# UNIMODAL MAPS AND HIERARCHICAL MODELS

## A PRELIMINARY DRAFT, NOT FOR DISTRIBUTION

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Contents



**Foreword.** This note came from an attempt by the author to understand the basis for the analogy between renormalization and universality in statistical mechanics, and in one-dimensional dynamics, which ever since the original works of Feigenbaum [Feig], and Collet and Tresser [CT] has motivated the development of the latter. We present the evidence that the two fields are related directly. We begin by discussing the best understood statistical-mechanical model of phase transitions - the Dyson's Hierarchical model studied by Bleher and Sinai [BS1, BS2] and others. We then proceed to discuss various constructions of Hamiltonians corresponding to renormalizable unimodal maps, beginning with the works of [VSK], and [Sul1], and show how some recent works in the field may be used to construct a thermodynamical analogue of the Feigenbaum-Collet-Tresser renormalization.

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This note is in a preliminary stage, and should be treated with caution. The author also requests that you do not circulate it. All comments and suggestions are most welcome.

## 1. Renormalization in Statistical Mechanics

1.1. **A brief review of phase transitions.** The purpose of this section is to broadly describe the phenomenology of phase transitions in statistical mechanics, and in this way to set the stage for further discussion. We will not attempt to give an introduction to the subject of statistical mechanics here, and will instead refer the reader to one of the numerous textbooks on the subject. We recall that the central object of study of the theory is the definition and properties of Gibbs probability distributions. For an infinite set $\Lambda \subset \mathbb{R}^d$ (most often $\Lambda$ is the lattice $\mathbb{Z}^d$, but not in the examples considered in this note), which is invariant under the action of the symmetry $G$, the set of spins on $\Lambda$ is the collection $\Phi$ of functions mapping $\Lambda$ to some set $A$ (which for our purposes can be assumed to be finite). A Hamiltonian $\mathcal{H}$ is a $G$-equivariant functional $\Phi \to \mathbb{R}$, which should be interpreted as the energy of the spin configuration $\phi$. In the case when $\Lambda$ is finite, the Gibbs measure $\mu_{\mathcal{H},\beta}$ defined for every $\beta > 0$ is the probability distribution on $\Phi$ given by

$$\mu_{\mathcal{H},\beta}(\phi) = \frac{\exp(-\beta\mathcal{H}(\phi))}{\sum_{\phi\in\Phi}\exp(-\beta\mathcal{H}(\phi))} \tag{1.1}$$

When $\Lambda$ is infinite, one chooses a sequence of finite subsets $\Lambda^n \to \Lambda$ and considers the set of measures obtained as weak limits of the expressions of the form (1.1) with $\phi$ restricted to $\Lambda^n$. Such measures form a convex set, and the extremal points of this set are called the Gibbs measures. Other quantities described the system can be introduced using a similar thermodynamical limit procedure. For instance, the function

$$\mathbf{f}(\beta) = \lim_{n\to\infty} |\Lambda^n|^{-1} \log \sum_{\phi|_{\Lambda^n}} \exp(-\beta\mathcal{H}(\phi))$$

is the free energy per unit volume which relates the canonical and microcanonical ensembles.

One possible way to define a phase transition, is as a value of $\beta = \beta_{\text{cr}}$ for which the structure of the set of Gibbs distributions changes. There are two kinds of phase transitions: in the first kind there is a discontinuous change in the thermodynamical parameters defining the system; in the second the change is continuous, but not smooth. It is the latter that will concern us. The simplest example, the kind of which occurs in ferromagnetics, would be a system in which for $\beta < \beta_{\text{cr}}$ there is a single Gibbs distribution, and for $\beta > \beta_{\text{cr}}$ there are two. The phenomenological picture of a phase transition is characterized by the appearance of critical exponents. Let us describe a typical such picture, following the exposition in Sinai's book [Sin]. For the Gibbs distribution $\mu_{\text{cr}}$ corresponding to $\beta = \beta_{\text{cr}}$ the correlation length becomes infinite, and we have

$$E_{\mu_{\text{cr}}}\phi(x)\cdot\phi(y) = \frac{\text{const}}{||x-y||^{\xi}}$$

for the two-point correlation function. This implies, in particular, that for a finite volume $V \subset \Lambda$

$$E_{\mu_{\rm cr}}\left(\sum_{x\in V}\phi(x)\right)^2 \sim \text{const}\cdot|V|^\alpha, \text{ for } \alpha=\alpha(\xi).$$

On the other hand, for $\beta < \beta_{\rm cr}$ the corresponding right-hand side is $\sigma(\beta)|V|$. One then expects that

$$\sigma(\beta)\sim \text{const}\cdot(\beta_{\rm cr}-\beta)^{-\gamma} \underset{\beta\to\beta_{\rm cr}}{\longrightarrow} \infty,$$

which gives another example of a critical exponent. Also, for $\beta > \beta_{\rm cr}$ the two Gibbs distributions produce two distinct average values for the spin $\phi(x)$. Denoting them $a_1(\beta)$ and $a_2(\beta)$ we expect that

$$|a_1(\beta)-a_2(\beta)|\sim \text{const}\cdot|\beta-\beta_{\rm cr}|^\omega.$$

The list of critical exponents can be continued, there are also various algebraic connections between them, which we will not dwell upon. The main goal of understanding a phase transition lies in estimating the values of the critical exponents. To this end a renormalization transformation $\mathcal{R}$ is introduced. Without attempting to give a general definition (indeed, finding an appropriate definition is usually the crux of the matter!) let us summarize its properties. Th operator $\mathcal{R}$ acts on Hamiltonians $\mathcal{H}$ by averaging out some of the degrees of freedom in the system. It is expected to preserve the sets of Gibbs measures, and the correlation speed $\xi$. On an appropriate Banach manifold of Hamiltonians it is expected to have a fixed point $\mathcal{H}_*$ which is hyperbolic. In the simplest scenario, it has a single unstable eigenvalue $\lambda\in\mathbb{C}\setminus\bar{\mathbb{D}}$, whose value determines all other critical exponents.

To conclude this section let us discuss hierarchical models, which are the main subject of this note. They are somewhat different from the usual lattice models. We will not attempt to give a definitive description of such models here, but just summarize the common features of the models that we will consider. A model is hierarchical if the symmetry group $G$ acts on $\Lambda$ by piecewise-linear transformations which properly map $\Lambda$ into itself. We thus obtain a hierarchy of images $\Lambda_{g_1}\subset \Lambda_{g_1g_2}\subset\cdots$. It is worth noting that in this context it is reasonable to expect $\mathcal{R}$ to be defined in such a way as to propagate a Hamiltonian from $\Lambda^n$ to a larger set $\cup_{g\in G}g^{-1}(\Lambda^n)$.

1.2. **Dyson's Hierarchical models.** The principal thermodynamical example which will concern us in the Dyson's Hierarchical models. These are well-studied objects, and their renormalization theory is presented in the books of Sinai [Sin], and Collet and Eckmann [CE]. Sinai's book, in particular, gives a clear and concise account of the theory, to which we do not have anything to add, therefore in this section we will simply recap its main points for future reference. The main results

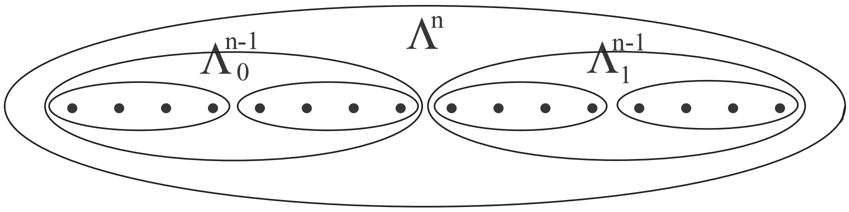


FIGURE 1

in the study of Dyson's Hierarchical models were obtained by Bleher and Sinai [BS1, BS2], these papers will serve as our general references.

Let us denote $\Lambda^n$ the set of binary sequences of length $n$

$$\Lambda^n = \{(x_1, x_2, \ldots, x_n)|\ x_i \in \{0,1\}\}.$$

This set is naturally partitioned in two, $\Lambda_0^{n-1} = (0, x_2, \ldots, x_n)$ and $\Lambda_1^{n-1} = (1, x_2, \ldots, x_n)$; and more generally, for every $k \leq n$ and every sequence $(y_1, \ldots, y_k) \in \{0,1\}^k$ we can define

$$\Lambda^{n-k}_{y_1,\ldots,y_k} = \{(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n)|\ x_i \in \{0,1\}\}$$

to partition $\Lambda^n$ into $2^{n-k}$ subsets of equal size. The space of spins on $\Lambda^n$ is $\Phi_n = \{\phi : \Lambda^n \to \{-1,1\}\}$. A Dyson's Hierarchical model is a Hamiltonial $\mathcal{H}^n : \Phi_n \to \mathbb{R}_{\geq 0}$. To define it we need the following set of data: $n_0 \leq n$; a "start-up" Hamiltonial $\mathcal{H}_{n_0} : \Phi_{n_0} \to \mathbb{R}_{\geq 0}$, and a number $c \in (1,2)$. The functional $\mathcal{H}^n$ is then defined inductively by the following formula:

$$\mathcal{H}^n(\phi|_{\Lambda^n}) = \mathcal{H}^{n-1}(\phi|_{\Lambda_0^{n-1}}) + \mathcal{H}^{n-1}(\phi|_{\Lambda_1^{n-1}}) + \frac{c^n}{2^{2n}}\left[\Sigma_{\zeta\in\Lambda^n}\phi(\zeta)\right]^2. \tag{1.2}$$

We shall denote the triple $(n_0, \mathcal{H}_{n_0}, c) = \Theta$, and sometimes write $\mathcal{H}^n_\Theta$ to denote the Dyson's Hamiltonian given by this data. It is natural to interpret the formula (1.2) as the definition of a renormalization transformation

$$\mathcal{R}_{\text{Dyson}} : \mathcal{H}^{n-1}_\Theta \to \mathcal{H}^n_\Theta.$$

Given a Hamiltonian $\mathcal{H}^n$ and a value $\beta > 0$ we obtain a probability measure $\mu_{\mathcal{H}^n,\beta}$ on $\Phi_n$ given by

$$\mu_{\mathcal{H}^n,\beta}(\phi) = \frac{\exp[-\beta\mathcal{H}^n(\phi)]}{\Sigma_{\phi\in\Phi_n}\exp[-\beta\mathcal{H}^n(\phi)]}.$$

For an even integer $t \in [-2^n, 2^n]$ let us define the function $f_n(t;\beta)$ as the probablity that the sum $\Sigma_{y\in\Lambda^n}\phi(y)$ takes the value $t$. Some calculations lead to the following recurrent relations:

$$(1.3)\qquad f_n(t;\beta) = P_n(\beta)\exp(\beta c^n 2^{-2n} t^2) \sum_{t_1=-2^{n-1}}^{2^{n-1}} f_{n-1}(t_1;\beta) f_{n-1}(t-t_1;\beta),$$

where $P_n(\beta)$ is a normalizing factor. Further considerations depend on the following heuristic understanding of the behaviour of the model near the critical $\beta_{cr}$. For $\beta < \beta_{cr}$ we expect that the typical values of the sum $\Sigma_{y\in\Lambda^n}\phi(y)$ are of order $t \sim 2^{n/2}$. For such $t$ the multiple $\exp(\beta c^n 2^{-2n} t^2) \to 1$ as $n \to \infty$, and the equation (1.3) for large $n$ behaves like the convolution.

On the other hand, for $\beta > \beta_{cr}$, the typical values of $t$ become of order $n$, and the multiple $\exp(\beta c^n 2^{-2n} t^2)$ becomes dominating. We then assume that at $\beta_{cr}$ the typical values of $t$ are such that $c^n 2^{-2n} t^2 \sim 1$. If that is so, it is natural to introduce a new variable $z$ by $t = z2^n c^{-n/2}$ and expect that $z \sim 1$ for $\beta = \beta_{cr}$. Setting $g_n(z;\beta) = f_n(z2^n c^{-n/2};\beta)2^{n+1}c^{-n/2}$, we have

$$g_n(z;\beta) = L_n(\beta) e^{\beta z^2} \sum_{(z_1+z_2)/2 = z/\sqrt{c}} g_{n-1}(z_1;\beta) g_{n-1}(z_2;\beta) c^{n/2} 2^{-n-1}.$$

Our assumptions lead us to expect that $g_n(z;\beta)$ has a limit $g(z;\beta)$ as $n \to \infty$. Passing to $n \to \infty$ in the above equation and dropping the normalizing factor we obtain the following integral equation for $g(z;\beta)$:

$$(1.4)\qquad g(z;\beta) = e^{\beta z^2} \int_{-\infty}^{\infty} g(\frac{z}{\sqrt{c}} + u;\beta) g(\frac{z}{\sqrt{c}} - u;\beta) du$$

Note that if $g(z;\beta_0)$ is a solution of (1.4) for some $\beta_0 > 0$, then

$$g(z;\beta) = g\left(\sqrt{\frac{\beta}{\beta_0}} z;\beta_0\right)\sqrt{\frac{\beta}{\beta_0}}.$$

Hence we may fix $\beta_0$ as convenient, and solve the integral equation to find a function of $z$.

**Solving the integral equation.** It is natural at this point to interpret the right side of the equation (1.4) as an integral operator $\mathcal{R}_{\text{int}} g(z;\beta)$ and also call it a renormalization transformation. Bleher and Sinai observe, that for every $c \in (1,2)$ the operator $\mathcal{R}_{\text{int}}$ has a Gaussian fixed point

$$g^{(0)}(z;\beta) = \sqrt{\frac{a_0}{\pi}} \exp[-a_0(\beta) z^2] \text{ where } a_0(\beta) = \beta c/(2-c).$$

They then show that in an appropriate Hilbert space of functions, for $c \in (\sqrt{2}, 2)$ this fixed point is hyperbolic with a single unstable eigenvalue $\lambda = 2/c > 1$. For $c < \sqrt{2}$ other unstable eigenvalues appear for the linearization of $\mathcal{R}_{\text{int}}$ at $g^{(0)}$. However, for $\epsilon$ small enough they are able to show for $c = \sqrt{2} - \epsilon$ the existence of a different fixed point $g_\epsilon(z;\beta)$ with a single unstable eigenvalue. The numerical evidence suggests that such a solution exists for every $c \in (1,2)$, but no proof of that exists as of yet.

For a given $n_0 \in \mathbb{N}$ the space of real-valued functionals on $\Phi_n$ is a real vector space; the space of start-up Hamiltonians $\mathcal{H}^{n_0}$ with the property that $\mathcal{H}^{n_0}(\phi) = \mathcal{H}^{n_0}(-\phi)$ is a closed subset of this space.

**Definition 1.1.** We will say that a fixed point $g(z;\beta)$ is thermodynamically stable if there exists $n_0$ and an open set $U$ of start-up Hamiltonians $\mathcal{H}^{n_0}$ such that the following holds.

- For each $\mathcal{H}^{n_0} \in U$ there exists $\beta_{\text{cr}}(\mathcal{H}^{n_0})$ such that the probability distribution $g_n(z;\beta_{\text{cr}})$ weakly converges to $g(z;\beta_{\text{cr}})$ as $n \to \infty$.
- For $\beta < \beta_{\text{cr}}$ there exists a function $h(\beta)$ such that as $n \to \infty$

$$f_n(t;\beta) \sim \frac{1}{\sqrt{2\pi h(\beta) 2^n}} \exp\left\{-\frac{t^2}{2h(\beta)2^n}\right\}$$

  for $t$ such that $|t2^{-n/2}| \leq A$ for every $A$. There exists $\gamma$ depending only on $g(z;\beta)$ such that $h(\beta) \sim \text{const} \cdot (\beta_{\text{cr}} - \beta)^{-\gamma}$ for $\beta \nearrow \beta_{\text{cr}}$.
- For $\beta > \beta_{\text{cr}}$ there exists a function $m(\beta)$ such that the average spint $2^{-n}\sum_{x\in\Lambda^n}\phi(x)$ converges to $\pm m(\beta)$ as $n \to \infty$ in probability generated by the Hamiltonian $\beta\mathcal{H}^n$. Moreover, there exists $\omega$ depending only on $g(z;\beta)$ such that $m(\beta) \sim \text{const} \cdot |\beta - \beta_{\text{cr}}|^{\omega}$.

One expects that for $c \in (\sqrt{2}, 2)$ the Gaussian fixed point $g^{(0)}$ is thermodynamically stable. This is indeed so, as was shown by Bleher and Sinai:

**Theorem 1.1.** *For every $\beta^{(0)}_{cr} > 0$, $\epsilon > 0$, $c \in (\sqrt{2}, 2)$ there exists $n_0 \in \mathbb{N}$ and an open set $U$ in the space of start-up Hamiltonians $\mathcal{H}^{n_0}$ such that for every $\mathcal{H}^{n_0} \in U$ there exists $\beta_{cr}(\mathcal{H}^{n_0})$ such that $|\beta_{cr} - \beta^{(0)}_{cr}|$; and $g_n(z;\beta_{cr})$ converges weakly to the Gaussian distribution. The Gaussian distribution is thermodynamically stable with respect to the start-up Hamiltonians in the set $U$.*

## 2. Unimodal maps

2.1. **A review of the Feigenbaum Renormalization Theory after Sullivan, McMullen, and Lyubich.** We briefly recall here the main results and conjectures of the celebrated Feigenbaum renormalization theory of unimodal maps. We will only discuss the case of period-doublings, for a more detailed exposition including a review of the general case we refer the reader to Lyubich's paper [Lyu1].

Let us fix $\gamma > 1$. Consider an even unimodal map $f : [-1,1] \to [-1,1]$, $f(-1) = f(1) = -1$. We will require that $f$ be $C^2$ smooth, except at the critical point. The latter will be further assumed to be of order $\gamma$, that is $f(x) = \phi(|x|^\gamma)$ near 0, where $\gamma > 1$, and $\phi$ is a local diffeomorphism.

**Definition 2.1.** The map $f$ is Feigenbaum-Collet-Tresser (FCT) renormalizable if the following holds: the critical value $f(0)$ lies above the diagonal $x = y$, which means, in particular, that $f$ has a fixed point $p_f = 1/\alpha_f \in (0, 1)$; and the iterate $f^2$ maps the interval $[-p_f, p_f]$ into itself.

If $f$ is FCT renormalizable, then the rescaled second iterate

$$-\alpha_f f \circ f(-x/\alpha_f) : [-1, 1] \to [-1, 1]$$

is again a unimodal map, we shall call it the FCT renormalization of $f$, and denote it $\mathcal{R}_{FCT} f$.

We shall say that a map $f$ is of Feigenbaum combinatorial type if the above procedure may be repeated indefinitely.

**Feigenbaum-Collet-Tresser renormalization hyperbolicity conjecture.** *For every $\gamma > 1$ there exists a Banach manifold $\mathcal{B}_\gamma$ of unimodal maps such that $\mathcal{R}_{FCT}$ is a smooth mapping of an open subset of $\mathcal{B}_\gamma$ into $\mathcal{B}_\gamma$. The mapping $\mathcal{R}_{FCT}$ has a fixed point $f_\gamma \in \mathcal{B}_\gamma$ with the following hyperbolicity property: $D_{f_\gamma}\mathcal{R}_{FCT}$ is a compact operator with a single eigenvalue $\delta_\gamma > 0$ outside the closed unit disk, and the rest of the spectrum inside $\mathbb{D}$. The stable manifold of $f_\gamma$ consists of unimodal maps with Feigenbaum combinatorics, and for every $f$ with Feigenbaum combinatorics and the critical point of order $\gamma$,*

$$\mathcal{R}^n_{FCT} f \to f_\gamma$$

*in the uniform topology.*

The eigenvalue $\delta_\gamma$ is called the Feigenbaum exponent, and the limit

$$\alpha = \lim_{n\to\infty} \alpha_{\mathcal{R}^n_{FCT} f}$$

is called the Feigenbaum scaling factor.

**The real renormalization theory.** The main achievement of this theory is Lanford's computer-assisted proof [Lan] of the existence of a fixed point of $\mathcal{R}_{FCT}$ for $\gamma = 2$ with the right hyperbolicity properties. Without the assistance of a computer, it has been shown that the renormalizations $\{\mathcal{R}^n_{FCT} f\}$ belong to a compact set in the $C^1$ topology. In particular, there exists a bound $C = C(\gamma) > 1$ such that $1/C < 1/\alpha_g < C$ for $g = \mathcal{R}^n_{FCT}$ for all $n$ sufficiently large. As a consequence, the $\omega$-limit set of the critical point of $f$ is a dyadic Cantor set with bounded geometry.

**The complex renormalization theory.** This theory has been developed by Sullivan [Sul2, MvS], and later by McMullen [McM], and Lyubich [Lyu1, Lyu2]. It applies when $\gamma$ is an even integer, we will only discuss the case $\gamma = 2$. Denote $\mathcal{A}_D$ the Banach manifold of analytic maps $f$ defined in a real-symmetric topological disk $D \subset \mathbb{C}$ with the sup norm, and such that 0 is a simple critical point of $f$.

**Renormalization hyperbolicity theorem.** *There exists a unique fixed point $f_*$ of $\mathcal{R}_{FCT}$ with $\gamma = 2$. For every unimodal $f$ with $\gamma = 2$ and Feigenbaum combinatorics, $\mathcal{R}^n_{FCT} f \to f_*$ in the $C^1$ topology. The mapping $f_*$ is an analytic mapping of a domain $D \subset \mathbb{C}$ which is quadratic-like (that is $f_* : D \to f_*(D)$ is a double-covering, branched at 0, and $D \Subset f_*(D)$). There is an open subset $\mathcal{QL} \subset \mathcal{A}_D$ which consists of quadratic-like maps, such that $\mathcal{R}_{FCT}$ extends to an analytic operator of an open neighborhood of $f_*$ in $\mathcal{QL}$ to $\mathcal{QL}$, such that $D_{f_*}\mathcal{R}_{FCT}$ has the right hyperbolicity properties.*

2.2. **Vul-Sinai-Khanin Hamiltonians.** The paper of Vul, Sinai, and Khanin [VSK] [1] offered a direct link between the Feigenbaum renormalization and the world of statistical mechanics. We will take it as the departure point for our investigation, and will briefly recap the main points of the construction below. We will begin by fixing an even unimodal mapping $f : [-1,1] \to [-1,1]$ with Feigenbaum combinatorics, with a critical exponent $\gamma > 1$ and will assume that the Feigenbaum-Collet-Tresser Hyperbolicity Conjecture holds for $\gamma$. That is, our statements will be founded for $\gamma = 2\ell,\ \ell \in \mathbb{N}$, and should be treated as conjectures for other values of $\gamma$. We note that the authors of [VSK] actually take $f$ to be a Feigenbaum renormalization fixed point, however, such a restriction is not necessary for their construction.

Let us begin by introducing some supporting notation. We will denote $\Delta_0^{(n)} \ni 0$ the symmetric renormalization interval of level $n$, that is, $\Delta_0^{(1)} = [-p_f, p_f]$, $f(\Delta_0^{(1)}) = \Delta_1^{(1)}$, and $f(\Delta_0^{(1)}) = [p_f, f(0)] \subset \Delta_0^{(1)}$. The cycle of intervals of level $n$ will be denoted $\Delta_k^{(n)} = f^k(\Delta_0^{(n)})$. The invariant Cantor set of $f$ is then the intersection

$$\mathbf{C}_f = \omega_f(0) = \bigcap_{n\geq 1} \bigcup_{k=0}^{2^n-1} \Delta_k^{(n)}.$$

For an integer number $k \geq 0$ let us also denote

$$\mathrm{Bin(k)} = \underline{\epsilon_0\epsilon_1\epsilon_2\cdots\epsilon_{n-1}},\ \epsilon_i \in \{0,1\}$$

its binary extension, $k = \epsilon_0 + \epsilon_1 \cdot 2 + \cdots + \epsilon_{n-1} \cdot 2^{n-1}$. We obviously have:

**Proposition 2.1.** *Suppose* $\Delta_k^{(n)} \supset \Delta_{(m)}^{n+1}$. *If* $\mathrm{Bin(k)} = \underline{\epsilon_0\epsilon_1\epsilon_2\cdots\epsilon_{n-1}}$, *then* $\mathrm{Bin(m)} = \underline{\epsilon_0\epsilon_1\epsilon_2\cdots\epsilon_n}$.

In view of the previous proposition, we have a well-defined one-to-one map $\Psi : x \mapsto \underline{\epsilon_0\epsilon_1\cdots\epsilon_n\cdots}$ from $\mathbf{C}_f$ to the space of one-sided binary sequences $\Sigma_{\{0,1\}}$.

**Proposition 2.2.** *The map $\Psi$ conjugates the action of $f$ to the adding one transformation*

$$\mathrm{Add}_1 : \underline{\epsilon_0\epsilon_1\cdots\epsilon_n\cdots} \mapsto \underline{\epsilon_0\epsilon_1\cdots\epsilon_n\cdots} + \underline{100000\cdots}.$$

[1] In Russian the order of the names becomes alphabetic: Вул, Синай, Ханин.

Let us define the lattices $\Lambda = \mathbb{N}$ and $\Lambda^n = \{1, \ldots, n\}$, and the spaces of spins $\Phi = \{\phi : \Lambda \to \{0,1\}\}$, $\Phi_n = \{\phi : \Lambda^n \to \{0,1\}\}$. The Vul-Sinai-Khanin Hamiltonian which we define below is a functional $\mathcal{H}_{VSK} : \Phi \to \mathbb{R}_{\geq 0}$. Let $\Delta^{(n)}_{p\cdot 2^{n-k}+1} \subset \Delta^{(n-1)}_{p'\cdot 2^{n-k-1}+1}$ for $0 \leq p < 2^k$. The map $\Psi$ carries them respectively into

$$\underline{\overbrace{1000\cdots 0}^{n-k}\epsilon^k\epsilon^{k-1}\cdots\epsilon^1} \text{ and } \underline{\overbrace{1000\cdots 0}^{n-k}\epsilon^k\epsilon^{k-1}\cdots\epsilon^2}$$

noindent Let

$$\log\left(\frac{|\Delta^{(n)}_{p\cdot 2^{n-k}+1}|}{|\Delta^{(n-1)}_{p'\cdot 2^{n-k-1}+1}|}\right) \equiv U^n_k(\epsilon^1, \epsilon^2, \ldots, \epsilon^k).$$

**Proposition 2.3.** *For a fixed sequence $\epsilon^1, \epsilon^2, \ldots, \epsilon^k$ the function $U^n_k(\epsilon^1, \epsilon^2, \ldots, \epsilon^k)$ has a limit as $n \to \infty$ which we denote $U_k(\epsilon^1, \epsilon^2, \ldots, \epsilon^k)$. In fact, if $\alpha = \lim_{n\to\infty} |\Delta^{(n)}_0|/|\Delta^{(n+1)}_0|$ then*

$$|U^n_k(\epsilon^1, \epsilon^2, \ldots, \epsilon^k) - U_k(\epsilon^1, \epsilon^2, \ldots, \epsilon^k)| \leq \text{const}\cdot\alpha^{-2(n-k)}\alpha^{2k}. \tag{2.1}$$

*Moreover, the functions $U_k(\epsilon^1, \epsilon^2, \ldots, \epsilon^k)$ converge to a limit as $k \to \infty$, which will be denoted $U(\epsilon^1, \epsilon^2, \ldots, \epsilon^k, \ldots)$, and*

$$|U(\epsilon^1, \epsilon^2, \ldots, \epsilon^k, \ldots) - U_k(\epsilon^1, \epsilon^2, \ldots, \epsilon^k)| \leq \text{const}\cdot(2\alpha^{-1})^k. \tag{2.2}$$

We have a natural identification of $\Phi$ with $\Sigma_{0,1}$. In view of this, the authors of [VSK] proposed to view the function $U(\epsilon^1, \epsilon^2, \ldots, \epsilon^k, \ldots)$ as the interaction potential of the first coordinate with the remaining coordinates. They have thus obtained a sequence of Hamiltonians $\mathcal{H}^n_{VSK} : \Phi_n \to \mathbb{R}_{\geq 0}$ given by

$$\mathcal{H}^n_{VSK}(\epsilon_1, \ldots, \epsilon_n) = -\Sigma^n_{s=1} U(\epsilon_s, \epsilon_{s-1}, \ldots, \epsilon_2, 1, 0, \ldots, 0, \ldots).$$

In view of (2.2) the Hamiltonian $\mathcal{H}_{VSK} : \Phi \to \mathbb{R}_{\geq 0}$ may be naturally defined as the limit of the above sequence of finite Hamiltonians. Observe that the Feigenbaum-Collet-Tresser renormalization hyperbolicity implies that

**Proposition 2.4.** *The values of the Hamiltonians $\mathcal{H}^n_{VSK}$ and $\mathcal{H}_{VSK}$ do not depend on the choice of a particular mapping $f$, but only on its universality class, that is, the value of the critical exponent $\gamma > 0$.*

This property of $\mathcal{H}_{VSK}$ makes it a convenient tool of the study of the geometric properties of $\mathbf{C}_f$, as will be seen below. However, the independence from the choice of $f$ makes $\mathcal{H}_{VSK}$ a poor candidate for defining a statistical-mechanical analogue of the Feigenbaum renormalization. We will dwell on the later point further, for the moment, let us proceed with describing the results of [VSK].

The following estimate of [VSK] is fundamental for their analysis: there exists a constant $C > 1$ such that for every $\beta > 0$

$$C^{-1} \leq \frac{|\Delta^{(n)}_k|^\beta}{\exp\{-\beta\mathcal{H}^n_{VSK}(\epsilon_0, \ldots, \epsilon_{n-1})\}} \leq C, \tag{2.3}$$

where $\mathrm{Bin}(k)=\underline{\epsilon_0\ldots\epsilon_{n-1}}$, and $\epsilon_0=1$. The free energy of $\mathcal{H}_{VSK}$ is, by definition, the following limit

$$\mathbf{f}(\beta)=\lim_{n\to\infty}\frac{1}{n}\log\left[\Sigma_{\epsilon_2,\ldots,\epsilon_n}\exp\left\{\beta\Sigma_{s=1}^n U(\epsilon_s,\epsilon_{s-1},\ldots,\epsilon_2,1,0,0,\ldots,0,\ldots)\right\}\right].$$

This limit is seen to exist, and to produce a smooth and monotonously decreasing function with $\mathbf{f}(0)=\log 2$, $\lim_{\beta\to\infty}\mathbf{f}(\beta)=-\infty$. The unique value of $\beta_0>0$ for which $\mathbf{f}(\beta_0)=0$ corresponds to a phase transition. For this value of $\beta$ the sum in the square brackets is bounded by above and below by two positive constants, which in conjunction with (2.3) can be shown to imply that

$$\mathrm{Hdim}(\mathbf{C}_f)=\beta_0.$$

**Sullivan's observation on $C^{1+\alpha}$-self-similar Cantor sets.** The existence of the limiting scaling ratios $U(\epsilon^1,\epsilon^2,\ldots,\epsilon^n,\ldots)$ may be seen at this point as a property intrinsic to the Feigenbaum functional equation. However, their nature is more general as seen from the work of Sullivan [Sul1]. Firstly, let $g$ be a fixed point of the Feigenbaum renormalization transformation with the critical exponent $\gamma$, set

$$\sigma(x)=\begin{cases}-\alpha x, & x\in\Delta_0^{(1)}\\ -\alpha g(x), & x\in\Delta_1^{(1)}\end{cases}\tag{2.4}$$

where $\alpha$ is the Feigenbaum scaling factor, as before. The function $\sigma(x)$ will play a significant role in what follows, at the moment, however, let us simply note that it is smooth, has no critical points, and leaves $\mathbf{C}_g$ invariant. The reader is invited to verify the last property, which is a ready consequence of the Feigenbaum functional equation. More specifically, the function $\sigma(x)$ shifts the binary sequences in $\Sigma_{0,1}$, parametrizing the points in $\mathbf{C}_f$ to the left, thus transforming $(\epsilon^1,\epsilon^2,\ldots)$ into $(\epsilon^2,\epsilon^3,\ldots)$. Sullivan then asks a question, when would the shift operation on a Cantor set be realized as a smooth mapping. The answer comes in the form of a theorem. Before formulating it, let us give several definitions. Let us consider a general binary Cantor set $\mathbf{C}$ which is the intersection of nested intervals $\cap_{n\geq 1}\cup_{k=0}^{2^n-1}\Delta_k^{(n)}$ indexed in such a way that Proposition 2.1 holds. Recall that the Cantor set $\mathbf{C}$ has bounded geometry if for every finite binary sequence $\omega_n=\{\epsilon^i\}_{i=1}^n$ the triple of scaling ratios

$$\mathfrak{g}_n(\omega_n)=\left\{\frac{|\Delta_{\underline{\epsilon^n\epsilon^{n-1}\ldots\epsilon^1}}|}{|\Delta_{\underline{\epsilon^n\epsilon^{n-1}\ldots\epsilon^1 1}}|},\frac{|\Delta_{\underline{\epsilon^n\epsilon^{n-1}\ldots\epsilon^1}}|}{|\Delta_{\underline{\epsilon^n\epsilon^{n-1}\ldots\epsilon^1 0}}|},\frac{|\Delta_{\underline{\epsilon^n\epsilon^{n-1}\ldots\epsilon^1}}|}{|G_{\underline{\epsilon^n\epsilon^{n-1}\ldots\epsilon^1}}|}\right\}$$

is bounded from above and below by positive constants independent of $\omega_n$ and $n$, where $G_{\underline{\epsilon^n\epsilon^{n-1}\ldots\epsilon^1}}$ is the middle gap between the two sub-intervals of $\Delta_{\underline{\epsilon^n\epsilon^{n-1}\ldots\epsilon^1}}$. For $\beta\in(0,1]$ let $C^{1+\beta}$ denote the class of smooth local diffeomorphisms with $\beta$-Hölder continuous derivatives. Change of coordinates by such diffeomorphisms have the following effect on the scaling ratios:

**Proposition 2.5.** *Suppose* **C** *has bounded geometry. Then for every* $\beta > 0$, *the values of the scaling ratios* $\mathfrak{g}_n(\omega_n)$ *become independent of* $C^{1+\beta}$*-coordinate changes exponentially fast in* $n$.

Let us say that the set of scaling functions $\{\mathfrak{g}_n(\epsilon^1, \epsilon^2, \epsilon^3, \dots, \epsilon^n)\}$ for the Cantor set **C** is Hölder continuous if the value of each scaling function is determined exponentially fast in $k$ by the first $k$ arguments. In particular, in this case, we have limiting scaling functions $\mathfrak{g}(\epsilon^1, \epsilon^2, \dots, \epsilon^k, \dots)$ associated to every sequence in $\Sigma_{0,1}$. Moreover, if **C** has bounded geometry, Proposition 2.5 implies that these scaling functions are independent of $C^{1+\beta}$ coordinate changes, and thus only depend on the choice of the $C^{1+\beta}$ differentialble structure on the Cantor set.

**Theorem 2.6.** *Suppose a bounded geometry Cantor set* **C** *is such that its scaling functions are Hölder continuous. Then the action of the binary shift on the Cantor set is realized as a* $C^{1+\beta}$*-expanding map in some smooth metric.*

*Conversely, if the shift on the Cantor set is* $C^{1+\beta}$ *smooth in a differentiable structure on the interval in which the geometry of the Cantor set is bounded, then its scaling functions are Hölder continuous, and, in particular, the limiting scaling functions* $\{\mathfrak{g}(\epsilon^1, \epsilon^2, \epsilon^3, \dots)\}$ *exist for every infinite sequence* $(\epsilon^1, \epsilon^2, \dots) \in \Sigma_{0,1}$.

Since the functions $U(\epsilon^1, \epsilon^2, \dots, \epsilon^n, \dots)$ introduced in [VSK] are nothing but logs of particular limiting scaling functions, the Proposition 2.3 is a corollary of the above theorem.

**"Naïve" Hamiltonians.** The above considerations, in particular, the estimate (2.3) suggest a different approach in associating Hamiltonians to a unimodal map $f$ with the Feigenbaum combinatorics. We may define $\mathcal{H}^n_N(f) : \Phi_n \to \mathbb{R}_{\geq 0}$ simply by

$$\mathcal{H}^n_N(f)(\epsilon_1, \dots, \epsilon_k) = -\log |\Delta^{(n)}_k|,$$

where $\text{Bin}(k) = \underline{\epsilon_1 \epsilon_2 \cdots \epsilon_k}$. Such "naïve" approach has the advantage that the Hamiltonians $\mathcal{H}^n_N(f)$ depend on the mapping $f$, and thus are eligible candidates for defining renormalization. We shall see in the next section, that more complex definitions are required, however, this is a step in the correct direction. For the moment, let us make note, that the estimate (2.3) may be shown to imply that denoting $\beta^n_0$ the critical values of $\beta$ for $\mathcal{H}^n_N(f)$, we have $\beta^n_0 \to \beta_0 = \text{Hdim}(\mathbf{C}_f)$. However, the Hamiltonians $\mathcal{H}^n_N(f)$ themselves do not converge to any particular limit.

2.3. **Martens' result in thermodynamical terms.** As we shall see in this section, a thermodynamical definition for renormalization of unimodal maps has already appeared in the literature. We will present an interpretation of a paper of Martens [Mar], which ties the Feigenbaum-Collet-Tresser Renormalization operator with a hierarchical renormalization transformation acting on Hamiltonians related to the ones considered above. The main result of Martens' work is the following theorem:

**Theorem 2.7.** *For every $\gamma > 1$, every $n \geq 2$ and every permutation $\tau \in S_n$, there exists an even unimodal map $f$ which is renormalizable with period $n$, and combinatorial type $\tau$, and whose critical point 0 has the order $\gamma$.*

While uniqueness of $f$ is not shown, an upshot of the argument is that $f$ belongs to a very restrictive class of maps (the so-called Epstein class). H. Epstein has obtained the same results by a different method in [Ep]. Of main interest to us will be not the theorem itself, but the rather remarkable method which Martens used to prove it. We will therefore confine ourselves exclusively to the case of the Feigenbaum combinatorics.

Let us begin by fixing $\gamma > 1$, and denoting $q_t(x) : [-1,1] \to [-1,1]$ for each $t \in [0,1]$ the unimodal mapping

$$q_t(x) = -2t|x|^\gamma + 2t - 1.$$

For a closed oriented interval $I = [a,b] \subset \mathbb{R}$ let us denote $\iota_I : \mathbb{R} \to \mathbb{R}$ the affine orientation preserving mapping which transforms $[-1,1]$ into $I$. An elementary property of the folding mappings $q_t$ which is useful to note is the following:

**Proposition 2.8.** *Let $J \equiv [-p,p]$ be a proper subinterval of $[-1,1]$, and $I = [q_t(p), l]$ such that $l \geq q_t(0) = 2t-1$. Then there exists $s = s(l,p)$ such that*

$$(\iota_I)^{-1} \circ q_t|_J \circ \iota_J = q_s.$$

Let $f = \phi \circ q_t$ be an even infinitely renormalizable unimodal mapping of the Feigenbaum combinatorial type, whose critical point has the order $\gamma$. As before, let us denote $\{\Delta_k^{(n)}\}_{k=0}^{2^n-1}$ the cycle of renormalization intervals of level $n$. The standard considerations of one-dimensional dynamics (see e.g. [MvS]) imply that the restrictions of $f$ to these intervals are approximated geometrically fast in $n$ by restrictions of $q_t$ in the smooth metric. This will motivate to some extent the following definition.

**Definition 2.2.** For every $n \in \mathbb{N}$ a Martens' Hamiltonian of level $n$ is a mapping $\mathcal{H}^n_{\text{Mar}} : \Phi_n \to [-1,1]^2$ which corresponds to every finite binary sequence $\{\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}\}$ a closed interval

$$\mathcal{H}^n_{\text{Mar}}(\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1} = \Delta_{\underline{\epsilon_0\epsilon_1\cdots\epsilon_{n-1}}} \subset [-1,1] \setminus \{0\}.$$

Similarly, a Martens' Hamiltonian $\mathcal{H}_{\text{Mar}} : \Phi \to [-1,1]$ corresponds an interval to each infinite binary sequence. An extended Martens' Hamiltonian $\tilde{\mathcal{H}}^n_{\text{Mar}}$ is a pair $(\mathcal{H}^n_{\text{Mar}}, t)$ with $t \in [0,1]$.

Let us denote $\mathbf{Mar}^n$ the set $([-1,0)^2 \cup (0,1]^2)^n$ viewed as the collection of Martens' Hamiltonians of level $n$. The extended Hamiltonians then correspond to points in $\mathbf{Mar}^n \cup [0,1]$. For the moment, we shall view them simply as topological spaces, whose topology in a natural way comes from the topology of $\mathbb{R}$. For an infinitely renormalizable mapping $f$ of Feigenbaum type we obtain a sequence of Martens' Hamiltonians $\mathcal{H}^n_{\rm Mar}(f)$ in an obvious fashion, taking as the intervals $\Delta^{(n)}_{\underline{\epsilon_0\epsilon_1\cdots\epsilon_{n-1}}}$ the elements of the cycle of renormalization of level $n$, indexed as before. To define an extended Hamiltonian $\tilde{\mathcal{H}}^n_{\rm Mar}(f)$, we set $\tilde{\mathcal{H}}^n_{\rm Mar}(f) = (\mathcal{H}^n_{\rm Mar}(f), s)$ where

$$(\iota_{\Delta_1^{(n)}})^{-1} \circ f|_{\Delta_0^{(0)}} \circ \iota_{\Delta_0^{(n)}}(0) = q_s(0).$$

Before proceeding any further, let us note that there is a natural way of associating a naïve Hamiltonian $\mathcal{H}^n_N(f)$ to $\mathcal{H}^n_{\rm Mar}(f)$ by a forgetful transformation $\varphi$ which replaces each of the intervals $\Delta^{(n)}_k$ with its length.

More importantly, there is also a way to recover a unimodal map from an extended Martens' Hamiltonian. To that end, let us define a composition operation $\mathrm{Comp}^n : \mathcal{H}^n_{\rm Mar} \mapsto \phi \in \mathrm{Diff}([-1,1])$ as follows.

**Definition 2.3.** The composition

$$\mathrm{Comp}^n(\mathcal{H}^n_{\rm Mar}) = \prod_{k=1}^{2^n \equiv 0 \bmod 2^n} \left( (\iota_{q_t(\Delta_k)})^{-1} \circ q_t|_{\Delta_k} \circ \iota_{\Delta_k} \right)$$

For an extended Martens Hamiltonian $\tilde{\mathcal{H}}^n_{\rm Mar} = (\mathcal{H}^n_{\rm Mar}, t)$ we get a unimodal mapping

$$f_{\tilde{\mathcal{H}}^n_{\rm Mar}} = \mathrm{Comp}^n(\mathcal{H}^n_{\rm Mar}) \circ q_t : [-1,1] \to [-1,1].$$

These definitions are natural in the following sense:

**Proposition 2.9.** *Let $g = q_{t*}$ be the map in the folding family with the Feigenbaum combinatorics. For every $n \in \mathbb{N}$ we have*

$$f_{\tilde{\mathcal{H}}^n_{Mar}(g)} = \mathcal{R}^n_{FCT}(g).$$

Of course, the composition operation behaves well with respect to the usual norm on $\mathbf{Mar}^n$. However, we will be interested also in making sense out of composing infinite Hamiltonians. To that end, a different notion of distance needs to be introduced. Let us recall that the nonlinearity of a $C^2$-diffeomorphism $\phi : [-1,1] \mapsto [-1,1]$ is

$$\eta(\phi)(x) = \frac{D^2\phi(x)}{D\phi(x)} = D|\ln D\phi(x)|.$$

This map is a bijection between $\mathrm{Diff}^2_+([-1,1])$ and the space of continuous functions $C([-1,1])$, the inverse being given by

$$\phi(\eta)(x) = 2\frac{\int_{-1}^x e^{\int_{-1}^s \eta(y)dy} ds}{\int_{-1}^1 e^{\int_{-1}^s \eta(y)dy} ds} - 1.$$

We will denote $||\cdot||$ the norm on $\mathrm{Diff}^2_+$ which is the pull-back of the sup-norm on $C([-1,1])$ under this bijection. Suppose $\mathcal{H}^n_{\rm Mar} : \{\epsilon_0, \ldots, \epsilon_{n-1}\} \to \Delta_{\underline{\epsilon_0\cdots\epsilon_{n-1}}}$ and

$\widehat{\mathcal{H}^n_{\mathrm{Mar}}} : \{\epsilon_0, \ldots, \epsilon_{n-1}\} \to \hat{\Delta}_{\underline{\epsilon_0 \cdots \epsilon_{n-1}}}$ are two Martens' Hamiltonians. We will define a pseudo-distance between them as

$$\mathrm{dist}(\mathcal{H}^n_{\mathrm{Mar}}, \widehat{\mathcal{H}^n_{\mathrm{Mar}}}) = \sum_k ||(\iota_{q_t(\Delta_k)})^{-1} \circ q_t|_{\Delta_k} \circ \iota_{\Delta_k} - (\iota_{q_t(\hat{\Delta}_k)})^{-1} \circ q_t|_{\hat{\Delta}_k} \circ \iota_{\hat{\Delta}_k}||.$$

It is clear that distinct Hamiltonians may become indistinguishable with respect to dist. In order to understand the situation better, let us note that

$$\eta(q_{1/2})(x) = (\gamma - 1)\frac{1}{x}.$$

Hence, the nonlinearity of each of the terms $(\iota_{q_t(\Delta_k)})^{-1} \circ q_t|_{\Delta_k} \circ \iota_{\Delta_k}$ for a Martens' Hamiltonian is an expression of the form $d/(ax+b)$. We may thus denote $\mathbf{Mar}^n_*$ the space of triples $a, b, d \geq 0$ such that $b^2 - a^2 > 0$, and naturally identify

$$\Psi : \mathbf{Mar}^n / \mathrm{dist} \to \mathbf{Mar}^n_*.$$

The distance induced by dist on $\mathbf{Mar}^n_*$ is seen to be equivalent to the $L^1$-distance on $\mathbb{R}^{3n}$ on compact sets. Let us endow the set **Mar** with the same notion of distance. Martens shows then that the composition operator Comp may be defined for bounded sets in **Mar** as limits of finite compositions. Further:

**Proposition 2.10.** *The composition operation* Comp *is continuous with respect to the above distance. Moreover, it is Lipschitz on every bounded set.*

*Remark* 2.1. It is appropriate to note here that there are other choices of distance on $\mathbf{Mar}^n$ which allow the composition to be extended to the case of infinitely many maps. One example would be to use the weigthed $L^1$-norm

$$||\mathcal{H}^n_{\mathrm{Mar}}||_\rho = \sum_k \sum_{x \in \partial\Delta_k} \rho^n |x|, \text{ for } \rho > 1.$$

Uniform boundedness in this norm would imply that the intervals shrink in size geometrically fast – while such an approach would be essentially equivalent to what we did above, it may still be technically useful.

**Renormalization.** Let us say that an extended Martens' Hamiltonian $\tilde{\mathcal{H}}^n_{\mathrm{Mar}} = (\mathcal{H}_{\mathrm{Mar}}, t)$ is renormalizable if the unimodal map $f_{\tilde{\mathcal{H}}^n_{\mathrm{Mar}}}$ is renormalizable with period 2 in the usual sense. In this case, $f_{\tilde{\mathcal{H}}_{\mathrm{Mar}}}$ has a pair of periodic intervals $J_0 = [-p, p] \ni 0$ and $J_1$ whose boundaries touch at a repelling fixed point $p > 0$.

**Definition 2.4.** The renormalization of a renormalizable extended Martens' Hamiltonian $\tilde{\mathcal{H}}^n_{\mathrm{Mar}} = (\mathcal{H}^n_{\mathrm{Mar}}, t)$ is a pair $\mathcal{R}\tilde{\mathcal{H}}^n_{\mathrm{Mar}} = (\widehat{\mathcal{H}^{n+1}_{\mathrm{Mar}}}, \hat{t})$ defined as follows.

- The intervals $\hat{\Delta}_{\underline{\epsilon_0 \cdots \epsilon_n}} = \widehat{\mathcal{H}^{n+1}_{\mathrm{Mar}}}$ are given by

$$\hat{\Delta}_{\underline{\epsilon_0 \cdots \epsilon_n}} = \left( \prod_{k > \underline{\epsilon_0 \cdots \epsilon_{n-1}}} (\iota_{q_t(\Delta_k)})^{-1} \circ q_t|_{\Delta_k} \circ \iota_{\Delta_k} \right)^{-1} J_{\epsilon_n}.$$

- The value of $\hat{t}$ is the critical value of the folding map $q_{\hat{t}}$ such that

$$\mathcal{R}_{FCT} f_{\tilde{\mathcal{H}}^n_{\text{Mar}}} = \text{Comp}^{n+1}(\widehat{\mathcal{H}^{n+1}_{\text{Mar}}}) \circ q_{\hat{t}}.$$

This definition is seen to naturally extend to the infinite Hamiltonians in **Mar**.

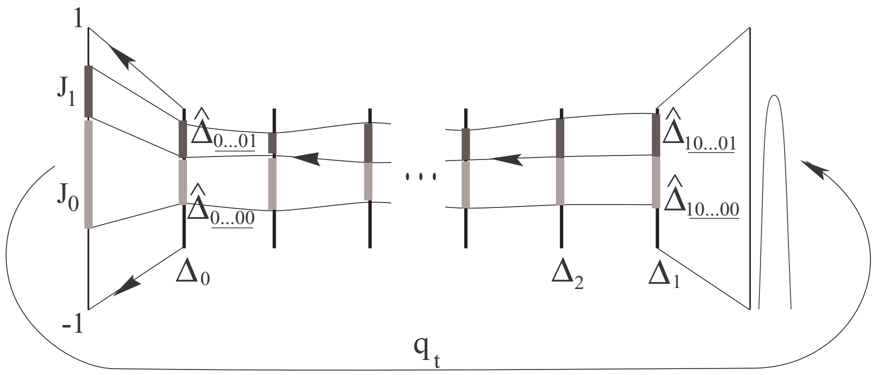


FIGURE 2

**Proposition 2.11.** *By virtue of the construction,* $\mathcal{R}_{FCT} f_{\tilde{\mathcal{H}}^n_{Mar}} = f_{\mathcal{R}\tilde{\mathcal{H}}^n_{Mar}}$.

**Proof of the existence of a fixed point.** The reader should see the virtue of using Martens' Hamiltonians to represent unimodal maps at this stage. While general unimodal mappings form an unwieldy infinite-dimensional real Banach space, those represented by Martens' Hamiltonians of level $n$ are an open subset of $\mathbb{R}^k$! The only obstacle to defining finite-dimensional approximation to the renormalization is that a Hamiltonian of level $n$ is transformed by $\mathcal{R}$ into a Hamiltonian of level $n+1$. A way to circumvent it is the following. Let us define a Martens' tree of depth $k$ as a sequence $\mathcal{T}^k = \{\mathcal{H}^n_{\text{Mar}}\}^k_{n=1}$, $k \in \mathbb{N} \cup \{\infty\}$ and an extended tree as a pair $(\mathcal{T}^k, t)$. Renormalization naturally acts on extended trees of infinite depth, and transforms trees of depth $k$ into trees of depth $k+1$. A truncation of level $l < k$ is the tree $\text{trunc}^l(\mathcal{T}^k) = \{\mathcal{H}^n_{\text{Mar}}\}^k_{n=1}$. Let us equip the set of trees with the same notion of distance, by summing up the distances between the levels. As seen above, modulo this distance, the space of trees may be identified with an open set in a real Banach space $\mathcal{B}$.

*Remark* 2.2. Let us note that using the norm $||\cdot||_\rho$ from Remark 2.1 to define the $L^1$ norm on trees has the following geometric advantage. If $\rho$ is sufficiently large, and $\mathcal{T}$ is the infinite tree corresponding to an infinitely renormalizable map, then all trees in its sufficiently small neighborhood may be interpreted as dyadic Cantor sets with bounded geometry.

Martens proved the following theorem:

**Theorem 2.12.** *For every $k > 1$ there exists an extended tree $(\mathcal{T}^k, t_k)$ such that*
$$\mathrm{trunc}^k(\mathcal{R}(\mathcal{T}^k, t_k)) = (\mathcal{T}^k, t_k).$$
*Moreover, there is an infinite depth tree $\mathcal{T}^\infty = \{\mathcal{H}^n_{Mar}\}$, and a subsequence $n_k$ such that*
$$t_{n_k} \to t, \text{ and } \mathrm{dist}(\mathcal{T}^{n_k}, \mathrm{trunc}^{n_k}(\mathcal{T}^\infty)) \to 0.$$
*Hence, the extended infinite tree $(\mathcal{T}^\infty, t)$ is a fixed point of $\mathcal{R}$.*

As a consequence, the unimodal map $f = \lim f_{(\mathcal{H}^{n_k}_{\mathrm{Mar}}, t)}$ is a fixed point of $\mathcal{R}_{FCT}$. At the heart of the argument is the following topological statement [2]:

**Lemma 2.13 ("Bottom goes down, top goes up").** *Suppose $D_n$ is a closed $n$-dimensional ball, and let $F : D_n \times [0,1] \to D_n \times \mathbb{R}$ be a continuous map. If*
$$F(D_n \times \{0\}) \subset D_n \times (-\infty, 0) \text{ and } F(D_n \times \{1\}) \subset D_n \times (1, \infty),$$
*then $F$ has a fixed point in $D_n \times [0,1]$.*

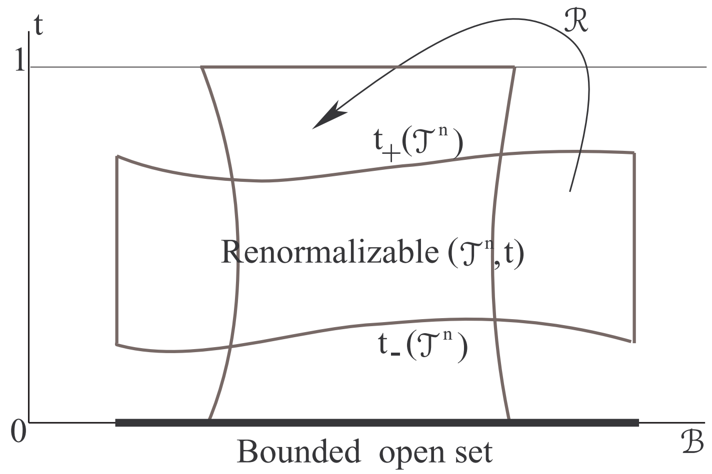


FIGURE 3

It is not difficult to understand where the bottom goes down, top goes up condition comes into play. If we denote $t_-(\mathcal{T}^k)$, $t_+(\mathcal{T}^k)$ the minimal and the maximal values of $t$ for which $(\mathcal{T}^k, t)$ is renormalizable, then $(\mathcal{T}^k, t)$ is renormalizable for all $t \in [t_-(\mathcal{T}^k), t_+(\mathcal{T}^k)]$, and the considerations of maximality imply that this interval of values is transformed by $\mathcal{R}$ into $[-1, 1]$. To show that there is an open and bounded set of trees which is mapped inside itself, Martens applies a version of real *a priori*

[2] Martens attributes the idea to Dennis Sullivan

bounds to get an *a priori* bound on the nonlinearity of renormalized maps. The same bounds also imply the existence of a convergent subsequence of $\mathcal{T}^k$ with the desired property. We will postpone our comments on the proof and on the method till the final chapter. At this point, let us note that we now have a new object to study - a termodynamical renormalization operator which is functorially related to the Feigenbaum-Collet-Tresser renormalization. As we shall see in the next section, there has been a previous instance when such a renormalization appeared in the literature.

2.4. **Geometric interpretation of renormalization: a Perron–Frobenius type operator.** We will discuss in this section the results of Jiang, Morita, and Sullivan from [JMS]. The central objective of [JMS] was to attempt to justify the existence of an expanding eigenvalue for the Feigenbaum-Collet-Tresser renormalization operator at a fixed point.
Let us fix $\gamma = 2\ell$, $\ell \in \mathbb{N}$, and let $g(x)$ denote the period doubling fixed point with critical point of order $\gamma$. Being an analytic mapping, $g(x)$ extends to an open topological disk $U$ of $[-1, 1]$. Let $\sigma(x)$ be the expanding mapping (2.4). Denote $J_0 = [g^2(0), -g^2(0)]$ and $J_1 = g(J_1) = [g^3(0), g(0)]$. Using the expansiveness of $\sigma$, we may select a real-symmetric topological disk $\Omega \subset g(U)$, $\Omega \Supset [-1, 1]$, small enough so that $J_0 \cup J_1 \Subset \Omega$, and there exist disjoint open neighborhoods $\Omega_i \Supset J_i$ such that $\Omega_i \Subset \Omega$, and $\sigma : \Omega_i \mapsto \Omega$ is a conformal mapping. Let us further denote by $V$ the connected component of $g^{-1}(\Omega)$ which contains the origin. Denote $\mathcal{A}_V$ the Banach space of analytic functions on $V$ continuous up to the boundary equipped with the uniform norm. Let $\mathcal{B}_V \subset \mathcal{A}_V$ be the Banach manifold of unimodal mappings $f$ with a single critical point at the origin of order $\gamma$ which have analytic extensions to $V$ continuous up to the boundary, and such that

$$f^3(0) > -f^2(0) > f^4(0) > 0 > f^2(0).$$

Let $\mathcal{V}^\omega_\Omega$ denote the space of real-symmetric analytic vector fields on $\Omega$ with a continuous extension to the boundary, again with a uniform norm. The following proposition is straightforward:

**Proposition 2.14.** *The mapping $g_*$ from $\mathcal{V}^\omega_\Omega$ into $T_g\mathcal{B}_V$ defined by*

$$g_*(v)(x) = v(g(x)) \text{ for } x \in \Omega \text{ and } v \in \mathcal{V}^\omega_\Omega$$

*is an isomorphism.*

**Definition 2.5.** The Perron-Frobenius type operator $\mathcal{L}_\sigma : \mathcal{V}^\omega_\Omega \to \mathcal{V}^\omega_\Omega$ is defined as

$$(\mathcal{L}_\sigma v)(z) = \sum_{w \in \sigma^{-1}(z)} \sigma'(w) v(w).$$

The next statement relates the Perron-Frobenius type operator $\mathcal{L}_\sigma$ with the Feigenbaum-Collet-Tresser renormalization:

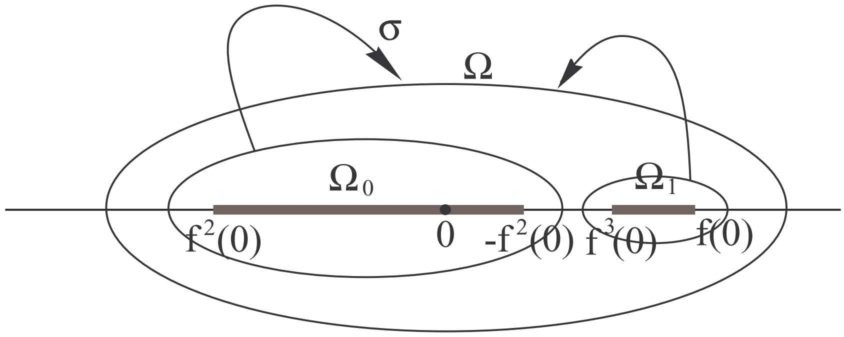


FIGURE 4

**Proposition 2.15.** *The operators $\mathcal{L}_\sigma$ and $D_g\mathcal{R}_{FCT}$ (the latter viewed as an operator on $T_g\mathcal{B}_V$) have the same eigenvalues except for the value 1. More precisely,*

$$\mathcal{L}_\sigma = (g_*)^{-1} \circ D_g\mathcal{R}_{FCT} \circ g_* + e_1,$$

*where $e_1$ is a projection onto the eigenspace of eigenvalue one generated by the vector $(g_*)^{-1}(g'(x)x - g(x))$.*

As a remark, the authors of [JMS] note that for every $m \in \mathbb{N}$ the vector

$$v_{2m-1} = (g_*)^{-1}[g'(x)x^{2m-1} - (g(x))^{2m-1}] \in \mathcal{V}^\omega_\Omega$$

is an eigenvector of $\mathcal{L}_\sigma$ with eigenvalue $\lambda_{2m-1} = \alpha^{-(2m-2)}$.
The statement of the last proposition should not come as a surprise after the results of the previous sections. Indeed, according to the Remark 2.2, Martens' trees in a small neighborhood of the tree of $g$ in an appropriate norm may be viewed as Cantor sets with bounded geometry. Since $\sigma$ acts as a shift on the Cantor set $\mathbf{C}_g$, the operator $\mathcal{L}_\sigma$ is naturally interpreted as a linearization of Martens' renormalization at the fixed point in a suitable smooth structure. We thus have a geometric interpretation for the statistical-mechanical renormalization of the previous section given by the embeddig of the invariant Cantor set in the plane.
Since the operator $\mathcal{L}_\sigma$ is not positive, the Ruelle-Perron-Frobenius theory does not apply to show the existence of the single expanding eigenvalue. The main result of [JMS] is the following:

**Theorem 2.16.** *Suppose $g : [-1, 1] \to [-1, 1]$ is a concave function. Then $\mathcal{L}_\sigma$ has an eigenvalue $\lambda > 1$.*

We remark that the assumption of concavity is supported by the numerical experiments for $\gamma = 2$, and is false if $\gamma$ is sufficiently large.

**Remark on critical circle maps.** We would like to remark here that a similar construction may be carried out for the commuting pair $(\eta, \xi)$ which is the golden-mean fixed point for the renormalization of critical circle maps. Namely, we may define an expanding map

$$\sigma(x) = \begin{cases} -\alpha x, & x \in [\xi(0), \xi \circ \eta(0)] \\ -\alpha \xi(x), x \in [\xi \circ \eta(0), 1 = \eta(0)] \end{cases}$$

where the scaling factor $\alpha = -1/\xi(0)$. In this case, the operator $\mathcal{L}_\sigma$ is negative,

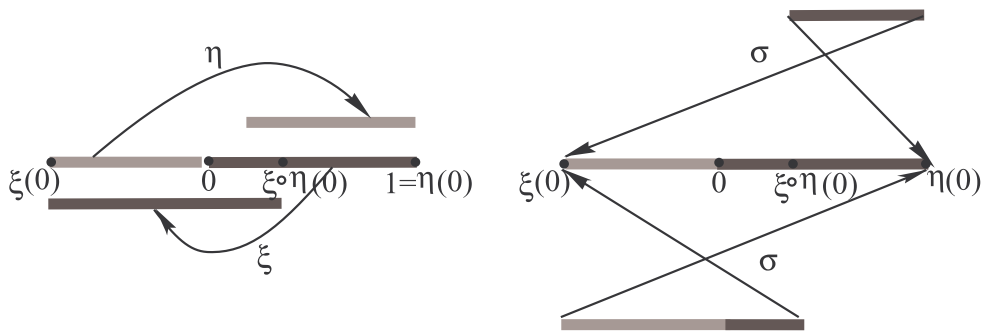


FIGURE 5

and the Ruelle-Perron-Frobenius theory applies to establish the existence of a single expanding eigenvalue. This aspect of the renormalization theory of critical circle maps, however, is quite trivial. On the other hand, constructing a Banach manifold in which the renormalization is an analytic operator is highly nontrivial. One may hope therefore that there is a possibility for an alternative approach to the hyperbolicity problem in the renormalization theory of critical circle maps – the study of the spectrum of a linear operator $\mathcal{L}_\sigma$ in a real Banch space.

2.5. **Concluding remarks.**

**Comparison of the two cases.** Let us summarize the obvious similarities between the two examples considered in this note: the Dyson's Hierarchical Hamiltonians, and the Hamiltonians for the unimodal maps. In both cases one begins by considering an action of a renormalization transformation acting on Hamiltonians: in the first instance it is $\mathcal{R}_{\text{Dyson}}$, in the second, the Martens' renormalization $\mathcal{R}$. For a fixed point of such a renormalization, the self-similarity produces functional equations for the thermodynamical parameters of the system (in the first case, $g(z;\beta)$, in the second, the unimodal map $f$). For the Dyson's models it is the equation (1.4), for the unimodal maps, the FCT fucntional equation. The equation is then interpreted as a renormalization operator acting on an appropriate function space. We may call these two approaches to renormalization a microscopic and a macroscopic ones. In both instances the macroscopic renormalization is studied by analytic methods: the properties of the Gauss integral operator in the first, the Sullivan-McMullen-Lyubich theory in the second. For the unimodal maps there is now Martens' proof of the existence of a fixed point for the microscopic renormalization, as well as the Jiang-Morita-Sullivan approach to proving its hyperbolicity. To our knowledge, such microscopic approach is missing in the Dyson's case.

| **Dyson's Hierarchical models** | **Unimodal maps** |
|---|---|
| Dyson's Hamiltonian $\mathcal{H}_\Theta$ | Martens' Hamiltonian $\tilde{\mathcal{H}}_{\text{Mar}}$ |
| Microscopic renormalization $\mathcal{R}_{\text{Dyson}}$ | Microscopic renormalization $\mathcal{R}$ |
| **?** | Martens' proof of the existence of fixed point; JMS analysis of hyperbolicity |
| Distribution $g(z;\beta)$ | Unimodal map $f$ |
| $\mathcal{R}_{\text{int}}g(z;\beta)=\int g(\frac{z}{\sqrt{c}}+u;\beta)g(\frac{z}{\sqrt{c}}-u;\beta)du$ | $\mathcal{R}_{FCT}(f)=-\alpha_f f\circ f(-x/\alpha_f)$ |
| Bleher-Sinai analysis of Gauss integral operator | Sullivan-McMullen-Lyubich theory |

**Some open problems.** Several questions on hierarchical models were posed in Sinai's book [Sin]. The main open problem is:

- Extend the existence of the hyperbolic fixed point of $\mathcal{R}_{int}$ with a single unstable eigenvalue to the whole interval $c\in(1,\sqrt{2})$.

In view of our analysis we will add:

- Find a critical fixed point of the microscopic renormalization. In a forthcoming article with M. Benedicks we will show how such a problem may be attacked with Martens' approach to renormalization.

In the case of the dynamical renormalization it is natural to pose the following problems:

- Develop a complete hyperbolic picture for the Martens' renormalization. A possible approach to this is given by the study of the Perron-Frobenius type operator of [JMS].
- Use the same approach, as indicated in the previous section, to construct an alternative renormalization theory for critical circle maps.

Finally, we would like to speculate that the microscopic approach to renormalization could prove useful in other contexts, both statistical-mechanical, and dynamical (such as the conservative mappings of $\mathbb{T}^2$, for instance).

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