinate.

[17]It is easy to see that this argument gives much more information on the size of X. One can see for instance that the Hausdorff dimension of X in λ_0 (defined as the infimum of the Hausdorff dimension of $X \cap \mathbb{D}_\epsilon(\lambda_0)$) is no greater than the Hausdorff dimension of K in $f_{\lambda_0}(0)$, which is known to be less than 1. Notice that X is essentially the set of non-regular non-recurrent parameters avoiding a definite neighborhood of 0. We should remark that these ideas show also that the Hausdorff dimension of the set of non-regular non-recurrent parameters is usually 1 except for some trivial situations.

Lemma 8.6. *Let f_λ be a non-trivial analytic family of unimodal maps. Then almost every parameter λ with $f_\lambda \in \mathcal{F}$ is simple.*

Proof. If this is not the case, we could find $C > 0$, $\rho < 1$, $m \geq 0$ and a set X of parameters of positive measure such that if $\lambda_0 \in X$ then

(1) $f_{\lambda_0} \in \mathcal{F}$ and is not simple (by hypothesis),
(2) f_λ is transverse at λ_0 (by Theorem 8.3),
(3) The sequence of parameter windows $J_n[\lambda_0]$ associated to λ_0 are defined for $n \geq m$ (by Theorem 7.4),
(4) If $n_{k,\lambda_0} - 1$ denotes the sequence of non-central levels of the principal nest of f_{λ_0} then for $n_{k,\lambda_0} \geq m$, $|J_{n_{k,\lambda_0}+1}[\lambda_0]|/|J_{n_{k,\lambda_0}}[\lambda_0]| < C\rho^k$ (by Theorem 7.4).

Consider now the set X_k, $k \geq m$ of parameters $\lambda_0 \in X$ such that the return of level n_{k,λ_0} is central. Let Δ_k be the union of $J_{n_{k,\lambda_0}}[\lambda_0]$, $\lambda_0 \in X_k$ and Π_k be the union of $J_{n_{k,\lambda_0}+1}[\lambda_0]$, $\lambda_0 \in X_k$.

Then each connected component $J_{n_{k,\lambda_0}}[\lambda_0]$ of Δ_k contains a single connected component $J_{n_{k,\lambda_0}+1}[\lambda_0]$ of Π_k, and thus $|\Pi_k|/|\Delta_k| < C\rho^k$, so that $|X_k| \leq |\Pi_k| < C\rho^k|\Delta_k| \leq C\rho^k|\Delta_m|$. On the other hand, $X \subset \cap_{k_0 \geq m} \cup_{k \geq k_0} X_k$ and thus, $|X| \leq \inf_{k_0 \geq m} \sum_{k \geq k_0} C\rho^k |\Delta_m| = 0$, contradiction. $\square$

The proof of Theorem A is concluded.

9. Proof of Theorem B

We will give now a proof of Theorem B using a parameter exclusion argument. In the first version of this work (in [Av1]), a different proof was given relying on the refined statistical analysis of [AM1], but we will give a much simpler argument based on a modified version of the quasisymmetric capacities of [AM1], which allows us to get rid of distortion estimates and at the same time to work with a fixed quasisymmetric constant.

9.1. **Measure estimate.** Define the modified γ-qs capacity of a set X in an interval I as

$$p_\gamma(X|I) = \sup \frac{|h_1 \circ h_2(X \cap I)|}{|h_1 \circ h_2(I)|}$$

where $h_1 : \mathbb{R} \to \mathbb{R}$ is γ-qs and $h_2 : I \to \mathbb{R}$ is a diffeomorphism (onto its image) with non-negative Schwarzian derivative.

Notice that if $F : T_1 \to T_2$ is a diffeomorphism with non-positive Schwarzian derivative and $X \subset T_1$ then

$$p_\gamma(X|T_1) \leq p_\gamma(F(X)|T_2).$$

This is the main advantage of modified quasisymmetric capacities over the traditional ones of [AM1].

By the Koebe Principle, if $h : I \to I$ is a diffeomorphism and has non-positive Schwarzian derivative then $h([-\epsilon, \epsilon]) = O(\epsilon)$. By Hölder continuity of γ-qs maps, we get

$$p_\gamma([-\epsilon, \epsilon]|[-1, 1]) = O(\epsilon^\kappa)$$

for some $0 < \kappa < 1$ depending on γ.

For a map $f \in \mathcal{F}$ with principal nest $\{I_n\}$, let s be as in Lemma 5.6, and let

$$\alpha_n = p_\gamma(s(\cup I_n^j)|s(I_n)).$$

Let us consider the components T_n^k of $(R_{n-1}|I_{n-1}^0)^{-1}(\cup I_{n-1}^j)$. We reserve the index 0 for the component containing 0, and the indexes -1 and 1 for the components of $(R_{n-1}|I_{n-1}^0)^{-1}(I_{n-1}^0)$. If $|k|>1$ then $R_{n-1}|T_n^k$ is a diffeomorphism onto some I_{n-1}^j, $j \neq 0$ and $R_{n-1}^2|T_n^k$ is a diffeomorphism onto I_{n-1}. Let

$$\epsilon_n = p_\gamma(s(\cup_{|k|>1} T_n^k)|s(I_n)).$$

Lemma 9.1. *If n is sufficiently large, $(1-\alpha_{n+1}) \geq (1-\epsilon_{n+1})(1-\alpha_n)$.*

Proof. If $|k|>1$ then $s(T_{n+1}^k)$ is taken to $s(I_n)$ by $s \circ R_n^2 \circ s^{-1}$ which has negative Schwarzian derivative for n big. In particular

$$p_\gamma(s(\cup I_{n+1}^j)|s(T_{n+1}^k)) \leq p_\gamma(s(\cup C_n^{\underline{d}})|s(I_n)) \leq \alpha_n.$$

Thus $p_\gamma(s(\cup I_{n+1}^j)|I_{n+1}) \leq \epsilon_{n+1} + (1-\epsilon_{n+1})\alpha_n$. □

Lemma 9.2. *If f is simple then the ϵ_n decay exponentially fast.*

Proof. If f is simple then $|s(I_{n+1})|/|s(I_n)|$ decays exponentially fast by Lemma 5.7. In particular, by the Koebe Principle, for each j, each of the connected components of $s(I_{n+1} \setminus I_{n+1}^j)$ is exponentially (in n) bigger than $s(I_{n+1}^j)$. This implies that, for each k, each component of $s(I_{n+2} \setminus T_{n+2}^k)$ is exponentially bigger than $s(T_{n+2}^k)$ (using the Koebe Principle), so $p_\gamma(s(T_{n+2}^k)|s(I_{n+2}))$ decays exponentially and so does ϵ_n. □

Lemma 9.3. *If $f \in \mathcal{F}$ does not admit a quasiquadratic renormalization then $\cup I_n^j$ is not dense in I_n, for n sufficiently big.*

Proof. Up to considering a renormalization or unimodal restriction, we may assume that f is non-renormalizable and does not admit unimodal restriction. It is easy to see that if $x \in I$ never enters I_1 then the iterates of x accumulate on an orientation preserving fixed point of f, and since f does not admit a unimodal restriction, we conclude that $x \in \partial I$.

Since f is not conjugate to a quadratic map, there exists an interval T whose orbit does not accumulate on the critical point (Lemma 5.9). Let n be biggest with the orbit of T intersecting I_n (T intersects I_1 by the previous discussion). Of course, the set of points which land on I_{n+1} does not intersect the orbit of T, and so is not dense in I_n.

It is easy to see that if the set of points in I_m which eventually land in I_{m+1} is not dense in I_n then $\cup I_{m+1}^j$ is not dense on I_{m+1}. In particular, by induction, $\cup I_m^j$ is not dense in I_m for $m \geq n+1$. □

Lemma 9.4. *If f does not admit a quasiquadratic renormalization then for n large enough, $\alpha_n < 1$.*

Proof. Let n be large enough such that there exists an open interval $E \subset I_n$ disjoint from $\cup I_n^j$, and $s \circ R_n \circ s^{-1}$ has negative Schwarzian derivative. We may assume that $E \subset T$, where $\overline{T} \subset \operatorname{int} I_n$ is a symmetric interval containing I_n^0. By the Koebe Principle, there exists $C>0$ such that if $h_2 : s(I_n) \to \mathbb{R}$ has non-positive Schwarzian derivative then $|h_2(s(E))| > C|h_2(s(T))|$. In particular, there exists $\epsilon > 0$ such that if $h_1 : \mathbb{R} \to \mathbb{R}$ is γ-qs, then, with $h = h_1 \circ h_2$, we have $|h(s(E))| > \epsilon |h(s(T))| \geq \epsilon |h(s(I_n^0))|$.

For $\underline{d} \in \Omega$, let $E^{\underline{d}} = (R_n^{\underline{d}})^{-1}(E)$. Since $(R_n^{\underline{d}})^{-1}$ has non-positive Schwarzian derivative, we see that for any h as above, $|h(s(E^{\underline{d}}))| > \epsilon |h(s(C_n^{\underline{d}}))|$. Notice that all the intervals $E^{\underline{d}}$, $\underline{d} \in \Omega$ are disjoint, and $\cup E^{\underline{d}}$ does not intersect $\cup C_n^{\underline{d}}$ so

$$p_\gamma(s(\cup C_n^{\underline{d}})|s(I_n)) \leq \frac{1}{1+\epsilon}.$$

By a previous argument of the proof of Lemma 9.1,

$$p_\gamma(s(\cup I_{n+1}^j)|s(T_{n+1}^k)) \leq p_\gamma(s(\cup C_n^{\underline{d}})|s(I_n)) < 1$$

for $|k| > 1$.

Thus, $p_\gamma(s(\cup I_{n+1}^j)|s(I_{n+1})) \leq \epsilon_{n+1} + (1-\epsilon_{n+1}) p_\gamma(s(\cup C_n^{\underline{d}})|s(I_n)) < 1$. $\square$

Lemma 9.5. *Let f_λ be a one-parameter non-trivial analytic family of unimodal maps satisfying the Phase-Parameter relation at a parameter λ_0 (in particular, $f = f_{\lambda_0}$ is simple). Assume that f does not admit quasiquadratic renormalization. Then λ_0 is not a density point of non-hyperbolic parameters*[18].

Proof. Let J_n and Ξ_n be as in the Topological Phase-Parameter relation. Since $|J_n| \to 0$, and $\lambda_0 \in \Xi_n(I_n^{\tau_n}) \subset J_n$, it is enough to show that then there exists $\alpha < 1$ such that

$$\limsup \frac{|\Xi_n(\cup C_n^{\underline{d}} \cap I_n^{\tau_n})|}{|\Xi_n(I_n^{\tau_n})|} \leq \alpha < 1.$$

Indeed, if $\lambda \notin \Xi_n(\cup C_n^{\underline{d}})$ then the critical point is non-recurrent. By Lemma 8.4, for almost every non-recurrent parameter, f_λ is hyperbolic.

Fix $1 < \hat{\gamma} < \gamma$ By PhPa1, $\Xi_n | K_n \cap I_n^{\tau_n}$ is $\hat{\gamma}$-qs for n big enough. On the other hand, for n big enough, $s^{-1}|s(I_n^{\tau_n})$ is C^1 close to being linear (because s is analytic, and in particular C^1, and $s(I_n^{\tau_n})$ is small). So $\Xi_n \circ s^{-1}|s(K_n \cap I_n^{\tau_n})$ is γ-qs for n big enough. In particular

$$\frac{|\Xi_n(\cup C_n^{\underline{d}} \cap I_n^{\tau_n})|}{|\Xi_n(I_n^{\tau_n})|} \leq \frac{|\Xi_n \circ s^{-1} s(\cup C_n^{\underline{d}} \cap I_n^{\tau_n})|}{|\Xi_n \circ s^{-1} s(I_n^{\tau_n})|} \leq p_\gamma(s(\cup C_n^{\underline{d}})|s(I_n^{\tau_n}) \leq \alpha_n.$$

By Lemmas 9.1, 9.2 and 9.3, $\alpha = \limsup \alpha_n < 1$. $\square$

Theorem A and Lemma 9.5 imply Theorem B for one-parameter families.

9.1.1. *Many parameters.* The argument of Lemma 8.1 implies the following:

Lemma 9.6. *Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a k-parameter non-trivial analytic family of unimodal maps. The set of parameters which are not Kupka-Smale or have a periodic or preperiodic critical point is contained in a countable union of analytic submanifolds of Λ, of codimension at least 1, and so has Lebesgue measure zero.*

Let us now show how the one-dimensional version of Theorem B implies the general case. Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a k-parameter analytic family of unimodal maps. By Lemma 9.6, we just have to show that for any Kupka-Smale parameter $\lambda_0 \in \operatorname{int} \Lambda$, there exists a small $\epsilon > 0$, such that, letting $B_\epsilon \subset \Lambda$ be the ball around λ_0 of radius ϵ, almost every parameter in B_ϵ is either regular or admits a quasiquadratic renormalization.

Using Theorem 5.10, if ϵ is sufficiently small, $\lambda \mapsto f_\lambda$ is an analytic map from B_ϵ to some open set $\mathcal{V}$ where the hybrid lamination $\mathcal{L}$ is defined. Let $\lambda_1 \in B_\epsilon$

[18]One can actually use those techniques to show that λ_0 is a density point of hyperbolic parameters, see Remark B.3 for the complex counterpart.

be a regular parameter. If L is a line in $\mathbb{R}^k$ through λ_1, then by Lemma 8.2, $L \cap B_\epsilon$ is not contained in the topological class of a non-regular parameter, and so regular parameters are dense in $L \cap B_\epsilon$. By the one-dimensional Theorem B, we see that almost every non-regular parameter in $L \cap B_\epsilon$ is quasiquadratic. By Fubini's Theorem, almost every non-regular parameter in B_ϵ is quasiquadratic.

This completes the proof of Theorem B.

10. Proof of corollaries

10.1. **Some conditions related to good ergodic properties.** Let us first recall the conditions on the critical orbit stated in the introduction. Let $f \in \mathbb{U}^2$. We say that f is *Collet-Eckmann* if the lower Lyapunov exponent of the critical value is bigger than zero:

$$\liminf \frac{\ln |Df^n(f(0))|}{n} > 0. \tag{10.1}$$

We say that f has *subexponential recurrence* if

$$\limsup \frac{-\ln |f^n(0)|}{n} = 0. \tag{10.2}$$

We say that f has *polynomial recurrence* if

$$\gamma = \limsup \frac{-\ln |f^n(0)|}{\ln(n)} < \infty, \tag{10.3}$$

and in this case, we call γ the *exponent* of the recurrence.

We introduce the following additional condition: f is called *Weakly Regular* if

$$\lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \sum_{\substack{1 \leq k \leq n \\ f^k(0) \in (-\delta, \delta)}} \ln |Df(f^k(0))| = 0. \tag{10.4}$$

Notice that polynomial recurrence is much stronger than subexponential recurrence.

Remark 10.1. Maps satisfying the Collet-Eckmann and the subexponential recurrence conditions have been intensively studied after the works of Benedicks and Carleson. Those two conditions give a very precise control of the critical orbit. They are not sufficient to show that f has good statistical properties however: one must also ask that f has a renormalization with all periodic orbits repelling (which is then conjugate to a quadratic polynomial). Under this additional assumption, it is possible to show that f has an absolutely continuous invariant measure (see [BY]).

In order to study further the properties of μ, it is convenient to consider the smallest periodic nice interval T of f (f is not infinitely renormalizable, since it has an absolutely continuous invariant measure). The first return map $f^m : T \to T$ can be then rescaled to a unimodal map $\hat{f}$, which also possess an absolutely continuous invariant measure $\hat{\mu}$.

Assuming that f is also Kupka-Smale and using Lemma 5.1, we see that the dynamics of f splits in a hyperbolic part, that describes points $x \in I$ which never enter $\operatorname{int} T$, and an interesting part described by $\hat{f}$.

The measurable dynamics of $\hat{f}$ are described by $\hat{\mu}$: for almost every $x \in I$ and any continuous function $\phi : I \to \mathbb{R}$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1} \phi(\hat{f}^k(x)) = \int \phi d\hat{\mu}.$$

Since $\hat{f}$ is non-renormalizable, it follows that $\hat{\mu}$ is supported on $[\hat{f}^2(0), \hat{f}(0)]$, and $(\hat{f}, \hat{\mu})$ is exponentially mixing[19] (see [Y]).

The condition of Weak Regularity is important to show that $(\hat{f}, \hat{\mu})$ is stochastically stable[20] (see [T2]). If we assume a little bit more smoothness, $f \in \mathbb{U}^3$, the Weak Regularity condition is not necessary, and it is possible to show that $(\hat{f}, \hat{\mu})$ is stochastically stable in a stronger sense[21] (see [BV]).

10.2. **Analytic families.** We will actually prove the following result, which is a more precise form of Corollaries C and E:

Theorem 10.1. *Let f_λ, be a non-trivial analytic family of unimodal maps. Then almost every non-regular parameter is Kupka-Smale and has a quasiquadratic renormalization which satisfies the Collet-Eckmann condition and is polynomially recurrent with exponent 1.*

Proof. We will prove the stated result for one-parameter families, the general case reducing to this one by the argument of §9.1.1.

By Theorems A and B of [AM1], the conclusion of the theorem holds for the quadratic family. However, the only properties of the quadratic family that are actually used in the proof is that it is an analytic family of quasiquadratic maps with negative Schwarzian derivative for which the Phase-Parameter relation holds at almost every parameter, see Remark 3.3 of that paper. Due to the work of Kozlovski, the hypothesis of negative Schwarzian derivative can also be removed (this can be checked directly using Lemma 5.6). Using our Theorem A, we get the result for analytic families of quasiquadratic maps.

Let us now consider the general case. By Theorem A, almost every non-regular parameter is simple, and by Theorem B, almost every non-regular parameter has a quasiquadratic renormalization. Let us fix such a parameter λ_0.

Let T be the smallest periodic nice interval for f_{λ_0} (of period m). For λ near λ_0, the interval T has a continuation $T[\lambda]$. Consider the analytic family $\hat{f}_\lambda = A[\lambda] \circ f_\lambda^m \circ A[\lambda]^{-1}$, $|\lambda - \lambda_0| < \epsilon$, where $A[\lambda] : T[\lambda] \to I$ is affine. Then $\hat{f}_\lambda$ is C^∞ close to $\hat{f}_{\lambda_0}$, which is quasiquadratic, so we conclude that for $\epsilon > 0$ small, $\hat{f}_\lambda$, $|\lambda - \lambda_0| < \epsilon$ is an analytic family of quasiquadratic maps. Since f_λ is non-trivial, $\hat{f}_\lambda$ is also non-trivial.

In particular, by the quasiquadratic case, for almost every λ near λ_0, $\hat{f}_\lambda$ is either regular or satisfy the Collet-Eckmann condition and its critical point is polynomially recurrent with exponent 1. In particular, the same holds for f_λ, which concludes the proof of the theorem. $\square$

[19] For a certain class of observables, for instance, of bounded variation.

[20] For a certain class of i.i.d. absolutely continuous stochastic perturbations, the perturbed system possess a stationary measure which is close to $\hat{\mu}$ in the weak topology.

[21] Densities of stationary measures of perturbed systems are close to the density of $\hat{\mu}$ in the L^1 sense.

Remark 10.2. Notice that the proof of Theorem A in [AM2] could not use directly the proof of [AM1] (the argument needs modifications which are dealt in the Appendix of [AM2]), since their main phase-parameter tool essentially amounts to comparing the phase-space of a non-trivial family with the parameter space of the quadratic family. This distorts the estimates and makes it impossible to obtain the exponent of the recurrence.

10.3. **Smooth families.** Recall that if $\Lambda \in \mathbb{R}^k$ is a bounded open connected domain with smooth boundary, $\mathbb{UF}^r(\Lambda)$ is the space of C^r families of unimodal maps parametrized by Λ, and is a Baire space.

Theorem 10.2. *Let f_λ, $\lambda \in \Lambda$ be a non-trivial family of unimodal maps. For every $\epsilon > 0$ there exists a neighborhood $\mathcal{V} \subset \mathbb{UF}^2(\Lambda)$ of f_λ such that if $g_\lambda \in \mathcal{V}$ then, outside a set of parameters λ of measure at most ϵ, g_λ is either regular or is Kupka-Smale and has a renormalization with all periodic orbits repelling satisfying the Collet-Eckmann, subexponential recurrence, and Weak Regularity conditions.*

Proof. Using Vitali's Covering Lemma, let $\{B_i\}$, $\{C_i\}$ be finite families of disjoint closed balls covering the parameter space up to a set of Lebesgue measure $\epsilon/2$ such that:
(1) For $\lambda \in B_i$, f_λ is regular;
(2) For $\lambda \in C_i$, there exists a nice interval $T_i[\lambda]$, which is periodic of period m_i, depending continuously on λ such that $f_\lambda^{m_i} : T_i[\lambda] \to T_i[\lambda]$ can be rescaled to a quasiquadratic map $\hat{f}_{i,\lambda}$.

It is easy to see that if g_λ is C^2 close to f_λ, then:
(1) For every $\lambda \in B_i$, g_λ is regular;
(2) For every $\lambda \in C_i$, there exists an interval $T_i^g[\lambda]$, depending continuously on λ, close to $T_i^g[\lambda]$, such that $g_\lambda^{m_i} : T_i^g[\lambda] \to T_i^g[\lambda]$ can be rescaled to a unimodal map $\hat{g}_{i,\lambda}$, and the family $\hat{g}_{i,\lambda}$ is C^2 close to $\hat{f}_{i,\lambda}$.

The family $\hat{f}_{i,\lambda}$ is non-trivial, so by Theorem B of [ALM], the set of parameters in C_i such that $\hat{g}_{i,\lambda}$ is either regular or has all periodic orbits repelling and satisfies the Collet-Eckmann, subexponential recurrence, and Weak Regularity conditions, has Lebesgue measure at least $|C_i|(1-\epsilon/4)$, provided g_λ is close enough to f_λ. The result follows. $\square$

Remark 10.3. In particular, if f_λ is a non-trivial analytic family of unimodal maps, almost every parameter is Weakly Regular.

Recall that by Remark 5.4, non-trivial analytic families are dense in $\mathbb{UF}^r(\Lambda)$. Using Theorem 10.2 and an easy Baire argument we obtain the following precise version of Corollary D:

Theorem 10.3. *In a generic family f_λ in $\mathbb{UF}^r(\Lambda)$, $r = 2, ..., \infty$ for almost every non-regular parameter $\lambda_0 \in \Lambda$, $f = f_{\lambda_0}$ is Kupka-Smale and has a renormalization which has all periodic orbits repelling and satisfies the Collet-Eckmann, subexponential recurrence, and Weak Regularity conditions.*

Appendix A. Hybrid classes

In this section we will give a global characterization of the leaves of the lamination $\mathcal{L}$ of Theorem 5.10.

Notice that the leaves of $\mathcal{L}$ are claimed to coincide with topological classes only in the non-regular case: the partition in topological classes is not a lamination because regular topological classes are open sets. It turns out that the behavior of the regular leaves of $\mathcal{L}$ can be quite arbitrary. In order to give a global characterization of the leaves of $\mathcal{L}$, we need to introduce once and for all an arbitrary, but fixed, way to refine the topological classes of regular maps. We shall call this refinement the *hybrid lamination*.

If f is non-regular, the hybrid class of f is just the set of all non-regular maps g which are topologically conjugate to f.

Let f be a regular map, and let A be the set of attracting periodic orbits of f and let $B = \{x \in I | f^n(x) \to A\}$ denote the basins of the attracting periodic orbits of f. Notice that if f is a regular map, there exists a minimal $m \geq 0$ such that $f^m(0)$ belongs to a periodic connected component of B. It is possible to show that if f is quasiquadratic, then $m = 0$. It turns out that if $m = 0$ (this case will be called essential), there is a natural way to refine the topological class of f: the hybrid class of f is the set of all regular maps g which are topologically conjugate to f and the multiplier of the periodic orbit that attracts 0 is the same for both maps (this definition agrees with the one of [ALM] in the quasiquadratic case).

In the non-essential case, there is no natural way to refine the topological class of f, so we fix an arbitrary way that works.

Definition A.1. Let f be a Kupka-Smale map. We say that a homeomorphism $h : I \to \mathbb{C}$ is f-admissible if the following holds. Let T be a periodic component of $B \setminus A$ which does not contain 0, and, writing $T = (a, b)$ with $|a| < |b|$, we have that the interval $[-a, a]$ is nice. Then h takes $d = (a + b)/2$ to $h(d) = (h(a) + h(b))/2$ and $h|[d, f^q(d)]$ is affine, where q is the period of T.

Definition A.2. Let f be a regular map of non-essential type. The hybrid class of f is defined as the set of all regular maps g such that there exists an f-admissible topological conjugacy between f and g.

The following proposition is elementary, and shows that the definition of hybrid class is minimally adequate:

Proposition A.1. *Let f be a regular map. Then its hybrid class intersects $\mathcal{U}_a$ in a codimension-one analytic submanifold.*

Moreover, with this definition, it is possible to prove the full Theorem 5.10 in the case of hyperbolic maps f. The case of infinitely renormalizable f can be dealt by reduction to the quasiquadratic case using renormalization (dealt in Theorem A of [ALM]), see Lemma 5.5.

A.1. **Persistent puzzle.** The remaining case of Theorem 5.10 is trickier and one needs to go into the proof of [ALM]. We will discuss here only the main modification one needs to make in order to adapt the argument. This modification concerns the main tool used in the finitely renormalizable case, the concept of persistent puzzle, whose definition needs to be adapted. We follow basically the approach of [Av1].

Assume that $f \in \mathcal{F}$. As in §6.1, fix a level $\mathbf{n}$ of the principal nest and assume that $|I_{\mathbf{n}}|/|I_{\mathbf{n-1}}|$ is very small. Let us consider the first landing map to $A^0 = I_{\mathbf{n}}$, the connected components of its domain are denoted A^j. Let A^1 be the component of $f(0)$, and let $A^1 = [l, r]$, with $l < r$. Let V^j be the complexification of the A^j

obtained as in Lemma 6.1. Let V be the union of all V^j such that $V^j \cap \mathbb{R} \subset [-1, r]$. We shall informally call V the *puzzle*.

Let $\mathcal{V} \subset \mathcal{A}_a$ be a real-symmetric neighborhood of f. We will say that the puzzle *persists* in $\mathcal{V}$ if there exists a real-symmetric holomorphic motion h over $\mathcal{V}$ given by a family of transition maps $h[f, g] = h_g : \mathbb{C} \to \mathbb{C}$, $g \in \mathcal{V}$ such that:

(1) $h_g|\mathbb{C} \setminus \Omega_a = \mathrm{id}$;
(2) $g \circ h_g|V \setminus V^0 = h_g \circ f$, $g \circ h_g|\partial V^0 = f$;
(3) $h_g|I$ is f-admissible and $g \circ h_g|([-1, r] \setminus V) = h_g \circ f$.

The following plays the role of Lemma 5.6 of [ALM].

Lemma A.2. *Let $f \in \mathcal{F} \cap \mathcal{U}_a$. If $|I_{\mathbf{n}}|/|I_{\mathbf{n}-1}|$ is sufficiently small, then there exists a neighborhood of f where the puzzle persists.*

The proof is the same as of Lemma 5.6 of [ALM], and we will not reproduce the whole argument here, but only comment the main steps:

(1) One considers a holomorphic motion h' of $[-1, r] \setminus V$ which is f-admissible and equivariant: $g \circ h'_g = h'_g \circ f$ (this holomorphic motion exists because the dynamics of $f|[-1, r] \setminus V$ is hyperbolic) over a small neighborhood of f.

(2) Using the Canonical Extension Lemma, we extend h' to a holomorphic motion defined also on $\partial f(V_0)$. Considering a slightly smaller neighborhood $\mathcal{V}'$ of f we may extend h' to $\mathbb{C} \setminus \Omega$ as id.

(3) One considers a holomorphic motion h^0 of $\overline{V}^0$ such that $g \circ h^0_g|\partial V^0 = h'_g \circ f$ over a neighborhood $\mathcal{V}^0$ of f.

(4) One notices that for each V^i, $i \neq 0$, we can define (uniquely) a holomorphic motion h^i on V^i as a lift of $h^0|V^0$ over a small neighborhood $\mathcal{V}^i$ of f.

(5) The (countably many) holomorphic motions h', h^i are defined apriori over different neighborhoods of f, but using again hyperbolicity of $f|[-1, r] \setminus V$, one sees that all those holomorphic motions are defined over a definite neighborhood of f.

(6) An estimate of hyperbolic geometry shows that the several regions of definition of those different holomorphic motions cannot collide in a slightly smaller neighborhood of f, so they define a common holomorphic motion which can be completed using the Canonical Extension Lemma and satisfies automatically (1), (2), and (3).

Remark A.1. The last condition of the definition of persistence defines uniquely h_g in $[-1, r] \setminus \overline{V}$. This set is empty in the quasiquadratic case (and so this condition does not appear in [ALM]). This (obvious) observation concerning the first step is the only formal difference in the proof, the remaining steps do not need to be modified.

Remark A.2. If f is a Kupka-Smale, finitely-renormalizable, non-hyperbolic map, with a non-recurrent critical point, a similar construction can be made. In this case, we take $T \subset T'$ nice intervals with preperiodic boundary such that 0 does not return to T' and $|T|/|T'|$ is very small. We let $A^0 = T$, and put A^1 as a domain of the first landing map to A^0 which is contained in $[f(0), f(0) + \epsilon]$, ϵ very small.

Remark A.3. If $g_1, g_2 \in \mathcal{V} \cap \mathcal{U}_a$ are regular maps in the same hybrid class then they are of non-essential type if and only if for all m sufficiently big,

$$h_{g_1}^{-1}(g_1^m(0)), h_{g_2}^{-1}(g_2^m(0)) \notin [-1, r] \setminus \overline{V}$$

(use the Schwarz Lemma). The definition of hybrid class implies

$$h_{g_1}^{-1}(g_1^m(0)) = h_{g_2}^{-1}(g_2^m(0)).$$

This is important for the application of the several pullback arguments of [ALM].

One obtains Theorem 5.10 in the finitely renormalizable, non-regular case by repetition of the proof of Theorem A of [ALM], taking into consideration the above remarks.

Appendix B. Non-renormalizable parameters in the Mandelbrot set

Let $p_c = z^2 + c$ and let $\mathcal{M}$ (the Mandelbrot set) be the set of parameters $c \in \mathbb{C}$ such that the orbit of 0 does not escape to infinity under iteration by p_c. The aim of this appendix is to show how the idea of the proof of Theorem B can be coupled with Lyubich's result of [L3] to obtain the following theorem:

Theorem B.1. *Let $\mathcal{NR}$ be the set of non-renormalizable quadratic parameters with recurrent critical point and no indifferent periodic orbits in the boundary of the Mandelbrot set. Then $\mathcal{NR}$ has Lebesgue measure* 0*.*

Theorem B.1 implies easily Shishikura's Theorem F stated in the introduction.

Remark B.1. The reduction of Theorem F to Theorem B.1 is obtained using the following three steps:

(1) It is easy to pass from the non-renormalizable case to the finitely renormalizable case using renormalization techniques: the (countably many) little copies of the Mandelbrot set are related by renormalization to the original Mandelbrot set by a quasiconformal (and thus absolutely continuous) transformation, see [L4]. Alternatively, we can also repeat the proofs for the little Mandelbrot copies.

(2) Quadratic polynomials with a neutral fixed point are contained in the boundary of the main cardioid of the Mandelbrot set, which is a real analytic curve (with one singularity) and thus has Lebesgue measure zero.

(3) The case of non-recurrent non-renormalizable polynomial without neutral fixed points can be treated easily using holomorphic motions, see our proof of Lemma 8.4 (it is enough to use that under those conditions the set of points that never enter a small neighborhood of 0 is a hyperbolic set and thus persistent[22]).

To prove Theorem B.1 we will make use of the Phase-Parameter estimates described in Lemma 3.1 and Lyubich's parapuzzle estimate (Theorem 4.3). Then, we will redo the estimates of Theorem B in the complex setting to show that non-renormalizable parameters have Lebesgue measure zero, because the critical point has a tendency to fall in the basin of infinity (in the same way that in the real setting the critical point has a tendency to fall in the basin of non-essential attractors).

Remark B.2. Lyubich has another proof of Theorem B.1, also based on [L3] and estimates on the area of the set of points that return to deep puzzle pieces. Graczyk and Swiatek have also obtained a different proof of Shishikura's Theorem.

[22]This actually holds for any non-renormalizable quadratic polynomial without neutral fixed points.

B.1. **Parapuzzle notation.** Let us fix $c_0 \in \mathcal{NR}$. By Theorem 4.3, there exists a neighborhood $\Lambda_1 \subset \mathbb{C}$ of c_0 and domains $0 \in U_1[\lambda] \subset \mathbb{C}$, $\lambda \in \Lambda_1$ such that the first return map to $U_1[\lambda]$ by p_λ induces a full R-family over Λ_1. To prove Theorem B.1, it is clearly sufficient to show that $\Lambda_1 \cap \mathcal{NR}$ has Lebesgue measure zero.

For $\lambda \in \mathcal{NR} \cap \Lambda_1$, we can define a R-chain over λ since the critical point is recurrent. Let us denote the parameter domains of this chain by $\Lambda_i[\lambda]$. Let $\mathcal{NR}^\infty \subset \mathcal{NR} \cap \Lambda_1$ be the set of parameters λ such that the chain $\mathcal{R}_i$ over λ has infinitely many central levels, and let $\mathcal{NR}^0$ be the complementary set in $\mathcal{NR} \cap \Lambda_1$. By Theorem 4.4, there exists a constant $C(\lambda) > 0$, $\lambda \in \mathcal{NR} \cap \Lambda_1$ such that $\mathrm{mod}(\Lambda_{n_k}[\lambda] \setminus \overline{\Lambda_{n_k+1}[\lambda]}) > C(\lambda)k$, where $n_k - 1$ counts the non-central levels of the chain. If $\lambda \in \mathcal{NR}^0$, we actually have linear growth of moduli (without passing through a subsequence), and by Lemma 3.1, conditions CPhPa1 and CPhPh1 are satisfied (with the dilatation parameter γ arbitrarily close to 1) for i sufficiently big.

B.2. **Finitely many central cascades.** The argument of Lyubich which shows that almost every real quadratic maps in $\mathcal{F}$ is simple applies in the complex setting and gives:

Lemma B.2. $|\mathcal{NR}^\infty| = 0$.

Proof. Let $\mathcal{NR}^\infty_\epsilon$ be the set of parameters $\lambda \in NR^\infty$ such that $C(\lambda) \geq \epsilon$. If $\mathcal{NR}^\infty$ has positive Lebesgue measure then we can select ϵ such that $\mathcal{NR}^\infty_\epsilon$ also has positive Lebesgue measure. Let $\mathcal{NR}^\infty_\epsilon(k) \subset \mathcal{NR}^\infty_\epsilon$ be the set of parameters such that the n_k level is central. If $\lambda \in \mathcal{NR}^\infty_\epsilon(k)$, $\mathcal{NR}^\infty_\epsilon(k) \cap \Lambda_{n_k}[\lambda] \subset \Lambda^0_{n_k}[\lambda]$, thus $|\mathcal{NR}^\infty_\epsilon(k) \cap \Lambda_{n_k}[\lambda]| \leq |\Lambda^0_{n_k}[\lambda]|$.

Since $C(\lambda) \geq \epsilon$, there exists δ and k_0 which only depend on ϵ such that if $k > k_0$ then $\Lambda_{n_k}[\lambda] \setminus \overline{\Lambda^0_{n_k}[\lambda]}$ contains a round annuli of moduli $k\delta$. This implies that $|\Lambda^0_{n_k}[\lambda]| \leq e^{-k\delta'}|\Lambda_{n_k}[\lambda]|$ for some δ' depending on δ. For each k, the domains $\Lambda_{n_k}[\lambda]$, $\lambda \in \mathcal{NR}^\infty_\epsilon(k)$ are either equal or disjoint, and their union has Lebesgue measure at most $|\Lambda_1|$, so $|\mathcal{NR}^\infty_\epsilon(k)|$ decays exponentially on k. It follows immediately that $\mathcal{NR}^\infty_\epsilon = \cap_{k\geq 1} \cup_{n \geq k} \mathcal{NR}^\infty_\epsilon(k)$ has Lebesgue measure zero, contradiction. $\square$

B.3. **Area estimate.** Let U be a bounded open set of $\mathbb{C}$ and Z be a measurable set of $\mathbb{C}$. Let
$$c_\gamma(Z|U) = \sup \frac{|h(Z \cap U)|}{|h(U)|}$$
where h ranges over all quasiconformal homeomorphisms $h : U \to \mathbb{C}$ with dilatation bounded by γ and such that $h(U)$ is bounded. The following two properties are immediate:

(1) If $V^j \subset U$ are disjoint open subsets and $Z \subset \cup V^j$ then
$$c_\gamma(Z|U) \leq \sup_j c_\gamma(Z|V^j) c_\gamma(\cup V^j|U).$$

(2) If $A, B \subset U$ are disjoint open subsets and $Z \subset A \cup B$ then
$$c_\gamma(Z|U) \leq c_\gamma(B|U) + (1 - c_\gamma(B|U))c_\gamma(Z|B).$$

Denote by $V^k_n[\lambda]$ the connected components of the preimages of
$$(R_{n-1}[\lambda]|U^0_n[\lambda])^{-1}(\cup U^j_{n-1}[\lambda]).$$
We reserve the index 0 for the component of 0, so that $0 \in V^0_n$. We also reserve the indexes -1 and 1 for the components of the preimages of $U_n[\lambda]$.

Fix some $\gamma > 1$. Let

$$\epsilon_n(\lambda) = c_\gamma(\cup_{|k|\le 1} V_n^k[\lambda] | U_n[\lambda]) \tag{B.1}$$

$$\alpha_n(\lambda) = c_\gamma(\cup_j U_n^j[\lambda] | U_n[\lambda]). \tag{B.2}$$

Lemma B.3. *Let $\lambda \in \mathcal{NR}^0$. Then $\alpha_2 < 1$.*

Proof. Notice that $\cup U_1^j[\lambda]$ is not dense in $U_1[\lambda]$ (otherwise the filled-in Julia set of p_λ would have to contain $U_1[\lambda]$, but in our situation the filled-in Julia set of p_λ has empty interior). Thus, there exists a domain $U_1^0[\lambda] \subset D[\lambda] \subset U_1[\lambda]$ such that $U_1[\lambda] \setminus \overline{D[\lambda]}$ is an annulus, and a non-empty open set $E[\lambda] \subset D[\lambda] \setminus \cup U_1^j[\lambda]$. By the Koebe distortion Lemma, if $h : U_1[\lambda] \to \mathbb{C}$ is a γ-qc map with bounded image then $|h(E[\lambda])| > C|h(U_1^0[\lambda])|$ for some constant $C > 0$.

For $\underline{d} \in \Omega$, let $E^{\underline{d}}[\lambda] = (R_1^{\underline{d}}[\lambda])^{-1}(E[\lambda])$. We conclude that, for any γ-qc map $h : U_1[\lambda] \to \mathbb{C}$ with bounded image, we have $|h(\cup E^{\underline{d}}[\lambda])| > C|h(\cup W_1^{\underline{d}}[\lambda])|$, so $c_\gamma(\cup W_1^{\underline{d}}[\lambda]|U_1[\lambda]) < 1$.

If $|k| > 1$ then $R_1^2[\lambda]|V_2^k[\lambda]$ is a diffeomorphism onto $U_1[\lambda]$ and we conclude that $c_\gamma(\cup U_2^j[\lambda]|V_2^k[\lambda]) = c_\gamma(\cup W_1^{\underline{d}}[\lambda]|U_1[\lambda])$.

Thus $c_\gamma(\cup U_2^j[\lambda]|U_2[\lambda]) \le \epsilon_2 + (1-\epsilon_2)c_\gamma(\cup W_1^{\underline{d}}[\lambda]|U_1[\lambda]) < 1$. $\square$

Lemma B.4. *If $\lambda \in \mathcal{NR}^0$ then $\epsilon_n(\lambda) \to 0$ exponentially fast.*

Proof. Notice that if $R_{n-1}[\lambda](V_n^k[\lambda]) = U_{n-1}^j[\lambda]$ then

$$\mathrm{mod}(U_n[\lambda] \setminus \overline{V_n^k[\lambda]}) \ge \mathrm{mod}(U_{n-1}[\lambda] \setminus \overline{U_{n-1}^j[\lambda]})/3,$$

$$\mathrm{mod}(U_{n-1}[\lambda] \setminus \overline{U_{n-1}^j[\lambda]}) \ge \mathrm{mod}(U_{n-2}[\lambda] \setminus \overline{U_{n-2}^0[\lambda]})/2.$$

For $\lambda \in \mathcal{NR}^0$, $\mathrm{mod}(U_{n-2}[\lambda] \setminus \overline{U_{n-2}^0[\lambda]})$ grows linearly in n, so $\inf_k \mathrm{mod}(U_n[\lambda] \setminus \overline{V_n^k[\lambda]})$ also grows linearly, and this implies exponential decay of $\sup_k c_\gamma(V_n^k[\lambda]|U_n[\lambda])$, which implies exponential decay of ϵ_n. $\square$

Lemma B.5. *If $\lambda \in \mathcal{NR}^0$ then $\alpha(\lambda) = \sup_{n\ge 2} \alpha_n(\lambda) < 1$.*

Proof. Indeed, if $|k| > 1$ then $R_n^2[\lambda]|V_{n+1}^k[\lambda]$ is a diffeomorphism onto $U_n[\lambda]$. In particular, $c_\gamma(\cup U_{n+1}^j[\lambda]|V_{n+1}^k[\lambda]) \le c_\gamma(\cup U_n^j[\lambda]|U_n[\lambda]) = \alpha_n(\lambda)$. Thus

$$c_\gamma(\cup U_{n+1}^j[\lambda]|U_{n+1}[\lambda] \setminus \overline{\cup_{|k|\le 1} V_{n+1}^k[\lambda]}) \le \alpha_n(\lambda),$$

which implies $\alpha_{n+1}(\lambda) \le \epsilon_{n+1}(\lambda) + (1-\epsilon_{n+1}(\lambda))\alpha_n(\lambda)$ and

$$1 - \alpha_{n+1}(\lambda) \ge (1-\epsilon_{n+1}(\lambda))(1-\alpha_n(\lambda)).$$

If $\lambda \in \mathcal{NR}^0$, $\epsilon_n(\lambda)$ decays exponentially (Lemma B.4) and $\alpha_2(\lambda) < 1$ (Lemma B.3), so the result follows. $\square$

If $\mathcal{NR}^0$ has positive measure, there exists $\alpha > 0$, $k > 0$ and a positive measure set X such that for $\lambda \in X$, $\alpha(\lambda) < \alpha$ and for $n > k$ the estimate CPhPa1 of Lemma 3.1 is valid with a constant smaller than γ.

Let $Y \supset X$ be an open set such that $\alpha|Y| < |X|$. For every parameter $\lambda \in X$, let $\mu(\lambda)$ be the smallest $j > k$ such that $\lambda \in Z[\lambda] = \Lambda_j^{\tau_j(\lambda)}[\lambda] \subset Y$ (such a j exists since $\cap \Lambda_j[\lambda] = \{\lambda\}$). The resulting collection of parameter domains $Z[\lambda]$, $\lambda \in X$ are either disjoint or equal. To reach a contradiction, it is enough to show that $\alpha|Z[\lambda]| \geq |X \cap Z[\lambda]|$, for in this case $\alpha|Y| \geq |X|$. But this is an immediate consequence of CPhPa1, for

$$\frac{|X \cap Z[\lambda]|}{|Z[\lambda]|} \leq c_\gamma(\cup W^{\underline{d}}_{\mu(\lambda)}|U^{\tau_{\mu(\lambda)}(\lambda)}_{\mu(\lambda)}) \leq c_\gamma(\cup U^j_{\mu(\lambda)}|U_{\mu(\lambda)}) \leq \alpha,$$

since $\tau_{\mu(\lambda)} \neq 0$ by hypothesis (notice that we even have $|\mathcal{M} \cap Z[\lambda]|/|Z[\lambda]| \leq \alpha$, that is, a definite proportion of parameters in $Z[\lambda]$ have escaping critical point).

Remark B.3. Our estimates can be easily pushed further to obtain more precise results. For instance, it is clear that

$$\alpha_{n+1} \leq \epsilon_{n+1} + (1 - \epsilon_{n+1})\epsilon_n \sum_{k=0}^{\infty} \alpha_n^k \leq \epsilon_{n+1} + \frac{\epsilon_n}{1-\alpha},$$

so $\alpha_n \to 0$ (exponentially fast) for all parameters in $\mathcal{NR}^0$. This in turn can be used to show that each parameter in $\mathcal{NR}^0$ is a density point of the complement of $\mathcal{M}$[23].

References

[AsM] Astala, K.; Martin, G. J. Holomorphic motions. Papers on analysis, 27–40, Rep. Univ. Jyvskyl Dep. Math. Stat., 83, Univ. Jyvskyl, Jyvskyl, 2001.

[Av1] A. Avila. Bifurcations of unimodal maps: the topologic and metric picture. IMPA Thesis (2001) (http://www.math.sunysb.edu/∼artur/).

[Av2] A. Avila. Bifurcations d'applications unimodales. C. R. Acad. Sci. Paris, Ser. I 334 (2002), 483-488.

[ALM] A. Avila, M. Lyubich and W. de Melo. Regular or stochastic dynamics in real analytic families of unimodal maps. Invent. Math. 154 (2003), 451-550.

[AM1] A. Avila, C. G. Moreira. Statistical properties of unimodal maps: the quadratic family. Preprint (http://www.arXiv.org). To appear in Annals of Math.

[AM2] A. Avila and C. G. Moreira. Statistical properties of unimodal maps: smooth families with negative Schwarzian derivative. Astérisque 286 (2003), 81-118.

[AM3] A. Avila and C. G. Moreira. Statistical properties of unimodal maps: physical measures, periodic orbits and pathological laminations. Preprint (http://www.arXiv.org). Submitted for publication.

[BV] V. Baladi and M. Viana. Strong stochastic stability and rate of mixing for unimodal maps. Ann. Scient. Éc. Norm. Sup., v. 29 (1996), 483-517.

[BY] M. Benedicks and L.-S. Young. Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps. Ergodic Theory Dynam. Systems 12 (1992), no. 1, 13–37.

[BR] L. Bers & H.L. Royden. Holomorphic families of injections. Acta Math., v. 157 (1986), 259-286.

[D] A. Douady. Prolongement de mouvements holomorphes (d'après Slodkowski et autres). Astérisque, v. 227, 7-20 (1995).

[GSS] J. Graczyk, D. Sands and G. Swiatek. Schwarzian derivative in unimodal dynamics. C. R. Acad. Sci. Paris.

[H] Hubbard, J. H. Local connnectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. Topological methods in modern mathematics (Stony Brook, NY, 1991), 467–511, Publish or Perish, Houston, TX, 1993.

[23]This last result does not hold for all parameters in $\mathcal{NR}$: there are parameters $c \in \mathcal{NR}^\infty$ (well approximated by cusps of little Mandelbrot sets) such that $\limsup |\mathbb{D}_\epsilon(c) \cap \mathcal{M}|/|\mathbb{D}_\epsilon(c)| = 1$. Our techniques show however that for every $c \in \mathcal{NR}^\infty$, $\alpha_{n_k+1} \to 0$ (to see this, one needs to do the area estimate jumping through central cascades, using a combinatorial procedure similar to Theorem 4.6). In particular, the upper density of the complement of the Mandelbrot set is one at any $c \in \mathcal{NR}$: $\liminf |\mathbb{D}_\epsilon(c) \cap \mathcal{M}|/|\mathbb{D}_\epsilon(c)| = 0$. This result also follows from Graczyk-Swiatek's proof of Shishikura's Theorem (personal communication by Jacek Graczyk).

[K1] O.S. Kozlovski. Getting rid of the negative Schwarzian derivative condition. Ann. of Math. (2) 152 (2000), no. 3, 743–762.

[K2] O.S. Kozlovski. Axiom A maps are dense in the space of unimodal maps in the C^k topology. Ann. of Math. (2) 157 (2003), no. 1, 1–43.

[L1] M. Lyubich. Combinatorics, geometry and attractors of quasi-quadratic maps. Ann. Math, **140** (1994), 347-404.

[L2] M. Lyubich. Dynamics of quadratic polynomials, I-II. Acta Math., **178** (1997), 185-297.

[L3] M. Lyubich. Dynamics of quadratic polynomials, III. Parapuzzle and SBR measure. Astérisque, v. 261 (2000), 173 - 200.

[L4] M. Lyubich. Feigenbaum-Coullet-Tresser Universality and Milnor's Hairiness Conjecture. Ann. Math. v. 149 (1999), 319 - 420.

[L5] M. Lyubich. Almost every real quadratic map is either regular or stochastic. Ann. of Math. (2) 156 (2002), no. 1, 1-78.

[M] R. Mañé. Hyperbolicity, sinks and measures for one-dimensional dynamics. Comm. Math. Phys. 100 (1985), 495-524.

[MSS] R. Mañé, P. Sad & D. Sullivan. On the dynamics of rational maps, Ann. scient. Ec. Norm. Sup., **16** (1983), 193-217.

[MN] Martens, Marco; Nowicki, Tomasz Invariant measures for typical quadratic maps. Géométrie complexe et systèmes dynamiques (Orsay, 1995). Astrisque No. 261 (2000), xiii, 239–252.

[MS] W. de Melo & S. van Strien. One-dimensional dynamics. Springer, 1993.

[NS] T. Nowicki and D. Sands. Non-uniform hyperbolicity and universal bounds for S-unimodal maps. Invent. Math. 132 (1998), no. 3, 633–680.

[P] Palis, Jacob A global view of dynamics and a conjecture on the denseness of finitude of attractors. Géométrie complexe et systèmes dynamiques (Orsay, 1995). Astérisque No. 261 (2000), xiii–xiv, 335–347.

[S] D. Sands. Misiurewicz maps are rare. Comm. Math. Phys. 197 (1998), no. 1, 109–129.

[Sh] M. Shishikura. Topological, geometric and complex analytic properties of Julia sets. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 886–895, Birkhäuser, Basel, 1995.

[Sl] Z. Slodkowsky. Holomorphic motions and polynomial hulls. Proc. Amer. Math. Soc., **111** (1991), 347-355.

[T2] M. Tsujii. Small random perturbations of one dimensional dynamical systems and Margulis-Pesin entropy formula. Random & Comput. Dynamics. Vol.1 No.1 59-89, (1992).

[Y] L.-S. Young. Statistical properties of dynamical systems with some hyperbolicity. Ann. Math. (2) 147, No.3, 585-650 (1998).

Collège de France – 3 Rue d'Ulm, 75005 Paris – France.
E-mail address: `avila@impa.br`

IMPA – Estr. D. Castorina 110, 22460-320 Rio de Janeiro – Brazil.
E-mail address: `gugu@impa.br`