

# On invariant measures of finite affine type tilings

Samuel Petite \*

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## Abstract

In this paper, we consider tilings of the hyperbolic 2-space  $\mathbb{H}^2$ , built with a finite number of polygonal tiles, up to affine transformation. To such a tiling  $T$ , we associate a space of tilings: the continuous hull  $\Omega(T)$  on which the affine group acts. This space  $\Omega(T)$  inherits a solenoid structure whose leaves correspond to the orbits of the affine group. First we prove the finite harmonic measures of this laminated space correspond to finite invariant measures for the affine group action. Then we give a complete combinatorial description of these finite invariant measures. Finally we give examples with an arbitrary number of ergodic invariant probability measures.

## 1 Introduction

Let  $N$  be either the hyperbolic 2-space  $\mathbb{H}^2$ , identified with the upper half complex plane:  $\{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$  with the metric  $ds^2 = \frac{dx^2 + dy^2}{y^2}$ , or the Euclidean plane  $\mathbb{R}^2$ .

A tiling  $T = \{t_1, \dots, t_n, \dots\}$  of  $N$ , is a collection of convex compact polygons  $t_i$  with geodesic borders, called *tiles*, such that their union is the whole space  $N$ , their interiors are pairwise disjoint and they meet full edge to full edge. Let  $\mathbf{G}$  denote a Lie group of isometries of  $N$  preserving the orientation. A tiling is said of  *$\mathbf{G}$ -finite type* if there exists a finite number of polygons  $\{p_1, \dots, p_n\}$  called *prototiles* such that each  $t_i$  is the image of one of these polygons by an element of  $\mathbf{G}$ . For instance, when  $F$  is a fundamental domain of a discrete cocompact group  $G$  of isometries of  $N$ , then  $\{\gamma(F), \gamma \in G\}$  is a tiling of  $N$ . However the set of finite type tilings is much richer than the one given by discrete cocompact groups. When  $N = \mathbb{R}^2$ , R. Penrose [15] gave an example whose set of prototiles is made with teen rhombi: the Penrose's tiling. When  $N = \mathbb{H}^2$ , Penrose also constructed a finite type tiling made with a single prototile which is not stable for any Fuchsian group. This example is the typical example of tilings studied in this paper. The construction goes as follows.

Let  $P$  be the convex polygon with vertices  $A_p$  with affix  $(p-1)/2 + i$  for  $1 \leq p \leq 3$  and  $A_4 : 2i + 1$  and  $A_5 : 2i$  (see figure 1):  $P$  is a polygon with 5 geodesic edges. Consider the two maps:

$$R : z \mapsto 2z \text{ and } S : z \mapsto z + 1.$$

The hyperbolic Penrose's tiling is defined by  $\mathcal{T} = \{R^k \circ S^n P \mid n, k \in \mathbb{Z}\}$  (see figure 2). This tiling is an example of  $\mathcal{P}$ -finite type tiling where  $\mathcal{P}$  denote the group of *affine maps i.e.* isometries of  $\mathbb{H}^2$  of the kind  $z \mapsto az + b$  with  $a, b$  reals and  $a > 0$ .

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\*Institut de Mathématiques de Bourgogne, U.M.R. 5584 du CNRS, Université de Bourgogne, B.P. 47870-21078 Dijon Cedex France; E-mail: samuel.petite@u-bourgogne.fr

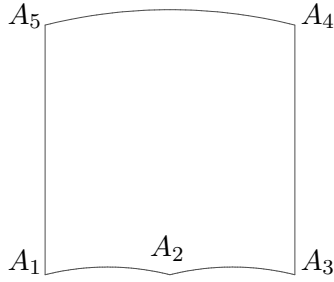


Figure 1: The prototile  $P$

The argument of Penrose is a homological one: he associates with the edge  $A_4A_5$  a positive charge and two negative charges with edges  $A_1A_2$ ,  $A_2A_3$ . If  $\mathcal{T}$  was stable for a Fuchsian group, then  $P$  would tile a compact surface. Since the edge  $A_4A_5$  can meet only the edges  $A_1A_2$  or  $A_2A_3$ , the surface has a neutral charge. This is in contradiction with the fact  $P$  is negatively charged.

G. Margulis and S. Mozes [12] have generalized this construction to build a family of prototiles which cannot be used to tile a compact surface. Notice the group of isometries which preserves  $\mathcal{T}$  is generated by the transformation  $R$ . In order to break this symmetry, it is possible to decorate prototiles to get a new finite type tiling which is not stable for any non trivial isometry (we say in this case that the tiling is *aperiodic*). Using the same procedure, C. Goodman-Strauss [10] construct a set of polygons which can tile  $\mathbb{H}^2$  only in an aperiodic way.

To understand the combinatorial properties of a tiling, it is useful to associate with this tiling, a set of tilings that we can study both from a geometric and dynamical point of view. The image of a  $\mathbf{G}$  finite type tiling  $T$  by an element of  $\mathbf{G}$  is again a  $\mathbf{G}$  finite type tiling. We consider a compact metric space  $\Omega(T)$ , which is the completion of the set of tilings image of  $T$  by elements of  $\mathbf{G}$ , for a natural metrizable topology defined in section 2. The space  $\Omega(T)$  is called the *continuous hull* of  $T$ . The group  $\mathbf{G}$  acts continuously on this space. In this paper we are mainly interested in the situation when the  $\mathbf{G}$ -action on the hull is free (without fixed point). This is the case for the  $\mathcal{P}$ -action on the hulls of examples in [10] as well as for the translation group action on the hull of the Euclidean Penrose's tiling. In this case, the  $\mathbf{G}$ -action induces a specific laminated structure on the hull: a  $\mathbf{G}$ -solenoid structure, where leaves are orbits for the group  $\mathbf{G}$ -action (see section 2). The combinatorics properties of the tiling  $T$  are related to geometrical properties of  $\Omega(T)$  and dynamical properties of  $(\Omega(T), \mathbf{G})$ . In particular, the distribution of tiles of the tiling, which is our main interest for this paper, can be described by the statistical properties of the leaves of the solenoid.

On one hand, these properties can be grasped from a dynamical point of view. When the group  $\mathbf{G}$  is amenable, the  $\mathbf{G}$ -action possesses finite invariant measures. R. Benedetti, J.-M. Gambaudo [2] and L. Sadun [17], show that a  $\mathbf{G}$ -solenoid can be seen as a projective limit  $\lim_{\leftarrow} (\mathcal{B}_n, \pi_n)$  of branched manifold  $\mathcal{B}_n$ . Furthermore, when the group  $\mathbf{G}$  is unimodular (for example when  $N = \mathbb{R}^2$  and  $\mathbf{G}$  is the translation group), authors of [2] prove that the notions of transverse invariant measure, foliated cycle and finite  $\mathbf{G}$  invariant measure, are equivalent. Thanks to this, they characterize finite  $\mathbf{G}$ -invariant measures as the elements of a projective limit of cones in the  $\dim \mathbf{G}$ -homology groups of the branched manifolds  $\mathcal{B}_n$ . When the group  $\mathbf{G}$  is amenable and not unimodular (this is the case when  $\mathbf{G}$  is the affine group  $\mathcal{P}$ ), their

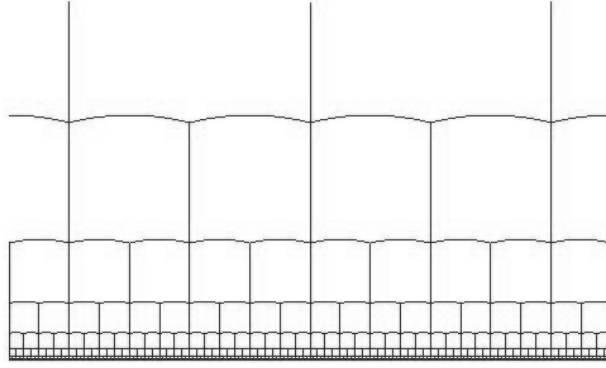


Figure 2: The hyperbolic Penrose's tiling

results do not apply. Actually, we prove that on  $\mathcal{P}$ -solenoid there is no transverse invariant measure (Proposition 3.1).

On the other hand, statistical properties of the leaves can be studied through a geometric point of view. Following the work of L. Garnett [7] on foliations, we can consider harmonic currents on the hull (such currents always exist on laminations). A riemannian metric on the leaves yields a correspondence between harmonic currents and finite harmonic measures and these measures give statistical properties of random path in a leaf of Brownian motions. More particularly, harmonic measures enable to define the average time of a generic path crossing an open subset of the hull. We prove that, for a  $\mathcal{P}$ -solenoid, both geometrical and dynamical approaches are related:

**Theorem 1.1** *A finite measure on a  $\mathcal{P}$ -solenoid is harmonic if and only if it is invariant for the affine group action.*

By using the structure of projective limit  $\lim_{\leftarrow}(\mathcal{B}_n, \pi_n)$  of a  $\mathcal{P}$ -solenoid, we give a characterization of harmonic measures of a  $\mathcal{P}$ -solenoid:

**Theorem 1.2** *There exists a sequence of linear morphisms  $A_n$  such that the set of harmonic measures is isomorphic to the projective limit of cones in 2 chains spaces of branched manifold  $\mathcal{B}_n$ ,  $\lim_{\leftarrow}(\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+, A_n)$ .*

The linear morphisms  $A_n$  will be defined in section 4. We deduce from Theorem 1.2 that the number of ergodic invariant probability measures on the solenoid is bounded from above by the maximal number of faces of the branched manifolds. Finally we prove, by giving explicit examples:

**Proposition 1.3** *For any integer  $r \geq 1$ , there exists a  $\mathcal{P}$ -finite type tiling  $T$  such that the  $\mathcal{P}$ -action on  $\Omega(T)$  is free and minimal (all orbits are dense) and has  $r$  invariant ergodic probability measures.*

This paper is organized as follows. In section 2, we recall some standard background on the tiling spaces, their solenoid structures and their description as projective limits of branched manifolds. Section 3 is devoted to harmonic currents and foliated cycles. We prove

here that there exists no foliated cycle for a  $\mathcal{P}$ -solenoid. In Section 4, we prove Theorem 1.1 and Theorem 1.2. The last section, is devoted to the construction of examples which prove Proposition 1.3.

## 2 Background on tiling spaces

We recall here different useful notions defined in [11] and [2]

### 2.1 Action on the hull

Let  $\mathbf{G}$  be the subgroup of isometries acting transitively, freely and preserving the orientation of the surface  $N$ , thus  $\mathbf{G}$  is a Lie group homeomorphic to  $N$ . The metric on  $N$  gives a left multiplicative invariant metric on  $\mathbf{G}$ . We fix a point  $O$  in  $N$  that we call *origin*.

For a tiling  $T$  of  $\mathbf{G}$  finite type and an isometry  $p$  in  $\mathbf{G}$ , the image of  $T$  by  $p^{-1}$  is again a tiling of  $N$  of finite affine type. We denote by  $T.\mathbf{G}$  the set of tilings which are image of  $T$  by isometries in  $\mathbf{G}$ . The affine group  $\mathbf{G}$  acts on this set by the right action:

$$\begin{aligned} \mathbf{G} \times T.\mathbf{G} &\longrightarrow T.\mathbf{G} \\ (p, T') &\longrightarrow T'.p = p^{-1}(T') \end{aligned}$$

We equip  $T.\mathbf{G}$  with a metrizable topology, finer as one induced by the standard hyperbolic metric. A base of neighborhoods is defined as follows: two tilings are close one of the other if they agree, on a big ball of  $N$  centered at the origin, up to an isometry in  $\mathbf{G}$  close to the identity. This topology can be generated by the metric  $\delta$  on  $T.\mathbf{G}$  defined by :

for  $T$  and  $T'$  be two tilings of  $T.\mathbf{G}$ , let

$$A = \{\epsilon \in [0, 1] \mid \exists g \in B_\epsilon(Id) \subset \mathbf{G} \text{ s.t. } (T.g) \cap B_{1/\epsilon}(O) = T' \cap B_{1/\epsilon}(O)\}$$

where  $B_{1/\epsilon}(O)$  is the set of points  $x \in N$  such that  $d(x, O) < 1/\epsilon$ .

we define :

$$\delta(T, T') = \inf A \text{ if } A \neq \emptyset$$

$$\delta(T, T') = 1 \text{ else.}$$

The *continuous hull* of the tiling  $T$ , is the metric completion of  $T.\mathbf{G}$  for the metric  $\delta$ . We denote it by  $\Omega(T)$ . Actually this space is a set of tilings of  $N$  of  $\mathbf{G}$ -finite type. A *patch* of a tiling  $T$  is a finite set of tiles of  $T$ . It is straightforward to check that patches of tilings in  $\Omega(T)$  are copies of patches of  $T$ . The set  $\Omega(T)$  is then a compact metric set and the action of  $\mathbf{G}$  can be extended to a continuous right action on this space. The dynamical system  $(\Omega(T), \mathbf{G})$  has a dense orbit (the orbit of  $T$ ).

We fix in each prototile *prot* of  $T$ , a marked point  $x_{prot}$  in its interior. Consequently, each tile  $t$  of a tiling  $T' \in \Omega(T)$  admits a distinguished point  $x_t$ . Let  $\Omega_0(T)$  denote the set of tilings of  $\Omega(T)$  such that one  $x_t$  coincides with the origin  $O$ . With the induced topology,  $\Omega_0(T)$  is compact and completely disconnected.

**Definition 2.1** *A tiling  $T$  satisfies the repetitivity condition if for all patch  $P$ , there exists a real  $R(P)$  such that every ball of  $N$  with radius  $R(P)$  intersected with the tiling  $T$  contains a copy of the patch  $P$ .*

This definition can be interpreted from a dynamical point of view (see for instance [11]).

**Proposition 2.2** *The dynamical system  $(\Omega(T), \mathbf{G})$  is minimal (all orbits are dense) if and only if the tiling  $T$  satisfies the repetitivity condition.*

We call a tiling *non-periodic* if the action of  $\mathbf{G}$  on  $\Omega(T)$  is free: for all  $p \neq Id$  of  $\mathbf{G}$  and all tilings  $T'$  of  $\Omega(T)$  we have  $T'.p \neq T'$ . It is straightforward to show that when the stabilizer of  $T$  is reduced to the identity ( $T$  is aperiodic) and  $T$  is repetitive then  $T$  is non periodic. In this case the space  $\Omega_0(T)$  is a Cantor set.

For example when  $N = \mathbb{R}^2$  and  $\mathbf{G}$  is the translation group, the Euclidean Penrose's tiling is a non-periodic repetitive tiling of  $\mathbb{R}^2$  finite type. When  $N = \mathbb{H}^2$  and  $\mathbf{G}$  is the affine group  $\mathcal{P}$ , we saw that the hyperbolic Penrose's tiling is not aperiodic, however we shall construct in the last section examples of repetitive and aperiodic affine finite type tilings (with specific ergodic properties).

## 2.2 Structure of $\mathbf{G}$ -solenoid

### 2.2.1 Solenoids

Let  $M$  be a compact metric space, suppose there exists a covering of  $M$  by open set  $U_i$ , called *boxes*, and homeomorphisms called *charts*  $h_i : U_i \rightarrow V_i \times C_i$  where  $V_i$  is an open set of  $\mathbf{G}$ , considered as a Lie group, and  $C_i$  is a totally disconnected compact metric space. The collection of open set and homeomorphisms  $(U_i, h_i)$  is called an *atlas* of a  $\mathbf{G}$ -solenoid if the *transition map*  $h_{i,j} = h_i \circ h_j^{-1}$ , on their domains of definitions, read:

$$h_{i,j}(x, c) = (f_{i,j}.x, g_{i,j}(c))$$

where  $f_{i,j}.x$  means the multiplication of  $x \in V_j$  with an element  $f_{i,j}$  of  $\mathbf{G}$ , independent of  $x$  and  $c \in C_j$ ; and  $g_{i,j}$  is a continuous map from  $C_j$  to  $C_i$  independent of  $x$ .

Two atlases are *equivalent* if their union is again an atlas. We will call a compact metric space  $M$  with an equivalence class of atlas, a  *$\mathbf{G}$ -solenoid*.

The transition maps structure provides the following important notions:

1. *slices* and *leaves*: a slice is a set of the kind  $h_i^{-1}(V_i \times \{c\})$ . The *leaves* are the union of the slices which intersect. The global space  $M$  is laminated by these leaves. Leaves are differentiable manifolds of dimension 2. A  $\mathbf{G}$ -solenoid  $M$  is called *minimal* if every leaf of  $M$  is dense in  $M$ .
2. *Vertical germs*: it is a set of the kind  $h_i^{-1}(\{x\} \times C_i)$ . Transition maps map vertical germs onto vertical germs, and thus this notion is well defined (independently of the charts).

These transition maps enable to define right multiplication by an element of  $\mathbf{G}$  close to the identity. We suppose furthermore that each leaf is diffeomorphic to  $N$  and that this local  $\mathbf{G}$  right action on a leaf extends to a free  $\mathbf{G}$  right action on  $M$ . Leaves correspond to orbits of the action of  $\mathbf{G}$  by right multiplication. This action is minimal if and only if the  $\mathbf{G}$ -solenoid is minimal.

Furthermore this action has locally constant return times: if an orbit (or a leaf) intersects two verticals  $V$  and  $V'$  at points  $v$  and  $v.g$  where  $g \in \mathbf{G}$ , then for any point  $w$  of  $V$  close enough to  $v$ ,  $w.g$  belongs to  $V'$ .

It turns out that the hull of a tiling has a laminated structure (see for instance É. Ghys [8]). More precisely, in [2] authors prove that the hull  $\Omega(T)$  of a non periodic  $\mathbf{G}$  finite type tiling  $T$ , has a  $\mathbf{G}$ -solenoid structure. The boxes of  $\Omega(T)$  are homeomorphic to spaces  $V_i \times C_i$  where  $V_i$  is an open subset of  $\mathbf{G} \simeq N$  and  $C_i$  is a closed and open subset of  $\Omega_0(T)$ . The charts are the inverse of the maps  $f_i : V_i \times C_i \rightarrow U_i \subset \Omega(T)$  with  $f_i(z, T') = z^{-1}(T')$ .

The action of the group  $\mathbf{G}$  on the solenoid coincides with the action of this group on the hull. This  $\mathbf{G}$ -action is *expansive*: there exists a positive real  $\epsilon$  such that for every points  $T_1$  and  $T_2$  in the same vertical in  $\Omega(T)$ , if  $\delta(T_1.g, T_2.g) < \epsilon$  for every  $g \in \mathbf{G}$ , then  $T_1 = T_2$ .

If furthermore  $T$  verifies the repetitivity condition, the hull  $\Omega(T)$  is minimal, and the transversal in any point in any box is homeomorphic to a Cantor set.

### 2.2.2 Branched manifolds and projective limits

A *box decomposition* of a solenoid  $M$  is a finite collection of charts  $B_1, \dots, B_n$  such that: any two boxes are disjoint and the closure of the union of all boxes is the whole space  $M$ ; furthermore each  $B_i$  is homeomorphic to a space  $V_i \times C_i$ , with  $C_i$  a totally disconnected set and  $V_i$  an open convex geodesic polyhedron in  $N$ . The *vertical boundary* of  $B_i$  is the set homeomorphic to  $\partial V_i \times C_i$ .

The hull of a finite affine type tiling has a natural box decomposition, where boxes are homeomorphic to the product of a prototile of the tiling times a disconnected set. Boxes are sets of tilings having the same tile on the origin. We say that this box decomposition is *associated* to tiles of the tiling.

Let us consider a box decomposition on  $M$ . We consider now the equivalence relation  $\sim$  generated by the relation  $\approx$ :

$$x \approx y \Leftrightarrow x \text{ and } y \text{ belong to the closure of the same box and are in the same vertical.}$$

Let  $B$  be the quotient space  $M/\sim$  and let  $p$  be the projection of  $M$  onto  $B$ . Authors of [2] prove that the set  $B$  with the quotient topology, has a differentiable structure and is a branched manifold, a structure by R. Williams (see [22]). Actually, in the proof of Theorem 1.2 we will only use the simplex structure of  $B$ .

**Example:** consider a non-periodic tiling  $T$  which is a decorated hyperbolic Penrose's tiling (see section 5). The set of prototiles is a finite union of different copies of  $P$ . Let us consider now the box decomposition of  $\Omega(T)$  associated to its prototiles. The quotient space  $\Omega(T)/\sim$  is then homeomorphic to the collapsing of prototiles along edges. Points on prototiles are identified if somewhere, on  $T$ , their copies meet (see [1]). For the Penrose's tiling  $\mathcal{T}$ , this identification leads to a branched manifold  $\mathcal{N}$  homeomorphic to  $P$  with edges identified as follows: edges  $A_1A_2$ ,  $A_2A_3$  and  $A_4A_5$  are identified themselves and edge  $A_4A_1$  is identified with  $A_5A_3$ . This space is homeomorphic to the mapping torus of the application  $x \mapsto 2x \bmod 1$  on the circle  $S^1 \simeq \mathbb{R}/\mathbb{Z}$ . There is a natural projection of  $\Omega(T)/\sim$  onto  $\mathcal{N}$ .

We say that the box decomposition  $\mathbf{B}_2$  is *zoomed out* of the box decomposition  $\mathbf{B}_1$  if:

1. for each point  $x$  in a box  $B_1$  in  $\mathbf{B}_1$  and in a box  $B_2$  in  $\mathbf{B}_2$ , the vertical of  $x$  in  $B_2$  is contained in the vertical of  $x$  in  $B_1$ .
2. the vertical boundaries of the boxes of  $\mathbf{B}_2$  are contained in the vertical boundaries of the boxes of  $\mathbf{B}_1$ .

3. for each box  $B_2$  in  $\mathbf{B}_2$ , there exists a box  $B_1$  in  $\mathbf{B}_1$  such that  $B_1 \cap B_2 \neq \emptyset$  and the vertical boundary of  $B_1$  doesn't intersect the vertical boundary of  $B_2$ .
4. if a vertical in the vertical boundary of a box in  $\mathbf{B}_1$  contains a point in a vertical boundary of a box in  $\mathbf{B}_2$ , then it contains the whole vertical.

A *tower system* of a solenoid  $M$  is a sequence of box decompositions  $(\mathbf{B}_n)_{n \geq 1}$ , such that for any  $n \geq 1$ ,  $\mathbf{B}_{n+1}$  is zoomed out of  $\mathbf{B}_n$ . In [2] it is proved that any  $\mathcal{P}$ -solenoid admits a tower system  $(\mathbf{B}_n)_n$ .

From above, for every  $n$ , there exists a branched manifold  $\mathcal{B}_n$  associated to the box decomposition  $\mathbf{B}_n$  and a projection  $p_n : M \rightarrow \mathcal{B}_n$ . By definition, the set of verticals of boxes of  $\mathbf{B}_{n+1}$  is included in the set of verticals of  $\mathbf{B}_n$ , this induces a natural map  $\pi_n : \mathcal{B}_{n+1} \rightarrow \mathcal{B}_n$  such that  $p_n = \pi_n \circ p_{n+1}$ .

**Theorem 2.3** (R. Benedetti, J.M. Gambaudo) *A  $\mathbf{G}$ -solenoid  $M$ , always posses a tower system  $(\mathbf{B}_n)_{n \geq 1}$ , and  $M$  is homeomorphic to the projective limit  $\lim_{\leftarrow} (\mathcal{B}_n, \pi_n)$*

We recall that  $\lim_{\leftarrow} (\mathcal{B}_n, \pi_n)$  is a subspace of  $\prod \mathcal{B}_n$  defined by  $\{(x_n) \in \prod \mathcal{B}_n \mid x_n = \pi_n(x_{n+1})\}$  and equipped with the topology induced by the product topology.

### 3 Foliated cycles and harmonic currents

#### 3.1 Foliated cycles

The leaves of a  $\mathbf{G}$  solenoid  $M$  carry a 2-manifold structure. Following [8], we call *k-differential form* the data, in any box, of a family of real  $k$ -differential forms ( $\mathcal{C}^\infty$ ) on slices  $V_i \times \{c\}$  which depends continuously of the parameter  $c$  (in the  $\mathcal{C}^\infty$ -topology) and such that each family is mapped onto each other by the transition maps. We denote by  $A^k(M)$  the set of  $k$ -differential forms on  $M$ . The differentiation along leaves gives an operator  $d : A^k(M) \rightarrow A^{k+1}(M)$ .

*Foliated cycles*, introduced by D. Sullivan [20], are a continuous linear forms  $A^2(M) \rightarrow \mathbb{R}$  which are positive on positive forms and vanish on exact forms.

**Proposition 3.1** *A  $\mathcal{P}$ -solenoid does not admit a foliated cycle.*

In order to prove this result, let us introduce the following definition.

**Definition 3.2** *A finite transverse invariant measure on  $M$  is the data of a finite positive measure  $\mu_i$  on each set  $C_i$  such that for any borelian set  $B$  in some  $C_i$  which is contained in the definition set of the transition map  $g_{ij}$  then*

$$\mu_i(B) = \mu_j(g_{ij}(B))$$

The data of a transverse invariant measure for a given atlas provides another invariant transverse measure for any equivalent atlas and thus gives an invariant measure on each verticals. Thus it makes sense to consider a transverse invariant measure  $\mu^t$  of a  $\mathcal{P}$ -solenoid. It turns out that finite transverse invariant measures are in one-to-one correspondence with foliated cycles (also called *Ruelle-Sullivan current*) and that conversely any foliated cycle implied the existence of a transverse invariant measure.

**Proof of Proposition 3.1:** if  $\mu^t$  is a finite invariant transverse measure of a  $\mathcal{P}$ -solenoid  $\Omega$  and  $\lambda$  is a left invariant Haar measure on borelian sets of  $\mathcal{P}$  (for example the measure induced by the standard metric on  $\mathbb{H}^2$ ). We can define a global finite measure  $\mu$  on  $\Omega$  as follows. On a box  $U_i \times C_i$  we consider the product measure  $\lambda \otimes \mu^t$ , which is well defined thanks the invariance properties of considered measures. Up to multiplication by a scalar, we can suppose the measure  $\mu$  is a probability measure on  $\Omega$ . As  $\mathcal{P}$  acts on  $\Omega$ , any element  $g$  of  $\mathcal{P}$  defines an homeomorphism of  $\Omega$  denoted  $\tau_g$ .

Let  $f$  be a continuous function on  $\Omega$  with value in  $\mathbb{R}$  with support included in a box  $B \simeq U \times C$ . Thanks the locally constant return times property, we can decompose  $B$  into a finite disjoint union of boxes  $b_i \simeq U \times C_i$  where  $C_i$  is a closed and open subset of  $C$ , such that  $b_i$  and  $\tau(b_i)$  are included in the same box  $D_i$ . We consider now the probability measure  $\tau_g * \mu$  obtained by the transport of  $\mu$  by  $\tau_g$ . We have

$$\int f d\tau_g * \mu = \sum_i \int_{b_i} f d\tau_g * \mu.$$

In each box  $D_i$ ,  $\int_{b_i} f d\tau_g * \mu = \int_{D_i} f(\tau_g^{-1}(x)) \lambda \otimes \mu^t$ . For a point  $(z, c) \in U \times C_i$ , we have  $\tau_g^{-1}((z, c)) = (z.g^{-1}, c)$  where for  $z = (x, y)$  in  $\mathbb{H}^2$  and  $g^{-1}$  is the transformation  $z \mapsto az + b$ , the point  $z.g^{-1} = (x + by, ay)$ . Therefore we obtain  $\int_{b_i} f d\tau_g * \mu = a \int_{b_i} f d\mu$  and

$$\int f d\tau_g * \mu = a \int f d\mu. \quad (\#)$$

By taking a partition of the unity associated with open sets of an atlas, it is possible to prove the equality  $(\#)$  holds true for any continuous function  $f : \Omega \rightarrow \mathbb{R}$ . Thus the measure  $\tau_g * \mu$  is the measure  $a\mu$ . This is a contradiction with the fact that  $\mu$  is a probability measure.  $\square$

**Remark 1** When the Lie group  $\mathbf{G}$  is unimodular, a  $\mathbf{G}$ -solenoid admits foliated cycles, which are characterized in [2].

**Remark 2** The existence of a foliated cycle is a very strong hypothesis. The non existence of foliated cycle gives information on geometric behavior of leaves. Following J. Plante [16], it implies the exponential growth for every leaf of a  $\mathcal{P}$ -solenoid.

### 3.2 Harmonic currents

Harmonic currents were introduced by L. Garnett in [7]. The Laplacian  $\Delta$  in the leaf direction induces an operator  $\Delta : A^0(M) \rightarrow A^2(M)$  and its image  $(Im\Delta)$  is contained in the space of exact forms. A *harmonic current* is a continuous operator  $A^2(M) \rightarrow \mathbb{R}$  strictly positive on strictly positive form and null on  $Im\Delta$ . Foliated cycles are then specific harmonic current. Any lamination and in particular any  $\mathbf{G}$ -solenoid admits a harmonic current ([7]).

As for foliated cycles it is possible to associate to a harmonic current  $I$  a finite positive measure on  $M$ . We choose a metric on the tangent bundle of  $M$ . This defined a 2 differential form along the leaves, which enables us to identify  $A^2(M)$  with the space of  $(\mathcal{C}^\infty)$  functions on  $M$ . Thanks to the positivity of  $I$ , it can be extended to a linear form on space of functions on  $M$  and it defines then a finite positive measure  $\mu$  on  $M$ . These measures  $\mu$  are called *harmonic measures* and are characterized by the following property. For any bounded



measurable function  $f$  on  $M$ , smooth in the leaf direction, the integral  $\int \Delta f d\mu$  is null, where  $\Delta$  denotes the Laplacian in the leaf direction.

L. Garnett [7] gives the local structure of such measures. In a box  $U_i \simeq V_i \times C_i$  a harmonic measure  $\mu$  disintegrates into a probability measure  $\nu_i$  on  $C_i$  times the measure  $f_i(z, c)dz$  where  $dz$  denotes the Riemannian leaf measure and  $f_i : V_i \times C_i \rightarrow \mathbb{R}^+$  denotes a function defined for almost all  $c$  of  $C_i$  and harmonic on all the slices  $V_i \times \{c\}$ . Thus for any Borelian  $B$  included in  $U_i$ :

$$\mu(B) = \iint_B f_i(z, c) dz d\nu_i(c)$$

This local decomposition is not unique. If two decompositions  $\mu_i, f_i$  and  $\mu'_i, f'_i$  define the same measure, then it exists a measurable application  $\delta_i : C_i \rightarrow \mathbb{R}_*^+$  such that  $\mu_i = \delta_i^{-1}(c)\mu'_i$  and  $f_i(z, c) = \delta_i(c)f'_i(z, c)$ .

Thus if we fix an atlas of  $M$ , harmonic functions  $f_i(z, c)$  defined on slices are equal on intersecting slices up to a positive constant. Since in our case, leaves have no topology, it is possible to extend the harmonic function  $f_i(z, c)$  defined on a slice, into a positive harmonic function on all the leaf.

**Remark 3** *For a  $\mathbb{R}^2$ -solenoid, leaves are then homeomorphic to the plane. The harmonic function obtained is positive and defined on all the plane then it is a constant function. The harmonic measure associated with is then locally disintegrated into a measure  $\mu_i$  on  $C_i$  times the Riemannian measure, and  $\mu_i$  is then a transverse invariant measure.*

### 3.3 Harmonic measures and ergodic theorem

Let  $x \in M$  be a point of the hull and let  $\Gamma_x$  be the set  $\{\gamma : \mathbb{R}^+ \rightarrow L_x \text{ continuous } |\gamma(0) = x, \gamma(\mathbb{R}^+) \subset L_x\}$  where  $L_x$  is the leaf passing through  $x$ . The set  $\Gamma_x$  is the set of continuous path beginning at  $x$  and strictly include in  $L_x$ . We equip this set with the topology of uniform convergence on compact sets. On the space of borelians, there exists a natural finite measure  $w_x$  called the *Wiener* measure. This measure is defined so that the motion  $\Gamma_x \times \mathbb{R}^+ : (\gamma, t) \mapsto \gamma(t) \in L_x$  is a brownian motion.

Let  $\Gamma = \bigsqcup_{x \in \Omega(T)} \Gamma_x$  be the set of continuous paths of  $M$  strictly included in leaves, we equip again this set with the topology of uniform convergence on compact sets. If  $\mu$  is a finite measure on  $M$ , then  $\bar{\mu} = w_x \otimes \mu(x)$  is a finite measure on  $\Gamma$ .

The semi-group  $\mathbb{R}^+$  acts on the space  $\Gamma$  by time translations: for  $\tau > 0$  and  $\gamma \in \Gamma$  we define the semi-group of transformations  $S_\tau$  with  $S_\tau(\gamma)(s) = \gamma(s + \tau)$ . It is straightforward to check transformations  $S_\tau$  preserve  $\bar{\mu}$  if and only if  $\mu$  is a harmonic measure. This comes the Wiener measure is built with the heat kernel. For a harmonic measure  $\mu$ , we can apply the Birkhoff ergodic theorem.

**Theorem 3.3** (L. Garnett) *For any bounded continuous function  $f$  from  $M$  to  $\mathbb{R}$  the limit  $l(x, \gamma) = \lim_{n \rightarrow \infty} 1/n \sum_{i=0}^{n-1} f(\gamma(i))$  exists for  $\mu$  almost all points  $x$  and  $w_x$  almost all paths  $\gamma$  of  $\Gamma_x$ .*

*This limit is constant on leaves of  $M$  and  $l(x, \gamma)$  is constant for  $w_x$  almost path  $\gamma$ .*

*Furthermore  $\int l(x) d\mu(x) = \int f(x) d\mu(x)$ .*

Thanks to this theorem, we can define the average time of a generic path  $\gamma$  crossing a Borelian set  $B$  of  $M$  [8]. It is the limit  $\lim_{T \rightarrow \infty} 1/T \int_0^T \chi_B(\gamma(t)) dt$  where  $dt$  denotes the Lebesgue measure and  $\chi_B$  the indicative function of  $B$ .

## 4 Invariant measures for the action

In this section we characterize invariant measures for the  $\mathcal{P}$ -action on a  $\mathcal{P}$ -solenoid  $M$ .

### 4.1 Proof of Theorem 1.1

These measures are defined on the Borel tribe of the hull  $M$ . A measure  $m$  is *invariant* if for any  $g \in \mathcal{P}$  and any measurable set  $B \subset M$ ,  $m(B.g) = m(B)$ .

Since the group  $\mathcal{P}$  is the extension of two Abelian groups,  $\mathcal{P}$  is amenable, and the set of invariant measures is a closed non-empty set for the weak topology. Actually, for a  $\mathcal{P}$ -solenoid invariant measures and harmonic measures are the same (Theorem 1.1).

First let us prove that a harmonic measure of  $M$  is an invariant finite measure for the  $\mathcal{P}$ -action. We will use the lemma:

**Lemma 4.1** *Let  $H : \mathbb{H}^2 \rightarrow \mathbb{R}$  be a positive harmonic map. If the quotient  $\frac{H(x,y)}{y}$  is uniformly bounded, then  $H(x,y) = \alpha y$  for some real  $\alpha$ .*

**Proof :** It is a consequence of the Pick's formula (see [4] for example). Any positive harmonic map  $H$  reads  $H(x,y) = \alpha y + \int_{-\infty}^{\infty} \frac{y}{(s-x)^2 + y^2} d\lambda(s)$  where  $\lambda$  is a positive measure on  $\mathbb{R}$  defined for any real  $a < b$  by:

$$\lambda([a,b]) = \lim_{y \rightarrow 0} \frac{1}{b-a} \int_{x=a}^b H(x,y) dx$$

and  $dx$  denotes here the standard Lebesgue measure on the real line. The fact the quotient  $\frac{H(x,y)}{y}$  is uniformly bounded implies the measure  $\lambda$  is null.  $\square$

Let  $\mu$  be a harmonic measure of  $M$  and let  $\phi$  be a continuous positive function with support included in a box  $B \simeq U \times C$  of  $M$ . We identify the Lie group  $\mathcal{P}$  with  $\mathbb{H}^2$  and consider the function  $F : \mathcal{P} \rightarrow \mathbb{R}$  defined by  $F(\tau) = \int \phi d(\tau * \mu)$  where  $\tau * \mu$  denotes the measure transported via the action of  $\tau$ . Fix an element  $\tau$  of  $\mathcal{P}$  and a small positive real  $\epsilon$ . Thanks the locally constant return times property, we can decompose  $B$  into a finite disjoint union of boxes  $b_i \simeq U \times C_i$  with  $C_i$  a closed and open subset of  $C$  with a diameter smaller than  $\epsilon$ ; such that for each  $i$ ,  $b_i$  and  $b_i.\tau^{-1}$  are included in a same box  $D_i$ . By taking  $\epsilon$  small enough, for every element  $g$  of a neighborhood of  $\tau$ , we have also that  $b_i$  and  $b_i.g^{-1}$  are included in  $D_i$ .

Therefore

$$F(\tau) = \sum_i \int_{b_i} \phi d\tau * \mu$$

In each box  $D_i$ , the measure  $\mu$  reads  $f_i(z,t) dz d\nu_i(t)$  with  $f_i$  a harmonic map in  $z$ . Then

$$\begin{aligned} \int_{b_i} \phi dg * \mu &= \int_{D_i} \phi(z.g^{-1}, t) f_i(z, t) dz d\nu_i \\ &= \int_{D_i} \phi(z, t) f_i(z.g, t) \frac{dz}{a} d\nu_i \end{aligned}$$

where  $g$  is the map  $z \mapsto az + b$ . We recall here for  $z = (x, y)$  in  $\mathbb{H}^2$ ,  $z.g = (x + by, ay)$ .

As shown in section 3.2, the map  $f_i(., t)$  for a fixed  $t$ , can be extended to a harmonic map on the whole half plane  $\mathbb{H}^2$ . The map  $g \mapsto f_i(z.g, t)$  is defined on  $\mathcal{P}$  and it is straightforward

to check it is a harmonic map. Thus the bounded map  $g \in \mathbb{H}^2 \rightarrow \int_{b_i} \phi dg * \mu \in \mathbb{R}$  reads  $(x, y) \mapsto \frac{H(x, y)}{y}$  with  $H$  a positive harmonic map. The lemma 4.1 enables us to conclude the function  $F$  is constant.

For a continuous function  $\phi$ , by taking a partition of the unity associated with a cover of  $M$  by the open set of an atlas, we can prove the value  $\int \phi d(\tau * \mu)$  is independent of  $\tau$ , this concludes the first part of the proof.

Conversely let us prove that finite invariant measures are harmonic measures. This can be seen in the local expression of an invariant measure.

**Lemma 4.2** *If a measure  $m$  on  $M$  is an invariant measure for the right  $\mathcal{P}$ -action then in each box, the measure  $m$  disintegrates into a transversal sum of leaf measures, where almost every leaf measure is a right invariant Haar measure of  $\mathcal{P}$ .*

**Proof :** Fix a box  $V \times C$ , we decompose  $m$  in this box into a transversal measure  $\nu$  on  $C$  and a system of leaf measure  $\sigma_c$  on  $V \times \{c\}$  for each  $c$  of  $C$ . Hence we have for any measurable function  $f$  with support included in the box,

$$\int f dm = \int_C \int_V f(z, c) d\sigma_c(z) d\nu(c).$$

We fix a point  $x$  of the box and a closed neighborhood  $K$  included in the box. Let  $A$  be the set of bounded measurable functions with support in  $K$ . If  $m$  is  $\mathcal{P}$ -invariant for any  $f \in A$  and for any  $g \in \mathcal{P}$  s.t.  $K.g$  is included in the box,  $\int f(x) - f(x.g) dm(x) = 0$ .

We can decompose  $f = f_1 + f_2$  where  $f_1$  is the restriction of  $f$  to slices for which  $\int_V f(x) - f(x.g) d\sigma_c > 0$ ; and  $f_2$  is the restriction of  $f$  to slices for which the integral is negative. If  $m$  is invariant, then  $\int f_i(x) - f_i(x.g) dm(x) = 0$  and thus

$$\nu\{c \in C \mid \int_V f_i(x) - f_i(x.g) d\sigma_c \neq 0\} = 0 \quad \text{for } i = 1, 2.$$

It follows that when  $m$  is invariant, for  $\nu$  almost all  $c$  in  $C$ ,  $\int f(x) - f(x.g) d\sigma_c = 0$ . Therefore, by identifying the leaf with the Lie group  $\mathcal{P}$ , for  $\nu$  almost all  $c$ ,  $\sigma_c$  is a right invariant Haar measure.  $\square$

When identifying the Lie group  $\mathcal{P}$  with  $\mathbb{H}^2$ , a right invariant measure reads  $\frac{\lambda}{y} dx dy$  for some constant  $\lambda > 0$ . Therefore an invariant measure  $m$  on  $M$  can be written in a box  $\lambda_c \frac{dx dy}{y} d\nu(c)$ , where  $c \in C \mapsto \lambda_c \in \mathbb{R}^+$  is a measurable map. Then the measure  $m$  is harmonic. This ends the proof of Theorem 1.1.

As we know, the local decomposition of an invariant measure  $m$  is not unique. If  $\frac{\lambda_c}{y} dx dy d\nu(c)$  and  $\frac{\lambda'_c}{y} dx dy d\nu'(c)$  are two decompositions of the same measure  $m$ , the measures  $\nu$  and  $\nu'$  are in the same class, and thus there exists a positive measurable map defined almost everywhere  $\delta : C \rightarrow \mathbb{R}_*^+$  such that  $\nu = \frac{1}{\delta(\cdot)} \nu'$  and  $\lambda_c = \delta(c) \lambda'_c$ .

An important consequence is that the value  $\int_C \lambda_c d\sigma(c)$  is well defined. Consider  $f$  the positive function  $\mathbb{H}^2 \rightarrow \mathbb{R}$  defined by  $f(x, y) = \int_C \lambda_c d\sigma(c).y$ , then the measure of a cylinder  $A \times C$  (where  $A$  is a measurable set of  $V$ ) of the box is  $m(A \times C) = \int_A f(x, y) \frac{dx dy}{y^2}$ . We will use this function to characterize invariant measures.

## 4.2 Combinatorics of the invariant measures

For a branched manifold  $\mathcal{B}$ , let us denote by  $\mathcal{C}_2(\mathcal{B}, \mathbb{R})$  the finite dimensional  $\mathbb{R}$  module space with basis the 2 faces of  $\mathcal{B}$ . Its elements are called 2 *chains*. Let  $\mathcal{C}_2(\mathcal{B}, \mathbb{R})^+$  be the cone of vectors of  $\mathcal{C}_2(\mathcal{B}, \mathbb{R})$  with positive coefficients, and let  $\mathbf{P}(\mathcal{B}, \mathbb{R})$  be the intersection of  $\mathcal{C}_2(\mathcal{B}, \mathbb{R})^+$  and the closed unit sphere centered in the origin for the norm  $|(b_1, \dots, b_q)|_1 = \sum_i |b_i|$ . We denote by  $\mathcal{M}(M)$  the set of finite positive measure of  $M$  invariant for the  $\mathcal{P}$ -action.

We consider first a box decomposition of the  $\mathcal{P}$ -solenoid  $M$ . With each box  $B$  and for an invariant measure  $m$ , we have seen that we can associate a non negative number  $b = \int_C \lambda_c d\sigma(c) > 0$ . The identification of elements belonging to the same vertical of the box decomposition leads to a fibration  $p$  of  $M$  over a branched manifold  $\mathcal{B}$ . We associate to the interior  $F_i$  of a 2-face of  $\mathcal{B}$  a box  $B_i = p^{-1}(F_i)$  with the fibration and then we consider the 2-chain  $\sum_i b_i \overline{F_i} \in \mathcal{C}_2(\mathcal{B}, \mathbb{R})^+$ . Therefore the fibration  $p : M \rightarrow \mathcal{B}$  induces a linear map  $p_* : \mathcal{M}(M) \rightarrow \mathcal{C}_2(\mathcal{B}, \mathbb{R})^+$ .

If we consider now a tower decomposition  $(\mathbf{B}_n)_n$ , we obtain a sequence of fibration  $p_n$  over branched manifolds  $\mathcal{B}_n$  and a sequence of map  $\pi_n : \mathcal{B}_{n+1} \rightarrow \mathcal{B}_n$  such that  $p_n = \pi_n \circ p_{n+1}$  and  $M \simeq \lim_{\leftarrow} (\mathcal{B}_n, \pi_n)$ . These maps induce linear maps  $(p_n)_* : \mathcal{M}(M) \rightarrow \mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+$ .

The relation between  $(p_n)_*(m)$  and  $(p_{n+1})_*(m)$  can be described as follows. We denote by  $B_i^n \simeq F_i^n \times C_i^n$  the boxes of  $\mathbf{B}_n$ , where the index  $i$  is an enumeration of these boxes. Let  $f_i(x, y)$  be the function  $(x, y) \mapsto \int_{C_i^n} \lambda_{ic}^n d\sigma_i^n(c) \cdot y = b_i^n y$  for a local decomposition of the measure  $m$ . The intersection of  $B_i^n$  and  $B_j^{n+1}$  is either empty or a disjoint union of boxes  $\bigsqcup_l D_{ij}^l$ . In the non trivial case, there exists transition maps  $h_{ij}^l : D_{ij}^l \cap B_i^n \rightarrow B_j^{n+1}$ , with  $h_{ij}^l(z, c) = (g_{ij}^l \cdot z, \gamma_{ij}(c))$  and  $g_{ij}^l \in \mathcal{P}$ .

Thus for any cylinder  $A \times C_i^n$  of  $B_i^n$  we have

$$\begin{aligned} m(A \times C_i^n) &= \sum_j \sum_l m(h_{ij}^l((A \times C_i^n) \cap D_{ij}^l)) \\ &= \sum_j \sum_l \int_{g_{ij}^l(A)} f_j(x, y) \frac{dx dy}{y^2} \\ &= \sum_j \sum_l \int_A \alpha(g_{ij}^l) b_j^{n+1} \frac{dx dy}{y} \end{aligned}$$

where  $\alpha$  is the morphism  $\alpha(z \mapsto az + b) = a$

$$= \sum_j \sum_l \alpha(g_{ij}^l) \int_A f_j(x, y) \frac{dx dy}{y^2}.$$

Since this is true for any  $A \subset V_i^n$ , we have the relation:

$$b_i^n = \sum_j \sum_l \alpha(g_{ij}^l) b_j^{n+1} = \sum_j b_j^{n+1} \sum_l \alpha(g_{ij}^l).$$

Let us denote  $p(n)$  the dimension of  $\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})$  and  $A_n$  the  $p(n) \times p(n+1)$  matrix with positive coefficients  $a_{i,j}^n = \sum_l \alpha(g_{ij}^l)$  when  $B_i^n$  and  $B_j^{n+1}$  intersect and 0 otherwise. We have the relation  $(p_n)_*(m) = A_n((p_{n+1})_*(m))$ , and thus the sequence  $((p_n)_*(m))_n$  is an element of  $\lim_{\leftarrow} (\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+, A_n)$ . This enables us to extend maps  $(p_n)_*$  to a map

$$p_* : \mathcal{M}(M) \rightarrow \lim_{\leftarrow} (\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+, A_n).$$

It is obvious that  $p_*$  maps the set of probability invariant measures to the set  $\lim_{\leftarrow}(\mathbf{P}(\mathcal{B}_n, \mathbb{R}), A_n)$ .

Actually this linear map is an isomorphism whose inverse can be constructed as follows. Let  $(v_n)_n$  be an element of  $\lim_{\leftarrow}(\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+, A_n)$ . We consider the family of cylinder  $A$  such that there exists a box  $B_i^n \simeq V_i^n \times C_i^n$  where  $A \subset B_i^n$  and  $A \simeq A_i^n \times C_i^n$  for some measurable subset  $A_i^n$  of  $V_i^n$ . Let  $m(A)$  be the value  $\int_{A_i^n} b_i^n \frac{dx dy}{y}$  where  $v_n = (b_1^n, \dots, b_i^n, \dots, b_{p(n)}^n)$ . Thanks to the relations between  $v_n$  and  $v_{n+1}$ , the value  $m(A)$  is well defined and can be extended by additivity to the  $\sigma$ -algebra generated by cylinders  $A$ . This set is big enough so that its  $\sigma$ -algebra is actually the Borel tribe. It is then straightforward to check that  $p_*(m) = (v_n)_n$ . Furthermore, since  $m$  disintegrates locally into a transverse measure times a measure of the kind  $by \frac{dx dy}{y^2}$  on the slices,  $m$  is a harmonic measure, then from Theorem 1.1  $m$  is also an invariant measure.

The above result can be summarized in the following theorem which is an explicit reformulation of Theorem 1.2 :

**Theorem 4.3** *If  $M$  is a  $\mathcal{P}$  solenoid and  $M$  is homeomorphic to a projective limit of branched manifolds  $\mathcal{B}_n$ ,  $\lim_{\leftarrow}(\mathcal{B}_n, p_n)$ .*

*Then:  $\mathcal{M}(M)$  is homeomorphic to*

*$\lim_{\leftarrow}(\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+, A_n)$ , where  $A_n$  is a matrix with positive coefficients  $A_n : \mathbb{R}^{p(n+1)} \rightarrow \mathbb{R}^{p(n)}$  with  $\dim \mathcal{C}_2(\mathcal{B}_n, \mathbb{R}) = p(n)$ .*

*The restriction to the set of invariant probability measure is then homeomorphic to  $\lim_{\leftarrow}(\mathbf{P}(\mathcal{B}_n, \mathbb{R})^+, A_n)$ .*

This last theorem allows us to exhibit some criteria to bound the number of invariant probabilities.

**Proposition 4.4** *With the same conditions as in Theorem 4.3.*

1. *If the number of faces of  $\mathcal{B}_n$  are uniformly bounded by  $N$ , then there is at most  $N$  ergodic invariant probability measures.*
2. *If furthermore  $M$  is minimal and the linear map  $A_n$  are uniformly bounded, then there is a unique invariant probability measure.*

**Proof :** Without loose of generality, we may assume that for all  $n \geq 1$ ,  $\dim_{\mathbb{R}} \mathcal{C}_2(\mathcal{B}_n, \mathbb{R}) = N$ . Let us consider  $N$  sequences  $(w_j^n)_n \in \Pi_n \mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+$  for  $j \in \{1, \dots, N\}$  where  $w_j^n = (w_{j,1}^n, \dots, w_{j,i}^n, \dots, w_{j,N}^n)$  and  $w_{j,i}^n = 0$  if  $j \neq i$  and 1 otherwise.

Fix an integer  $n$ , for any  $j$  in  $\{1, \dots, N\}$  and  $m > n$  let  $w_j^{nm} = A_n \circ \dots \circ A_{m-1}(w_j^m)$ . Up to a choice of a subsequence, we can suppose that the sequences  $(w_j^{nm})_{m>n}$  converge to  $w_j \in \mathbf{P}(\mathcal{B}_1, \mathbb{R})$ . Let us denote  $proj_n$  the projection of the product  $\Pi_n \mathcal{C}_2(\mathcal{B}_n, \mathbb{R})$  onto  $\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})$ , and  $Prob_n = proj_n(\lim_{\leftarrow}(\mathbf{P}(\mathcal{B}_n, \mathbb{R}), A_n))$ . The set  $Prob_n$  is a convex set and if  $H_m$  is the convex hull of  $\{w_j^{nm} | j = 1, \dots, N\}$ , we have  $Prob_n = \bigcap_{m>n} A_n \circ \dots \circ A_{m-1}(H_m)$ . Therefore  $Prob_n$  is the convex hull of  $\{w_j | j = 1, \dots, N\}$ . Suppose now there is more than  $N$  ergodic invariant probabilities then for  $n$  big enough, there would be more than  $N$  extremal points in  $Prob_n$ , a contradiction.

In order to prove the second statement, we show that for any  $n$ ,  $Prob_n$  is reduced to a point. For this we define the hyperbolic distance between two points  $x, y$  in  $\mathbf{P}(\mathcal{B}_n, \mathbb{R})$ .

$$d_h(x, y) = -\ln \frac{(m+l).(m+r)}{l.r}$$

where  $m$  is the Euclidean length of the segment  $[x, y]$  and  $l, r$  are the length of connected components of  $S \setminus [x, y]$  where  $S$  is the largest line segment containing  $[x, y]$  in  $\mathbf{P}(\mathcal{B}_n, \mathbb{R})$ . It is straightforward to check positive matrices contract this distance and the minimality of the action implies the positivity of matrices. Since linear maps  $A_n$  are uniformly bounded and defined on space with bounded dimension, the contraction is uniform. Therefore  $Prob_n = \bigcap_{m>n} A_n \circ \dots \circ A_{m-1}(\mathbf{P}(\mathcal{B}_m, \mathbb{R}))$  is reduced to a point.  $\square$

## 5 Examples and proof of Proposition 1.3

We give an example of a non periodic repetitive  $\mathcal{P}$  finite type tiling with exactly  $r$  ergodic invariant probability measures, for any integer  $r > 0$ .

The idea is to decorate the Penrose's tiling with a non periodic bi-infinite sequence. We choose a sequence such that the action of the shift on the closure  $X$  of the orbit for the action, is minimal and has  $r$  ergodic invariant probability measures.

First, consider the case  $r \geq 2$ . Let  $\Sigma$  be the set  $\{1, \dots, r\}$ . We associate to each symbol in  $\Sigma$  a different color. Let  $P$  be the polygon defined in the introduction to build the Penrose's tiling. Let  $R$  and  $S$  be the affine maps defined in the introduction. For an element  $i$  of  $\Sigma$ , let  $P_i$  be the prototile  $P$  painted in the color  $i$ . To a sequence  $w = (w_k)_{k \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ , we associate the decorated tiling  $\mathcal{T}(w)$  of finite affine type, with prototiles  $P_i$  for  $i$  in  $\Sigma$ , defined by

$$\mathcal{T}(w) = \{R^q \circ S^n(P_{w_q}) | n, q \in \mathbb{Z}\}.$$

Its tiles are isometric to  $P$  and its stabilizer is included in  $\langle R \rangle$ . To a sequence  $(w_n)_{n \in \mathbb{Z}}$  the shift  $\sigma$  associates the sequence  $(w'_n)_{n \in \mathbb{Z}}$  where  $w'_n = w_{n+1}$ . Thus we have  $\mathcal{T}(w).R = \mathcal{T}(\sigma(w))$ . Therefore if the sequence  $w$  is not periodic for the action of the shift, then  $\mathcal{T}(w)$  is not stable for any element of  $\mathcal{P}$ .

The product space  $\Sigma^{\mathbb{Z}}$  is equipped with the product topology and is a Cantor set. Let  $X$  denote the closure of the orbit of  $w$  by the action of the shift  $\sigma$ :  $X = \overline{\{\sigma^n(w), n \in \mathbb{Z}\}}$ . The set  $X$  is a compact metric space stable under the action of  $\sigma$ . When the dynamical system  $(X, \sigma)$  is minimal then  $\Omega(\mathcal{T}(w))$  is minimal.

In [23], S. Williams generalizes an example of J. C. Oxtoby ([14]) and defines a Toeplitz sequence  $w \in \Sigma^{\mathbb{Z}}$  for which the action of the shift is minimal and has  $r$  ergodic probability measures. We recall here the definition of this sequence.

Consider the sequence of natural numbers  $(p_i)_{i \in \mathbb{N}}$  with  $p_0 = 3$  and  $p_{i+1} = 3^i \cdot p_i$  and the sequence  $s_i \equiv i \pmod{r} \in \Sigma$  for  $i \in \mathbb{N}$ .

Define then the sequence  $w = (w_q)_{q \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$  by inductive steps. The first step (step 1) is to set  $w_q = s_1$  for all  $q \equiv 0$  or  $-1 \pmod{p_1}$ . In general for  $i \in \mathbb{N}$ ,  $k$  in  $\mathbb{Z}$ , let  $J(i, k)$  denote the set of integers  $q \in [kp_i, (k+1)p_i)$  for which  $w_q$  has been not yet defined at the end of the step  $i$ . The step  $(i+1)$  is to set  $w_q = s_{i+1}$  for  $q \in J(i, k)$  with  $k \equiv -1$  or  $0 \pmod{3^i}$ .

The dynamical system  $(X, \sigma)$  is minimal and  $X$  is a Cantor set.

Let us define now a sequence of atlas of words for the sequence  $w$ . Let  $\mathcal{A}_0$  be the set of words  $\{s_i, i = 1 \dots r\}$ . Let  $\mathcal{A}_1$  be the set of words  $\{s_1 s_i^{p_1-2} s_1, i = 1, \dots, r\}$ , where

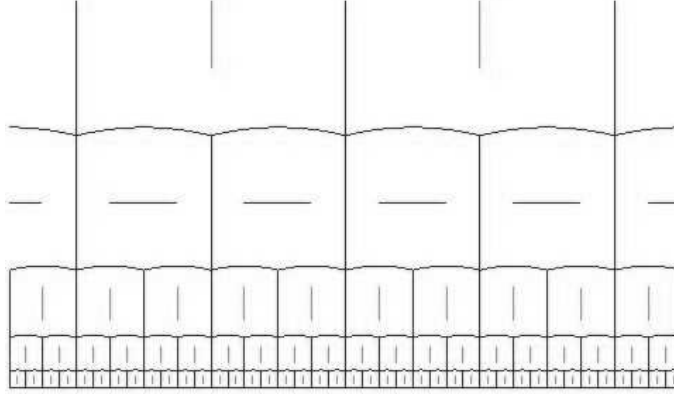


Figure 3: Decorated Penrose's tiling associated to an Oxtoby's sequence

for two words  $a$  and  $b$ ,  $ab$  denotes the concatenation of the two words and  $a^q$  denotes the concatenation of  $q$  times the word  $a$ . In the general case for any integer  $q \geq 1$ , we denote by  $p_{q,i}$   $i \in \{1, \dots, r\}$  the word of  $\mathcal{A}_q$  indexed by  $i$  and for  $q > 1$ ,  $\mathcal{A}_q$  is the set of words  $\{p_{q-1,s_q}(p_{q-1,i})^{3^{q-1}-2}p_{q-1,s_q}, i = 1, \dots, r\}$ . For any  $q \in \mathbb{N}$  the sequence  $w$  is a bi-infinite sequence of words of  $\mathcal{A}_q$ .

The *suspension* of the action of  $\sigma$  on  $X$ , is the quotient space  $\mathcal{X} = \mathbb{R} \times X / \sigma$  where points  $(t, x)$  and  $(s, x')$  are identified if  $s - t \in \mathbb{Z}$  and  $x = \sigma^{s-t}(x')$ . The natural  $\mathbb{R}$ -action by time translation on the space  $\mathbb{R} \times X$  induces a  $\mathbb{R}$ -action on the suspension. It turns out that the suspension  $\mathbb{R} \times X / \sigma$  is a  $\mathbb{R}$ -solenoid ([2]) which has exactly  $r$  invariant ergodic probability measures ([23]). For any  $q \geq 0$ ,  $\mathcal{A}_q$  defines a box decomposition of the suspension  $\mathcal{X}$ . Each box is identified with a unique word of  $\mathcal{A}_q$ .

We will construct a tower system for  $\Omega(\mathcal{T}(w))$  associated to the former box decompositions of the suspension, thanks to a collection of patches for the tiling  $\mathcal{T}(w)$ . For a word  $b = w_{i_0} \dots w_{i_0+l}$  of  $w$ , let  $\mathcal{P}a(b)$  be the patch  $\bigcup_{j=0}^l \{R^{-j} \circ S^k(P_{w_{i_0+j}}) \text{ for } k = 0, \dots, j\}$  of  $\mathcal{T}(w)$ . Now let us consider for  $q \geq 0$  the collection of patches  $\mathcal{P}a_q = \{\mathcal{P}a(p_{q,i}), \text{ for } i = 1, \dots, r\}$ . For any  $q$ , the tiling  $\mathcal{T}(w)$  is an union of elements of  $\mathcal{P}a_q$ , copies of patches meeting only on their borders. Remark that all the patches of  $\mathcal{P}a_q$  have the same size and actually, the box decompositions of  $\Omega(\mathcal{T}(w))$  associated to  $\mathcal{P}a_q$  define a tower system of the hull.

If we denote by  $\sim_q$  the relation generated by the identification of borders of patches of  $\mathcal{P}a_q$  which meet somewhere in the tiling  $\mathcal{T}(w)$  and  $\mathcal{B}_q = \bigsqcup_{i=1}^r \mathcal{P}a_{p_{q,i}} / \sim_q$ , we have applications  $\pi_q$  such that:

$$\Omega(\mathcal{T}(w)) \simeq \varprojlim (\mathcal{B}_q, \pi_q).$$

Now we construct a natural continuous map  $h$  from  $\Omega(\mathcal{T}(w))$  onto  $\mathcal{X}$ . For an element  $g : z \mapsto az + b$  of the group  $\mathcal{P}$ , we define  $h(\mathcal{T}(w).g) = [(\log_2(a), w)] \in \mathcal{X}$  where  $[(t, x)]$  denotes the class of the element  $(t, x)$  in  $\mathbb{R} \times X$  for the relation defined by  $\sigma$ . The map  $h$  is then a continuous map from  $\mathcal{T}(w).\mathcal{P}$  to  $\mathcal{X}$ . Remark that if the origin  $O$  lies in a copy of a patch  $\mathcal{P}a(p_{q,i})$  for some  $q \geq 1$  and  $i \in \Sigma$  in the tiling  $\mathcal{T}(w).g$ , then  $O$  lies also in a copy of the patch  $\mathcal{P}a(p_{q,i})$  in the tiling  $\mathcal{T}(\sigma^n(w))$ , where  $n$  denotes the integer part of  $\log_2(a)$ . Thus the origin of the sequence  $\sigma^n(w)$  lies in the word  $p_{q,i}$ . As  $h(\mathcal{T}(w).g) = [(\log_2(a) - n, \sigma^n(w))]$ , we get

that  $h(\mathcal{T}(w).g)$  is in the box of the suspension defined by the word  $p_{q,i}$ . It follows that for any  $q \geq 1$ , the map  $h$  sends the restriction to the orbit of  $\mathcal{T}(w)$  of the box associated to the patch  $\mathcal{P}a(p_{q,i})$  to the box of the suspension associated to the word  $p_{q,i}$ . Thus the map  $h$  is uniformly continuous.

It follows that  $h$  can be extended to a map from  $\Omega(\mathcal{T}(w))$  onto  $\mathcal{X}$  also denoted  $h$ . It is straightforward to check that each fiber of the map  $h$  is stable under the action of the group  $\mathcal{N} = \{z \mapsto z + t, t \in \mathbb{R}\}$ . Furthermore, as  $\mathcal{P}$  is an extension over  $\mathcal{N}$  and the the group  $\{z \mapsto az, a > 0\}$ , the action of the group  $\mathcal{P}$  preserves the set of fibers. Then the  $\mathcal{P}$ -action on the hull  $\Omega(\mathcal{T}(w))$  defines through the application  $h$ , a  $\mathcal{P}$ -action on the suspension  $\mathcal{X}$  and  $h$  is a semi-conjugacy from the hull  $\Omega(\mathcal{T}(w))$  to  $\mathcal{X}$ . The group  $\mathcal{N}$  acts trivially on  $\mathcal{X}$ . The invariant measures for the  $\mathcal{P}$ -action on  $\mathcal{X}$  are the invariant measures for the  $\mathbb{R}$ -action. We claim that the map  $h$  sends the invariant measures of the hull onto the invariant measures of the suspension.

To prove this, we use a Følner's base of  $\mathcal{P}$  that we denote  $(A_n)_n$  and a right multiplicative invariant Haar measure on  $\mathcal{P}$  that we denote  $\lambda$ . Let  $\mu$  be a ergodic invariant probability measure for the  $\mathcal{P}$ -action on  $\mathcal{X}$ . By the ergodic theorem, there exists a point  $x$  in the suspension such that the sequence of probability measures  $\mu_n = \frac{1}{\lambda(A_n)} \int_{A_n} \delta_{g.x} d\lambda(g)$  converges, when  $n$  grows to infinity, to the measure  $\mu$ . Let  $y$  be a point in  $\Omega(\mathcal{T}(w))$  such that  $h(y) = x$ . Then, up to the choice of a subsequence, the sequence of probability measures on  $\Omega(\mathcal{T}(w))$   $\nu_n = \frac{1}{\lambda(A_n)} \int_{A_n} \delta_{g.y} d\lambda(g)$  converges to a probability measure  $\nu$  invariant for the  $\mathcal{P}$ -action. As  $h * \nu_n = \mu_n$ , we get  $h * \nu = \mu$ . It follows that the map  $h$  sends the set of invariant measures of  $\Omega(\mathcal{T}(w))$  onto the set of invariant measures of  $\mathcal{X}$ . Furthermore the map  $h$  sends ergodic measures on ergodic measures. Then  $\Omega(\mathcal{T}(w))$  has at least  $r$  independent ergodic probability measures. From Proposition 4.4, we also know that the hull  $\Omega(\mathcal{T}(w))$  admits at most  $r$  invariant ergodic probability measures. Thus there are exactly  $r$  probability measures.

To obtain an example of a minimal  $\mathcal{P}$ -solenoid with a single  $\mathcal{P}$ -invariant probability measure, we use the same strategy as before. We keep the same notations as the case  $r = 2$  but we define an other Toeplitz sequence  $w$  on which the shift action is free, minimal and uniquely ergodic ([9]). We consider the *substitution*  $\mathfrak{S}$  over the alphabet  $\Sigma = \{1, 2\}$  defined by  $\mathfrak{S}(1) = 112$ ,  $\mathfrak{S}(2) = 122$ . Using the extension of the substitution over the words by the concatenation, we can iterate the substitution. The sequence  $w$  is then the bi-infinite sequence defined by:

$$w = \lim_n \overleftarrow{\mathfrak{S}^n(2)} \cdot \lim_n \overrightarrow{\mathfrak{S}^n(1)},$$

where the dot  $\cdot$  is placed between the 0 and  $-1$  coordinate.

Let  $\mathcal{A}_0$  be the set  $\{1, 2\}$ , and for any integer  $q \geq 1$ , let  $\mathcal{A}_q$  be the atlas of words  $\{\mathfrak{S}^{q-1}(1)\mathfrak{S}^{q-1}(i)\mathfrak{S}^{q-1}(2), i = 1, 2\}$  for the sequence  $w$ . The sequence  $w$  is a bi-infinite sequence of words of  $\mathcal{A}_q$ . Now let us consider the collection of patches  $\mathcal{P}a_q = \{\mathcal{P}a(wo), wo \in \mathcal{A}_q\}$ . For any  $q \geq 0$ , the tiling  $\mathcal{T}(w)$  is an union of elements of  $\mathcal{P}a_q$  and the box decompositions of  $\Omega(\mathcal{T}(w))$  associated to  $\mathcal{P}a_q$  define a tower system of the hull. The hull  $\Omega(\mathcal{T}(w))$  is then homeomorphic to  $\lim_{\leftarrow} (\mathcal{B}_q, \pi_q)$  where  $\mathcal{B}_q = \bigsqcup_{wo \in \mathcal{A}_q} \mathcal{P}a(wo) / \sim_q$ .

By Theorem 4.3, the space of invariant measures  $\mathcal{M}(\Omega(\mathcal{T}(w)))$  is isomorphic to  $\lim_{\leftarrow} (\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+, A_n)$ . A simple calculation shows that the linear applications  $A_n$  are defined by the matrices:

$$A_n = \begin{pmatrix} 1 + 2^{-3^n+1} & 1 \\ 2^{-3^n+2} & 2^{-3^n+1} + 2^{-3^n+2} \end{pmatrix}.$$



Proposition 4.4 enables us to conclude that the hull  $\Omega(\mathcal{T}(w))$  admits only one  $\mathcal{P}$ -invariant probability measure.

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