GAFA, Geom. funct. anal.
Vol. 10 (2000) 977 – 1027
1016-443X/00/050977-51 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

ADIABATIC LIMITS AND SPECTRAL SEQUENCES FOR RIEMANNIAN FOLIATIONS

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Abstract

For general Riemannian foliations, spectral asymptotics of the Laplacian is studied when the metric on the ambient manifold is blown up in directions normal to the leaves (adiabatic limit). The number of "small" eigenvalues is given in terms of the differentiable spectral sequence of the foliation. The asymptotics of the corresponding eigenforms also leads to a Hodge theoretic description of this spectral sequence. This is an extension of results of Mazzeo-Melrose and R. Forman.

1 Introduction and Main Results

Let $\mathcal{F}$ be a C^∞ foliation on a closed Riemannian manifold (M, g), and let $T\mathcal{F} \subset TM$ denote the subbundle of vectors tangent to the leaves. Then the metric g can be written as an orthogonal sum, $g = g_\perp \oplus g_F$, with respect to the decomposition $TM = T\mathcal{F}^\perp \oplus T\mathcal{F}$; i.e., $g_\perp, g_F$ are the restrictions of g to $T\mathcal{F}^\perp, T\mathcal{F}$, respectively. By introducing a parameter $h > 0$, we can define a family of metrics

$$g_h = h^{-2} g_\perp \oplus g_F\,. \tag{1.1}$$

The "limit" of the Riemannian manifolds (M, g_h) as $h \downarrow 0$ is what is known as the *adiabatic limit*. Observe that, in a foliation chart, the plaques get further from each other as $h \downarrow 0$. This form of the adiabatic limit was introduced by E. Witten in [W] for Riemannian bundles over the circle. Witten investigated the limit of the eta invariant of the Dirac operator. This question was also considered in [BiF1,2] and [C2], and extended to general Riemannian bundles in [BiC] and [D1].

Partially supported by Xunta de Galicia, grant XUGA20701B97.

Partially supported by Dirección General de Enseñanza Superior e Investigación Científica (Spain), sabbatical grant SAB1995-0717.

New properties of adiabatic limits were discovered by Mazzeo and Melrose for the case of Riemannian bundles, relating them to the Leray spectral sequence [MaM]. This work was used in [D1], and further developed by R. Forman in [F], where the very general setting of any pair of complementary distributions is considered. Nevertheless the most interesting results of [F] are only proved for foliations satisfying very restrictive conditions. The ideas from [MaM] and [F] were also applied in the case of the contact-adiabatic (or sub-riemannian) limit by Z. Ge [G1,2] and M. Rumin [Ru].

For a general C^∞ foliation $\mathcal{F}$ on M, the role of (the differentiable version of) the Leray spectral sequence is played by the so called *differentiable spectral sequence* (E_k, d_k), which converges to the de Rham cohomology of M. The definition of (E_k, d_k) is given by filtering the de Rham complex (Ω, d) of M as in the bundle case: A differential form ω of degree r is said to be of filtration $\geq k$ if it vanishes whenever $r-k+1$ of the vectors are tangent to the leaves; that is, roughly speaking, if ω is of degree $\geq k$ transversely to the leaves. Moreover the C^∞ topology of Ω induces a topological vector space structure on each term E_k such that d_k is continuous. A subtle problem here is that E_k may not be Hausdorff [H]. So it makes sense to consider the subcomplex given by the closure of the trivial subspace, $\bar{0}_k \subset E_k$, as well as the quotient complex $\widehat{E}_k = E_k/\bar{0}_k$, whose differential operator will be also denoted by d_k.

The differentiable spectral sequence is known to satisfy certain good properties for the so called *Riemannian foliations*, which are the foliations with "rigid transverse dynamics"; i.e., foliations with isometric holonomy for some Riemannian metric on smooth transversals. A characteristic property of Riemannian foliations is the existence of a so called *bundle-like metric* on the ambient manifold, which means that the foliation is locally defined by Riemannian submersions [R], [Mo1,2]. For such foliations, each term E_k is Hausdorff of finite dimension if $k \geq 2$, and $H(\bar{0}_1) = 0$ [M], [AK]. So $E_k \cong \widehat{E}_k$ for $k \geq 2$. Moreover it was recently proved by X. Masa and the first author that, for $k \geq 2$, the terms E_k are homotopy invariants of Riemannian foliations [AM] – this generalizes previous work showing the topological invariance of the so called *basic cohomology* [EN].

Besides the requirement that $\mathcal{F}$ has to be a Riemannian foliation, the mentioned restrictive hypothesis of Forman in [F] is that the positive spectrum of the "leafwise Laplacian" on Ω must be bounded away from zero. (The leafwise Laplacian is what will be denoted by Δ_0 in this paper.) Both conditions together are so strong that the only examples we know are Riemannian foliations with compact leaves; i.e., Seifert bundles. The purpose of our paper is to generalize Forman's work to arbitrary Riemannian foliations. To state our first main result, let Δ_{g_h} denote the Laplacian defined by g_h on differential forms, and let

$$0 \leq \lambda_0^r(h) \leq \lambda_1^r(h) \leq \lambda_2^r(h) \leq \cdots$$

denote its spectrum on Ω^r, taking multiplicities into account. It is well known that the eigenvalues of the Laplacian on differential forms vary continuously under continuous perturbations of the metric [C1], and thus the "branches" of eigenvalues $\lambda_i^r(h)$ depend continuously on $h > 0$. In this paper, we shall only consider the "branches" $\lambda_i^r(h)$ that are convergent to zero as $h \downarrow 0$; roughly speaking, the "small" eigenvalues. The asymptotics as $h \downarrow 0$ of these metric invariants is related to the differential invariant $\widehat{E}_1^r$ and the homotopy invariants E_k^r, $k \geq 2$, as follows.

Theorem A. *With the above notation, for Riemannian foliations on closed Riemannian manifolds we have*

$$\dim \widehat{E}_1^r = \sharp \left\{ i \bigm| \lambda_i^r(h) \in O(h^2) \text{ as } h \downarrow 0 \right\}, \tag{1.2}$$

$$\dim E_k^r = \sharp \left\{ i \bigm| \lambda_i^r(h) \in O(h^{2k}) \text{ as } h \downarrow 0 \right\}, \quad k \geq 2\,. \tag{1.3}$$

As a part of the proof of Theorem A, and also because of its own interest, we shall also study the asymptotics of eigenforms of Δ_{g_h} corresponding to "small" eigenvalues. This study was begun in [MaM] for the case of Riemannian bundles, and continued in [F] for general complementary distributions. From both [MaM] and [F], certain rescaling Θ_h of differential forms, depending on $h > 0$, is crucial to study this asymptotics.

The following well-known technicality will be useful to explain Θ_h. The decomposition $TM = T\mathcal{F}^\perp \oplus T\mathcal{F}$ induces a bigrading

$$\bigwedge TM^* = \bigoplus_{u,v} \Big(\bigwedge^u T\mathcal{F}^{\perp *} \otimes \bigwedge^v T\mathcal{F}^* \Big); \tag{1.4}$$

roughly speaking, here u denotes transverse degree and v tangential degree. Then a bigrading of Ω is defined by considering C^∞ sections of (1.4); i.e., each $\Omega^{u,v}$ is the space of C^∞ sections of $\bigwedge^u T\mathcal{F}^{\perp *} \otimes \bigwedge^v T\mathcal{F}^*$. Then the de Rham derivative and coderivative decompose as the sum of bihomogeneous components,

$$d = d_{0,1} + d_{1,0} + d_{2,-1}\,, \quad \delta = \delta_{0,-1} + \delta_{-1,0} + \delta_{-2,1}\,, \tag{1.5}$$

where the double subindex denotes the corresponding bidegree (see e.g. [A1]); observe that $d_{i,j}^* = \delta_{-i,-j}$.

Now define $\Theta_h \omega = h^u \omega$ if $\omega \in \bigwedge TM^*$ is of transverse degree u. As pointed out in [MaM] and [F], such a Θ_h is an isometry of Riemannian vector bundles $(\bigwedge TM^*, g_h) \to (\bigwedge TM^*, g)$, where g, g_h also denote the metrics induced by g, g_h on $\bigwedge TM^*$. So we get an isomorphism, also denoted by Θ_h, between the corresponding Hilbert spaces of L^2 sections because the volume elements induced by the metrics g_h are multiples of each other. Thus our setting is moved via Θ_h to the fixed Hilbert space of square integrable differential forms on M with the inner product induced by g; this Hilbert space is denoted by $\mathbf{\Omega}$ in this paper. Concretely, we have the "rescaled derivative" $d_h = \Theta_h d \Theta_h^{-1}$, whose g-adjoint is the "rescaled coderivative" $\delta_h = \Theta_h \delta_{g_h} \Theta_h^{-1}$. It is easy to verify that

$$d_h = d_{0,1} + h d_{1,0} + h^2 d_{2,-1} \tag{1.6}$$

directly from (1.5) and the definition of Θ_h. Thus

$$\delta_h = \delta_{0,-1} + h\delta_{-1,0} + h^2 \delta_{-2,1}\,. \tag{1.7}$$

(Another way to check (1.7) is by proving directly that

$$\delta_{g_h} = \delta_{0,-1} + h^2\delta_{-1,0} + h^4 \delta_{-2,1}\,.)$$

The "rescaled Laplacian"

$$\Delta_h = \Theta_h \Delta_{g_h} \Theta_h^{-1} = d_h\delta_h + \delta_h d_h$$

is elliptic and essentially self-adjoint in $\mathbf{\Omega}$. Moreover Δ_h has the same spectrum as Δ_{g_h}, and eigenspaces of Δ_{g_h} are transformed into eigenspaces of Δ_h by Θ_h. We shall prove that eigenspaces of Δ_h corresponding to "small" eigenvalues are convergent as $h \downarrow 0$ when the metric g is bundle-like, and the limit is given by a nested sequence of bigraded subspaces,

$$\Omega \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \mathcal{H}_3 \supset \cdots \supset \mathcal{H}_\infty\,.$$

The definition of $\mathcal{H}_1, \mathcal{H}_2$ was already given in [AK] as a Hodge theoretic approach to (E_1, d_1) and (E_2, d_2), which is based on our study of leafwise heat flow. The other spaces $\mathcal{H}_k$ are defined in this paper as an extension of this Hodge theoretic approach to the whole spectral sequence (E_k, d_k) (see sections section 2.2 and section 5.1 for the precise definition of $\mathcal{H}_k$). In particular,

$$\mathcal{H}_1 \cong \widehat{E}_1\,, \quad \mathcal{H}_k \cong E_k\,, \quad k = 2, 3, \dots, \infty\,, \tag{1.8}$$

as bigraded topological vector spaces. Thus this sequence stabilizes (we mean $\mathcal{H}_k = \mathcal{H}_\infty$ for k large enough) because the differentiable spectral sequence is convergent in a finite number of steps. The convergence of eigenforms corresponding to "small" eigenvalues is precisely stated in the following result, where $L^2\mathcal{H}_1$ denotes the closure of $\mathcal{H}_1$ in $\mathbf{\Omega}$.

Theorem B. *For any Riemannian foliation on a closed manifold with a bundle-like metric, let ω_i be a sequence in Ω^r such that $\|\omega_i\| = 1$ and*

$$\langle \Delta_{h_i}\omega_i, \omega_i\rangle \in o\big(h_i^{2(k-1)}\big) \tag{1.9}$$

for some fixed integer $k \geq 1$ *and some sequence* $h_i \downarrow 0$. *Then some subsequence of the* ω_i *is strongly convergent, and its limit is in* $L^2\mathcal{H}_1^r$ *if* $k = 1$, *and in* $\mathcal{H}_k^r$ *if* $k \geq 2$.

To simplify notation let $m_1^r = \dim \widehat{E}_1^r$, and let $m_k^r = \dim E_k^r$ for each $k = 2, 3, \ldots, \infty$. Thus Theorem A establishes $\lambda_i^r(h) \in O\left(h^{2k}\right)$ for $i \leq m_k^r$, yielding $\lambda_i^r(h) \equiv 0$ for i large enough. For every $h > 0$, consider the nested sequence of graded subspaces

$$\Omega \supset \mathcal{H}_1(h) \supset \mathcal{H}_2(h) \supset \mathcal{H}_3(h) \supset \cdots \supset \mathcal{H}_\infty(h)\,,$$

where $\mathcal{H}_k^r(h)$ is the space generated by the eigenforms of Δ_h corresponding to eigenvalues $\lambda_i^r(h)$ with $i \leq m_k^r$; in particular, we have $\mathcal{H}_k(h) = \mathcal{H}_\infty(h) = \ker \Delta_h$ for k large enough. Set also $\mathcal{H}_k(0) = \mathcal{H}_k$. We have $\dim \mathcal{H}_k^r(h) = m_k^r$ for all $h > 0$, so the following result is a sharpening of Theorem A.

Corollary C. *For any Riemannian foliation on a closed manifold with a bundle-like metric and* $k = 2, 3, \ldots, \infty$, *the assignment* $h \mapsto \mathcal{H}_k^r(h)$ *defines a continuous map from* $[0, \infty)$ *to the space of finite dimensional linear subspaces of* $\mathbf{\Omega}^r$ *for all* $r \geq 0$. *If* $\dim \widehat{E}_1^r < \infty$, *then this also holds for* $k = 1$.

In Corollary C, the continuity of $h \mapsto \mathcal{H}_k^r(h)$ for $h > 0$ is a particular case of the general property that eigenspaces of the Laplacian on closed Riemannian manifolds vary continuously as subspaces of $\mathbf{\Omega}$ when the metric is perturbed C^0-continuously [C1], [BD]. On the other hand, the continuity of $h \mapsto \mathcal{H}_k^r(h)$ at $h = 0$ is a direct consequence of Theorem B.

With an analogous aim, other nested sequences of bigraded subspaces were introduced by Mazzeo–Melrose in [MaM] and by Forman in [F], which are respectively denoted by

$$\Omega \supset \mathfrak{h}_1 \supset \mathfrak{h}_2 \supset \mathfrak{h}_3 \supset \cdots \supset \mathfrak{h}_\infty\,, \quad \Omega \supset \mathfrak{H}_1 \supset \mathfrak{H}_2 \supset \mathfrak{H}_3 \supset \cdots \supset \mathfrak{H}_\infty$$

in this paper. These sequences are defined in the following way. According to the expressions (1.6) and (1.7), we can consider d_h and δ_h as polynomials on the variable h whose coefficients are the differential operators $d_{i,j}$ and $\delta_{i,j}$. Thus d_h and δ_h canonically become operators on the polynomial algebra $\Omega[h]$, and Δ_h as well. Then each $\mathfrak{h}_k$ is the space of differential forms $\omega \in \Omega$ with some extension $\tilde{\omega}(h) \in \Omega[h]$ satisfying

$$\Delta_h \tilde{\omega}(h) \in h^k \Omega[h]\,, \tag{1.10}$$

where *extension* means $\tilde{\omega}(0) = \omega$. And each $\mathfrak{H}_k$ is the space of differential forms $\omega \in \Omega$ with some extension $\tilde{\omega}(h) \in \Omega[h]$ satisfying

$$d_h \tilde{\omega}(h) \in h^k \Omega[h]\,, \quad \delta_h \tilde{\omega}(h) \in h^k \Omega[h]\,. \tag{1.11}$$

The sequence $\mathcal{H}_k$ also fits in this kind of description as follows (this is a direct consequence of Theorem 5.1): Each $\mathcal{H}_k$ is the space of differential forms $\omega \in \Omega$ having sequences of extensions $\tilde{\omega}_i^1(h), \tilde{\omega}_i^2(h) \in \Omega[h]$ satisfying

$$d_h\tilde{\omega}_i^1(h) + h^k\Omega[h] \longrightarrow 0\,, \quad \delta_h\tilde{\omega}_i^2(h) + h^k\Omega[h] \longrightarrow 0 \tag{1.12}$$

in $\Omega[h]/h^k\Omega[h]$ as $i \to \infty$. From (1.6), (1.7), (1.11) and (1.12) it easily follows that

$$\mathfrak{H}_k \subset \mathfrak{h}_k \subset \mathfrak{H}_{[k/2]}\,, \tag{1.13}$$

$$\mathfrak{H}_1 = \mathcal{H}_1\,, \quad \mathfrak{H}_k \subset \mathcal{H}_k\,, \quad k \geq 2\,. \tag{1.14}$$

For the case of Riemannian bundles, Mazzeo and Melrose prove in [MaM] that the sequence $\mathfrak{h}_k$ stabilizes, and $\mathfrak{h}_\infty$ is the limit of the spaces $\ker\Delta_h$ as $h \downarrow 0$. And for foliations under the restrictive hypothesis of [F], Forman proves that the sequence $\mathfrak{H}_k$ is a Hodge theoretic version of the spectral sequence (E_k, d_k), and describes the limit of the eigenspaces of Δ_h corresponding to "small" eigenvalues. This improves the results of Mazzeo–Melrose by (1.13). But Forman's sequence $\mathfrak{H}_k$ does not have the same important properties for general Riemannian foliations and bundle-like metrics, as follows from the following result, where the notation $\mathcal{H}_k(g)$ and $\mathfrak{H}_k(g)$ is used to emphasize the dependence of $\mathcal{H}_k$ and $\mathfrak{H}_k$ on the metric g – of course, each $\mathcal{H}_k(g)$ is independent of g up to isomorphism by (1.8).

Theorem D. *Let $\mathcal{F}$ be a Riemannian foliation of dimension p on a closed manifold M. We have:*

(i) *There is a bundle-like metric g on M such that $\mathfrak{H}_2^{0,p}(g) = \mathcal{H}_2^{0,p}(g)$.*

(ii) *If $\bar{0}_1^{0,p} \neq 0$, then there is a bundle-like metric g' on M such that $\mathfrak{H}_2^{0,p}(g') = 0$.*

The condition $\bar{0}_1^{0,p} \neq 0$ holds for Kronecker's flows on T^2 whose slope is a Liouville number [He], [Ro]. This was generalized to linear foliations on tori of arbitrary dimension in [ArS]. Moreover $E_2^{0,p} \cong \mathbb{R}$ in these examples [M], [A2]. Therefore Theorem D implies that, in these examples, the dimension of $\mathfrak{H}_2^{0,p}(g)$ changes when appropriately varying the metric g. Thus $\mathfrak{H}_2^{0,p}(g) \ncong E_2^{0,p}$ for appropriate choices of g; that is, [F, Corollary 4.4] is not completely right with that generality – the possibility that E_1 may not be Hausdorff is not considered in that paper. So far it is unknown which topological or geometric conditions imply $\bar{0}_1 \neq 0$ for general Riemannian foliations, but the above examples suggest that this may happen "generically".

A simple argument shows that $\mathfrak{H}_k^r = \mathcal{H}_k^r$ if $h \mapsto \mathcal{H}_k^r(h)$ is a C^∞ map: In this case, any $\omega \in \mathcal{H}_k^r$ has an extension depending smoothly on $h \geq 0$, whose Taylor polynomial of degree k at zero is easily seen to satisfy (1.11), yielding $\omega \in \mathfrak{H}_k^r$. Therefore, since both $\mathcal{H}_k$ and $\mathfrak{H}_k$ obviously stabilize at $k = 2$ for flows on surfaces, Theorem D shows that the map $h \mapsto \mathcal{H}_\infty^1(h)$

is not C^∞ at $h = 0$ for Kronecker's flows on T^2 whose slope is a Liouville number and appropriate bundle-like metrics. So [MaM, Corollary 18] and [F, Corollary 5.22] have no direct generalizations to arbitrary Riemannian foliations and bundle-like metrics.

Nevertheless, the arguments of Forman in [F] are right when $\bar{0}_1 = 0$. In particular, Sections 2–4 in [F] show that, in this case, $\mathfrak{H}_k \cong E_k$ as bigraded vector spaces. (Indeed [F, Lemma 2.7] is a version of this isomorphism – it must be pointed out that the notation used in [F] is very different from ours.) Therefore, by (1.8) and (1.14), Forman's arguments prove the following.

Theorem E. *Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold M. If $\bar{0}_1 = 0$, then $\mathfrak{H}_k(g) = \mathcal{H}_k(g)$ for every $k \geq 1$ and any bundle-like metric g on M.*

Theorem D(ii) is a partial reciprocal of Theorem E, and we could conjecture that its statement holds for any bidegree, but we do not pursue such a result in this paper. A similar question can be raised about Theorem D(i).

The following are the main ideas of the proofs in this paper. The proof of "$\leq$" in (1.2) (Theorem A) has three main ingredients. The first one is a variational formula for the spectral distribution function of the Laplacian, which is a consequence of the Hodge decomposition, and was used by Gromov and Shubin in another setting [GrS]. The second ingredient is a direct sum decomposition that holds for general spectral sequences – it is a kind of (only linear) Hodge decomposition. The relation between this decomposition and the formula of Gromov–Shubin can be easily seen, and leads to the proof. But this cannot be directly applied to the differentiable spectral sequence (E_k, d_k) because of some technical difficulty (Remark 3). For this reason, we introduce the third ingredient: The L^2 *spectral sequence* $(\mathbf{E}_k, \mathbf{d}_k)$, which is another spectral sequence defined in the very same way as (E_k, d_k) but using square integrable differential forms. This change of spectral sequence can be made because we show that $\mathbf{E}_k \cong E_k$ for Riemannian foliations and $k \geq 2$. The proof of this isomorphism heavily depends on the Hodge theoretic approach of the terms E_1 and E_2 that follows from our work [AK] on leafwise heat flow.

The rest of Theorem A is an easy consequence of Theorem B, which in turn is proved by characterizing the terms $\mathcal{H}_k$ in the appropriate way to apply certain estimation of Δ_h – this estimation is similar to what was done by Forman in [F].

Theorem D follows easily from the above theorems and other well-known results about Riemannian foliations.

As a possible application of this theory, it could be possible to relate the spectral sequence (E_k, d_k) of a Riemannian foliation $\mathcal{F}$ with the curvature of a bundle-like metric g on the ambient manifold M. The simpler relationship we have in mind is a kind of Bochner type theorem for the terms E_k. This would be a generalization of the Bochner type theorem for the basic cohomology (recall that the basic cohomology is equal to $E_2^{\cdot,0}$) proved in [MiRT], and the proof could be as follows. In the Weitzenbock formula for each metric g_h of the family (1.1), it seems that the curvature term can be written as a polynomial on h whose coefficients are given by bihomogeneous components of the same curvature term for g (with respect to the bigrading (1.4)). Thus, by Theorem A and arguing as in the proof of Bochner theorem, if some of these components are positive, then vanishing and collapsing statements would follow for the spectral sequence.

Another more involved relation between (E_k, d_k) and the curvature of g could be obtained as follows. On the one hand, our main results establish relations between (E_k, d_k) and the spectral asymptotics of the metrics g_h as $h \downarrow 0$. On the other hand, a well-known procedure yields spectral invariant expressions of the curvature of g_h. Such expressions are the coefficients in the asymptotic expansion of the heat kernel of g_h along the diagonal of $M \times M$ when time goes to zero – we are considering the heat equation on differential forms. As above, it seems that such spectral invariants of g_h can be written as polynomials on h whose coefficients are defined by bihomogeneous components of the same invariants for the original metric g. To sum up, we would have connections between (E_k, d_k), the spectral asymptotics of g_h, the curvature of g_h and components of the curvature of g. So, by keeping track of the powers of h through these connections, some formulae hopefully could be proved relating the spectral sequence of Riemannian foliations with components of the curvature of bundle-like metrics.

We also hope that the methods developed in this paper will be useful to analyze the contribution of the spectral sequence of $\mathcal{F}$ to adiabatic limits of eta-invariants and analytic torsion, generalizing results that have been proved for fiber bundles [D1,2], [LST], [DM].

Finally, let us mention that a closely related study is done in [K], where the second author proves an asymptotic formula for the eigenvalue distribution function of Δ_{g_h} in adiabatic limits for Riemannian foliations. That work establishes relationships with the spectral theory of leafwise Laplacian and with the noncommutative spectral geometry of foliations.

2 Differentiable Spectral Sequence

In this section, we mainly recall known properties of the differentiable spectral sequence (E_k, d_k) of foliations, with special emphasis on its topology.

The definition and basic properties of (E_k, d_k) are recalled in section 2.1, and section 2.2 is devoted to the particular case of Riemannian foliations. Specially, in section 2.2, we recall from [AK] the Hodge theoretic approach of the terms $\widehat{E}_1, E_2$. This approach will play an important role in this paper: It is the key to studying the L^2 spectral sequence of Riemannian foliations, some of its simple consequences will be used later too, and it is a first step in the construction of our Hodge theoretic nested sequence.

2.1 General properties. Let $(\mathcal{A}, d)$ be a complex with a finite decreasing filtration

$$\mathcal{A} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_q \supset \mathcal{A}_{q+1} = 0$$

by differential subspaces; i.e., $d(\mathcal{A}_k) \subset \mathcal{A}_k$ for all k. Recall that the induced spectral sequence (E_k, d_k) is defined in the following standard way [Mc]:

$$\begin{aligned} Z_k^{u,v} &= \mathcal{A}_u^{u+v} \cap d^{-1}\left(\mathcal{A}_{u+k}^{u+v+1}\right), & Z_\infty^{u,v} &= \mathcal{A}_u^{u+v} \cap \ker d\,, \\ B_k^{u,v} &= \mathcal{A}_u^{u+v} \cap d\left(\mathcal{A}_{u-k}^{u+v-1}\right), & B_\infty^{u,v} &= \mathcal{A}_u^{u+v} \cap \operatorname{im} d\,, \\ E_k^{u,v} &= \frac{Z_k^{u,v}}{Z_{k-1}^{u+1,v-1} + B_{k-1}^{u,v}}, & E_\infty^{u,v} &= \frac{Z_\infty^{u,v}}{Z_\infty^{u+1,v-1} + B_\infty^{u,v}}\,. \end{aligned}$$

In particular $Z_0^{u,v} = Z_{-1}^{u,v} = \mathcal{A}_u^{u+v}$. We assume $B_{-1}^{u,v} = 0$, so $E_0^{u,v} = \mathcal{A}_u^{u+v}/\mathcal{A}_{u+1}^{u+v}$. Also, we have $B_u^{u,v} = B_\infty^{u,v}$ and $Z_{q-u+1}^{u,v} = Z_\infty^{u,v}$ since the filtration of $\mathcal{A}$ is of length $q+1$. Each homomorphism $d_k : E_k^{u,v} \to E_k^{u+k,v-k+1}$ is canonically induced by d.

Now let $\mathcal{F}$ be a C^∞ foliation of codimension q on a closed manifold M, and (Ω, d) the de Rham complex of M. The *differentiable spectral sequence* (E_k, d_k) of $\mathcal{F}$ is defined by the decreasing filtration by differential subspaces

$$\Omega = \Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_q \supset \Omega_{q+1} = 0\,,$$

where the space of r-forms of filtration degree $\geq k$ is given by

$$\Omega_k^r = \left\{ \omega \in \Omega^r \;\middle|\; \begin{array}{l} i_X\omega = 0 \quad \text{for all} \quad X = X_1 \wedge \cdots \wedge X_{r-k+1}\,, \\ \text{where the} \quad X_i \quad \text{are vector fields tangent} \\ \text{to the leaves} \end{array} \right\}.$$

Moreover, the C^∞ topology of Ω canonically induces a topology on each $E_k^{u,v}$, which becomes a topological vector space. Then each d_k is continuous on $E_k = \bigoplus_{u,v} E_k^{u,v}$ with the product topology. Thus, for each k, we have two new bigraded complexes: the closure of the trivial subspace $\bar{0}_k \subset E_k$ and the quotient $\widehat{E}_k = E_k/\bar{0}_k$.

Assume M is endowed with a Riemannian metric, and let $\pi_{u,v} : \Omega \to \Omega^{u,v}$ denote the induced projection defined by the bigrading of Ω. Define the topological vector spaces

$$z_k^{u,v} = \pi_{u,v}(Z_k^{u,v})\,, \quad b_k^{u,v} = \pi_{u,v}(B_k^{u,v})\,, \quad e_k^{u,v} = z_k^{u,v}/b_k^{u,v}\,, \quad e_k = \bigoplus_{u,v} e_k^{u,v}\,.$$

Observe that

$$\Omega_k = \bigoplus_{u\geq k} \Omega^{u,\cdot}\,, \tag{2.1}$$

yielding

$$Z_k^{u,v} \cap \ker \pi_{u,v} = Z_{k-1}^{u+1,v-1}\,.$$

Thus the projection $\pi_{u,v}$ induces a continuous linear isomorphism $E_k^{u,v} \xrightarrow{\cong} e_k^{u,v}$. The operator on e_k that corresponds to d_k on E_k by the above linear isomorphisms will be denoted by d_k as well. We also consider the closure of the trivial subspace, $\bar{o}_k \subset e_k$, and the quotient $\hat{e}_k = e_k/\bar{o}_k$. We are going to show that d_k is continuous on e_k for $k = 0,1$, and thus $\bar{o}_k$ and $\hat{e}_k$ become bigraded complexes in a canonical way. But, for $k \geq 2$, we do not know whether d_k is continuous on e_k, and whether d_k induces differentials on $\bar{o}_k$ and $\hat{e}_k$. This holds at least for Riemannian foliations as easily follows from Theorem 2.2(vii) in section 2.2.

By comparing bihomogeneous components in the equality $d^2 = 0$ we get (see e.g. [A1]):

$$\left.\begin{aligned} d_{0,1}^2 = d_{2,-1}^2 = d_{0,1}d_{1,0} + d_{1,0}d_{0,1} = 0\,, \\ d_{1,0}d_{2,-1} + d_{2,-1}d_{1,0} = d_{1,0}^2 + d_{0,1}d_{2,-1} + d_{2,-1}d_{0,1} = 0\,. \end{aligned}\right\} \tag{2.2}$$

The term $d_{2,-1}$ is of order zero, and vanishes if and only if $T\mathcal{F}^\perp$ is completely integrable. Moreover from (2.1) we get

$$Z_0^{u,v} = \Omega_u^{u+v}\,, \tag{2.3}$$

$$B_0^{u,v} = d_{0,1}\left(\Omega^{u,v-1}\right) \oplus \Omega_{u+1}^{u+v}\,, \tag{2.4}$$

$$Z_1^{u,v} = \left(\Omega^{u,v} \cap \ker d_{0,1}\right) \oplus \Omega_{u+1}^{u+v} \tag{2.5}$$

as topological vector spaces. So

$$z_0^{u,v} = \Omega^{u,v}\,, \quad b_0^{u,v} = d_{0,1}(\Omega^{u,v-1})\,, \quad z_1^{u,v} = \Omega^{u,v} \cap \ker d_{0,1}\,, \tag{2.6}$$

and the continuous linear isomorphisms $E_k^{u,v} \xrightarrow{\cong} e_k^{u,v}$, induced by $\pi_{u,v}$, are homeomorphisms too for $k = 0,1$. Thus $\bar{0}_1 \cong \bar{o}_1$ and $\widehat{E}_1 \cong \hat{e}_1$ as topological vector spaces, and $\bar{o}_1$ and $\hat{e}_1$ become bigraded complexes with the differential induced by d_1. For this reason, using the spaces $e_1, \bar{o}_1, \hat{e}_1$ is rather redundant; we have introduced these spaces to be compared with the corresponding ones for the L^2 spectral sequence (section 3), where this does not obviously hold. Furthermore (2.3)–(2.5) yield

$$(e_0, d_0) = (\Omega, d_{0,1}) , \tag{2.7}$$

and a canonical isomorphism

$$(e_1, d_1) \cong \big(H(\Omega, d_{0,1}), d_{1,0*}\big) \tag{2.8}$$

of topological complexes. Nevertheless, we cannot go further keeping full control of the topology. In fact, with this generality, we do not know whether the continuous linear isomorphism $E_2^{u,v} \stackrel{\cong}{\to} e_2^{u,v}$, induced by $\pi_{u,v}$, is a homeomorphism, nor whether the canonical continuous linear isomorphisms $E_2 \stackrel{\cong}{\to} H(E_1, d_1)$ and $e_2 \stackrel{\cong}{\to} H(e_1, d_1)$ are homeomorphisms.

2.2 Hodge theory of the terms E_1 and E_2 for Riemannian foliations. Here, $\mathcal{F}$ is assumed to be a Riemannian foliation and the metric bundle-like.

The de Rham coderivative δ decomposes as the sum of bihomogeneous components $\delta_{i,j} = d^*_{-i,-j}$, and the operators

$$D_0 = d_{0,1} + \delta_{0,-1} , \quad \Delta_0 = D_0^2 = d_{0,1}\delta_{0,-1} + \delta_{0,-1}d_{0,1}$$

are essentially self-adjoint in $\mathbf{\Omega}$ [Ch]. But D_0 and Δ_0 are not elliptic on M – avoiding the trivial case where $q = 0$. The closures of d, δ, $d_{0,1}$, $\delta_{0,-1}$, D_0 and Δ_0 in $\mathbf{\Omega}$ will be denoted by $\mathbf{d}$, $\boldsymbol{\delta}$, $\mathbf{d}_{0,1}$, $\boldsymbol{\delta}_{0,-1}$, $\mathbf{D}_0$ and $\mathbf{\Delta}_0$, respectively. Then we have the orthogonal decomposition

$$\mathbf{\Omega} = \ker \mathbf{\Delta}_0 \oplus \mathrm{cl}_0(\mathrm{im}\,\mathbf{d}_{0,1}) \oplus \mathrm{cl}_0(\mathrm{im}\,\boldsymbol{\delta}_{0,-1}) , \tag{2.9}$$

where cl_0 denotes closure in $\mathbf{\Omega}$. Moreover

$$\ker \mathbf{\Delta}_0 = \ker \mathbf{D}_0 = \ker \mathbf{d}_{0,1} \cap \ker \boldsymbol{\delta}_{0,-1} , \tag{2.10}$$
$$\mathrm{cl}_0(\mathrm{im}\,\mathbf{\Delta}_0) = \mathrm{cl}_0(\mathrm{im}\,\mathbf{D}_0) = \mathrm{cl}_0(\mathrm{im}\,\mathbf{d}_{0,1}) \oplus \mathrm{cl}_0(\mathrm{im}\,\boldsymbol{\delta}_{0,-1}) .$$

Thus let Π, P and Q denote the orthogonal projections of $\mathbf{\Omega}$ onto $\ker \mathbf{\Delta}_0$, $\mathrm{cl}_0(\mathrm{im}\,\mathbf{d}_{0,1})$ and $\mathrm{cl}_0(\mathrm{im}\,\boldsymbol{\delta}_{0,-1})$, respectively, and set $\widetilde{\Pi} = \mathrm{id} - \Pi$, $\widetilde{P} = \mathrm{id} - P$ and $\widetilde{Q} = \mathrm{id} - Q$. We shall also use the notation $W^k\Omega$ for the kth Sobolev space completion of Ω, and let cl_k denote closure in $W^k\Omega$. Thus $\mathbf{\Omega} = W^0\Omega$.

Theorem 2.1 (Álvarez-Kordyukov [AK]). *For each $k \in \mathbb{Z}$, decomposition* (2.9) *restricts to $W^k\Omega$; i.e.,*

$$W^k\Omega = \ker(\Delta_0 \text{ in } W^k\Omega) \oplus \mathrm{cl}_k(\mathrm{im}\, d_{0,1}) \oplus \mathrm{cl}_k(\mathrm{im}\,\delta_{0,-1})$$

as topological vector space. Thus (2.9) *also restricts to C^∞ differential forms; i.e.,*

$$\Omega = \ker \Delta_0 \oplus \overline{\mathrm{im}\, d_{0,1}} \oplus \overline{\mathrm{im}\,\delta_{0,-1}}$$

with respect to the C^∞ topology, where the bar denotes C^∞ closure in Ω. In particular Π, P and Q preserve Ω.

From (2.7), (2.8) and Theorem 2.1, we get a canonical isomorphism $\ker\Delta_0 \cong \hat{e}_1$ of topological vector spaces, induced by the inclusion

$$\Omega^{u,v} \cap \ker\Delta_0 \hookrightarrow \Omega^{u,v} \cap \ker d_{0,1} = z_1^{u,v}\,.$$

So $\ker\Delta_0 \cong \widehat{E}_1$ as topological vector spaces. As in [AK], let

$$\mathcal{H}_1 = \ker\Delta_0 = \ker D_0 = \ker d_{0,1} \cap \ker\delta_{0,-1}\,,$$

$$\widetilde{\mathcal{H}}_1 = \overline{\operatorname{im}\Delta_0} = \overline{\operatorname{im} D_0} = \overline{\operatorname{im} d_{0,1}} \oplus \overline{\operatorname{im}\delta_{0,-1}}\,,$$

and let $L^2\mathcal{H}_1 = \operatorname{cl}_0(\mathcal{H}_1)$ and $L^2\widetilde{\mathcal{H}}_1 = \operatorname{cl}_0(\widetilde{\mathcal{H}}_1)$. From (2.10) and Theorem 2.1 we get

$$\ker\mathbf{\Delta}_0 = \ker\mathbf{D}_0 = L^2\mathcal{H}_1\,. \tag{2.11}$$

Since Δ_0 is bihomogeneous of bidegree $(0,0)$, the bigrading of Ω restricts to a bigrading of $\mathcal{H}_1$. Moreover, by (2.7), (2.8) and Theorem 2.1, the operator d_1 on $\hat{e}_1$ corresponds to the map $\Pi d_{1,0}$ on $\mathcal{H}_1$, which will be also denoted by d_1. Hence $H^u(\mathcal{H}_1^{\cdot,v}, d_1) \cong H^u(\hat{e}_1^{\cdot,v}) \cong H^u(\widehat{E}_1^{\cdot,v})$. Since $\delta_1 = \Pi\delta_{-1,0}$ is adjoint of d_1 in $\mathcal{H}_1$, the operators $D_1 = d_1 + \delta_1$ and $\Delta_1 = D_1^2 = d_1\delta_1 + \delta_1 d_1$ on $\mathcal{H}_1$ are symmetric. Now, let $\mathcal{H}_2 = \ker\Delta_1$, which inherits the bigrading from Ω because Δ_1 is bihomogeneous of bidegree $(0,0)$.

We also define maps $\tilde{d}_1$ and $\tilde{\delta}_1$ on $\widetilde{\mathcal{H}}_1$ as follows. First we define the following bigrading on $\widetilde{\mathcal{H}}_1$:

$$\widetilde{\mathcal{H}}_1^{u,v} = \overline{d_{0,1}(\Omega^{u,v-1})} \oplus \overline{\delta_{0,-1}(\Omega^{u+1,v})}\,.$$

Let $\widetilde{\Pi}_{\cdot,v}$ be the projection of Ω onto $\widetilde{\mathcal{H}}_1^{\cdot,v}$, and set $\tilde{d}_1 = \widetilde{\Pi}_{\cdot,v} d$ and $\tilde{\delta}_1 = \widetilde{\Pi}_{\cdot,v}\delta$ on $\widetilde{\mathcal{H}}_1^{\cdot,v}$, which are adjoint of each other. Consider also the symmetric operators $\widetilde{D}_1 = \tilde{d}_1 + \tilde{\delta}_1$ and $\widetilde{\Delta}_1 = \widetilde{D}_1^2$ on $\widetilde{\mathcal{H}}_1$.

The closures of d_1, δ_1, D_1 and Δ_1 in $L^2\mathcal{H}_1$, and of $\tilde{d}_1$, $\tilde{\delta}_1$, $\widetilde{D}_1$ and $\widetilde{\Delta}_1$ in $L^2\widetilde{\mathcal{H}}_1$, will be respectively denoted by $\mathbf{d}_1$, $\boldsymbol{\delta}_1$, $\mathbf{D}_1$, $\mathbf{\Delta}_1$, $\tilde{\mathbf{d}}_1$, $\tilde{\boldsymbol{\delta}}_1$, $\widetilde{\mathbf{D}}_1$ and $\widetilde{\mathbf{\Delta}}_1$.

The following theorem collects the main results of [AK, Section 7].

Theorem 2.2 (Álvarez-Kordyukov [AK]). *We have:*

(i) *The operators D_1 and Δ_1 are essentially self-adjoint in $L^2\mathcal{H}_1$, and the operators $\widetilde{D}_1$ and $\widetilde{\Delta}_1$ are essentially self-adjoint in $L^2\widetilde{\mathcal{H}}_1$.*

(ii) *The spectrums of $\mathbf{D}_1$, $\mathbf{\Delta}_1$, $\widetilde{\mathbf{D}}_1$ and $\widetilde{\mathbf{\Delta}}_1$ are discrete subsets of $\mathbb{R}$ given by eigenvalues of finite multiplicity.*

(iii) *We have the Hodge type decompositions*

$$L^2\mathcal{H}_1 = \ker\mathbf{\Delta}_1 \oplus \operatorname{im}\mathbf{d}_1 \oplus \operatorname{im}\boldsymbol{\delta}_1\,,$$

$$L^2\widetilde{\mathcal{H}}_1 = \operatorname{im}\tilde{\mathbf{d}}_1 \oplus \operatorname{im}\tilde{\boldsymbol{\delta}}_1\,,$$

as Hilbert spaces with the L^2 norm, and moreover

$$\ker\mathbf{\Delta}_1 = \ker\mathbf{D}_1 = \ker\mathbf{d}_1 \cap \ker\boldsymbol{\delta}_1\,,$$

$$\operatorname{im}\mathbf{\Delta}_1=\operatorname{im}\mathbf{D}_1=\operatorname{im}\mathbf{d}_1\oplus\operatorname{im}\boldsymbol{\delta}_1\,,$$
$$\ker\widetilde{\mathbf{\Delta}}_1=\ker\widetilde{\mathbf{D}}_1=0\,,\quad \operatorname{im}\mathbf{\Delta}_1=\operatorname{im}\mathbf{D}_1=L^2\widetilde{\mathcal{H}}_1\,.$$

Furthermore the operators $\mathbf{\Delta}_1$ and $\widetilde{\mathbf{\Delta}}_1$ satisfy Garding type inequalities [AK, Corollary 7.3]. Thus $\ker\mathbf{\Delta}_1=\mathcal{H}_2$, and the above decompositions restrict to C^∞ differential forms; i.e.,

$$\mathcal{H}_1=\ker\Delta_1\oplus\operatorname{im}d_1\oplus\operatorname{im}\delta_1\,,$$
$$\widetilde{\mathcal{H}}_1=\operatorname{im}\tilde{d}_1\oplus\operatorname{im}\tilde{\delta}_1\,,$$

as topological vector spaces with the C^∞ topology, as well as with the restriction of the L^2 norm topology.

(iv) *The space $\mathcal{H}_2$ is of finite dimension, and the inclusion $\mathcal{H}_2\hookrightarrow\mathcal{H}_1$ induces isomorphisms*

$$\mathcal{H}_2^{u,v}\xrightarrow{\cong}H^u(\mathcal{H}_1^{\cdot,v},d_1)\cong H^u(\hat{e}_1^{\cdot,v})\cong H^u(\widehat{E}_1^{\cdot,v})\,.$$

(v) *We have $\tilde{d}_1^2=0$ and $H(\widetilde{\mathcal{H}}_1,\tilde{d}_1)=0$.*

(vi) *Each map $\widetilde{\mathcal{H}}_1^{\cdot,v}\to\widetilde{\mathcal{H}}_1^{\cdot,v}=\bar{o}_1^{\cdot,v}\cong\bar{0}_1^{\cdot,v}$, defined by the canonical projection*

$$\overline{d_{0,1}(\Omega^{\cdot,v-1})}\oplus\overline{\delta_{0,-1}(\Omega^{\cdot,v})}\longrightarrow\overline{d_{0,1}(\Omega^{\cdot,v-1})}/d_{0,1}(\Omega^{\cdot,v-1})\,,$$

induces an isomorphism

$$0=H^u(\widetilde{\mathcal{H}}_1^{\cdot,v},d_1)\cong H^u(\bar{o}_1^{\cdot,v})\cong H^u(\bar{0}_1^{\cdot,v})\,.$$

(vii) *All the following bigraded topological vector spaces are Hausdorff of finite dimension and isomorphic to each other by maps that are either canonical or induced by the projections $\pi_{u,v}$: $H(\hat{e}_1)$, $H(e_1)$, e_2, $H(\widehat{E}_1)$, $H(E_1)$ and E_2.*

In Theorem 2.2, the triviality of $H(\bar{0}_1)$ in property (vi) was originally shown by Masa [M], as well as property (vii), which is a consequence.

Lemma 2.3. *The following properties are satisfied:*

(i) *We have*

$$d_{1,0}P=Pd_{1,0}P\,,\qquad d_{1,0}\widetilde{Q}=\widetilde{Q}d_{1,0}\widetilde{Q}\,,\qquad Qd_{1,0}=Qd_{1,0}Q\,,$$
$$\widetilde{P}d_{1,0}=\widetilde{P}d_{1,0}\widetilde{P}\,,\qquad \delta_{-1,0}Q=Q\delta_{-1,0}Q\,,\qquad \delta_{-1,0}\widetilde{P}=\widetilde{P}\delta_{-1,0}\widetilde{P}\,,$$
$$P\delta_{-1,0}=P\delta_{-1,0}P\,,\qquad \widetilde{Q}\delta_{-1,0}=\widetilde{Q}\delta_{-1,0}\widetilde{Q}\,.$$

(ii) *We have*

$$\widetilde{P}d_{1,0}P=Qd_{1,0}\widetilde{Q}=\widetilde{Q}\delta_{-1,0}Q=P\delta_{-1,0}\widetilde{P}=0\,.$$

Proof. The equalities involving $d_{1,0}$ in property (i) follow from (2.2) since

$$P(\Omega)=\overline{d_{0,1}(\Omega)}\,,\quad \widetilde{Q}(\Omega)=\ker d_{0,1}\,.$$

The other equalities in property (i) are obtained by taking adjoints, and property (ii) is a direct consequence of property (i). □

LEMMA 2.4. *The following properties are satisfied:*

(i) *The following operators on Ω define bounded operators on $\mathbf{\Omega}$:*

$$\widetilde{\Pi} d_{1,0}\Pi\,,\qquad \Pi d_{1,0}\widetilde{\Pi}\,,\qquad \widetilde{\Pi}\delta_{-1,0}\Pi\,,\qquad \Pi\delta_{-1,0}\widetilde{\Pi}\,,$$
$$\widetilde{Q} d_{1,0}Q\,,\qquad P d_{1,0}\widetilde{P}\,,\qquad \widetilde{P}\delta_{-1,0}P\,,\qquad Q\delta_{-1,0}\widetilde{Q}\,.$$

(ii) *The following operators on Ω define bounded operators on $\mathbf{\Omega}$ too:*

$$\widetilde{\Pi} d\Pi\,,\qquad \Pi d\widetilde{\Pi}\,,\qquad \widetilde{\Pi}_{\cdot,v+1} d\widetilde{\Pi}_{\cdot,v}\,,\qquad \widetilde{\Pi}_{\cdot,v-1} d\widetilde{\Pi}_{\cdot,v}\,,$$
$$\widetilde{\Pi}\delta\Pi\,,\qquad \Pi\delta\widetilde{\Pi}\,,\qquad \widetilde{\Pi}_{\cdot,v+1}\delta\widetilde{\Pi}_{\cdot,v}\,,\qquad \widetilde{\Pi}_{\cdot,v-1}\delta\widetilde{\Pi}_{\cdot,v}\,.$$

(iii) *We have*

$$\operatorname{dom}\mathbf{d}_1 = L^2\mathcal{H}_1\cap\operatorname{dom}\mathbf{d}\,,\qquad \operatorname{dom}\boldsymbol{\delta}_1 = L^2\mathcal{H}_1\cap\operatorname{dom}\boldsymbol{\delta}\,,$$
$$\operatorname{dom}\tilde{\mathbf{d}}_1 = L^2\widetilde{\mathcal{H}}_1\cap\operatorname{dom}\mathbf{d}\,,\qquad \operatorname{dom}\tilde{\boldsymbol{\delta}}_1 = L^2\widetilde{\mathcal{H}}_1\cap\operatorname{dom}\boldsymbol{\delta}\,.$$

Proof. Set $D_\perp = d_{1,0} + \delta_{-1,0}$. Then, by Remark 3.7 and the proof of Lemma 7.2 in [AK], the operators

$$[D_\perp,\Pi]\,,\quad \widetilde{\Pi}_{\cdot,v-1}D_\perp\widetilde{\Pi}_{\cdot,v}\,,\quad (\mathrm{id}-\widetilde{\Pi}_{\cdot,v})D_\perp\widetilde{\Pi}_{\cdot,v}$$

on Ω define bounded operators on $\mathbf{\Omega}$. This easily yields property (i). Now properties (ii) and (iii) follows from property (i) since $d_{2,-1}$ and $\delta_{-2,1}$ are of order zero, and $d_{0,1}$ and $\delta_{0,-1}$ vanish on $\mathcal{H}_1$ and preserve each $\widetilde{\mathcal{H}}_1^{\cdot,v}$. □

3 L^2 Spectral Sequence

The L^2 spectral sequence $(\mathbf{E}_k,\mathbf{d}_k)$ is introduced in this section. Its definition and basic properties are stated for arbitrary foliations (section 3.1), but a deeper study is only achieved for Riemannian foliations (section 3.2). In this case, we prove that $(\mathbf{E}_k,\mathbf{d}_k)$ is "almost" isomorphic to the differentiable spectral sequence (E_k,d_k). This will allow to use $(\mathbf{E}_k,\mathbf{d}_k)$ as an intermediate step to establish a first relationship of (E_k,d_k) with the spectral asymptotics in the adiabatic limit.

The rather atypical notation for the L^2 spectral sequence and related objects was chosen with the aim of simplifying complicated expressions.

3.1 General properties. For a C^∞ foliation $\mathcal{F}$ on a closed manifold M, what we call the L^2 spectral sequence of $\mathcal{F}$ is also a spectral sequence $(\mathbf{E}_k,\mathbf{d}_k)$ converging to the de Rham cohomology of M; in fact, it converges to the L^2 cohomology of M, but both cohomologies are canonically isomorphic since M is closed. Recall that $\mathbf{\Omega}$ denotes the Hilbert space of square integrable differential forms on M, and $\mathbf{d}$ denotes the closure of d in $\mathbf{\Omega}$.

Also, let $\mathbf{\Omega}_k$ be the closure of Ω_k in $\mathbf{\Omega}$, and consider the decreasing filtration of the complex $(\operatorname{dom}\mathbf{d},\mathbf{d})$ by the differential subspaces $\mathbf{\Omega}_k\cap\operatorname{dom}\mathbf{d}$. We define $(\mathbf{E}_k,\mathbf{d}_k)$ to be the corresponding spectral sequence. Since the inclusion $\Omega\hookrightarrow\operatorname{dom}\mathbf{d}$ obviously is a homomorphism of filtered complexes, it induces a canonical homomorphism $(E_k,d_k)\to(\mathbf{E}_k,\mathbf{d}_k)$ of spectral sequences. We point out that, by the compactness of M, the filtered complex $(\operatorname{dom}\mathbf{d},\mathbf{d})$ is well defined independently of any metric, and thus so is the L^2 spectral sequence $(\mathbf{E}_k,\mathbf{d}_k)$.

Each $\mathbf{E}_1^{u,v}$ is a topological vector space with the topology induced by the L^2 norm of $\mathbf{\Omega}$, and consider the product topology on $\mathbf{E}_1=\bigoplus_{u,v}\mathbf{E}_1^{u,v}$.

The notation $Z_k^{u,v}$ and $B_k^{u,v}$ of section 2.1 will be used for the spaces involved in the definition of the differentiable spectral sequence of $\mathcal{F}$, and the corresponding spaces for the L^2 spectral sequence will be denoted by $\mathbf{Z}_k^{u,v}$ and $\mathbf{B}_k^{u,v}$. We have

$$\begin{aligned}\mathbf{Z}_k^{u,v}&=\mathbf{\Omega}_u^{u+v}\cap\mathbf{d}^{-1}\left(\mathbf{\Omega}_{u+k}^{u+v+1}\right), &\mathbf{Z}_\infty^{u,v}&=\mathbf{\Omega}_u^{u+v}\cap\ker\mathbf{d}\,,\\ \mathbf{B}_k^{u,v}&=\mathbf{\Omega}_u^{u+v}\cap\mathbf{d}\left(\mathbf{\Omega}_{u-k}^{u+v-1}\cap\operatorname{dom}\mathbf{d}\right), &\mathbf{B}_\infty^{u,v}&=\mathbf{\Omega}_u^{u+v}\cap\operatorname{im}\mathbf{d}\,.\end{aligned}$$

As in the case of the differentiable spectral sequence, let $\pi_{u,v}:\mathbf{\Omega}\to\mathbf{\Omega}^{u,v}$ be the canonical projection defined by the bigrading of $\mathbf{\Omega}$; i.e., $\pi_{u,v}:\mathbf{\Omega}\to\mathbf{\Omega}^{u,v}$ is the continuous extension of $\pi_{u,v}:\Omega\to\Omega^{u,v}$. Consider also the topological vector spaces

$$\mathbf{z}_k^{u,v}=\pi_{u,v}\left(\mathbf{Z}_k^{u,v}\right),\ \mathbf{b}_k^{u,v}=\pi_{u,v}\left(\mathbf{B}_k^{u,v}\right),\ \mathbf{e}_k^{u,v}=\mathbf{z}_k^{u,v}/\mathbf{b}_{k-1}^{u,v},\ \mathbf{e}_k=\bigoplus_{u,v}\mathbf{e}_k^{u,v}$$

for $k=0,1,\dots,\infty$, with the topology induced by the L^2 norm of $\mathbf{\Omega}$. We clearly have $\mathbf{Z}_k^{u,v}\cap\ker\pi_{u,v}=\mathbf{Z}_{k-1}^{u+1,v-1}$, and thus each projection $\pi_{u,v}$ induces a continuous linear isomorphism $\mathbf{E}_k^{u,v}\xrightarrow{\cong}\mathbf{e}_k^{u,v}$. Via these isomorphisms, the differential $\mathbf{d}_k$ on $\mathbf{E}_k$ induces a differential on $\mathbf{e}_k$ that will be denoted by $\mathbf{d}_k$ as well. We also have canonical continuous homomorphisms $e_k^{u,v}\to\mathbf{e}_k^{u,v}$.

In general, the L^2 spectral sequence is more difficult to deal with than the differentiable spectral sequence. For example, we do not know whether the continuous linear isomorphism $\mathbf{E}_1^{u,v}\xrightarrow{\cong}\mathbf{e}_1^{u,v}$, induced by $\pi_{u,v}$, is a homeomorphism with this generality. Also, the useful expressions (2.3)–(2.8) do not hold for the L^2 spectral sequence; indeed, for $r=u+v$, instead of (2.3)–(2.5) we have

$$\mathbf{Z}_0^{u,v}=\mathbf{\Omega}_u^r\cap\operatorname{dom}\mathbf{d}\,,\tag{3.1}$$

$$\mathbf{B}_0^{u,v}=\mathbf{d}(\mathbf{\Omega}_u^{r-1}\cap\operatorname{dom}\mathbf{d})\,,\tag{3.2}$$

$$\mathbf{Z}_1^{u,v}=\left((\mathbf{\Omega}^{u,v}\cap\ker\mathbf{d}_{0,1})+\mathbf{\Omega}_{u+1}^r\right)\cap\operatorname{dom}\mathbf{d}\,.\tag{3.3}$$

For this reason, it will be useful to introduce the spaces

$$D^{u,v} = \pi_{u,v}(\mathbf{\Omega}^r_u \cap \operatorname{dom} \mathbf{d}) \subset \mathbf{\Omega}^{u,v}\,, \quad r = u+v\,,$$

which satisfy

$$(V+\mathbf{\Omega}^r_{u+1}) \cap \operatorname{dom}\mathbf{d} = \big((V\cap D^{u,v})+\mathbf{\Omega}^r_{u+1}\big)\cap \operatorname{dom}\mathbf{d}, \quad r=u+v. \tag{3.4}$$

for any subspace $V \subset \mathbf{\Omega}^{u,v}$.

Observe that the canonical homomorphism $E_0^{u,v} \to \mathbf{E}_0^{u,v}$ is injective with dense image because it is just the inclusion $Z_0^{u,v} \hookrightarrow \mathbf{Z}_0^{u,v}$, whose image is dense by (2.3) and (3.1). With this generality, at least injectivity holds for $E_1 \to \mathbf{E}_1$ too, as asserted by the following result.

Lemma 3.1. *The canonical homomorphism $E_1 \to \mathbf{E}_1$ is injective.*

Proof. For $r = u+v$ we have

$$\begin{aligned}
\mathbf{Z}_0^{u+1,v-1} + \mathbf{B}_0^{u,v} &= (\mathbf{\Omega}^r_{u+1}\cap \operatorname{dom}\mathbf{d}) + \mathbf{d}(\mathbf{\Omega}^{r-1}_u \cap \operatorname{dom}\mathbf{d})\\
&= \big(\mathbf{\Omega}^r_{u+1} + \mathbf{d}(\mathbf{\Omega}^{r-1}_u\cap\operatorname{dom}\mathbf{d})\big)\cap\operatorname{dom}\mathbf{d}\\
&= (\mathbf{d}_{0,1}D^{u,v-1} + \mathbf{\Omega}^r_{u+1})\cap\operatorname{dom}\mathbf{d}
\end{aligned} \tag{3.5}$$

by (3.2), (3.1), and since $\operatorname{im}\mathbf{d}\subset\operatorname{dom}\mathbf{d}$. Then

$$Z_1^{u,v}\cap(\mathbf{Z}_0^{u+1,v-1}+\mathbf{B}_0^{u,v}) = Z_0^{u+1,v-1} + B_0^{u,v}$$

by (2.4), (2.3) and (2.5), and the result follows. □

Lemma 3.2. *We have $D^{u,v}\subset\operatorname{dom}\mathbf{d}_{0,1}$.*

Proof. Take any $\alpha\in D^{u,v}$. For $r=u+v$, there exists some $\beta\in\mathbf{\Omega}^r_{u+1}$ such that $\alpha+\beta\in\operatorname{dom}\mathbf{d}$. So $\pi_{u,v}\mathbf{d}(\alpha+\beta)$ is defined in $\mathbf{\Omega}^{u,v}$. But $\pi_{u,v}\mathbf{d}(\alpha+\beta)=\mathbf{d}_{0,1}\alpha$ because $\alpha+\beta\in\mathbf{\Omega}^r_u$. □

Lemma 3.3. *We have*

$$\pi_{u,v}(\mathbf{Z}_1^{u,v}) = D^{u,v}\cap\ker\mathbf{d}_{0,1}\,,\quad \pi_{u,v}(\mathbf{B}_0^{u,v}) = \mathbf{d}_{0,1}D^{u,v-1}\,,$$

and thus

$$\mathbf{e}_1^{u,v} = \frac{D^{u,v}\cap\ker\mathbf{d}_{0,1}}{\mathbf{d}_{0,1}D^{u,v-1}}\,.$$

Proof. For $r=u+v$, we have

$$\begin{aligned}
\pi_{u,v}(\mathbf{Z}_1^{u,v}) &= \pi_{u,v}\big(((\mathbf{\Omega}^{u,v}\cap\ker\mathbf{d}_{0,1})+\mathbf{\Omega}^r_{u+1})\cap\operatorname{dom}\mathbf{d}\big)\,, && \text{by (3.3)}\,,\\
&= \pi_{u,v}\big(((D^{u,v}\cap\ker\mathbf{d}_{0,1})+\mathbf{\Omega}^r_{u+1})\cap\operatorname{dom}\mathbf{d}\big)\,, && \text{by (3.4)}\,,\\
&= D^{u,v}\cap\ker\mathbf{d}_{0,1}\,,\\
\pi_{u,v}(\mathbf{B}_0^{u,v}) &= \pi_{u,v}\big(\mathbf{d}(\mathbf{\Omega}^{r-1}_u\cap\operatorname{dom}\mathbf{d})\big)\,, && \text{by (3.2)}\,,\\
&= \pi_{u,v}\big((\mathbf{d}_{0,1}D^{u,v-1}+\mathbf{\Omega}^r_{u+1})\cap\operatorname{dom}\mathbf{d}\big)\,, && \text{by (3.4)}\,,\\
&= \mathbf{d}_{0,1}D^{u,v-1}\,.
\end{aligned}$$

□

As for the differentiable spectral sequence, let $\bar{\mathbf{0}}_1 \subset \mathbf{E}_1$ and $\bar{\mathbf{o}}_1 \subset \mathbf{e}_1$ be the closures of the corresponding trivial subspaces, which are bigraded subspaces with bigraded quotients $\widehat{\mathbf{E}}_1 = \mathbf{E}_1/\bar{\mathbf{0}}_1$ and $\hat{\mathbf{e}}_1 = \mathbf{e}_1/\bar{\mathbf{o}}_1$. Lemma 3.3 has the following direct consequence.

Corollary 3.4. *We have*

$$\bar{\mathbf{o}}_1^{u,v} = \frac{D^{u,v} \cap \mathrm{cl}_0(\mathbf{d}_{0,1} D^{u,v-1})}{\mathbf{d}_{0,1} D^{u,v-1}} = \frac{D^{u,v} \cap \mathrm{cl}_0(d_{0,1} \Omega^{u,v-1})}{\mathbf{d}_{0,1} D^{u,v-1}} ,$$

$$\hat{\mathbf{e}}_1^{u,v} = \frac{D^{u,v} \cap \ker \mathbf{d}_{0,1}}{D^{u,v} \cap \mathrm{cl}_0(\mathbf{d}_{0,1} D^{u,v-1})} = \frac{D^{u,v} \cap \ker \mathbf{d}_{0,1}}{D^{u,v} \cap \mathrm{cl}_0(d_{0,1} \Omega^{u,v-1})} .$$

The map $\mathbf{d}_1$, either on $\mathbf{E}_1$ or on $\mathbf{e}_1$, may not be continuous. So $\bar{\mathbf{0}}_1$, $\widehat{\mathbf{E}}_1$, $\bar{\mathbf{o}}_1$ and $\hat{\mathbf{e}}_1$ may not have canonical structures of bigraded complexes in general. However we shall show that this holds for Riemannian foliations in section 3.2.

3.2 L^2 spectral sequence of Riemannian foliations.

Theorem 3.5. *Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold M. Then the canonical map $E_k \to \mathbf{E}_k$ is injective with dense image for $k = 0, 1$, and is an isomorphism of topological vector spaces for $k \geq 2$. In particular $\mathbf{E}_k$ is Hausdorff of finite dimension for $k \geq 2$.*

The goal of this subsection is to prove Theorem 3.5. Thus, from now on, assume $\mathcal{F}$ is a Riemannian foliation. Since its statement is independent of any metric on M, we can take a bundle-like metric on M to prove it.

In Theorem 3.5, the case $k = 0$ is obvious, and the case $k = 1$ follows directly from Lemma 3.1 and the following lemma.

Lemma 3.6. *The space $Z_1^{u,v}$ is dense in $\mathbf{Z}_1^{u,v}$.*

Proof. Since the orthogonal projection

$$\widetilde{Q} : \mathbf{\Omega}^{u,v} \longrightarrow \mathbf{\Omega}^{u,v} \cap \ker \mathbf{d}_{0,1}$$

preserves smoothness on M, the result follows by (2.5) and (3.3). □

The proof of Theorem 3.5 for $k \geq 2$ requires much more work than Lemma 3.6. To establish this, we shall use the Hodge theoretic approach to e_1 and e_2 from section 2.2, and a similar approach to $\mathbf{e}_1$ and $\mathbf{e}_2$. To begin with, we show that $\mathbf{d}_1$ preserves $\bar{\mathbf{o}}_1$.

Lemma 3.7. *We have $\mathbf{d}_1(\bar{\mathbf{o}}_1) \subset \bar{\mathbf{o}}_1$.*

Proof. Take any $\alpha \in D^{u,v} \cap \mathrm{cl}_0(d_{0,1}\Omega^{u,v-1})$, and fix some $\beta \in \mathbf{\Omega}^r_{u+1}$ with $\alpha + \beta \in \mathrm{dom}\, \mathbf{d}$, where $r = u + v$. We know that $\pi_{u+1,v}\mathbf{d}(\alpha + \beta) \in D^{u+1,v}$

by Lemma 3.3. On the other hand, if $\bar{\mathbf{d}}_{0,1}$ and $\bar{\mathbf{d}}_{1,0}$ denote the extensions of $d_{0,1}$ and $d_{1,0}$ to continuous maps $\mathbf{\Omega} \to W^{-1}\Omega$, we have

$$\pi_{u+1,v}\mathbf{d}(\alpha+\beta) = \bar{\mathbf{d}}_{1,0}\alpha + \bar{\mathbf{d}}_{0,1}\beta_1 \in \bar{\mathbf{d}}_{1,0}\alpha + \bar{\mathbf{d}}_{0,1}\mathbf{\Omega}^{u+1,v-1},$$

where $\beta_1 = \pi_{u+1,v-1}\beta \in \mathbf{\Omega}^{u+1,v-1}$, and

$$\bar{\mathbf{d}}_{1,0}\alpha \in \bar{\mathbf{d}}_{1,0}\big(\operatorname{cl}_0(d_{0,1}\Omega^{u,v-1})\big) \subset \operatorname{cl}_{-1}(d_{0,1}\Omega^{u+1,v-1}).$$

Hence

$$\pi_{u+1,v}\mathbf{d}(\alpha+\beta) \in D^{u+1,v} \cap \operatorname{cl}_{-1}(d_{0,1}\Omega^{u+1,v-1}) = D^{u+1,v} \cap \operatorname{cl}_0(d_{0,1}\Omega^{u+1,v-1})$$

by Theorem 2.1. Therefore the result follows by Lemma 3.3 and Corollary 3.4. $\square$

Now $\bar{\mathbf{o}}_1$ and $\hat{\mathbf{e}}_1$ canonically are bigraded complexes by Lemma 3.7, and we have the short exact sequence

$$0 \longrightarrow \bar{\mathbf{o}}_1 \longrightarrow \mathbf{e}_1 \longrightarrow \hat{\mathbf{e}}_1 \longrightarrow 0,$$

which induces long exact sequences

$$\cdots \to H^u(\bar{\mathbf{o}}_1^{\cdot,v}) \to H^u(\mathbf{e}_1^{\cdot,v}) \to H^u(\hat{\mathbf{e}}_1^{\cdot,v}) \to H^{u+1}(\bar{\mathbf{o}}_1^{\cdot,v}) \to \cdots. \tag{3.6}$$

Lemma 3.8. *We have*

$$D^{u,v} \cap \ker \mathbf{d}_{0,1} = (D^{u,v} \cap L^2\mathcal{H}_1) \oplus \big(D^{u,v} \cap \operatorname{cl}_0(d_{0,1}\Omega^{u,v-1})\big)$$

as topological vector spaces, and moreover

$$D^{u,v} \cap L^2\mathcal{H}_1 = L^2\mathcal{H}_1^{u,v} \cap \operatorname{dom} \mathbf{d}_1.$$

Proof. The inclusion "$\supset$" of the first equality is obvious, and the inclusion "$\supset$" of the second equality follows from Lemma 2.4(iii).

To prove the inclusion "$\subset$" of the first equality, by (2.9) it is enough to prove that $\Pi\alpha \in D^{u,v}$ for all $\alpha \in D^{u,v} \cap \ker \mathbf{d}_{0,1}$. This obviously holds if we prove $\Pi\alpha \in \operatorname{dom} \mathbf{d}_1$ for every such an α since the inclusion "$\supset$" of the second equality is already proved. This also proves the inclusion "$\subset$" of the second equality by taking $\alpha \in L^2\mathcal{H}_1$.

Thus take any $\alpha \in D^{u,v} \cap \ker \mathbf{d}_{0,1}$. Then there is some $\beta \in \mathbf{\Omega}^r_{u+1}$ such that $\alpha+\beta \in \operatorname{dom}\mathbf{d}$, where $r = u+v$. Write $\beta = \beta_1+\beta_2$ with $\beta_1 \in \mathbf{\Omega}^{u+1,v-1}$ and $\beta_2 \in \mathbf{\Omega}^r_{u+2}$. Thus, since $\alpha \in \ker \mathbf{d}_{0,1}$, we get

$$\mathbf{\Omega}^{u+1,v} \ni \Pi\pi_{u+1,v}\mathbf{d}(\alpha+\beta) = \Pi(\bar{\mathbf{d}}_{1,0}\alpha + \bar{\mathbf{d}}_{0,1}\beta_1) = \Pi\bar{\mathbf{d}}_{1,0}\alpha.$$

Here we consider Π and $\pi_{u+1,v}$ as bounded operators on $W^{-1}\Omega$. But

$$\Pi\bar{\mathbf{d}}_{1,0}\alpha = \Pi\bar{\mathbf{d}}_{1,0}\Pi\alpha + \Pi\bar{\mathbf{d}}_{1,0}P\alpha$$

because $Q\alpha = 0$, and

$$\Pi\bar{\mathbf{d}}_{1,0}P\alpha = \Pi\widetilde{P}\bar{\mathbf{d}}_{1,0}P\alpha \in \mathbf{\Omega}$$

by Lemma 2.4(i). Therefore $\Pi\bar{\mathbf{d}}_{1,0}\Pi\alpha \in \mathbf{\Omega}$, yielding $\Pi\alpha \in \operatorname{dom}\mathbf{d}_1$ as desired. $\square$

Corollary 3.9. *The inclusions* $L^2\mathcal{H}_1^{u,v}\cap\operatorname{dom}\mathbf{d}_1 \hookrightarrow D^{u,v}\cap\ker\mathbf{d}_{0,1}$ *induce an isomorphism of* $(\operatorname{dom}\mathbf{d}_1,\mathbf{d}_1)$ *onto the quotient complex* $\hat{\mathbf{e}}_1$, *which is also an isomorphism of topological vector spaces.*

Proof. This follows from Corollary 3.4 and Lemma 3.8. □

Corollary 3.10. *Each inclusion* $\mathcal{H}_2^{u,v} \hookrightarrow D^{u,v}\cap\ker\mathbf{d}_{0,1}$ *induces an isomorphism* $H^u(\hat{\mathbf{e}}_1^{\cdot,v})\cong\mathcal{H}_2^{u,v}$ *of topological vector spaces. In particular* $H(\hat{\mathbf{e}}_1)$ *is Hausdorff of finite dimension.*

Proof. This follows from Corollary 3.9 and Theorem 2.2(iii),(iv). □

The canonical homomorphism $e_1\to\mathbf{e}_1$ is obviously continuous. Hence it induces homomorphisms of complexes $\bar{o}_1\to\bar{\mathbf{o}}_1$ and $\hat{e}_1\to\hat{\mathbf{e}}_1$, and homomorphisms $H(\bar{o}_1)\to H(\bar{\mathbf{o}}_1)$ and $H(\hat{e}_1)\to H(\hat{\mathbf{e}}_1)$ in cohomology.

Corollary 3.11. *The canonical map* $H(\hat{e}_1)\to H(\hat{\mathbf{e}}_1)$ *is an isomorphism of topological vector spaces.*

Proof. This follows from Theorem 2.2(iv) and Corollary 3.10. □

We also need a Hodge theoretic study of certain complex whose cohomology is isomorphic to $H(\bar{\mathbf{o}}_1)$. To simplify notation let

$$\mathcal{Z}_v=\bigoplus_u \mathbf{Z}_1^{u,v}\,,\quad \mathcal{B}_v=\bigoplus_u\left(\mathbf{Z}_0^{u-1,v+1}+\mathbf{B}_0^{u,v}\right),$$

which are subcomplexes of $(\operatorname{dom}\mathbf{d},\mathbf{d})$. (This notation is used in [AK] for the C^∞ versions of these complexes.) Then

$$\bar{\mathbf{o}}_1^{\cdot,v}=\operatorname{cl}_0(\mathcal{B}_v)/\mathcal{B}_v\,. \tag{3.7}$$

Observe that $\mathcal{Z}_{v-1}\subset\mathcal{B}_v$.

Lemma 3.12. *The quotient complex* $\mathcal{B}_v/\mathcal{Z}_{v-1}$ *is acyclic. Thus the quotient map* $\operatorname{cl}_0(\mathcal{B}_v)/\mathcal{Z}_{v-1}\to\operatorname{cl}_0(\mathcal{B}_v)/\mathcal{B}_v=\bar{\mathbf{o}}_1^{\cdot,v}$ *induces an isomorphism in cohomology.*

Proof. The result follows from (3.3) and (3.5) with easy arguments (see Lemma 2.5 in [Se] and Lemma 7.4 in [AK]). □

Set

$$\begin{gathered}\widetilde{\mathbf{\Omega}}^{u,v}=\mathbf{\Omega}^{u,v}+\mathbf{\Omega}^{u+1,v-1}\,,\\ \tilde{\pi}_{u,v}=\pi_{u,v}+\pi_{u+1,v-1}:\mathbf{\Omega}\longrightarrow\widetilde{\mathbf{\Omega}}^{u,v}\,,\\ \widetilde{D}^{u,v}=\tilde{\pi}_{u,v}(\mathbf{\Omega}_u^r\cap\operatorname{dom}\mathbf{d})\,,\quad r=u+v\,.\end{gathered}$$

We have

$$\operatorname{cl}_0(\mathcal{B}_v)\cap\ker\tilde{\pi}_{u,v}\subset\mathcal{Z}_{v-1}\cap\ker\tilde{\pi}_{u,v}\,.$$

Hence, for each topological vector space

$$\tilde{\mathbf{e}}_1^{u,v} = \frac{\tilde{\pi}_{u,v}(\mathrm{cl}_0(\mathcal{B}_v))}{\tilde{\pi}_{u,v}(\mathcal{Z}_{v-1})}\,,$$

the projection $\tilde{\pi}_{u,v}$ induces a continuous linear isomorphism

$$\mathrm{cl}_0(\mathcal{B}_v^r)/\mathcal{Z}_{v-1}^r \xrightarrow{\cong} \tilde{\mathbf{e}}_1^{u,v}\,, \quad r = u + v\,. \tag{3.8}$$

Let $\tilde{\mathbf{d}}_1$ be the operator on $\tilde{\mathbf{e}}_1 = \bigoplus_{u,v} \tilde{\mathbf{e}}_1^{u,v}$ that corresponds to the differential operator on the quotient complex $\mathrm{cl}_0(\mathcal{B}_v)/\mathcal{Z}_{v-1}$ by the above isomorphisms. Observe that $\tilde{\mathbf{d}}_1$ is given as follows: if $\alpha \in \mathrm{cl}_0(\mathcal{B}_v)$, and $[\tilde{\pi}_{u,v}\alpha] \in \tilde{\mathbf{e}}_1^{u,v}$ denotes the class defined by $\tilde{\pi}_{u,v}\alpha$, then $\tilde{\mathbf{d}}_1[\tilde{\pi}_{u,v}\alpha] = [\tilde{\pi}_{u+1,v}\mathbf{d}\alpha]$.

The spaces $\widetilde{D}^{u,v}$ and $D^{u,v}$ have similar properties. For instance, for any subspace $V \subset \widetilde{\mathbf{\Omega}}^{u,v}$ we have

$$(V + \mathbf{\Omega}_{u+2}^r) \cap \mathrm{dom}\,\mathbf{d} = \big((V \cap \widetilde{D}^{u,v}) + \mathbf{\Omega}_{u+2}^r\big) \cap \mathrm{dom}\,\mathbf{d}\,,\ r = u+v\,. \tag{3.9}$$

Lemma 3.13. *For $r = u + v$, we have*

$$\tilde{\pi}_{u,v}(\mathrm{cl}_0(\mathcal{B}_v)) = \widetilde{D}^{u,v} \cap \mathrm{cl}_0(\mathcal{B}_v)\,,$$
$$\tilde{\pi}_{u,v}(\mathcal{Z}_{v-1}) = D^{u+1,v-1} \cap \mathcal{Z}_{v-1} = D^{u+1,v-1} \cap \ker \mathbf{d}_{0,1}\,,$$

and thus

$$\tilde{\mathbf{e}}_1^{u,v} = \frac{\widetilde{D}^{u,v} \cap \mathrm{cl}_0(\mathcal{B}_v)}{D^{u+1,v-1} \cap \ker \mathbf{d}_{0,1}}\,.$$

Proof. This easily follows from (3.4) and (3.9). □

The following result and Lemma 3.8 are similar, as well as their proofs.

Lemma 3.14. *We have*

$$\widetilde{D}^{u,v} \cap \mathrm{cl}_0(\mathcal{B}_v) = (\widetilde{D}^{u,v} \cap L^2\widetilde{\mathcal{H}}_1) \oplus (D^{u+1,v-1} \cap \ker \mathbf{d}_{0,1})$$

as topological vector spaces, and moreover

$$\widetilde{D}^{u,v} \cap L^2\widetilde{\mathcal{H}}_1 = L^2\widetilde{\mathcal{H}}_1^{u,v} \cap \mathrm{dom}\,\tilde{\mathbf{d}}_1\,.$$

Proof. The inclusion "$\supset$" of the first equality is obvious, and the inclusion "$\supset$" of the second equality follows from Lemma 2.4(iii).

To prove the inclusion "$\subset$" of the first equality, by (2.9) it is enough to prove that $\widetilde{\Pi}\alpha \in \widetilde{D}^{u,v}$ for all $\alpha \in \widetilde{D}^{u,v} \cap \mathrm{cl}_0(\mathcal{B}_v)$. This obviously holds if we prove $\widetilde{\Pi}\alpha \in \mathrm{dom}\,\tilde{\mathbf{d}}_1$ for every such an α since the inclusion "$\supset$" of the second equality is already proved. This also proves the inclusion "$\subset$" of the second equality by taking $\alpha \in L^2\widetilde{\mathcal{H}}_1$.

Thus take any $\alpha \in \widetilde{D}^{u,v} \cap \mathrm{cl}_0(\mathcal{B}_v)$. Then there is some $\beta \in \mathbf{\Omega}_{u+2}^r$ such that $\alpha + \beta \in \mathrm{dom}\,\mathbf{d}$, where $r = u + v$. Write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in \mathbf{\Omega}^{u,v}$ and $\alpha_2 \in \mathbf{\Omega}^{u+1,v-1}$, and let $\bar{\mathbf{d}} : \mathbf{\Omega} \to W^{-1}\Omega$ denote the continous extension of $\mathbf{d}$. Since $\alpha \in \mathrm{cl}_0(\mathcal{B}_v)$ and $\bar{\mathbf{d}}\beta \in \ker \widetilde{\Pi}_{\cdot,v}$, where $\widetilde{\Pi}_{\cdot,v}$ is considered as a projection in $W^{-1}\Omega$, we get

$$\widetilde{\boldsymbol{\Omega}}^{u+1,v} \ni \widetilde{\Pi}_{\cdot,v}\mathbf{d}(\alpha+\beta) = \widetilde{\Pi}_{\cdot,v}\bar{\mathbf{d}}\alpha\,.$$

But

$$\widetilde{\Pi}_{\cdot,v}\bar{\mathbf{d}}\alpha = \widetilde{\Pi}_{\cdot,v}\bar{\mathbf{d}}\widetilde{\Pi}_{\cdot,v}\alpha + \widetilde{\Pi}_{\cdot,v}\bar{\mathbf{d}}\Pi\alpha_2 + \widetilde{\Pi}_{\cdot,v}\bar{\mathbf{d}}\widetilde{\Pi}_{\cdot,v-1}\alpha_2$$

because $\alpha \in \widetilde{D}^{u,v} \cap \mathrm{cl}_0(\mathcal{B}_v)$, and

$$\widetilde{\Pi}_{\cdot,v}\bar{\mathbf{d}}\Pi\alpha_2 + \widetilde{\Pi}_{\cdot,v}\bar{\mathbf{d}}\widetilde{\Pi}_{\cdot,v-1}\alpha_2 \in \boldsymbol{\Omega}$$

by Lemma 2.4(ii). Therefore $\widetilde{\Pi}_{\cdot,v}\bar{\mathbf{d}}\widetilde{\Pi}_{\cdot,v}\alpha \in \boldsymbol{\Omega}$, yielding $\widetilde{\Pi}_{\cdot,v}\alpha \in \mathrm{dom}\,\tilde{\mathbf{d}}_1$ as desired. □

Consider the projection

$$\widetilde{D}^{u,v} \cap \mathrm{cl}_0(\mathcal{B}_v) \longrightarrow \widetilde{D}^{u,v} \cap L^2\widetilde{\mathcal{H}}_1 = L^2\widetilde{\mathcal{H}}_1^{u,v} \cap \mathrm{dom}\,\tilde{\mathbf{d}}_1$$

defined by Lemma 3.14, which is obviously an orthogonal projection.

Corollary 3.15. *The inclusions* $L^2\widetilde{\mathcal{H}}_1^{u,v} \cap \mathrm{dom}\,\tilde{\mathbf{d}}_1 \hookrightarrow \widetilde{D}^{u,v} \cap \mathrm{cl}_0(\mathcal{B}_v)$ *induce an isomorphism* $(\mathrm{dom}\,\tilde{\mathbf{d}}_1, \tilde{\mathbf{d}}_1) \xrightarrow{\cong} (\tilde{\mathbf{e}}_1, \tilde{\mathbf{d}}_1)$ *of bigraded complexes and topological vector spaces.*

Proof. This follows from Lemmas 3.13 and 3.14. □

Corollary 3.16. *We have* $H(\bar{\mathbf{o}}_1) = 0$.

Proof. This follows from (3.7), (3.8), Lemma 3.12, Corollary 3.15 and Theorem 2.2(v). □

Corollary 3.17. *The canonical map* $H(\mathbf{e}_1) \to H(\hat{\mathbf{e}}_1)$ *is an isomorphism of topological vector spaces. In particular* $H(\mathbf{e}_1)$ *is Hausdorff of finite dimension.*

Proof. The canonical map $H(\mathbf{e}_1) \to H(\hat{\mathbf{e}}_1)$ is a linear isomorphism by Corollary 3.16 and the exactness of (3.6). Moreover it is obviously continuous. Then it is also an homeomorphism because $H(\hat{\mathbf{e}}_1)$ is a Hausdorff topological vector space of finite dimension. □

Corollary 3.18. *The canonical map* $H(e_1) \to H(\mathbf{e}_1)$ *is an isomorphism of topological vector spaces.*

Proof. By the commutativity of the diagram

$$\begin{array}{ccc} H(e_1) & \longrightarrow & H(\mathbf{e}_1) \\ \big\downarrow & & \big\downarrow \\ H(\hat{e}_1) & \longrightarrow & H(\hat{\mathbf{e}}_1)\,, \end{array}$$

where all maps are canonical, the result follows directly from Theorem 2.2 (vii), Corollaries 3.11 and 3.17. □

COROLLARY 3.19. *The canonical map $E_2 \to \mathbf{E}_2$ is an isomorphism of topological vector spaces.*

Proof. Consider the compositions

$$E_2 \longrightarrow H(E_1) \longrightarrow H(e_1)\,, \quad \mathbf{E}_2 \longrightarrow H(\mathbf{E}_1) \longrightarrow H(\mathbf{e}_1)\,,$$

where the first map of each composition is canonical, and the second one is canonically induced by the projections $\pi_{u,v}$. The first composition is an isomorphism of topological vector spaces by Theorem 2.2(vii), and we know that the second composition is a continuous linear isomorphism (section 3.1). Then the second composition is also an homeomorphism because $H(\mathbf{e}_1)$ is Hausdorff of finite dimension by Corollary 3.17. So the result follows from Corollary 3.18 and the commutativity of the diagram

$$\begin{array}{ccc} E_2 & \longrightarrow & H(e_1) \\ \downarrow & & \downarrow \\ \mathbf{E}_2 & \longrightarrow & H(\mathbf{e}_1)\,, \end{array}$$

where the horizontal arrows denote the above compositions, and the vertical arrows denote canonical maps. □

Now Theorem 3.5 for $k \geq 2$ follows from Corollary 3.19 because the canonical map $(E_k, d_k) \to (\mathbf{E}_k, \mathbf{d}_k)$ is a homomorphism of spectral sequences.

4 L^2 Spectral Sequence and Small Eigenvalues

The relationship of the L^2 spectral sequence with small eigenvalues is the main result of this section, stated in section 4.1. It is proved for arbitrary foliations when some very technical condition is satisfied. Such a condition holds for Riemannian foliations, but it could hold with greater generality; this suggests that the L^2 spectral sequence could be the right object to consider for a possible generalization of this paper to arbitrary foliations.

The proof of the main result, given in section 4.4, consists of analyzing the connection between other two independent results: A variational formula for the spectral distribution function that was used by Gromov–Shubin in [GrS], which is recalled in section 4.2, and a direct sum decomposition of general spectral sequences, which is described in section 4.3. This connection is rather natural and easy to understand because both of these results are somehow related with the Hodge decomposition of differential forms. Nevertheless, there is a technical difficulty in this proof, which is the reason we were not able to relate the differentiable spectral sequence directly with small eigenvalues (without using the L^2 spectral sequence).

4.1 Main results. Let $\mathcal{F}$ be a C^∞ foliation on a closed manifold M with a Riemannian metric g, and consider the family of metrics g_h, $h > 0$, which were defined in (1.1) and give rise to the adiabatic limit. As in section 1, let Δ_{g_h} denote the Laplacian on Ω defined by g_h, and

$$0 \le \lambda_0^r(h) \le \lambda_1^r(h) \le \lambda_2^r(h) \le \cdots$$

its spectrum on Ω^r, taking multiplicity into account. The following result suggests that, with this generality, the number of small eigenvalues of Δ_h may be more related with the L^2 spectral sequence than with the differentiable one. Nevertheless, so far we do not know about the relevance its hypothesis for non-Riemannian foliations.

Theorem 4.1. *Let $\mathcal{F}$ be a C^∞ foliation on a closed Riemannian manifold. If $\mathbf{Z}_{k-1}^{u+1,v-1} + \mathbf{Z}_\infty^{u,v}$ is closed in $\mathbf{Z}_k^{u,v}$ for all u, v, with $r = u + v$, then*

$$\dim \mathbf{E}_k^r \le \sharp\big\{i \mid \lambda_i^r(h) \in O(h^{2k}) \quad \text{as} \quad h \downarrow 0\big\}$$

for all r.

The following more understandable result is a direct consequence of Theorem 4.1 because

$$\frac{\mathbf{Z}_\ell^{u,v}}{\mathbf{Z}_{\ell-1}^{u+1,v-1} + \mathbf{Z}_\infty^{u,v}}$$

is a quotient of $\mathbf{E}_\ell^{u,v}$.

Corollary 4.2. *Let $\mathcal{F}$ be a C^∞ foliation on a closed Riemannian manifold. If $\mathbf{E}_k$ is Hausdorff of finite dimension, then*

$$\dim \mathbf{E}_\ell^r \le \sharp\big\{i \mid \lambda_i^r(h) \in O(h^{2l}) \quad \text{as} \quad h \downarrow 0\big\}, \quad \ell \ge k\,.$$

Remark 1. Observe that, by Theorem 3.5, Corollary 4.2 holds for Riemannian foliations and $k = 2$, and inequality "$\le$" of (1.2) in Theorem A follows.

The proof of Theorem 4.1 is given in section 4.4, and its two main ingredients are described in sections section 4.2 and section 4.3: the variational formula of the spectral distribution function used by Gromov–Shubin, and the direct sum decomposition for general spectral sequences.

4.2 Spectral distribution function. For a closed Riemannian manifold (M, g), let $N^r(\lambda)$ denote the spectral distribution function of the Laplacian Δ on Ω^r; i.e. $N^r(\lambda)$ is the number of eigenvalues of Δ on Ω^r which are $\le \lambda$, taking multiplicity into account. Recall that $\mathbf{\Omega}$ denotes the Hilbert space of square integrable differential forms with the inner product induced by g, and $\mathbf{d}$ the closure of the de Rham derivative d in $\mathbf{\Omega}$. Let $\bar{\mathbf{d}} : \operatorname{dom}\mathbf{d}/\ker\mathbf{d} \to \mathbf{\Omega}$ denote the map induced by $\mathbf{d}$, and consider the quotient Hilbert norm on $\mathbf{\Omega}/\ker\mathbf{d}$. The following variational expression of $N^r(\lambda)$ is a consequence of the Hodge decomposition of $\mathbf{\Omega}$.

Proposition 4.3 (Gromov–Shubin [GrS]). *We have*

$$N^r(\lambda) = F^{r-1}(\lambda) + \beta^r + F^r(\lambda)\,,$$

where β^r is the rth Betti number of M, and

$$F^r(\lambda) = \sup_L \dim L\,,$$

with L ranging over the closed subspaces of $\operatorname{dom}\mathbf{d}/\ker\mathbf{d}$ *satisfying*

$$\left\|\bar{\mathbf{d}}\zeta\right\| \le \sqrt{\lambda}\,\|\zeta\| \quad \textit{for all} \quad \zeta \in L\,.$$

Now take again a C^∞ foliation $\mathcal{F}$ on M. Then, for each metric g_h of the family (1.1) that gives rise to the adiabatic limit, the spectral distribution function of Δ_{g_h} will be denoted by $N_h^r(\lambda)$, and decomposes as

$$N_h^r(\lambda) = F_h^{r-1}(\lambda) + \beta^r + F_h^r(\lambda)\,,$$

according to Proposition 4.3.

Suppose $\mathcal{F}$ is of codimension q, and let $\|\ \|_h$ be the norm induced by g_h on $\mathbf{\Omega}$. The following equality will be also used to prove Theorem 4.1:

$$\|\omega\|_h = h^{-q/2}h^u\,\|\omega\| \quad \text{if} \quad \omega \in \Omega^{u,v}\,. \tag{4.1}$$

This follows from two observations. First, if the metrics induced by g and g_h on $\bigwedge TM^*$ are also denoted by g and g_h, then $g_h = h^{2u}g$ on forms with transverse degree u. And second, assuming M is oriented, the volume forms μ and μ_h, induced by g and g_h, satisfy $\mu_h = h^{-q}\mu$ since volume forms are of transverse degree q.

By using Proposition 4.3 in the same spirit of [GrS], we could prove that the asymptotics of the $\lambda_i^r(h)$, as $h \downarrow 0$, are C^∞ homotopy invariants of $\mathcal{F}$ (with respect to the appropriate definition of homotopy between foliations). However, for our purposes in this paper, it will be enough to prove that the asymptotics of the $\lambda_i^r(h)$ are independent of the choice of the given metric g on M. This will not be used to prove Theorem 4.1 but will play an important role to finish the proof of Theorem A in section 5.2. Such independence of g is proved in the following way. Let g' be another metric on M with corresponding 1-parameter family of metrics g'_h, and let $\|\ \|'$ and $\|\ \|'_h$ denote the corresponding norms on $\mathbf{\Omega}$. Compactness of M implies the existence of some $C > 0$ such that

$$C^{-1}\,\|\omega\| \le \|\omega\|' \le C\,\|\omega\|$$

for all $\omega \in \mathbf{\Omega}$, yielding

$$C^{-1} \|\omega\|_h \leq \|\omega\|'_h \leq C \|\omega\|_h \tag{4.2}$$

for all $\omega \in \mathbf{\Omega}$ and $h > 0$ by (4.1). Let $N_h'^r(\lambda)$ be the spectral distribution function of $\Delta_{g'_h}$ on Ω^r, and let

$$N_h'^r(\lambda) = F_h'^{r-1}(\lambda) + \beta^r + F_h'^r(\lambda)$$

be its decomposition according to Proposition 4.3. Then

$$F_h'^r(C^{-4}\lambda) \leq F_h^r(\lambda) \leq F_h'^r(C^4\lambda)$$

for all $\lambda \geq 0$ $h > 0$ by (4.2) and the definition of F_h^r and $F_h'^r$. Thus

$$N_h'^r(C^{-4}\lambda) \leq F_h^r(\lambda) \leq N_h'^r(C^4\lambda)\,, \tag{4.3}$$

yielding the metric independence of the asymptotics of the $\lambda_i^r(h)$.

4.3 Direct sum decomposition of spectral sequences. In this subsection we consider the general setting where (E_k, d_k) is the spectral sequence induced by an arbitrary complex $(\mathcal{A}, d)$ with a finite decreasing filtration

$$\mathcal{A} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_q \supset \mathcal{A}_{q+1} = 0$$

by differential subspaces.

Lemma 4.4. *The following properties are satisfied:*

(i) *There is a (non-canonical) isomorphism*

$$\begin{aligned} \mathcal{A}^r &\cong E_\infty^r \oplus \bigoplus_\ell \left((E_\ell^r \cap \operatorname{im} d_\ell) \oplus \frac{E_\ell^r}{E_\ell^r \cap \ker d_\ell} \right) \\ &= \bigoplus_{u+v=r} \left(E_\infty^{u,v} \oplus \bigoplus_\ell \left((E_\ell^{u,v} \cap \operatorname{im} d_\ell) \oplus \frac{E_\ell^{u,v}}{E_\ell^{u,v} \cap \ker d_\ell} \right) \right). \end{aligned}$$

(ii) *The isomorphism in (i) can be chosen so that $\mathcal{A}_k^r$ corresponds to*

$$\bigoplus_{u\geq k,\ u+v=r} \left(E_\infty^{u,v} \oplus \bigoplus_\ell \left((E_\ell^{u,v} \cap \operatorname{im} d_\ell) \oplus \frac{E_\ell^{u,v}}{E_\ell^{u,v} \cap \ker d_\ell} \right) \right).$$

(iii) *The isomorphism in (i) can be chosen so that the only possibly nontrivial components of the operator corresponding to d by (i) are the isomorphisms*

$$\bar{d}_\ell : \frac{E_\ell^{u,v}}{E_\ell^{u,v} \cap \ker d_\ell} \longrightarrow E_\ell^{u+\ell,v-\ell+1} \cap \operatorname{im} d_\ell$$

canonically defined by d_ℓ.

Before proving Lemma 4.4, we state three corollaries that will be needed in the proof of Proposition 4.3.

Corollary 4.5. *There is a (non-canonical) isomorphism*

$$E_k^r \cong E_\infty^r \oplus \bigoplus_{\ell\geq k}\left((E_\ell^r \cap \operatorname{im} d_\ell) \oplus \frac{E_\ell^r}{E_\ell^r \cap \ker d_\ell}\right).$$

Proof. This is a direct consequence of Lemma 4.4. □

Let

$$m_k^r = \dim \bigoplus_{\ell\geq k} \frac{E_\ell^r}{E_\ell^r \cap \ker d_\ell}.$$

Corollary 4.6. *We have*

$$\dim E_k^r = m_k^{r-1} + H^r(\mathcal{A}, d) + m_k^r.$$

Proof. This follows from Corollary 4.5 since each d_ℓ induces isomorphisms

$$\frac{E_\ell^r}{E_\ell^r \cap \ker d_\ell} \cong E_\ell^{r+1} \cap \operatorname{im} d_\ell. \qquad \square$$

Corollary 4.7. *For $r = u+v$, there is a subspace $L_k^{u,v} \subset \mathcal{A}^r/(\mathcal{A}^r \cap \ker d)$ such that:*

(i) *We have*

$$\frac{Z_k^{u,v} + (\mathcal{A}^r \cap \ker d)}{\mathcal{A}^r \cap \ker d} = L_k^{u,v} \oplus \frac{Z_{k-1}^{u+1,v-1} + (\mathcal{A}^r \cap \ker d)}{\mathcal{A}^r \cap \ker d}$$

as vector spaces. In particular $\bar{d}(L_k^{u,v}) \subset \mathcal{A}_{u+k}^{r+1}$.

(ii) *The direct sum $L_k^r = \bigoplus_{u+v=r} L_k^{u,v}$ makes sense in $\mathcal{A}^r/(\mathcal{A}^r \cap \ker d)$, and we have $\dim L_k^r = m_k^r$.*

Proof. From Lemma 4.4 we get a (non-canonical) isomorphism

$$\frac{\mathcal{A}^r}{\mathcal{A}^r \cap \ker d} \cong \bigoplus_\ell \frac{E_\ell^r}{E_\ell^r \cap \ker d_\ell}. \tag{4.4}$$

Then let $L_k^{u,v}$ be the subspace of $\mathcal{A}^r/(\mathcal{A}^r \cap \ker d)$ that corresponds to

$$\bigoplus_{\ell\geq k} \frac{E_\ell^{u,v}}{E_\ell^{u,v} \cap \ker d_\ell}$$

by (4.4). Then property (i) easily follows from Lemma 4.4, and property (ii) is obvious; in fact, L_k^r corresponds to

$$\bigoplus_{\ell\geq k} \frac{E_\ell^r}{E_\ell^r \cap \ker d_\ell}$$

by (4.4). □

Remark 2. By Corollary 4.7(i), the canonical isomorphism

$$\frac{Z_k^{u,v}}{Z_{k-1}^{u+1,v-1} + Z_\infty^{u,v}} \xrightarrow{\cong} \frac{Z_k^{u,v} + (\mathcal{A}^r \cap \ker d)}{Z_{k-1}^{u+1,v-1} + (\mathcal{A}^r \cap \ker d)} \tag{4.5}$$

yields

$$\frac{Z_k^{u,v}}{Z_{k-1}^{u+1,v-1}+Z_\infty^{u,v}} \cong L_k^{u,v}\,.$$

When applying Corollary 4.7 to the L^2 spectral sequence of a C^∞ foliation, the subspaces $L_k^r \subset \operatorname{dom}\mathbf{d}/\ker\mathbf{d}$ of Corollary 4.7 will be the spaces L needed to apply Proposition 4.3.

The rest of this section will be devoted to prove Lemma 4.4. To begin with, we have [Mc]

$$E_\ell^{u,v}\cap d_\ell(E_\ell) = \frac{B_\ell^{u,v}+Z_{\ell-1}^{u+1,v-1}}{Z_{\ell-1}^{u+1,v-1}+B_{\ell-1}^{u,v}}\,,$$

$$E_\ell^{u,v}\cap \ker d_\ell = \frac{Z_{\ell+1}^{u,v}+Z_{\ell-1}^{u+1,v-1}}{Z_{\ell-1}^{u+1,v-1}+B_{\ell-1}^{u,v}}\,.$$

So

$$E_\ell^{u,v}\cap d_\ell(E_\ell) \cong \frac{B_\ell^{u,v}}{B_{\ell+1}^{u+1,v-1}+B_{\ell-1}^{u,v}}\,, \tag{4.6}$$

$$\frac{E_\ell^{u,v}}{E_\ell^{u,v}\cap\ker d_\ell} \cong \frac{Z_\ell^{u,v}}{Z_{\ell-1}^{u+1,v-1}+Z_{\ell+1}^{u,v}} \tag{4.7}$$

canonically. Here, isomorphism (4.7) is obvious, and (4.6) follows since

$$B_{\ell-1}^{u,v}\subset B_\ell^{u,v}\,,\quad B_\ell^{u,v}\cap Z_{\ell-1}^{u+1,v-1} = B_{\ell+1}^{u+1,v-1}\,.$$

Consider the following chain of inclusions for $0\le u\le q$ and $r=u+v$:

$$\left.\begin{aligned} \mathcal{A}_{u+1}^r \subset \mathcal{A}_{u+1}^r + B_0^{u,v} \subset \mathcal{A}_{u+1}^r + B_1^{u,v} \subset \cdots\\ \cdots \subset \mathcal{A}_{u+1}^r + B_\infty^{u,v} \subset \mathcal{A}_{u+1}^r + Z_\infty^{u,v} \subset \cdots\\ \cdots \subset \mathcal{A}_{u+1}^r + Z_2^{u,v} \subset \mathcal{A}_{u+1}^r + Z_1^{u,v} \subset \mathcal{A}_u^r\,. \end{aligned}\right\} \tag{4.8}$$

The inclusions in (4.8) have the following quotients:

$$\frac{\mathcal{A}_{u+1}^r + B_\ell^{u,v}}{\mathcal{A}_{u+1}^r + B_{\ell-1}^{u,v}} \cong \frac{B_\ell^{u,v}}{B_{\ell+1}^{u+1,v-1}+B_{\ell-1}^{u,v}}\,, \tag{4.9}$$

$$\frac{\mathcal{A}_{u+1}^r + Z_\infty^{u,v}}{\mathcal{A}_{u+1}^r + B_\infty^{u,v}} \cong \frac{Z_\infty^{u,v}}{Z_\infty^{u+1,v-1}+B_\infty^{u,v}} = E_\infty^{u,v}\,, \tag{4.10}$$

$$\frac{\mathcal{A}_{u+1}^r + Z_\ell^{u,v}}{\mathcal{A}_{u+1}^r + Z_{\ell+1}^{u,v}} \cong \frac{Z_\ell^{u,v}}{Z_{\ell-1}^{u+1,v-1}+Z_{\ell+1}^{u,v}}\,, \tag{4.11}$$

where these isomorphisms are canonical because

$$\begin{aligned} B_{\ell-1}^{u,v}\subset B_\ell^{u,v}\,,&\qquad B_\ell^{u,v}\cap\mathcal{A}_{u+1}^r = B_{\ell+1}^{u+1,v-1}\,,\\ B_\infty^{u,v}\subset Z_\infty^{u,v}\,,&\qquad Z_\infty^{u,v}\cap\mathcal{A}_{u+1}^r = Z_\infty^{u+1,v-1}\,,\\ Z_{\ell+1}^{u,v}\subset Z_\ell^{u,v}\,,&\qquad Z_\ell^{u,v}\cap\mathcal{A}_{u+1}^r = Z_{\ell-1}^{u+1,v-1}\,. \end{aligned}$$

The direct sum decomposition in property (i) will depend on the choice of linear complements for the inclusions in (4.8):

$$\begin{aligned}\mathcal{A}^r_{u+1}+B^{u,v}_\ell&=U^{u,v}_\ell\oplus(\mathcal{A}^r_{u+1}+B^{u,v}_{\ell-1})\,,\\ \mathcal{A}^r_{u+1}+Z^{u,v}_\infty&=V^{u,v}\oplus(\mathcal{A}^r_{u+1}+B^{u,v}_\infty)\,,\\ \mathcal{A}^r_{u+1}+Z^{u,v}_\ell&=W^{u,v}_\ell\oplus(\mathcal{A}^r_{u+1}+Z^{u,v}_{\ell+1})\,.\end{aligned}$$

On the one hand, since the chains in (4.8) form a filtration of $\mathcal{A}^r$ when varying u, we have

$$\mathcal{A}^r=\bigoplus_{u+v=r}\Big(V^{u,v}\oplus\bigoplus_\ell(U^{u,v}_\ell\oplus W^{u,v}_\ell)\Big)\tag{4.12}$$

as vector space. On the other hand, according to the canonical isomorphisms (4.9), (4.10) and (4.11), the spaces $U^{u,v}_\ell$, $V^{u,v}$ and $W^{u,v}_\ell$ can be chosen so that

$$U^{u,v}_\ell\subset B^{u,v}_\ell\,,\quad V^{u,v}\subset Z^{u,v}_\infty\,,\quad W^{u,v}_\ell\subset Z^{u,v}_\ell\,,\tag{4.13}$$

yielding direct sum decompositions

$$B^{u,v}_\ell=U^{u,v}_\ell\oplus\Big(B^{u+1,v-1}_{\ell+1}+B^{u,v}_{\ell-1}\Big)\,,\tag{4.14}$$

$$Z^{u,v}_\infty=V^{u,v}\oplus(Z^{u+1,v-1}_\infty+B^{u,v}_\infty)\,,\tag{4.15}$$

$$Z^{u,v}_\ell=W^{u,v}_\ell\oplus(Z^{u+1,v-1}_{\ell-1}+Z^{u,v}_{\ell+1})\,.\tag{4.16}$$

Hence

$$U^{u,v}_\ell\cong E^{u,v}_\ell\cap\operatorname{im}d_\ell\,,\tag{4.17}$$

$$V^{u,v}\cong E^{u,v}_\infty\,,\tag{4.18}$$

$$W^{u,v}_\ell\cong\frac{E^{u,v}_\ell}{E^{u,v}_\ell\cap\ker d_\ell}\tag{4.19}$$

by (4.6), (4.7) and (4.14)–(4.16). Therefore property (i) follows from (4.12) and (4.17)–(4.19).

Property (ii) follows from (4.12) because

$$U^{u,v}_\ell\,,\ V^{u,v}\,,\ W^{u,v}_\ell\subset\mathcal{A}^r_u\,.$$

Now property (iii) is obviously equivalent to the existence of $U^{u,v}_\ell$, $V^{u,v}$ and $W^{u,v}_\ell$ as above satisfying

$$d\left(U^{u,v}_\ell\right)=d\left(V^{u,v}\right)=0\,,\quad d\left(W^{u,v}_\ell\right)=U^{u+\ell,v-\ell+1}_\ell\,.\tag{4.20}$$

The first equality of (4.20) holds by (4.13). We shall also check that, once the $W^{u,v}_\ell$ is given satisfying (4.16), the $U^{u,v}_\ell$ defined by (4.20) satisfies (4.14). This follows because d canonically induces a map

$$\hat d_\ell:\frac{Z^{u,v}_\ell}{Z^{u+1,v-1}_{\ell-1}+Z^{u,v}_{\ell+1}}\longrightarrow\frac{B^{u+\ell,v-\ell+1}_\ell}{B^{u+\ell+1,v-\ell}_{\ell+1}+B^{u+\ell,v-\ell+1}_{\ell-1}}\,,$$

which corresponds to the isomorphism $\bar{d}_\ell$ via (4.7) and (4.6). So $\hat{d}_\ell$ is an isomorphism as well, and thus the above $U_\ell^{u,v}$ satisfies (4.14) as desired. This finishes the proof of Lemma 4.4.

4.4 Proof of Theorem 4.1. Assume $\mathbf{Z}_{k-1}^{u+1,v-1} + \mathbf{Z}_\infty^{u,v}$ is closed in $\mathbf{Z}_k^{u,v}$ for all u, v. We shall need the following abstract result.

LEMMA 4.8. *Let L be a real complete metrizable topological vector space, and $V, W \subset L$ linear subspaces. If $V \cap W = 0$, V is closed in L, and W is closed in $V + W$, then $V + W = V \oplus W$ as topological vector spaces.*

Proof. We have $(V + W) \cap \overline{W} = W$ since W is closed in $V + W$, yielding $V \cap \overline{W} = 0$ because $V \cap W = 0$. So $V + \overline{W} = V \oplus \overline{W}$ as topological vector spaces because all spaces involved are closed subspaces of L (see for instance [S, Corollary 3 of Theorem 2.1, Chapter III, page 78]). Now the result follows easily. □

LEMMA 4.9. *For $u + v = r$, the space $(\mathbf{\Omega}^r \cap \ker \mathbf{d}) + \mathbf{\Omega}_{u+1}^r$ is a closed subspace of $\mathbf{Z}_k^{u,v} + (\mathbf{\Omega}^r \cap \ker \mathbf{d}) + \mathbf{\Omega}_{u+1}^r$.*

Proof. The space $\mathbf{\Omega}^r \cap \ker \mathbf{d}$ is closed in $\mathbf{\Omega}$ since $\mathbf{d}$ is a closed operator, and thus so is its subspace $\mathbf{Z}_\infty^{u,v} = \mathbf{\Omega}_u \cap (\mathbf{\Omega}^r \cap \ker \mathbf{d})$. Hence $\mathbf{\Omega}^r \cap \ker \mathbf{d} = V \oplus \mathbf{Z}_\infty^{u,v}$ as Hilbert spaces, where V is the orthogonal complement of $\mathbf{Z}_\infty^{u,v}$ in $\mathbf{\Omega}^r \cap \ker \mathbf{d}$; in particular V is closed in $\mathbf{\Omega}$ too. Obviously,

$$(\mathbf{\Omega}^r \cap \ker \mathbf{d}) + \mathbf{\Omega}_{u+1}^r = V + \mathbf{Z}_\infty^{u,v} + \mathbf{\Omega}_{u+1}^r \,,$$
$$\mathbf{Z}_k^{u,v} + (\mathbf{\Omega}^r \cap \ker \mathbf{d}) + \mathbf{\Omega}_{u+1}^r = V + \mathbf{Z}_k^{u,v} + \mathbf{\Omega}_{u+1}^r \,.$$

On the other hand we clearly have

$$\mathbf{Z}_\infty^{u,v} + \mathbf{\Omega}_{u+1}^r = \mathbf{\Omega}_u \cap \left((\mathbf{\Omega}^r \cap \ker \mathbf{d}) + \mathbf{\Omega}_{u+1}^r\right) \,,$$
$$\mathbf{Z}_k^{u,v} + \mathbf{\Omega}_{u+1}^r = \mathbf{\Omega}_u \cap \left(\mathbf{Z}_k^{u,v} + (\mathbf{\Omega}^r \cap \ker \mathbf{d}) + \mathbf{\Omega}_{u+1}^r\right) \,,$$

and thus $\mathbf{Z}_\infty^{u,v} + \mathbf{\Omega}_{u+1}^r$ and $\mathbf{Z}_k^{u,v} + \mathbf{\Omega}_{u+1}^r$ are respectively closed in $(\mathbf{\Omega}^r \cap \ker \mathbf{d}) + \mathbf{\Omega}_{u+1}^r$ and $\mathbf{Z}_k^{u,v} + (\mathbf{\Omega}^r \cap \ker \mathbf{d}) + \mathbf{\Omega}_{u+1}^r$. Therefore Lemma 4.8 yields

$$(\mathbf{\Omega}^r \cap \ker \mathbf{d}) + \mathbf{\Omega}_{u+1}^r = V \oplus \left(\mathbf{Z}_\infty^{u,v} + \mathbf{\Omega}_{u+1}^r\right) \,,$$
$$\mathbf{Z}_k^{u,v} + (\mathbf{\Omega}^r \cap \ker \mathbf{d}) + \mathbf{\Omega}_{u+1}^r = V \oplus \left(\mathbf{Z}_k^{u,v} + \mathbf{\Omega}_{u+1}^r\right) \,,$$

as topological vector spaces, and the result follows. □

REMARK 3. In the proof of Lemma 4.9, the existence of V so that $\mathbf{\Omega}^r \cap \ker \mathbf{d} = V \oplus \mathbf{Z}_\infty^{u,v}$ as Hilbert spaces is the technical difficulty we were not able to solve without using square integrable differential forms; that is, we do not know if $\Omega^r \cap \ker d = V \oplus Z_\infty^{u,v}$ as topological vector spaces for some subspace V. This is the whole reason of introducing the L^2 spectral sequence in this paper.

Also, observe that the formula of Gromov–Shubin uses square integrable differential forms. Thus it can be more easily related to the L^2 spectral sequence than to the differentiable one. Though this is a minor problem that could be easily solved in the setting of C^∞ differential forms.

We shall use the notation

$$X_u^r = \bigoplus_{a\leq u} \Omega^{a,r-a}\,, \quad \rho_u^r = \sum_{a\leq u} \pi_{a,r-a} : \mathbf{\Omega}^r \longrightarrow X_u^r\,.$$

With respect to the inner product in $\mathbf{\Omega}$ induced by g or any g_h, the space X_u^r is the orthogonal complement of $\mathbf{\Omega}_{u+1}^r$ in $\mathbf{\Omega}^r$, and ρ_u^r is an orthogonal projection.

COROLLARY 4.10. *For $u+v=r$, the space $\rho_u^r(\mathbf{\Omega}^r \cap \ker \mathbf{d})$ is closed in $\rho_u^r(\mathbf{Z}_k^{u,v} + (\mathbf{\Omega}^r \cap \ker \mathbf{d}))$.*

Proof. This follows from Lemma 4.9 since we clearly have

$$(\mathbf{\Omega}^r \cap \ker \mathbf{d}) + \mathbf{\Omega}_{u+1}^r = \rho_u^r(\mathbf{\Omega}^r \cap \ker \mathbf{d}) \oplus \mathbf{\Omega}_{u+1}^r\,,$$
$$\mathbf{Z}_k^{u,v} + (\mathbf{\Omega}^r \cap \ker \mathbf{d}) + \mathbf{\Omega}_{u+1}^r = \rho_u^r\left(\mathbf{Z}_k^{u,v} + (\mathbf{\Omega}^r \cap \ker \mathbf{d})\right) \oplus \mathbf{\Omega}_{u+1}^r\,,$$

as topological vector spaces. □

Recall that $\bar{\mathbf{d}} : \operatorname{dom} \mathbf{d}/\ker \mathbf{d} \to \operatorname{im} \mathbf{d}$ denotes the map induced by $\mathbf{d}$, and let $L_k^{u,v}$ and L_k^r be the spaces introduced in Corollary 4.7 in section 4.3 for the particular case of the L^2 spectral sequence of $\mathcal{F}$.

LEMMA 4.11. *We have*

$$\left\|\bar{\mathbf{d}}\zeta\right\|_h \leq h^{-q/2} h^{u+k} \left\|\bar{\mathbf{d}}\zeta\right\|$$

for all $\zeta \in L_k^{u,v}$ and $0 < h \leq 1$.

Proof. This follows directly from Corollary 4.7 and (4.1). □

Let $\|\cdot\|$ and $\|\cdot\|_h$ also stand for the quotient Hilbert norms on $\mathbf{\Omega}/\ker \mathbf{d}$ induced by the norms $\|\cdot\|$ and $\|\cdot\|_h$ on $\mathbf{\Omega}$, respectively. In particular we have the restrictions of $\|\cdot\|$ and $\|\cdot\|_h$ to each subspace $L_k^{u,v} \subset \mathbf{\Omega}/\ker \mathbf{d}$.

LEMMA 4.12. *For each subspace $K \subset L_k^{u,v}$ of finite dimension there is some $C_K' > 0$, depending on K, such that*

$$h^{-q/2} h^u \|\zeta\| \leq C_K' \|\zeta\|_h$$

for all $\zeta \in K$ and $0 < h \leq 1$.

Proof. Let $u+v=r$. The restriction $\rho_u^r : \mathbf{Z}_k^{u,v} + (\mathbf{\Omega}^r \cap \ker \mathbf{d}) \to X_u^r$ induces a homomorphism

$$\bar\rho_u^r : \frac{\mathbf{Z}_k^{u,r-u} + (\mathbf{\Omega}^r \cap \ker \mathbf{d})}{\mathbf{\Omega}^r \cap \ker \mathbf{d}} \longrightarrow \frac{X_u^r}{\rho_u^r\left(\mathbf{\Omega}^r \cap \ker \mathbf{d}\right)}\,.$$

We clearly have

$$\ker \bar\rho^r_u = \frac{\mathbf{\Omega}^r_{u+1} + (\mathbf{\Omega}^r \cap \ker \mathbf{d})}{\mathbf{\Omega}^r \cap \ker \mathbf{d}} \,. \tag{4.21}$$

So $\bar\rho^r_u$ induces a continuous linear isomorphism

$$\frac{\mathbf{Z}^{u,v}_k + (\mathbf{\Omega}^r \cap \ker \mathbf{d})}{\mathbf{Z}^{u+1,v-1}_{k-1} + (\mathbf{\Omega}^r \cap \ker \mathbf{d})} \xrightarrow{\cong} \operatorname{im} \bar\rho^r_u = \frac{\rho^r_u\left(\mathbf{Z}^{u,v}_k + (\mathbf{\Omega}^r \cap \ker \mathbf{d})\right)}{\rho^r_u(\mathbf{\Omega}^r \cap \ker \mathbf{d})} \,.$$

Observe that $\operatorname{im} \bar\rho^r_u$ is a Hausdorff topological vector space by Corollary 4.10, and thus $\|\cdot\|$ and $\|\cdot\|_h$ induce norms on $\operatorname{im} \bar\rho^r_u$ that will be also denoted by $\|\cdot\|$ and $\|\cdot\|_h$, respectively. By (4.21) and Corollary 4.7, the homomorphism $\bar\rho^r_u$ restricts to an injection $\bar\rho^r_u : L^{u,v}_k \to \operatorname{im} \bar\rho^r_u$. Since ρ^r_u is an orthogonal projection for any metric g_h, we easily get

$$\|\bar\rho^r_u \zeta\|_h \le \|\zeta\|_h \quad \text{for all} \quad \zeta \in L^{u,v}_k \,. \tag{4.22}$$

Here, we use the norm on $\operatorname{im} \bar\rho^r_u$ in the left hand side of (4.22), and the norm on $\mathbf{\Omega}/\ker \mathbf{d}$ in its right hand side. Observe that, by (4.1),

$$h^{-q/2} h^u \|\omega\| \le \|\omega\|_h \quad \text{for all} \quad \omega \in X^r_u \quad \text{and} \quad 0 < h \le 1 \,,$$

yielding

$$h^{-q/2} h^u \|\xi\| \le \|\xi\|_h \quad \text{for all} \quad \xi \in \operatorname{im} \bar\rho^r_u \text{ and } 0 < h \le 1 \,. \tag{4.23}$$

Moreover, since K is of finite dimension, $\operatorname{im} \bar\rho^r_u$ is Hausdorff, and the restriction $\bar\rho^r_u : L^{u,v}_k \to \operatorname{im} \bar\rho^r_u$ is injective, we get the existence of some $C'_K > 0$ so that

$$\|\zeta\| \le C'_K \|\bar\rho^r_u \zeta\| \quad \text{for all} \quad \zeta \in K \,. \tag{4.24}$$

So

$$\begin{aligned} h^{-q/2} h^u \|\zeta\| &\le C'_K h^{-q/2} h^u \|\bar\rho^r_u \zeta\| \,, && \text{by (4.24)} \,, \\ &\le C'_K \|\bar\rho^r_u \zeta\|_h \,, && \text{by (4.23)} \,, \\ &\le C'_K \|\zeta\|_h \,, && \text{by (4.22)} \,, \end{aligned}$$

for all $\zeta \in K$ and $0 < h \le 1$ as desired. □

COROLLARY 4.13. *For each subspace $K \subset L^r_k$ of finite dimension there is some $C_K > 0$, depending on K, such that*

$$\left\|\bar{\mathbf{d}}\zeta\right\|_h \le C_K h^k \|\zeta\|_h$$

for all $\zeta \in K$ and $0 < h \le 1$.

Proof. Since K is of finite dimension, there is some constant C''_K, depending on K, so that

$$\left\|\bar{\mathbf{d}}\zeta\right\| \le C''_K \|\zeta\| \qquad \text{for all} \quad \zeta \in K \,. \tag{4.25}$$

Because $L^r_k = \bigoplus_{u+v=r} L^{u,v}_k$, any finite dimensional subspace $K \subset L^r_k$ is contained in the sum of finite dimensional subspaces $K^{u,v} \subset L^{u,v}_k$, $u+v = r$.

Therefore we can assume K is contained in some $L_k^{u,v}$ with $u+v=r$. Then, for $\zeta \in K$ and $0 < h \leq 1$, we have

$$\begin{aligned}
\left\|\bar{\mathbf{d}}\zeta\right\|_h &\leq h^{-q/2}h^{u+k}\left\|\bar{\mathbf{d}}\zeta\right\|\,, && \text{by Lemma 4.11}\,,\\
&\leq C''_K h^{-q/2}h^{u+k}\left\|\zeta\right\|\,, && \text{by (4.25)}\,,\\
&\leq C'_K C''_K h^k \left\|\zeta\right\|_h\,, && \text{by Lemma 4.12}\,,
\end{aligned}$$

and the result follows with $C_K = C'_K C''_K$. □

Now the proof of Theorem 4.1 can be finished as follows. If $m_k^r < \infty$, then Corollary 4.13 holds for $K = L_k^r$, and thus

$$F_h^r(C_\ell h^{2k}) \geq m_k^r\,.$$

Therefore, in this case, Theorem 4.1 follows from Corollary 4.6 and Proposition 4.3.

If $m_k^r = \infty$, choose any sequence of finite dimensional subspaces $K_i \subset L_k^r$ so that $\dim K_i \uparrow \infty$. Then Corollary 4.13 gives a sequence $C_i > 0$ such that

$$F_h^r(C_i h^{2k}) \geq \dim K_i$$

for $0 < h \leq 1$. Hence Theorem 4.1 also follows in this case by Corollary 4.6 and Proposition 4.3.

5 Asymptotics of Eigenforms

The goal of this section is to finish the proofs of Theorems A and B. At this stage, an inequality in Theorem A easily follows from connections established between the differentiable spectral sequence, the L^2 spectral sequence and small eigenvalues. The reverse inequality, and the rest of Theorem A, will follow from Theorem B, which is proved first.

To begin with the proof of Theorem B, our Hodge theoretic nested sequence $\mathcal{H}_k$ is studied in section 5.1. Its definition is simply a continuation of our previous Hodge theoretic approach of the terms $\widehat{E}_1, E_2$; this is only a simple application of linear algebra because $\mathcal{H}_2$ is Hausdorff of finite dimension. But we need a description of the sequence $\mathcal{H}_k$ (Theorem 5.1) whose proof is long and difficult; it is so because we have to keep track of the contributions of $\bar{0}_1$ in all terms $\mathcal{H}_k$ up to the limit.

The proof of Theorem B, and thus of Theorem A, is finished in section 5.2 by using the above description of $\mathcal{H}_k$ and certain estimates of the rescaled Laplacian Δ_h; these estimates are rather similar to some estimates of Forman in [F].

In the whole of this section, $\mathcal{F}$ is assumed to be a Riemannian foliation and the metric bundle-like.

5.1 The Hodge theoretic nested sequence. So far we have constructed bigraded subspaces $\mathcal{H}_1, \mathcal{H}_2 \subset \Omega$, which are respectively isomorphic to $\hat{e}_1, e_2$ as bigraded topological vector spaces by Theorem 2.1 and Theorem 2.2(iv). We continue constructing subspaces $\mathcal{H}_k \subset \Omega$ and isomorphisms $e_k \cong \mathcal{H}_k$ by induction on k as follows. Suppose we have constructed $\mathcal{H}_k$ and an explicit isomorphism $e_k \cong \mathcal{H}_k$ for some $k \geq 2$. Then the homomorphism d_k corresponds to some homomorphism on $\mathcal{H}_k$ that will be denoted by d_k as well. Thus $\mathcal{H}_k$ becomes a finite dimensional complex. Let δ_k be the adjoint of d_k on the finite dimensional Hilbert space $\mathcal{H}_k$, and set $\Delta_k = d_k\delta_k + \delta_k d_k$ and $\mathcal{H}_{k+1} = \ker \Delta_k = \ker d_k \cap \ker \delta_k$. We have the orthogonal decomposition

$$\mathcal{H}_k = \mathcal{H}_{k+1} \oplus \operatorname{im} d_k \oplus \operatorname{im} \delta_k ,$$

yielding

$$e_{k+1} \cong H(e_k, d_k) \cong H(\mathcal{H}_k, d_k) \cong \mathcal{H}_{k+1} ,$$

which completes the induction step. So $(\mathcal{H}_k, d_k)$ is, by definition, some kind of a Hodge theoretic version of the sequence $(\hat{e}_1, d_1), (e_2, d_2), (e_3, d_3), \dots$, and thus of the sequence $(\widehat{E}_1, d_1), (E_2, d_2), (E_3, d_3), \dots$ as well by Theorem 2.1 and Theorem 2.2(vii). Furthermore each Δ_k is bihomogeneous of bidegree $(0, 0)$, and thus $\mathcal{H}_k$ inherits the bigrading from $\mathbf{\Omega}$, which clearly corresponds to the bigrading of E_k and e_k. Observe that the nested sequence

$$\Omega \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \mathcal{H}_3 \supset \mathcal{H}_4 \supset \cdots$$

stabilizes at most at the $(q+1)$th step since so does E_k. Then its final term $\mathcal{H}_{q+1} = \mathcal{H}_{q+2} = \cdots$ will be denoted by $\mathcal{H}_\infty$, and we have $E_\infty \cong e_\infty \cong \mathcal{H}_\infty$.

We shall need a better understanding of the new terms $\mathcal{H}_k$ for $k > 2$. Precisely, we shall use the following result.

Theorem 5.1. *Let $k \geq 3$ and $\omega \in \mathcal{H}_2^{u,v}$. Then $\omega \in \mathcal{H}_k^{u,v}$ if and only if there are sequences $\alpha_i = \sum_{a>0} \alpha_i^a$ and $\beta_i = \sum_{a>0} \beta_i^a$, where $\alpha_i^a \in \Omega^{u+a,v-a}$ and $\beta_i^a \in \Omega^{u-a,v+a}$, such that*

$$\pi_{u+a,v-a+1} d(\omega + \alpha_i) \longrightarrow 0 , \quad \pi_{u-a,v+a-1} \delta(\omega + \beta_i) \longrightarrow 0$$

strongly in $\mathbf{\Omega}$ for $0 < a < k$.

The rest of this section will be devoted to prove Theorem 5.1. To begin with, the nested sequence $\mathcal{H}_k$ is most properly a Hodge theoretic version of another sequence of bigraded topological complexes $(\hat{e}_{1,k}, d_k)$, which are defined as follows by induction on $k \geq 1$. First, let $\hat{e}_{1,1} = \hat{e}_1$ and $\hat{e}_{1,2} = H(\hat{e}_1)$ with the induced topology in cohomology. We have an explicit isomorphism $e_2 \cong \hat{e}_{1,2}$ of bigraded topological vector spaces given by Theorem 2.2(vii). Now suppose that, for some fixed $k \geq 2$, we have defined $\hat{e}_{1,k}$ with an explicit isomorphism $e_k \cong \hat{e}_{1,k}$ of bigraded topological vector spaces. Then $\hat{e}_{1,k}$ becomes a topological complex via this isomorphism, and define $\hat{e}_{1,k+1} = H(\hat{e}_{1,k})$. Furthermore the composition $e_{k+1} \cong H(e_k) \cong \hat{e}_{1,k+1}$ is an explicit isomorphism of bigraded topological vector spaces.

Lemma 5.2. *For $k \geq 1$, we have a canonical isomorphism*

$$\hat{e}_{1,k}^{u,v} \cong \frac{z_k^{u,v} + \overline{b_0^{u,v}}}{b_{k-1}^{u,v} + \overline{b_0^{u,v}}} \tag{5.1}$$

of topological vector spaces. Moreover, for $k \geq 2$, the above isomorphism $e_k^{u,v} \cong \hat{e}_{1,k}^{u,v}$ corresponds to the canonical map

$$\frac{z_k^{u,v}}{b_{k-1}^{u,v}} \longrightarrow \frac{z_k^{u,v} + \overline{b_0^{u,v}}}{b_{k-1}^{u,v} + \overline{b_0^{u,v}}} \tag{5.2}$$

when applying (5.1).

Proof. The result is proved by induction on k. First, the case $k = 1$ is trivial.

Second, the kernel and the image of d_1 in $e_1^{u,v}$ respectively are $z_2^{u,v}/b_0^{u,v}$ and $b_1^{u,v}/b_0^{u,v}$, whose canonical projections in $\hat{e}_1^{u,v} = z_1^{u,v}/\overline{b_0^{u,v}}$ are

$$\frac{z_2^{u,v} + \overline{b_0^{u,v}}}{\overline{b_0^{u,v}}}, \quad \frac{b_1^{u,v} + \overline{b_0^{u,v}}}{\overline{b_0^{u,v}}}, \tag{5.3}$$

yielding the canonical isomorphism (5.1) for $k = 2$. Since the isomorphism $e_2^{u,v} \cong \hat{e}_{1,2}^{u,v}$ is canonically defined, it corresponds to the canonical map (5.2) for $k = 2$.

Now assume the result holds for $k = \ell \geq 2$ and we prove it for $k = \ell + 1$. The kernel and the image of d_ℓ in $e_\ell^{u,v}$ respectively are $z_{\ell+1}^{u,v}/b_{\ell-1}^{u,v}$ and $b_\ell^{u,v}/b_{\ell-1}^{u,v}$, whose images by the canonical isomorphism (5.2) for $k = \ell$ are

$$\frac{z_{\ell+1}^{u,v} + \overline{b_0^{u,v}}}{b_{\ell-1}^{u,v} + \overline{b_0^{u,v}}}, \quad \frac{b_\ell^{u,v} + \overline{b_0^{u,v}}}{b_{\ell-1}^{u,v} + \overline{b_0^{u,v}}}. \tag{5.4}$$

These spaces respectively correspond to the kernel and the image of d_ℓ in $\hat{e}_{1,\ell}^{u,v}$ by (5.1), yielding the canonical isomorphism (5.1) for $k = \ell + 1$. Again, because the isomorphism $e_{\ell+1}^{u,v} \cong \hat{e}_{1,\ell+1}^{u,v}$ is canonically defined, it is given by the canonical map (5.2) for $k = \ell + 1$. □

We shall consider each isomorphism (5.1) as an equality from now on.

For $k \geq 1$, let Π_k denote the orthogonal projections $\mathbf{\Omega} \to \mathcal{H}_k$; in particular, $\Pi_1 = \Pi$ with this notation. Let also $P_0 = P$, $Q_0 = Q$ and, for $k \geq 1$, let P_k and Q_k be the orthogonal projections of $\mathbf{\Omega}$ onto $d_k(\mathcal{H}_k)$ and $\delta_k(\mathcal{H}_k)$. Finally let $\bar{P}_k = \sum_{0\leq\ell\leq k} P_\ell$ and $\bar{Q}_k = \sum_{0\leq\ell\leq k} Q_\ell$ for $k \geq 0$.

Lemma 5.3. *For $k \geq 1$, Π_k induces an isomorphism $\hat{e}_{1,k}^{u,v} \xrightarrow{\cong} \mathcal{H}_k^{u,v}$, whose composition with the canonical isomorphism $e_k^{u,v} \xrightarrow{\cong} \hat{e}_{1,k}^{u,v}$ is the above isomorphism $e_k^{u,v} \cong \mathcal{H}_k^{u,v}$.*

Proof. Observe that the first part of the statement means that we have an orthogonal decomposition

$$z_k^{u,v} + \overline{b_0^{u,v}} = \mathcal{H}_k^{u,v} \oplus (b_{k-1}^{u,v} + \overline{b_0^{u,v}})\,. \tag{5.5}$$

Again the result follows by induction on k. We have an orthogonal decomposition

$$z_1^{u,v} = \mathcal{H}_1^{u,v} \oplus \overline{b_0^{u,v}} \tag{5.6}$$

by Theorem 2.1. Thus the isomorphism $\hat{e}_{1,1} = \hat{e}_1^{u,v} \xrightarrow{\cong} \mathcal{H}_1^{u,v}$ is induced by the orthogonal projection Π_1 onto $\mathcal{H}_1$. On the other hand, the kernel and image of d_1 in $\mathcal{H}_1^{u,v}$ respectively correspond by this isomorphism to the kernel and image of d_1 on $\hat{e}_1^{u,v}$, which are respectively given by (5.3). So the kernel and image of d_1 in $\mathcal{H}_1^{u,v}$ are the orthogonal projections $\Pi_1(z_2^{u,v} + \overline{b_0^{u,v}})$ and $\Pi_1(b_1^{u,v} + \overline{b_0^{u,v}})$, respectively. Hence, by definition, $\mathcal{H}_2^{u,v}$ is the orthogonal complement of $\Pi_1(b_1^{u,v} + \overline{b_0^{u,v}})$ in $\Pi_1(z_2^{u,v} + \overline{b_0^{u,v}})$, which is equal to the orthogonal complement of $b_1^{u,v} + \overline{b_0^{u,v}}$ in $z_2^{u,v} + \overline{b_0^{u,v}}$ by (5.6) since

$$\overline{b_0^{u,v}} \subset b_1^{u,v} + \overline{b_0^{u,v}} \subset z_2^{u,v} + \overline{b_0^{u,v}} \subset z_1^{u,v}\,.$$

Thus the result follows for $k = 2$.

Now suppose the statement holds for $k = \ell \geq 2$. Then, via the isomorphism $\hat{e}_{1,\ell}^{u,v} \xrightarrow{\cong} \mathcal{H}_\ell^{u,v}$ induced by Π_ℓ, the kernel and image of d_ℓ in $\mathcal{H}_\ell^{u,v}$ respectively correspond to the kernel and image of d_ℓ in $\hat{e}_{1,\ell}^{u,v}$, which are given in (5.4). So the kernel and image of d_ℓ in $\mathcal{H}_\ell^{u,v}$ are the orthogonal projections $\Pi_\ell(z_{\ell+1}^{u,v} + \overline{b_0^{u,v}})$ and $\Pi_\ell(b_\ell^{u,v} + \overline{b_0^{u,v}})$, respectively. Hence, by definition, $\mathcal{H}_{\ell+1}^{u,v}$ is the orthogonal complement of $\Pi_\ell(b_\ell^{u,v} + \overline{b_0^{u,v}})$ in $\Pi_\ell(z_{\ell+1}^{u,v} + \overline{b_0^{u,v}})$, which is equal to the orthogonal complement of $b_\ell^{u,v} + \overline{b_0^{u,v}}$ in $z_{\ell+1}^{u,v} + \overline{b_0^{u,v}}$ by (5.5) for $k = \ell$ since

$$b_{\ell-1}^{u,v} + \overline{b_0^{u,v}} \subset b_\ell^{u,v} + \overline{b_0^{u,v}} \subset z_{\ell+1}^{u,v} + \overline{b_0^{u,v}} \subset z_\ell^{u,v} + \overline{b_0^{u,v}}\,.$$

Thus the result follows for $k = \ell + 1$. □

Remark 4. The inverse of the isomorphism $\hat{e}_{1,k}^{u,v} \xrightarrow{\cong} \mathcal{H}_k^{u,v}$ in Lemma 5.3 is obviously induced by the inclusion $\mathcal{H}_k^{u,v} \hookrightarrow z_k^{u,v} + \overline{b_0^{u,v}}$. So we can summarize

Lemmas 5.2 and 5.3 by saying that, for $k \geq 2$, the isomorphism $e_k^{u,v} \cong \mathcal{H}_k^{u,v}$ is given by the diagram

$$e_k^{u,v} = \frac{z_k^{u,v}}{b_{k-1}^{u,v}} \xrightarrow{\cong} \hat{e}_{1,k}^{u,v} = \frac{z_k^{u,v} + \overline{b_0^{u,v}}}{b_{k-1}^{u,v} + \overline{b_0^{u,v}}} \xleftarrow{\cong} \mathcal{H}_k^{u,v}\,, \tag{5.7}$$

where both isomorphisms are canonically induced by inclusions.

Remark 5. In general, we have $z_k^{u,v} \neq \mathcal{H}_k^{u,v} \oplus b_{k-1}^{u,v}$ because $\mathcal{H}_k^{u,v} \not\subset z_k^{u,v}$, but the orthogonal decomposition (5.5) always holds. This is the reason the nested sequence $\mathcal{H}_k$ is a Hodge theoretic version of the sequence $(\hat{e}_{1,k}, d_k)$ better than of the sequence $(\hat{e}_1, d_1), (e_2, d_2), (e_3, d_3), \dots$.

The following proposition is the key result to prove Theorem 5.1.

Proposition 5.4. *Let $\omega \in \mathcal{H}_k^{u,v}$ and $\gamma \in \mathcal{H}_k^{u+k,v-k+1}$ for $k \geq 2$. If there is a sequence $\alpha_i \in \Omega_{u+1}^{u+v}$ such that*

$$\pi_{u+a,v-a+1} d(\omega + \alpha_i) \longrightarrow 0\,, \quad 0 < a < k\,,$$
$$\bar{Q}_{k-2}\pi_{u+k,v-k+1} d(\omega + \alpha_i) \longrightarrow 0\,, \quad \Pi_k \pi_{u+k,v-k+1} d(\omega + \alpha_i) \longrightarrow \gamma$$

strongly in $\mathbf{\Omega}$, then $d_k\omega = \gamma$. Moreover, in this case the sequence α_i can be chosen so that

$$\pi_{u+a,v-a+1} d(\omega + \alpha_i) \longrightarrow 0\,, \quad 0 < a < k\,,$$
$$\pi_{u+k,v-k+1} d(\omega + \alpha_i) \to \gamma$$

with respect to the C^∞ topology in Ω.

The following slightly weaker result will be used as an intermediate step in the proof of Proposition 5.4.

Lemma 5.5. *Let $\gamma \in \mathcal{H}_k^{u+k,v-k+1}$ for $k \geq 2$. If there is some sequence $\alpha_i \in \Omega_{u+1}^{u+v}$, such that*

$$\pi_{u+a,v-a+1} d\alpha_i \longrightarrow 0\,, \quad 0 < a < k\,,$$
$$\bar{Q}_{k-2}\pi_{u+k,v-k+1} d\alpha_i \longrightarrow 0\,, \quad \Pi_k \pi_{u+k,v-k+1} d\alpha_i \longrightarrow \gamma$$

strongly in $\mathbf{\Omega}$, then $\gamma = 0$.

Both Proposition 5.4 and Lemma 5.5 will be proved simultaneously by induction on $k \geq 2$. For the case $k = 2$ we need the following.

Lemma 5.6. *We have $\Pi_2 \pi_{u+2,v-1} d\tilde{d}_1\beta = 0$ for any $\beta \in \widetilde{\mathcal{H}}_1^{u-1,v}$.*

Proof. Write $\beta = \beta' + \beta''$ with $\beta' \in P(\Omega^{u-1,v})$ and $\beta'' \in Q(\Omega^{u,v-1})$. Then

$$\begin{aligned}\Pi_2\pi_{u+2,v-2} d\tilde{d}_1\beta &= \Pi_2\big(d_{2,-1}(d_{1,0}\beta' + d_{0,1}\beta'') + d_{1,0}Q(d_{2,-1}\beta' + d_{1,0}\beta'')\big)\\ &= \Pi_2\big((d_{2,-1}d_{1,0} + d_{1,0}d_{2,-1})\beta' + (d_{2,-1}d_{0,1} + d_{1,0}^2)\beta''\big)\\ &\quad - \Pi_2 d_{1,0}\Pi(d_{2,-1}\beta' + d_{1,0}\beta'') - \Pi_2 d_{1,0}P(d_{2,-1}\beta' + d_{1,0}\beta'')\end{aligned}$$

$$= -\Pi_2 d_1 \Pi(d_{2,-1}\beta' + d_{1,0}\beta'') - \Pi_2 P d_{1,0} P(d_{2,-1}\beta' + d_{1,0}\beta'')$$
$$= 0$$

by (2.2), Lemma 2.3 and because $\Pi_2 d_{0,1} = \Pi_2 d_1 = \Pi_2 P = 0$. □

LEMMA 5.7. *Let α_i be a sequence in $\widetilde{\mathcal{H}}_1^{u,v}$ such that $\tilde{d}_1\alpha_i \to 0$ strongly in $\mathbf{\Omega}$. Then*

$$\Pi_2 \pi_{u+2,v-1} d\alpha_i \longrightarrow 0$$

strongly in $\mathbf{\Omega}$.

Proof. Since the image of $\tilde{d}_1$ is closed and equal to its kernel, the hypothesis implies the existence of a sequence $\beta_i \in \widetilde{\mathcal{H}}_1^{u-1,v}$ such that $\alpha_i + \tilde{d}_1\beta_i \to 0$ strongly in $\mathbf{\Omega}$. On the other hand we have

$$\Pi_2 \pi_{u+2,v-1} d = \Pi_2 d_{2,-1} \pi_{u,v} + \Pi_2 d_{1,0} \pi_{u+1,v-1}$$
$$= \Pi_2 d_{2,-1} \pi_{u,v} + \Pi_2 \Pi d_{1,0} Q \pi_{u+1,v-1}$$

on $\widetilde{\mathcal{H}}_1^{u,v}$, and thus the operator $\Pi_2 \pi_{u+2,v-1} d : \widetilde{\mathcal{H}}_1^{u,v} \to \mathcal{H}_2^{u+2,v-1}$ is bounded because $d_{2,-1}$ and $\Pi d_{1,0} Q$ are bounded operators in $\mathbf{\Omega}$ by Lemma 2.4. Therefore

$$\Pi_2 \pi_{u+2,v-1} d(\alpha_i + \tilde{d}_1\beta_i) \longrightarrow 0$$

strongly in $\mathbf{\Omega}$. Then the result follows directly from Lemma 5.6. □

Proof of Lemma 5.5 for the case $k = 2$. In this case we have $\gamma \in \mathcal{H}_2^{u+2,v-1}$ and $\alpha_i \in \Omega^{u+1,v-1}$, which satisfy

$$d_{0,1}\alpha_i \longrightarrow 0\,, \quad Q d_{1,0}\alpha_i \longrightarrow 0\,, \quad \Pi_2 d_{1,0}\alpha_i \longrightarrow \gamma$$

strongly in $\mathbf{\Omega}$. Since

$$\Pi d_{1,0} \Pi \alpha_i = d_1 \Pi \alpha_i \perp \mathcal{H}_2^{u+2,v-1}\,,$$

we get $\Pi_2 d_{1,0} \widetilde{\Pi} \alpha_i \to \gamma$ strongly in $\mathbf{\Omega}$. But $\Pi_2 d_{1,0} P\alpha_i = \Pi_2 P d_{1,0} P \alpha_i = 0$ by Lemma 2.3, and thus we get

$$\Pi_2 \pi_{u+2,v-1} d Q\alpha_i = \Pi_2 d_{1,0} Q\alpha_i \longrightarrow \gamma \tag{5.8}$$

strongly in $\mathbf{\Omega}$. Now observe that $Q\alpha_i \in \widetilde{\mathcal{H}}_1^{u,v}$, and

$$\tilde{d}_1 Q\alpha_i = \widetilde{\Pi}_{u+1,v} d Q\alpha_i = d_{0,1} Q\alpha_i + Q d_{1,0} Q \alpha_i = d_{0,1}\alpha_i + Q d_{1,0}\alpha_i \longrightarrow 0$$

because $d_{0,1} Q = d_{0,1}$ and by Lemma 2.3. Then the result follows by (5.8) and Lemma 5.7. □

Now let $\ell \geq 2$ and assume that Lemma 5.5 holds for $2 \leq k \leq \ell$. If $\ell > 2$, assume also that Proposition 5.4 holds for $2 \leq k < \ell$.

Proof of Proposition 5.4 for $k = \ell$. First, we check that the assignment $\omega \mapsto \gamma$, under the conditions in the statement, defines a map $\mathcal{H}_\ell^{u,v} \to \mathcal{H}_\ell^{u+\ell,v-\ell+1}$ – observe that, if such a map is well defined, it is obviously linear. Suppose there is another $\gamma' \in \mathcal{H}_\ell^{u+\ell,v-\ell+1}$ and another sequence $\alpha'_i \in \Omega_{u+1}^{u+v}$ such that

$$\pi_{u+a,v-a+1} d\alpha'_i \longrightarrow 0\,, \quad 0 < a < \ell\,,$$
$$\bar{Q}_{\ell-2}\pi_{u+\ell,v-\ell+1} d\alpha'_i \longrightarrow 0\,, \quad \Pi_\ell \pi_{u+\ell,v-\ell+1} d\alpha'_i \longrightarrow \gamma'$$

strongly in $\mathbf{\Omega}$. Then the sequence $\alpha_i - \alpha'_i \in \Omega_{u+1}^{u+v}$ satisfies

$$\pi_{u+a,v-a+1} d(\alpha_i - \alpha'_i) \longrightarrow 0\,, \quad 0 < a < \ell\,,$$
$$\bar{Q}_{\ell-2}\pi_{u+\ell,v-\ell+1} d(\alpha_i - \alpha'_i) \longrightarrow 0\,, \quad \Pi_\ell \pi_{u+\ell,v-\ell+1} d(\alpha_i - \alpha'_i) \longrightarrow \gamma - \gamma'$$

strongly in $\mathbf{\Omega}$. Therefore $\gamma = \gamma'$ by Lemma 5.5 for the case $k = \ell$.

Second, we prove that the above map $\mathcal{H}_\ell^{u,v} \to \mathcal{H}_\ell^{u+\ell,v-\ell+1}$ is d_ℓ; i.e. for each $\omega \in \mathcal{H}_\ell^{u,v}$, we prove the existence of a sequence $\alpha_i \in \Omega_{u+1}^{u+v}$ such that

$$\pi_{u+a,v-a+1} d(\omega + \alpha_i) \longrightarrow 0\,, \quad 0 \le a < \ell\,, \tag{5.9}$$
$$\bar{Q}_{\ell-2}\pi_{u+\ell,v-\ell+1} d(\omega + \alpha_i) \longrightarrow 0\,, \tag{5.10}$$
$$\Pi_\ell \pi_{u+\ell,v-\ell+1} d(\omega + \alpha_i) \longrightarrow d_\ell\omega \in \mathcal{H}_\ell^{u+\ell,v-\ell+1} \tag{5.11}$$

strongly in $\mathbf{\Omega}$. According to (5.7), for each $\omega \in \mathcal{H}_\ell^{u,v}$ there is a sequence $\omega_i \in z_\ell^{u,v}$ converging to ω with respect to the C^∞ topology and such that ω and all the ω_i define the same class $\hat{\zeta} \in \hat{e}_{1,\ell}^{u,v}$; thus all the ω_i define the same class $\zeta \in e_\ell^{u,v}$. By definition of $z_\ell^{u,v}$, there is another sequence $\alpha_i \in \Omega_{u+1}^{u+v}$ such that $\omega_i + \alpha_i \in Z_\ell^{u,v}$. So all the $\omega_i + \alpha_i$ define the same class $\xi \in E_\ell^{u,v}$, and the class $d_\ell\xi \in E_\ell^{u+\ell,v-\ell+1}$ is defined by any of the forms $d(\omega_i + \alpha_i) \in Z_\ell^{u+\ell,v-\ell+1}$. Thus

$$\pi_{u+a,v-a+1} d(\omega_i + \alpha_i) = 0\,, \quad 0 \le a < \ell\,,$$

and any of the forms

$$\pi_{u+\ell,v-\ell+1} d(\omega_i + \alpha_i) \in z_\ell^{u+\ell,v-\ell+1}$$

define the class $d_\ell\zeta \in e_\ell^{u+\ell,v-\ell+1}$ as well as the class $d_\ell\hat{\zeta} \in \hat{e}_{1,\ell}^{u+\ell,v-\ell+1}$, yielding

$$\pi_{u+a,v-a+1} d(\omega_i + \alpha_i) = 0\,, \quad 0 \le a < \ell\,,$$
$$\bar{Q}_{\ell-2}\pi_{u+\ell,v-\ell+1} d(\omega_i + \alpha_i) = 0\,,$$
$$\Pi_\ell \pi_{u+\ell,v-\ell+1} d(\omega_i + \alpha_i) = d_\ell\omega \in \mathcal{H}_\ell^{u+\ell,v-\ell-1}$$

independently of i. Then (5.9)–(5.11) follow by the C^∞ convergence $\omega_i \to \omega$, as desired.

Finally we prove the last part of the statement. Observe that, in fact, the above arguments yield C^∞ convergence in (5.9)–(5.11), and also the C^∞ convergence

$$\bar{Q}_{\ell-1}\pi_{u+\ell,v-\ell+1} d(\omega + \alpha_i) \longrightarrow 0\,.$$

For each i, take $\sigma_i^1 \in Q_1(\Omega^{u+\ell-1,v-\ell+1})$ satisfying

$$\Pi_1 d_{1,0}\sigma_i^1 = P_1\pi_{u+\ell,v-\ell+1}d(\omega+\alpha_i)\,, \tag{5.12}$$

and take $\sigma_i^0 \in Q_0(\Omega^{u+\ell,v-\ell})$ such that

$$d_{0,1}\sigma_i^0 + P_0 d_{1,0}\sigma_i^1 - P_0\pi_{u+\ell,v-\ell+1}d(\omega+\alpha_i) \longrightarrow 0 \tag{5.13}$$

with respect to the C^∞ topology. If $\ell > 2$, for each i and $m = 2,\dots,\ell-1$ take some $\sigma_i^m \in Q_m(\Omega^{u+\ell-m,v-\ell+m})$ such that

$$d_m\sigma_i^m = P_m\pi_{u+\ell,v-\ell+1}d(\omega+\alpha_i)\,.$$

By Proposition 5.4 for $k < \ell$ there are sequences $\tau_{i,j}^m \in \Omega_{u+\ell-m+1}^{u+v}$ such that

$$\pi_{u+\ell-m+a,v-\ell+m-a+1}d(\sigma_i^m + \tau_{i,j}^m) \longrightarrow 0\,, \quad 0 < a < m\,,$$
$$\pi_{u+\ell,v-\ell+1}d(\sigma_i^m + \tau_{i,j}^m) \longrightarrow P_m\pi_{u+\ell,v-\ell+1}d(\omega+\alpha_i)$$

with respect to the C^∞ topology in Ω. Then, for each i,m we can clearly choose j depending on i,m so that $\tau_i^m = \tau_{i,j}^m$ satisfies

$$\pi_{u+\ell-m+a,v-\ell+m-a+1}d(\sigma_i^m + \tau_i^m) \longrightarrow 0\,, \quad 0 < a < m\,, \tag{5.14}$$

$$\pi_{u+\ell,v-\ell+1}d(\sigma_i^m + \tau_i^m) - P_m\pi_{u+\ell,v-\ell+1}d(\omega+\alpha_i) \longrightarrow 0 \tag{5.15}$$

with respect to the C^∞ topology. Let

$$\beta_i = \alpha_i - \sigma_i^0 - \sigma_i^1 - \sum_{m=2}^{\ell-1}(\sigma_i^m + \tau_i^m) \in \Omega_{u+1}^{u+v}\,,$$

where the last term does not show up if $\ell = 2$. From (5.12)–(5.15) we get

$$\pi_{u+a,v-a+1}d(\omega_i + \beta_i) \longrightarrow 0\,, \quad 0 \le a < \ell\,,$$
$$\pi_{u+\ell,v-\ell+1}d(\omega_i + \beta_i) \longrightarrow d_\ell\omega$$

with respect to the C^∞ topology in Ω, and the proof is finished. □

We already know that both Proposition 5.4 and Lemma 5.5 hold for $k \le \ell$, and we have to prove Lemma 5.5 for $k = \ell+1$. The arguments will be similar to the case $k = 2$, and thus we need an appropriate version of Lemma 5.7. In particular, the generalization of $\widetilde{\mathcal{H}}_1^{u,v}$ that fits our needs turns out to be the following:

$$\widetilde{\mathcal{H}}_\ell^{u,v} = P_0(\Omega^{u,v}) \oplus \bigoplus_{0<a<\ell} \Omega^{u+a,v-a} \oplus \bar{Q}_{\ell-1}(\Omega^{u+\ell,v-\ell})\,.$$

Let also $\widetilde{\mathcal{H}}_\ell^{\cdot,v} = \bigoplus_u \widetilde{\mathcal{H}}_\ell^{u,v}$. We have orthogonal projections $\widetilde{\Pi}_{\ell;u,v} : \Omega \to \widetilde{\mathcal{H}}_\ell^{u,v}$ and $\widetilde{\Pi}_{\ell;\cdot,v} : \Omega \to \widetilde{\mathcal{H}}_\ell^{\cdot,v}$ given by

$$\widetilde{\Pi}_{\ell;u,v} = P_0\pi_{u,v} + \sum_{0<a<\ell}\pi_{u+a,v-a} + \bar{Q}_{\ell-1}\pi_{u+\ell,v-\ell}\,, \quad \widetilde{\Pi}_{\ell;\cdot,v} = \sum_u \widetilde{\Pi}_{\ell;u,v}\,,$$

and let $\tilde{d}_\ell = \widetilde{\Pi}_{\ell;\cdot,v}d : \widetilde{\mathcal{H}}_\ell^{\cdot,v} \to \widetilde{\mathcal{H}}_\ell^{\cdot,v}$.

LEMMA 5.8. *We have* $\tilde{d}_\ell^2 = 0$.

Proof. Consider the following subspaces of Ω^{u+v}:

$$\mathcal{A}_\ell^{u,v} = Z_\ell^{u,v} + \overline{B_0^{u,v}} + \Omega_{u+1}^{u+v} = (z_\ell^{u,v} + \overline{b_0^{u,v}}) \oplus \Omega_{u+1}^{u+v}\,,$$
$$\mathcal{B}^{u,v} = B_0^{u,v} + \Omega_{u+1}^{u+v} = b_0^{u,v} \oplus \Omega_{u+1}^{u+v}\,.$$

First, observe that each $\widetilde{\mathcal{H}}_\ell^{u,v}$ is the orthogonal complement of $\mathcal{A}_{\ell-1}^{u+\ell,v-\ell}$ in $\overline{\mathcal{B}^{u,v}}$, and thus $\widetilde{\mathcal{H}}_\ell^{\cdot,v}$ is the orthogonal complement of $\mathcal{A}_{\ell-1}^{\cdot,v-\ell} = \bigoplus_a \mathcal{A}_\ell^{a,v-\ell}$ in $\overline{\mathcal{B}^{\cdot,v}} = \bigoplus_a \overline{\mathcal{B}^{a,v}}$. So the inclusion $\widetilde{\mathcal{H}}_\ell^{\cdot,v} \hookrightarrow \overline{\mathcal{B}^{\cdot,v}}$ induces an isomorphism of topological vector spaces

$$\widetilde{\mathcal{H}}_\ell^{\cdot,v} \xrightarrow{\cong} \overline{\mathcal{B}^{\cdot,v}}/\mathcal{A}_{\ell-1}^{\cdot,v-\ell} \tag{5.16}$$

whose inverse is induced by the orthogonal projection $\widetilde{\Pi}_{\cdot,v} : \overline{\mathcal{B}^{\cdot,v}} \to \widetilde{\mathcal{H}}_\ell^{\cdot,v}$. Second, observe that both $\overline{\mathcal{B}^{u,v}}$ and $\mathcal{A}_{\ell-1}^{\cdot,v-\ell}$ are subcomplexes of (Ω, d). Moreover $\tilde{d}_\ell$ in $\widetilde{\mathcal{H}}_\ell^{\cdot,v}$ clearly corresponds to the differential map in the quotient complex $\overline{\mathcal{B}^{\cdot,v}}/\mathcal{A}_{\ell-1}^{\cdot,v-\ell}$ via (5.16), and the result follows. □

Since $H(\widetilde{\mathcal{H}}_1^{\cdot,v}, \tilde{d}_1) = 0$, the following two lemmas generalize Lemma 5.6.

LEMMA 5.9. *For any*

$$\beta \in \bigoplus_{a<\ell} \Omega^{u-1+a,v-a} + \bar{Q}_{\ell-1}(\Omega^{u-1+\ell,v-\ell})$$

we have

$$\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell} d \widetilde{\Pi}_{\ell;\cdot,v} d\beta = 0\,.$$

Proof. By the expression

$$\begin{aligned}&\bigoplus_{a<\ell} \Omega^{u-1+a,v-a} + \bar{Q}_{\ell-1}(\Omega^{u-1+\ell,v-\ell})\\ &\quad = \bigoplus_{a<\ell} \Omega^{u-1+a,v-a} + \widetilde{\mathcal{H}}_1^{u+\ell-2,v-\ell+1} + (Q_1 + \cdots + Q_{\ell-1})(\Omega^{u+\ell-1,v-\ell})\end{aligned}$$

it is enough to consider the following three cases. First, assume $\beta \in \bigoplus_{a<\ell} \Omega^{u-1+a,v-a}$ and let $\beta' = \pi_{u+\ell-2,v-\ell+1}\beta$. We clearly have

$$(d - \widetilde{\Pi}_{\ell;\cdot,v} d)\beta = (\mathrm{id} - \bar{Q}_{\ell-1}) d_{2,-1}\beta' = (Q_\ell + \Pi_{\ell+1} + \bar{P}_\ell) d_{2,-1}\beta'\,,$$

yielding

$$\begin{aligned}\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell} d \widetilde{\Pi}_{\ell;\cdot,v} d\beta &= -\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell} d (Q_\ell + \Pi_{\ell+1} + \bar{P}_\ell) d_{2,-1}\beta'\\ &= -\Pi_{\ell+1} d_{1,0}(Q_\ell + \Pi_{\ell+1} + \bar{P}_\ell) d_{2,-1}\beta'\\ &= -\Pi_{\ell+1} d_1 (Q_\ell + \Pi_{\ell+1} + P_1 + \cdots + P_\ell) d_{2,-1}\beta'\\ &\quad - \Pi_{\ell+1} d_{1,0} P_0 d_{2,-1}\beta'\\ &= 0\end{aligned}$$

by Lemma 2.3, and because $\Pi_{\ell+1}d_1=0$ and $\Pi_{\ell+1}P_0=0$.

Second, suppose $\beta\in\widetilde{\mathcal{H}}_1^{u+\ell-2,v-\ell+1}$ and write $\beta=\beta'+\beta''$ with

$$\beta'\in P_0(\Omega^{u+\ell-2,v-\ell+1})\,,\quad \beta''\in Q_0(\Omega^{u+\ell-1,v-\ell})\,.$$

We clearly have

$$(\widetilde{\Pi}_{\ell;\cdot,v}d-\tilde{d}_1)\beta=(\tilde{d}_\ell-\tilde{d}_1)\beta=(Q_1+\cdots+Q_{\ell-1})(d_{2,-1}\beta'+d_{1,0}\beta'')\,,$$

yielding

$$\begin{aligned}\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d\widetilde{\Pi}_{\ell;\cdot,v}d\beta&=\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d\tilde{d}_1\beta\\&\quad+\Pi_{\ell+1}d_{1,0}(Q_1+\cdots+Q_{\ell-1})(d_{2,-1}\beta'+d_{1,0}\beta'')\\&=\Pi_{\ell+1}\Pi_2\pi_{u+\ell+1,v-\ell}d\tilde{d}_1\beta\\&\quad+\Pi_{\ell+1}d_1(Q_1+\cdots+Q_{\ell-1})(d_{2,-1}\beta'+d_{1,0}\beta'')\\&=0\end{aligned}$$

by Lemma 5.6.

Third, assume $\beta\in(Q_1+\cdots+Q_{\ell-1})(\Omega^{u+\ell-1,v-\ell})$, which is contained in $\mathcal{H}_1^{u+\ell-1,v-\ell}$. Then the result follows because $\widetilde{\Pi}_{\ell;\cdot,v}d=\widetilde{\Pi}_{\ell;\cdot,v}d_1$ on $\mathcal{H}_1^{\cdot,v-\ell}$, and

$$d_1\mathcal{H}_1^{\cdot,v-\ell}\subset P_1(\Omega^{\cdot,v-\ell})\perp\widetilde{\mathcal{H}}_\ell^{\cdot,v}\,.\qquad\square$$

LEMMA 5.10. *For $\alpha\in\widetilde{\mathcal{H}}_\ell^{u,v}$, if $\tilde{d}_\ell\alpha=0$, then $\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d\alpha=0$.*

Proof. Write $\alpha=\alpha'+\alpha''+\alpha'''$ with $\alpha'\in P_0(\Omega^{u,v})$, $\alpha''\in\Omega^{u+1,v-1}$ and

$$\alpha'''\in\bigoplus_{2\le a<\ell}\Omega^{u+a,v-a}\oplus\bar{Q}_{\ell-1}(\Omega^{u+\ell,v-\ell})\,.$$

Observe that $\alpha'+Q_0\alpha''\in\widetilde{\mathcal{H}}_1^{\cdot,v}$. Since $\tilde{d}_\ell\alpha=0$ and $\tilde{d}_1=\widetilde{\Pi}_{1;\cdot,v}\tilde{d}_\ell$ on $\widetilde{\mathcal{H}}_1^{\cdot,v}$, we have $\tilde{d}_1(\alpha'+Q_0\alpha'')=0$. Thus there is some $\beta\in\widetilde{\mathcal{H}}_1^{u-1,v}$ with $\tilde{d}_1\beta=\alpha'+Q_0\alpha''$ because $H(\widetilde{\mathcal{H}}_1^{\cdot,v},\tilde{d}_1)=0$. Then $\alpha-\tilde{d}_\ell\beta\in\widetilde{\mathcal{H}}_\ell^{u,v}$ satisfies

$$\pi_{u,v}(\alpha-\tilde{d}_\ell\beta)=Q_0\pi_{u+1,v-1}(\alpha-\tilde{d}_\ell\beta)=0\,,$$

and moreover

$$\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d\alpha=\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d(\alpha-\tilde{d}_\ell\beta)$$

by Lemma 5.9. Therefore we can assume $\alpha'+Q_0\alpha''=0$, and thus $\alpha'=Q_0\alpha''=0$. With this assumption, it follows that $\alpha''=(\Pi_1+P_0)\alpha''$ and

$$d_1\Pi_1\alpha''=\Pi_1d_{1,0}\Pi_1\alpha''=\Pi_1d_{1,0}\alpha''=\Pi_1\pi_{u+2,v-1}d\alpha=\Pi_1\pi_{u+2,v-1}\tilde{d}_\ell\alpha=0$$

by Lemma 2.3, yielding $Q_1\alpha''=0$.

Take a sequence

$$\phi_i\in Q_0(\Omega^{u+1,v-2})\subset\widetilde{\mathcal{H}}_\ell^{u-1,v}$$

such that $d_{0,1}\phi_i$ is C^∞ convergent to $P_0\alpha''$. Then the sequence $\alpha - \tilde{d}_\ell\phi_i \in \widetilde{\mathcal{H}}_\ell^{u,v}$ satisfies

$$\begin{aligned}\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d\alpha &= \Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d(\alpha - \tilde{d}_\ell\phi_i)\\ &\longrightarrow \Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d(\Pi_1\alpha'' + \alpha''')\end{aligned}$$

by Lemma 5.9. So

$$\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d\alpha = \Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d(\Pi_1\alpha'' + \alpha''')\,,$$

and thus we can also assume $P_0\alpha'' = 0$.

For each $k = 1,\dots,\ell-1$ there is some $\sigma^k \in Q_k(\Omega^{u-k+1,v+k-2})$ with $d_k\sigma^k = P_k\alpha''$. As above, from the existence of such a σ^1 we can assume $P_1\alpha'' = 0$ by Lemma 5.9 since $d_1 = \pi_{u,v-1}\Pi_1\tilde{d}_\ell$ on $\mathcal{H}_1^{u,v-1}$. If $\ell > 2$, by Proposition 5.4 for $k = 2,\dots,\ell-1$ there is a sequence $\tau_i^k \in \Omega_{u-m+2}^{u+v-1}$ such that

$$\begin{gathered}\pi_{u-k+a+1,v+k-a-1}d(\sigma^k + \tau_i^k) \longrightarrow 0\,,\quad 0<a<k\,,\\ \pi_{u+1,v-1}d(\sigma^k + \tau_i^k) \longrightarrow P_k\alpha''\end{gathered}$$

with respect to the C^∞ topology in Ω. We can thus suppose $P_k\alpha'' = 0$ for such a k because

$$\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d\widetilde{\Pi}_{\ell;\cdot,v}d(\sigma^k + \tau_i^k) = 0$$

by Lemma 5.9. Therefore

$$\alpha'' \in \mathcal{H}_\ell^{u+1,v-1} \oplus \bigoplus_{k=2}^{\ell-1} Q_k(\Omega^{u+1,v-1})\,, \tag{5.17}$$

where the last term does not show up if $\ell = 2$.

Now the condition $\tilde{d}_\ell\alpha = 0$ can be written as

$$\pi_{u+1+a,v-a}d(\alpha''+\alpha''')=\bar{Q}_{\ell-1}\pi_{u+\ell+1,v-\ell}d(\alpha''+\alpha''')=0\,,\ 0<a<\ell\,. \tag{5.18}$$

Observe that (5.18) summarizes the conditions of the first part of Proposition 5.4 for $k = 2,\dots,\ell$, with $\omega = \alpha''$, the constant sequence $\alpha_i = \alpha'''$, and $\gamma = 0$ if $2 \le k < \ell$. Since $\alpha'' \in \mathcal{H}_2^{u+1,v-1}$ by (5.17), we get inductively on $k = 2,\dots,\ell-1$ that $\alpha'' \in \mathcal{H}_k^{u+1,v-1}$ and $d_k\alpha'' = 0$ by (5.17), (5.18) and Proposition 5.4. Hence $\alpha'' \in \mathcal{H}_\ell^{u+1,v-1}$ by (5.17), and thus

$$\Pi_\ell\pi_{u+\ell+1,v-\ell}d\alpha = \Pi_\ell\pi_{u+\ell+1,v-\ell}d(\alpha'' + \alpha''') = d_\ell\alpha'' \perp \mathcal{H}_{\ell+1}$$

by (5.18) and Proposition 5.4 for $k = \ell$, and the result follows. □

We also need the following Hodge theory for the complex $(\widetilde{\mathcal{H}}_\ell^{\cdot,v}, \tilde{d}_\ell)$. Let $\tilde{\delta}_\ell = \widetilde{\Pi}_{\ell;\cdot,v}\delta$ on $\widetilde{\mathcal{H}}_\ell^{\cdot,v}$, and set $\widetilde{D}_\ell = \tilde{d}_\ell + \tilde{\delta}_\ell$ and $\widetilde{\Delta}_\ell = \widetilde{D}_\ell^2 = \tilde{\delta}_\ell\tilde{d}_\ell + \tilde{d}_\ell\tilde{\delta}_\ell$. Such a $\tilde{\delta}_\ell$ is adjoint of $\tilde{d}_\ell$ in $\widetilde{\mathcal{H}}_\ell^{\cdot,v}$ with respect to the L^2 inner product, and thus $\widetilde{D}_\ell$ and $\widetilde{\Delta}_\ell$ are symmetric unbounded operators in the L^2 completion $L^2\widetilde{\mathcal{H}}_\ell^{\cdot,v}$.

LEMMA 5.11. *The operator $\widetilde{D}_\ell$ is essentially self-adjoint in $L^2\widetilde{\mathcal{H}}_\ell^{\cdot,v}$.*

Proof. By Theorem 2.2 in [Ch], $D = d + \delta$ is essentially self-adjoint in $\mathbf{\Omega}$. Then, by using e.g. Lemma XII.1.6–(c) in [DuS], so is $\widetilde{\Pi}_{\ell;\cdot,v} D \widetilde{\Pi}_{\ell;\cdot,v}$ because $\widetilde{\Pi}_{\ell;\cdot,v}$ is a bounded self-adjoint operator on $\mathbf{\Omega}$. But $\widetilde{\Pi}_{\ell;\cdot,v} D \widetilde{\Pi}_{\ell;\cdot,v}$ is equal to $\widetilde{D}_\ell$ in $L^2\widetilde{\mathcal{H}}_\ell^{\cdot,v}$ and vanishes in its orthogonal complement. Hence $\widetilde{D}_\ell$ is essentially self-adjoint. □

LEMMA 5.12. *$D\widetilde{\Pi}_{\ell;\cdot,v} - \widetilde{\Pi}_{\ell;\cdot,v} D \widetilde{\Pi}_{\ell;\cdot,v}$ defines a bounded operator on $\mathbf{\Omega}$.*

Proof. We have

$$\begin{aligned} D\widetilde{\Pi}_{\ell;\cdot,v} - \widetilde{\Pi}_{\ell;\cdot,v} D \widetilde{\Pi}_{\ell;\cdot,v} &= \widetilde{P}_0(\delta_{-1,0} + \delta_{-2,1}\pi_{\cdot,v-1}) + \delta_{-2,1}\pi_{\cdot,v} \\ &\quad + (\Pi_\ell + \bar{P}_{\ell-1})(d_{1,0}\pi_{\cdot,v-\ell} + d_{2,-1}\pi_{\cdot,v-\ell+1}) \\ &\quad + d_{2,-1}\pi_{\cdot,v-\ell}\,. \end{aligned}$$

But

$$\widetilde{P}_0\delta_{-1,0}\pi_{\cdot,v} = \widetilde{P}_0\delta_{-1,0}P_0\pi_{\cdot,v}\,, \quad P_0 d_{1,0}\pi_{\cdot,v-\ell} = P_0 d_{1,0}\widetilde{P}_0\pi_{\cdot,v-\ell}$$

on $\widetilde{\mathcal{H}}_\ell^{\cdot,v}$. Then the result follows by Lemma 2.4(i). □

For each positive integer r, define the norm $\|\cdot\|'_r$ on $\widetilde{\mathcal{H}}_\ell^{\cdot,v}$ by setting

$$\|\phi\|'_r = \left\|(\mathrm{id} + \widetilde{D}_\ell)^r\phi\right\|,$$

and let $W^k\widetilde{\mathcal{H}}_\ell^{\cdot,v}$ be the corresponding completion of $\widetilde{\mathcal{H}}_\ell^{\cdot,v}$. Then the following result follows directly from Lemma 5.12.

COROLLARY 5.13. *The restriction of each rth Sobolev norm $\|\cdot\|_r$ to $\widetilde{\mathcal{H}}_\ell^{\cdot,v}$ is equivalent to the norm $\|\cdot\|'_r$. Thus $W^k\widetilde{\mathcal{H}}_\ell^{\cdot,v}$ is the closure of $\widetilde{\mathcal{H}}_\ell^{\cdot,v}$ in $W^k\Omega$.*

COROLLARY 5.14. *The Hilbert space $L^2\widetilde{\mathcal{H}}_\ell^{\cdot,v}$ has a complete orthonormal system $\{\phi_i : i = 1, 2, \dots\} \subset \widetilde{\mathcal{H}}_\ell^{\cdot,v}$, consisting of eigenvectors of $\widetilde{\Delta}_\ell$, so that the corresponding eigenvalues satisfy $0 \le \lambda_1 \le \lambda_2 \le \cdots$ with $\lambda_i \uparrow \infty$ if $\dim \widetilde{\mathcal{H}}_\ell^{\cdot,v} = \infty$; thus all of these eigenvalues have finite multiplicity. We also have the orthogonal decomposition*

$$\widetilde{\mathcal{H}}_\ell^{\cdot,v} = (\ker \tilde{d}_\ell \cap \ker \tilde{\delta}_\ell) \oplus \operatorname{im} \tilde{d}_\ell \oplus \operatorname{im} \tilde{\delta}_\ell\,,$$

with

$$\begin{aligned} \ker \widetilde{\Delta}_\ell &= \ker \tilde{d}_\ell \cap \ker \tilde{\delta}_\ell\,, \\ \operatorname{im} \widetilde{\Delta}_\ell &= \operatorname{im} \tilde{d}_\ell \oplus \operatorname{im} \tilde{\delta}_\ell\,, \\ \ker d_\ell &= (\ker \tilde{d}_\ell \cap \ker \tilde{\delta}_\ell) \oplus \operatorname{im} \tilde{d}_\ell\,, \\ \ker \delta_\ell &= (\ker \tilde{d}_\ell \cap \ker \tilde{\delta}_\ell) \oplus \operatorname{im} \tilde{\delta}_\ell\,. \end{aligned}$$

Proof. Corollary 5.13 implies that each inclusion $W^{r+1}\widetilde{\mathcal{H}}_\ell^{\cdot,v} \hookrightarrow W^r\widetilde{\mathcal{H}}_\ell^{\cdot,v}$ is a compact operator, and $\bigcap_r W^r\widetilde{\mathcal{H}}_\ell^{\cdot,v} = \widetilde{\mathcal{H}}_\ell^{\cdot,v}$. Then the result follows by Proposition 2.44 in [AT] and Lemma 5.11. □

Contrary to the case of $(\widetilde{\mathcal{H}}_1^{\cdot,v}, \tilde{d}_1)$, it may easily happen that the complex $(\widetilde{\mathcal{H}}_\ell^{\cdot,v}, \tilde{d}_\ell)$ has non-trivial cohomology. But we still can finish the proof of Lemma 5.5.

Proof of Lemma 5.5 for the case $k = \ell + 1$. Observe that the strong convergence

$$\begin{gathered}\pi_{u+a,v-a+1}d\alpha_i \longrightarrow 0\,, \quad 0 < a \leq \ell\,,\\ \bar{Q}_{\ell-1}\pi_{u+\ell+1,v-\ell}d\alpha_i \longrightarrow 0\end{gathered}$$

in $\mathbf{\Omega}$ just means the strong convergence $\tilde{d}_\ell\widetilde{\Pi}_{\ell;u,v}\alpha_i \to 0$. Write $\widetilde{\Pi}_{\ell;u,v}\alpha_i = \phi_i + \psi_i$ with $\phi_i \in \ker \tilde{d}_\ell$ and $\psi_i \in \operatorname{im}\tilde{\delta}_\ell$, according to Corollary 5.14. Then $\tilde{d}_\ell\psi_i \to 0$ strongly in $\mathbf{\Omega}$ by Lemma 5.8, yielding $\psi_i \to 0$ strongly in $\mathbf{\Omega}$ by Corollary 5.14. Moreover

$$\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d\alpha_i = \Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d\widetilde{\Pi}_{\ell;u,v}\alpha_i = \Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d\psi_i \longrightarrow 0$$

by Lemma 5.10 and because any linear map $\widetilde{\mathcal{H}}_\ell^{\cdot,v} \to \mathcal{H}_{\ell+1}^{\cdot,v-\ell}$ is continuous with respect to the L^2 norms since $\mathcal{H}_{\ell+1}$ is of finite dimension. Therefore $\gamma = 0$ as desired. □

This finishes the proof of Proposition 5.4, which has the following consequence.

Corollary 5.15. *Let $\omega \in \mathcal{H}_k^{u,v}$ and $\gamma \in \mathcal{H}_k^{u-k,v+k-1}$ for $k \geq 2$. If there is a sequence $\beta_i \in \bigoplus_{a>0}\Omega^{u-a,v+a}$ such that*

$$\begin{gathered}\pi_{u-a,v+a-1}\delta(\omega+\beta_i) \longrightarrow 0\,, \quad 0 < a < k\,,\\ \bar{P}_{k-2}\pi_{u-k,v+k-1}\delta(\omega+\beta_i) \longrightarrow 0\,, \quad \Pi_k\pi_{u-k,v+k-1}\delta(\omega+\beta_i) \longrightarrow \gamma\end{gathered}$$

strongly in $\mathbf{\Omega}$, then $\delta_k\omega = \gamma$. Moreover, in this case the sequence β_i can be chosen so that

$$\begin{gathered}\pi_{u-a,v+a-1}\delta(\omega+\beta_i) \longrightarrow 0\,, \quad 0 < a < k\,,\\ \pi_{u-k,v+k-1}\delta(\omega+\beta_i) \longrightarrow \gamma\end{gathered}$$

with respect to the C^∞ topology in Ω.

Proof. We can assume that M is oriented by using the two fold covering of orientations with standard arguments. Then it is easy to check that the Hodge star operator, $\star : \Omega \to \Omega$, satisfies $\star\mathcal{H}_k = \mathcal{H}_k$, and $\star d_k = (-1)^{r+1}\delta_k\star$ on $\mathcal{H}_k^r$ for each integer r. Then the result follows from Proposition 5.4. □

Now Theorem 5.1 follows directly from Proposition 5.4 and Corollary 5.15 by induction on k.

5.2 Estimates of the rescaled Laplacian. The rescaled Laplacian Δ_h is the square of the "rescaled Dirac operator" $D_h = d_h + \delta_h$, which will be used here too. The sum of (1.6) and (1.7) gives

$$D_h = D_0 + hD_\perp + h^2 F\,, \tag{5.19}$$

where

$$D_0 = d_{0,1} + \delta_{0,-1}\,, \quad D_\perp = d_{1,0} + \delta_{-1,0}\,, \quad F = d_{2,-1} + \delta_{-2,1}\,,$$

Let also $\Delta_\perp = D_\perp^2$.

Lemma 5.16 (Álvarez-Kordyukov [AK, Remark 3.5]). *There is a zero order differential operator B on Ω such that*

$$D_\perp D_0 + D_0 D_\perp = BD_0 + D_0 B^*\,.$$

Recall that, for self-adjoint operators A, B in a Hilbert space H, the inequality $A \le B$ is defined in the sense of quadratic forms: $\langle Au, u\rangle \le \langle Bu, u\rangle$ for all $u \in H$.

Proposition 5.17. *There is some $C > 0$ such that*

$$\Delta_h \ge \tfrac{1}{2}\Delta_0 + \tfrac{1}{2}h^2\Delta_\perp - Ch^2$$

for h small enough.

Proof. Consider the operators B, F given by Lemma 5.16 and (5.19). Since B, F are of order zero, there is some $C' > 0$ such that $B^*B, F^2 \le C'$. Because D_0 is symmetric, we get

$$\begin{aligned} h\big|\langle (BD_0 + D_0B^*)\omega, \omega\rangle\big| &\le 2h\big|\langle D_0\omega, B\omega\rangle\big| \\ &\le 2h\|D_0\omega\|\,\|B\omega\| \\ &\le \tfrac{1}{4}\,\|D_0\omega\|^2 + 4h^2\,\|B\omega\|^2 \\ &= 2\left\langle \left(\tfrac{1}{4}\Delta_0 + 4h^2B^*B\right)\omega, \omega\right\rangle \end{aligned}$$

for all $\omega \in \Omega$, yielding

$$h\,|BD_0 + D_0B^*| \le \Delta_0 + h^2B^*B \le \Delta_0 + C'h^2.$$

Similarly we get

$$|FD_0 + D_0F| \le \Delta_0 + F^2 \le \Delta_0 + C'\,,$$

$$|FD_\perp + D_\perp F| \le \Delta_\perp + F^2 \le \Delta_0 + C'\,.$$

Therefore, from (5.19) and Lemma 5.16 we get

$$\begin{aligned} \Delta_h &= \Delta_0 + h^2\Delta_\perp + h^4F^2 + h(BD_0 + D_0B^*) \\ &\quad + h^2(D_0F + FD_0) + h^3(D_\perp F + FD_\perp) \\ &\ge \Delta_0 + h^2\Delta_\perp + h^4C' - \tfrac{1}{4}\Delta_0 - C'h^2 \\ &\quad - h^2(\Delta_0 + C') - h^3(\Delta_\perp + C') \end{aligned}$$

$$\geq \tfrac{1}{2}\Delta_0 + \tfrac{1}{2}h^2\Delta_\perp - Ch^2$$

for some $C > 0$ and all h small enough. □

Proof of Theorem B. In the case $k = 1$, (1.9) just means $\langle \Delta_{h_i}\omega_i, \omega_i\rangle \to 0$. Therefore

$$\left\langle \left(\tfrac{1}{2}\Delta_0 + \tfrac{1}{2}h_i^2\Delta_\perp - Ch_i^2\right)\omega_i, \omega_i \right\rangle \longrightarrow 0$$

by Proposition 5.17. Hence

$$\langle \Delta_0\omega_i, \omega_i\rangle \longrightarrow 0 \tag{5.20}$$

and $\langle \Delta_\perp\omega_i, \omega_i\rangle$ is uniformly bounded since both Δ_0 and $\Delta_\perp$ are positive operators. It follows that ω_i is uniformly bounded in $W^1\Omega$. Therefore some subsequence of ω_i is weakly convergent in $W^1\Omega$ (and thus strongly convergent in $\mathbf{\Omega}$) to some $\omega \in W^1\Omega$. From (5.20) we also get that $\|D_0\omega_i\| \to 0$. So $D_0\omega_i \to 0$ strongly in $\mathbf{\Omega}$, yielding $\omega \in \ker \mathbf{D}_0$ because $\mathbf{D}_0$ is a closed operator in $\mathbf{\Omega}$. But $\ker \mathbf{D}_0 = L^2\mathcal{H}_1$ by (2.11). Thus the result follows for $k = 1$.

For $k = 2$, it follows from (1.9) that

$$\|d_{h_i}\omega_i\| \in o(h_i)\,, \quad \|\delta_{h_i}\omega_i\| \in o(h_i)\,,$$

yielding that

$$\left(\tfrac{1}{h_i}d_{0,1} + d_{1,0} + h_i d_{2,-1}\right)\omega_i \longrightarrow 0\,, \quad \left(\tfrac{1}{h_i}\delta_{0,-1} + \delta_{-1,0} + h_i\delta_{-2,1}\right)\omega_i \longrightarrow 0\,,$$

strongly in $\mathbf{\Omega}$ by (1.6) and (1.7). Hence

$$\Pi\,(d_{1,0} + h_i d_{2,-1})\,\omega_i \longrightarrow 0\,, \quad \Pi\,(\delta_{-1,0} + h_i\delta_{-2,1})\,\omega_i \longrightarrow 0$$

strongly in $\mathbf{\Omega}$ as well, and thus so does the sequence $\Pi D_\perp\omega_i$. Then

$$D_1\Pi\omega_i = \Pi D_\perp\Pi\omega_i = \Pi D_\perp\omega_i - \Pi D_\perp\widetilde{\Pi}\omega_i \longrightarrow 0$$

strongly in $\mathbf{\Omega}$ by Lemma 2.4(i). It follows that $\omega \in \ker \mathbf{D}_1$ because $\mathbf{D}_1$ is a closed operator in $L^2\mathcal{H}_1$. But $\ker \mathbf{D}_1 = \mathcal{H}_2$ by Theorem 2.2(iii), and the result follows for $k = 2$.

For the case $k > 2$, we can assume $\omega_i \in \Omega^r$ and $\omega \in \mathcal{H}_2^{u,v}$ for some integers $u + v = r$. Let $\omega_i^a = \pi_{a,r-a}\omega_i$ for each integer a, and set

$$\omega_i' = \sum_{a\geq 0} h_i^{-a}\omega_i^{u+a}\,, \quad \omega_i'' = \sum_{a\geq 0} h_i^{-a}\omega_i^{u-a}\,.$$

Now, by Theorem 5.1, the result follows from the following claim.

Claim 1. *For $0 < a < k$, we have*

$$\pi_{u+a,v-a+1}d\omega_i' \longrightarrow 0\,, \quad \pi_{u-a,v+a-1}\delta\omega_i'' \longrightarrow 0\,,$$

strongly in $\mathbf{\Omega}$.

Clearly

$$\pi_{u,v+1}d\omega_i' = d_{0,1}\omega_i^u\,, \quad \pi_{u,v-1}\delta\omega_i'' = \delta_{0,-1}\omega_i^u\,.$$

Thus both of these components converge strongly to zero because $\omega \in L^2\mathcal{H}_1$.

To prove Claim 1 for other bihomogeneous components observe that, again from (1.9), both $\|d_{h_i}\omega_i\|$ and $\|\delta_{h_i}\omega_i\|$ are in $o(h_i^{k-1})$. Then

$$\|h_i^2 d_{2,-1}\omega_i^{b-2} + h_i d_{1,0}\omega_i^{b-1} + d_{0,1}\omega_i^b\| \in o(h_i^{k-1})\,, \tag{5.21}$$

$$\|h_i^2 \delta_{-2,1}\omega_i^{b+2} + h_i \delta_{-1,0}\omega_i^{b+1} + \delta_{0,-1}\omega_i^b\| \in o(h_i^{k-1})\,, \tag{5.22}$$

for every integer b, by considering bihomogeneous components of $d_{h_i}\omega_i$ and $\delta_{h_i}\omega_i$. Now

$$\pi_{u+1,v} d\omega_i' = d_{1,0}\omega_i^u + h_i^{-1} d_{0,1}\omega_i^{u+1}\,,$$
$$\pi_{u-1,v} \delta\omega_i'' = \delta_{-1,0}\omega_i^u + h_i^{-1} \delta_{0,-1}\omega_i^{u-1}\,.$$

Both of these components strongly converge to zero in $\mathbf{\Omega}$ too by (5.21) and (5.22), since so does $h_i d_{2,-1}\omega_i^{u-1}$ and $h_i \delta_{-2,1}\omega_i^{u+1}$ because $d_{2,-1}$ and $\delta_{-2,1}$ are of order zero and $\|\omega_i\| = 1$.

The other bihomogeneous components of $d\omega_i'$ and $\delta\omega_i''$ are the following ones, where $a \geq 2$,

$$\pi_{u+a,v-a+1} d\omega_i' = h_i^{-a+2} d_{2,-1}\omega_i^{u+a-2} + h_i^{-a+1} d_{1,0}\omega_i^{u+a-1} + h_i^{-a} d_{0,1}\omega_i^{u+a}\,,$$
$$\pi_{u-a,v+a-1} \delta\omega_i'' = h_i^{-a+2} \delta_{-2,1}\omega_i^{u-a+2} + h_i^{-a+1} \delta_{-1,0}\omega_i^{u-a+1} + h_i^{-a} \delta_{0,-1}\omega_i^{u-a}\,,$$

which strongly converge to zero in $\mathbf{\Omega}$ for $a < k$ by (5.21) and (5.22). This finishes the proof of Claim 1. □

Proof of Theorem A. First, we can assume the metric is bundle-like by (4.3). So we can apply the results of this section.

If we had a strict inequality "<" in (1.2) for some $k \geq 2$, by the isomorphism $\mathcal{H}_k^r \cong E_k^r$ there are sequences $\omega_i \in \Omega^r$ and $h_i \downarrow 0$ such that $\|\omega_i\| = 1$, $\omega_i \perp \mathcal{H}_k$, and

$$\langle \Delta_{h_i}\omega_i, \omega_i\rangle \in O(h_i^{2k})\,.$$

But then we get a contradiction by Theorem B. So inequality "$\geq$" holds in (1.2) for all $k \geq 2$.

The proof of "$\geq$" in (1.2) follows with the same arguments since $\widehat{E}_1^r \cong \mathcal{H}_1^r$, which is of finite dimension if and only if so is $L^2\mathcal{H}_1^r$.

For $k \geq 2$, inequality "$\leq$" of (1.2) in Theorem A follows directly from Corollary 4.2 and Theorem 3.5, as was pointed out in Remark 1.

Now observe that, for each $h > 0$ and each $\omega \in \mathcal{H}_1^r$, we have

$$D_h\omega = hD_\perp\omega + h^2 F\omega\,,$$

according to (5.19). Therefore the inequality "$\leq$" in (1.2) follows from the isomorphism $\mathcal{H}_1^r \cong \widehat{E}_1^r$ by using the well-known variational formula $N_h^r(\lambda) = \sup_V \dim V$, where V runs over the subspaces of Ω^r satisfying

$$\langle \Delta_h\omega, \omega\rangle \leq \lambda\,\|\omega\|^2$$

for all $\omega \in V$. □

6 Forman's Nested Sequence

This section is devoted to the proof of Theorem D. Thus let $\mathcal{F}$ be a Riemannian foliation of dimension p on a closed manifold M. We need the following characterization of $\mathfrak{H}_2$, which is weaker than (1.11) for $k=2$.

CLAIM 2. *A differential form $\omega \in \Omega$ is in $\mathfrak{H}_2$ if and only if it has extensions $\tilde{\omega}_1(h), \tilde{\omega}_2(h) \in \Omega[h]$ satisfying*

$$d_h\tilde{\omega}_1(h) \in h^2\Omega[h]\,, \quad \delta_h\tilde{\omega}_2(h) \in h^2\Omega[h]\,. \tag{6.1}$$

According to (1.11), it is enough to prove the "if" part of Claim 2. We can assume

$$\tilde{\omega}_1(h) = \omega + h\omega_1\,, \quad \tilde{\omega}_2(h) = \omega + h\omega_2$$

for some $\omega_1, \omega_2 \in \Omega$ because $d_h(h^2\Omega[h])$ and $\delta_h(h^2\Omega[h])$ are contained in $h^2\Omega[h]$. On the other hand, since $\mathfrak{H}_2$ is a bigraded subspace of Ω, we can suppose $\omega \in \Omega^{u,v}$ for some u, v. Then it easily follows from (6.1) that $\omega_1 \in \Omega^{u+1,v-1}$ and $\omega_2 \in \Omega^{u-1,v+1}$. Furthermore we can assume $\delta_{0,-1}\omega_1 = d_{0,1}\omega_2 = 0$ by Theorem 2.1. Hence the extension

$$\tilde{\omega}(h) = \omega + h(\omega_1 + \omega_2)$$

of ω is easily seen to satisfy (1.11) for $k=2$, and thus $\omega \in \mathfrak{H}_2$, finishing the proof of Claim 2

The statement of Claim 2 seems to hold also for $\mathfrak{H}_k$ with $k>2$, but the proof cannot be so easy.

By Theorem A and (1.14), we have $\mathfrak{H}_2^{0,p} = \mathcal{H}_2^{0,p} = 0$ if $E_2^{0,p} = 0$. Therefore we can assume $E_2^{0,p} \neq 0$ to prove Theorem D. According to [M] and [A2], this assumption implies that $\mathcal{F}$ is orientable and $E_2^{0,p} \cong \mathbb{R}$. So $\mathcal{H}_2^{0,p} \cong \mathbb{R}$ by Theorem A, and thus either $\mathfrak{H}_2^{0,p}(g) = 0$ or $\mathfrak{H}_2^{0,p}(g) = \mathcal{H}_2^{0,p}(g)$ by (1.14).

Recall from [Rum] that the *characteristic form*, determined by $\mathcal{F}$ and a metric g on M, is the unique differential form $\chi \in \Omega^{0,p}$ whose restriction to the leaves is the leafwise volume form. If g is a bundle-like metric, then $\delta_{0,-1}$ corresponds to the leafwise coderivative by restriction to the leaves [A2], [AK], yielding $\delta_{0,-1}\chi = 0$, and thus $\chi \in \mathfrak{H}_1^{0,p}(g)$.

To prove Theorem D(i) just choose the bundle-like metric g so that $d_{1,0}\chi = 0$, which can be done by using Sullivan's purification [Su] (see also [M] and [A2]). Hence $\chi \in \mathfrak{H}_2^{0,p}(g)$ by Claim 2, yielding $\mathfrak{H}_2^{0,p}(g) \neq 0$.

To prove Theorem D(ii), let us begin with a bundle-like metric g satisfying Theorem D(i), and the corresponding bigrading of Ω and decomposition of d and δ as sum of bihomogeneous components. The hypothesis $\bar{0}_1^{0,p} \neq 0$ means that $d_{0,1}\Omega^{0,p-1}$ is not closed in $\Omega^{0,p}$, and thus we can take some

$\alpha \in \overline{d_{0,1}\Omega^{0,p-1}} \setminus d_{0,1}\Omega^{0,p-1}$. Take also some $\epsilon > 0$ small enough so that $\chi + \epsilon\alpha = f\chi$ for some positive function f. Therefore $\chi' = f\chi$ is the characteristic form of some bundle-like metric g' on M. Such a g' can be chosen to define the same bigrading on Ω as g, yielding the same decomposition of d as sum of bihomogeneous components. We have $\chi' \in \mathfrak{H}_1^{0,p}(g') = \mathcal{H}_1^{0,p}(g')$. Moreover, since α defines a non-trivial class

$$[\alpha] \in \overline{d_{0,1}\Omega^{0,p-1}}/d_{0,1}\Omega^{0,p-1} = \bar{o}_1^{0,p} \cong \bar{0}_1^{0,p}$$

and since $H^0(\bar{o}_1^{\cdot,p}) = H^0(\bar{0}_1^{\cdot,p}) = 0$ by Theorem 2.2(vi), we get

$$0 \neq d_1[\alpha] = [d_{1,0}\alpha] \in \bar{o}_1^{1,p} \cong \bar{0}_1^{1,p}\,.$$

So

$$d_{1,0}\chi' = d_{1,0}(\chi + \epsilon\alpha) = \epsilon d_{1,0}\alpha \in \overline{d_{0,1}\Omega^{1,0}} \setminus d_{0,1}\Omega^{1,0}\,,$$

yielding $\chi' \in \mathcal{H}_2^{0,p} \setminus \mathfrak{H}_2^{0,p}(g')$. Therefore $\mathfrak{H}_2^{0,p}(g') \neq \mathcal{H}_2^{0,p}(g')$, and thus $\mathfrak{H}_2^{0,p}(g') = 0$.

Acknowledgement. The second author gratefully acknowledges the hospitality and support of the University of Santiago de Compostela. We also thank the referee for suggestions to improve the paper.

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Submitted: February 1999
Revised version: November 1999