DISTRIBUTIONAL BETTI NUMBERS OF TRANSITIVE FOLIATIONS OF CODIMENSION ONE

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Abstract. Let $\mathcal{F}$ be a transitive foliation of codimension one on a closed manifold M. This means that there is an infinitesimal transformation X of $(M,\mathcal{F})$ transverse to the leaves. The flow of X induces an $\mathbb{R}$-action on the reduced leafwise cohomology $\overline{H}(\mathcal{F})$. By using leafwise Hodge theory, the trace of this action on each $\overline{H}^i(\mathcal{F})$ can be defined as a distribution β^i_{dis} on $\mathbb{R}$, which is called distributional Betti number because it is kind of a finite measure of the "size" of $\overline{H}^i(\mathcal{F})$. So the corresponding distributional Euler characteristic, $\chi_{\text{dis}}(\mathcal{F})$, is a distribution on $\mathbb{R}$ too. This is relevant because $\overline{H}(\mathcal{F})$ may be of infinite dimension, even when the leaves are dense, and its Euler characteristic makes no sense in general. The singularity at 0 of $\chi_{\text{dis}}(\mathcal{F})$ is expressed in terms of the Connes' Λ-Euler characteristic, where Λ is the holonomy invariant transverse measure of $\mathcal{F}$ induced by the volume form dt on $\mathbb{R}$. Moreover the whole of $\chi_{\text{dis}}(\mathcal{F})$ is computed by showing a dynamical Lefschetz formula.

1. Introduction

Let M be a closed manifold and $\mathcal{F}$ a smooth foliation on M of codimension one. As usual, let $\mathfrak{X}(M,\mathcal{F}) \subset \mathfrak{X}(M)$ denote the Lie subalgebra of infinitesimal transformations of $(M,\mathcal{F})$, and $\mathfrak{X}(\mathcal{F}) \subset \mathfrak{X}(M,\mathcal{F})$ the ideal of vector fields tangent to the leaves. For any $X \in \mathfrak{X}(M,\mathcal{F})$, the corresponding flow maps leaves to leaves, and will be denoted by $X_t : (M,\mathcal{F}) \to (M,\mathcal{F})$, $t \in \mathbb{R}$.

The foliation $\mathcal{F}$ is called *transitive* when

$$T_xM = \{X(x) \mid X \in \mathfrak{X}(M,\mathcal{F})\}\ .$$

Since

$$T_x\mathcal{F} = \{Y(x) \mid Y \in \mathfrak{X}(\mathcal{F})\}\ ,$$

we get that $\mathcal{F}$ is transitive if and only if there is some $X \in \mathfrak{X}(M,\mathcal{F})$ transverse to the leaves; i.e.,

$$T_xM = \mathbb{R}\,X(x) \oplus T_x\mathcal{F}$$

for all $x \in M$. Then the orbits of X_t, $t \in \mathbb{R}$, are non-singular and transverse to the leaves. Note that

$$\mathfrak{X}(M,\mathcal{F})/\mathfrak{X}(\mathcal{F}) \cong \mathbb{R} \tag{1}$$

if the leaves are dense, which is the most interesting case.

The leafwise de Rham complex of $\mathcal{F}$, $(\Omega(\mathcal{F}), d_{\mathcal{F}})$, is the the restriction of the de Rham complex of M to the leaves; i.e., it is given by the smooth sections of the exterior vector bundle $\bigwedge T\mathcal{F}^*$ over M. Its cohomology is called the *leafwise cohomology* of $\mathcal{F}$, and will be denoted by $H(\mathcal{F})$. Moreover $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ is a topological complex with the C^∞ topology, and $H(\mathcal{F})$ is a topological vector space with the induced topology. It is well known that $H(\mathcal{F})$ may not be Hausdorff [14]. So it is interesting to consider its quotient over the closure of the trivial subspace, which is called the *reduced leafwise cohomology* of $\mathcal{F}$, and is denoted by $\overline{H}(\mathcal{F})$ in this paper.

Consider a Riemannian metric on M such that X is of norm one and orthogonal to the leaves. So all flow orbits are geodesics of speed one orthogonal to the leaves. This is what is called a *bundle-like* metric on M. Consider the induced Riemannian structure on the leaves, and let $\delta_{\mathcal{F}}, \Delta_{\mathcal{F}}$ be the leafwise coderivative and leafwise Laplacian on $\Omega(\mathcal{F})$, which are the restrictions to $\Omega(\mathcal{F})$ of the coderivative and Laplacian on the leaves. The kernel $\mathcal{H}(\mathcal{F})$ of $\Delta_{\mathcal{F}}$ is the space of harmonic forms on the leaves that are smooth on M. The L^2 inner product on M induces a Hilbert space structure in the space $L^2\Omega(\mathcal{F})$ of square integrable leafwise differential forms on M. Consider $\Delta_{\mathcal{F}}$ as an unbounded operator in $L^2\Omega(\mathcal{F})$ with domain $\Omega(\mathcal{F})$, and let $\overline{\Delta}_{\mathcal{F}}$ be its closure. It is well known that $\Delta_{\mathcal{F}}$ is symmetric on M when the metric is bundle-like (see, for instance, [5, 16]), so $\overline{\Delta}_{\mathcal{F}}$ is a self-adjoint operator. Let $\overline{\Pi}$ be the orthogonal projection $L^2\Omega(\mathcal{F}) \to \ker \overline{\Delta}_{\mathcal{F}}$. By [3], $\overline{\Pi}$ has the restriction $\Pi : \Omega(\mathcal{F}) \to \mathcal{H}(\mathcal{F})$, and there is an orthogonal decomposition

$$\Omega(\mathcal{F}) = \mathcal{H}(\mathcal{F}) \oplus \overline{\operatorname{im} d_{\mathcal{F}}} \oplus \overline{\operatorname{im} \delta_{\mathcal{F}}} ,$$

which can be called a leafwise Hodge decomposition. In particular, the inclusion $\mathcal{H}(\mathcal{F}) \subset \ker d_{\mathcal{F}}$ induces an isomorphism

$$\mathcal{H}(\mathcal{F}) \xrightarrow{\cong} \overline{H}(\mathcal{F}) , \tag{2}$$

whose inverse is induced by the orthogonal projection $\Pi : \ker d_{\mathcal{F}} \to \mathcal{H}(\mathcal{F})$.

For any function $f \in C^\infty_c(\mathbb{R})$, define an operator A_f on $\Omega(\mathcal{F})$ by the formula

$$A_f = \Pi \circ \int_{\mathbb{R}} X_t^* \cdot f(t)\, dt \circ \Pi ,$$

and let $A_f^{(i)}$ denote its restriction to $\Omega^i(\mathcal{F})$. Our first main result is the following.

Theorem 1.1. *For any function $f \in C^\infty_c(\mathbb{R})$, the operator A_f is of trace class, and the functional $f \mapsto \operatorname{Tr}\left(A_f^{(i)}\right)$ defines a distribution $\beta^i_{\mathrm{dis}}(\mathcal{F})$ on $\mathbb{R}$ for each i.*

The distributions $\beta^i_{\mathrm{dis}}(\mathcal{F})$ depend only on $\mathcal{F}$ and the class of X in $\mathfrak{X}(M,\mathcal{F})/\mathfrak{X}(\mathcal{F})$ (Lemma 2.3); thus, when the leaves are dense, they depend only on $\mathcal{F}$ up to linear isomorphisms of $\mathbb{R}$ by (1).

The usual dimension of the spaces $\overline{H}^i(\mathcal{F})$ can be infinite even when the leaves are dense [1, 2, 3]. So the Euler characteristic of $\overline{H}(\mathcal{F})$ can not be defined, and thus a leafwise Gauss-Bonnet theorem makes no sense in the usual way. This is surely a reason of the poor rôle played by the reduced leafwise cohomology in foliation theory, which should be similar to the important rôle played by de Rham cohomology of closed manifolds.

To have finite leafwise Betti numbers, they must be defined in another way, by using the another kind of dimension ("exotic dimension"). A solution was given by Connes for foliations with a holonomy invariant transverse measure Λ [8, 9]. In our case, Λ is the transverse Riemannian volume element, which corresponds to dt on $\mathbb{R}$. This Λ is used to make kind of an average on M of the "local dimension" of the space of square integrable harmonic forms on the leaves at each degree i, giving the finite Λ-Betti numbers $\beta^i_\Lambda(\mathcal{F})$, and thus a Λ-Euler characteristic $\chi_\Lambda(\mathcal{F})$. The technical difficulties of this idea are solved by using the noncommutative integration theory of Connes. But, if the leaves are not compact, the forms of our space $\mathcal{H}(\mathcal{F})$

are not square integrable on the leaves because they are smooth on M. So, a priori, the Λ-Betti numbers are not directly related with the reduced leafwise cohomology.

Now we give another "exotic" solution to the above problem. Observe that, for $f \in C^\infty_c(\mathbb{R})$ supported around 0, the operator A_f is kind of a diffusion of the orthogonal projection $\Pi : \Omega(\mathcal{F}) \to \mathcal{H}(\mathcal{F})$. So the germ of $\beta^i_{\text{dis}}(\mathcal{F})$ at 0 can be considered as a finite measure of the size of $\mathcal{H}^i(\mathcal{F})$, and thus of $\overline{H}^i(\mathcal{F})$ as well. For this reason, the germs at 0 of the distributions $\beta^i_{\text{dis}}(\mathcal{F})$ could be called distributional Betti numbers. But, for the sake of simplicity, the whole distributions $\beta^i_{\text{dis}}(\mathcal{F})$ will be called the *distributional Betti numbers* of $\mathcal{F}$, even though they should be better considered as Lefschetz numbers away from 0. We also define the *distributional Euler characteristic* of $\mathcal{F}$ by the formula

$$\chi_{\text{dis}}(\mathcal{F}) = \sum_i (-1)^i \, \beta^i_{\text{dis}}(\mathcal{F}) \ .$$

The following theorem describes the singularity of $\chi_{\text{dis}}(\mathcal{F})$ at 0 in terms of Connes' Λ-Euler characteristic $\chi_\Lambda(\mathcal{F})$. So Connes' Λ-Betti numbers are really strongly related with the reduced leafwise cohomology. The similar result was obtained in [18] when the flow is isometric.

Theorem 1.2. *In some neighborhood of* 0 *in* $\mathbb{R}$*, we have*

$$\chi_{\text{dis}}(\mathcal{F}) = \chi_\Lambda(\mathcal{F}) \cdot \delta_0 \ ,$$

where δ_0 *denotes the Dirac measure at* 0*.*

Recall that a closed orbit c of length l of the flow X_t on $(M, \mathcal{F})$ is called *simple* when

$$\det(\mathrm{id} - X^*_l : T_x\mathcal{F}^* \to T_x\mathcal{F}^*) \neq 0$$

for any $x \in c$. The following theorem proves, for this type of foliations, a conjecture stated by Deninger in [10]. Under some additional assumptions, it was proved in [11, 18].

Theorem 1.3. *Assume that all closed orbits of the flow* X_t *on* $(M, \mathcal{F})$ *are simple. Then we have*

$$\chi_{\text{dis}}(\mathcal{F}) = \sum_c l(c) \sum_{k=1}^{\infty} \operatorname{sign} \det \left(\mathrm{id} - X^*_{l(c)} : T_x\mathcal{F}^* \to T_x\mathcal{F}^* \right) \cdot \delta_{kl(c)}$$

on $\mathbb{R}_+$*, where* c *runs over all primitive closed orbits of the flow* X_t*,* $l(c)$ *denotes the length of* c*, and* x *is an arbitrary point of* c*.*

Of course, in Theorem 1.3, a symmetric formula for $\chi_{\text{dis}}(\mathcal{F})$ also holds in $\mathbb{R}_-$.

Observe that, if $\dim \mathcal{H}^i(\mathcal{F}) = \beta^i(\mathcal{F}) < \infty$, then $\beta^i_{\text{dis}}(\mathcal{F})$ is a smooth measure whose value at 0 is $\beta^i(\mathcal{F})\, dt$. On the other hand, when $\mathcal{H}(\mathcal{F})$ is of finite dimension, its Euler characteristic can be defined :

$$\chi(\mathcal{F}) = \sum_i (-1)^i \, \beta^i(\mathcal{F}) \ .$$

But, by Theorems 1.2 and 1.3, the distributional Euler characteristic $\chi_{\text{dis}}(\mathcal{F})$ is trivial if it is smooth, obtaining the following.

Corollary 1.4. *If* $\dim \mathcal{H}(\mathcal{F}) < \infty$*, then* $\chi_{\text{dis}}(\mathcal{F})$*,* $\chi_\Lambda(\mathcal{F})$ *and* $\chi(\mathcal{F})$ *vanish.*

Theorem 1.3 also has the following consequence.

Corollary 1.5. *Assume that all closed orbits of the flow X_t on $(M, \mathcal{F})$ are simple. If* $\dim \mathcal{H}^{\cdot}(\mathcal{F}) < \infty$, *then, for any* $l \in \mathbb{R}$,

$$\sum_c \frac{1}{\mu(c)} \operatorname{sign} \det(\mathrm{id} - X_l^* : T_x\mathcal{F}^* \to T_x\mathcal{F}^*) = 0 \,,$$

where c runs over all closed orbits of the flow X_t of period l, $\mu(c)$ denotes the multiplicity of c, and $x \in c$ is an arbitrary point.

When the dimension of $\mathcal{F}$ is two and the leaves are dense, it is possible to relate directly each distributional Betti number with the corresponding Λ-Betti number because we obviously have

$$\beta_\Lambda^0(\mathcal{F}) = \beta_\Lambda^2(\mathcal{F}) = 0 \,, \quad \dim \mathcal{H}^0(\mathcal{F}) = 1 \,, \quad \dim \mathcal{H}^2(\mathcal{F}) \le 1 \,.$$

So Theorem 1.2 has the following consequence.

Corollary 1.6. *Assume that $\mathcal{F}$ is of dimension two with dense leaves. Then the singular part of $\beta_{\mathrm{dis}}^i(\mathcal{F})$ around 0 is $\beta_\Lambda^i(\mathcal{F}) \cdot \delta_0$ for each degree i.*

It is possible that the statement of Corollary 1.6 holds in general. Indeed, a proof could be given by finding appropriate heat kernel estimates on the leaves. So we propose the following.

Question 1.7. *For each degree i, is it true that the singular part of $\beta_{\mathrm{dis}}^i(\mathcal{F})$ around 0 is $\beta_\Lambda^i(\mathcal{F}) \cdot \delta_0$?*

If this question has an affirmative answer, then $\dim \mathcal{H}^i(\mathcal{F}) = \infty$ whenever $\beta_\Lambda^i(\mathcal{F}) \neq 0$. This would mean that the existence of non-trivial square integrable harmonic i-forms on the leaves implies the existence of non-trivial harmonic i-forms on the leaves that are smooth on M. Similar results were shown in [1, 3], where integrable harmonic i-forms on the leaves are used instead of square integrable ones, which are much easier to find.

Let R_L be the curvature of the leafwise metric, and $\mathrm{Pf}(R_L/2\pi) \in \Omega^p(\mathcal{F})$ the leafwise Euler form, $p = \dim \mathcal{F}$. The product $\mathrm{Pf}(R_L/2\pi) \wedge \Lambda$ is a differential form of top degree on M. In particular,

$$\mathrm{Pf}(R_L/2\pi) \wedge \Lambda = \frac{1}{2\pi} K_{\mathcal{F}}(x)\, \omega_M(x)$$

if $\mathcal{F}$ is of dimension 2, where $K_{\mathcal{F}}$ is the Gauss curvature of the leaves and ω_M is the volume form on M. Then Theorem 1.2 and the foliation Gauss-Bonnet theorem from [8], which computes $\chi_\Lambda(\mathcal{F})$, have the following consequence.

Corollary 1.8. *We have*

$$\chi_{\mathrm{dis}}(\mathcal{F}) = \delta_0 \cdot \int_M \mathrm{Pf}(R_L/2\pi) \wedge \Lambda$$

around 0. *In particular, if $\mathcal{F}$ is of dimension two, then*

$$\chi_{\mathrm{dis}}(\mathcal{F}) = \delta_0 \cdot \frac{1}{2\pi} \int_M K_{\mathcal{F}}(x)\, \omega_M(x)$$

around 0.

Corollary 1.8 seems to be a powerful tool to produce examples of foliations with dense leaves on closed Riemannian manifolds with $\dim \mathcal{H}^i(\mathcal{F}) = \infty$; specially, if Question 1.7 has an affirmative answer.

There are obvious versions of these results with general coefficients, which were not considered here for the sake of simplicity.

This type of foliations are just Lie foliations of codimension one. So this is a particular case of our work on distributional Betti numbers for arbitrary Lie foliations [4]. It is worth to explain this particular case here because the arguments are much easier to understand, and moreover the codimension one case is relevant for Deninger's approach to Riemann Hypothesis [10, 11].

Finally, let us mention that our results are somehow related with the study of transversely elliptic operators for Lie group actions [6, 23, 20, 9, 15, 17].

2. Distributional Betti numbers

2.1. **Leafwise homotopies.** A C^∞ foliation map $f:(M,\mathcal{F})\to(M,\mathcal{F})$ induces a homomorphism of topological complexes, $f^*:\Omega(\mathcal{F})\to\Omega(\mathcal{F})$, by pulling-back differential forms. Then it also induces homomorphisms of graded topological vector spaces, $f^*:H(\mathcal{F})\to H(\mathcal{F})$ and $f^*:\overline{H}(\mathcal{F})\to\overline{H}(\mathcal{F})$.

Two maps C^∞ maps $f,f':(M,\mathcal{F})\to(M,\mathcal{F})$ are said to be leafwise homotopic if there is a C^∞ homotopy between them, $h_s:(M,\mathcal{F})\to(M,\mathcal{F})$, $s\in I=[0,1]$, such that each curve $s\mapsto h_s(x)$, $x\in M$, is contained in a leaf. Such a homotopy is called an *leafwise homotopy*, and the notation $f\simeq_{\mathcal{F}}f'$ will be used. Then the usual construction of an homotopy of de Rham complexes produces a linear continuous map $k:\Omega(\mathcal{F})\to\Omega(\mathcal{F})$, homogeneous of degree -1, such that

$$f^*-f'^*=k\circ d_{\mathcal{F}}+d_{\mathcal{F}}\circ k. \tag{3}$$

Moreover k depends continuously on the homotopy h_s with respect to the C^∞ topology. We get

$$f\simeq_{\mathcal{F}}f'\Longrightarrow\begin{cases} f^*=f'^*:H(\mathcal{F})\to H(\mathcal{F})\ ,\\ f^*=f'^*:\overline{H}(\mathcal{F})\to\overline{H}(\mathcal{F})\ ,\end{cases}$$

and thus

$$f\simeq_{\mathcal{F}}f'\Longrightarrow\Pi\circ f^*=\Pi\circ f'^*:\mathcal{H}(\mathcal{F})\to\mathcal{H}(\mathcal{F}) \tag{4}$$

by the isomorphism (2).

2.2. **Smoothing operators.** Let ω_M denote the Riemannian volume element of M, and $\omega_{\mathcal{F}}$ the Riemannian volume element of $\mathcal{F}$. A *smoothing operator* on $\Omega(\mathcal{F})$ is a linear map $P:\Omega(\mathcal{F})\to\Omega(\mathcal{F})$, continuous with respect to the C^∞ topology, given by

$$(P\alpha)(x)=\int_M k(x,y)\,\alpha(y)\,\omega_M(y)\ ,\quad \alpha\in\Omega(\mathcal{F})\ ,$$

where $k\in C^\infty(\bigwedge T\mathcal{F}^*\boxtimes\bigwedge T\mathcal{F})$ is called the smoothing kernel of P. So

$$k(x,y)\in\bigwedge T\mathcal{F}^*_x\otimes\bigwedge T\mathcal{F}_y\equiv\operatorname{Hom}\left(\bigwedge T\mathcal{F}^*_y,\bigwedge T\mathcal{F}^*_x\right)\ ,\quad (x,y)\in M\times M\ .$$

Any smoothing operator P is of trace class, and we have

$$\operatorname{Tr}P=\int_M\operatorname{tr}k(x,x)\,\omega_M(x)\ , \tag{5}$$

where k is its smoothing kernel.

Let $\Omega(\mathcal{F})'$ be the dual space of $\Omega(\mathcal{F})$; i.e., the space of continuous linear functionals $\Omega(\mathcal{F})\to\mathbb{R}$, equipped with the weak dual topology (or topology of pointwise convergence). Let $\mathcal{L}(\Omega(\mathcal{F})',\Omega(\mathcal{F}))$ denote the space of continuous linear operators

$\Omega(\mathcal{F})' \to \Omega(\mathcal{F})$, equipped with the topology of bounded convergence. Consider also the C^∞ topology on $C^\infty\left(\bigwedge T\mathcal{F}^* \boxtimes \bigwedge T\mathcal{F}\right)$. The following result is well known.

Lemma 2.1. *A continuous operator in $\Omega(\mathcal{F})$ is smoothing if and only if it extends to a bounded linear operator $\Omega(\mathcal{F})' \to \Omega(\mathcal{F})$. Furthermore the map*

$$\mathcal{L}\left(\Omega(\mathcal{F})', \Omega(\mathcal{F})\right) \to C^\infty\left(\bigwedge T\mathcal{F}^* \boxtimes \bigwedge T\mathcal{F}\right) ,$$

which assigns its kernel to each operator, is an isomorphism of topological vector spaces.

Of course, Lemma 2.1 can be stated in terms of Sobolev spaces $W^k\Omega(\mathcal{F})$ of leafwise differential forms; in particular, a continuous operator in $\Omega(\mathcal{F})$ is smoothing if and only if it extends to a bounded operator $W^k\Omega(\mathcal{F}) \to W^l\Omega(\mathcal{F})$ for all k, l.

A special type of smoothing operators on $\Omega(\mathcal{F})$ can be constructed as follows. A subspace $V \subset \mathfrak{X}(M)$ is called transitive if

$$T_xM = \{Z(x) \mid Z \in V\}$$

for all $x \in M$. Since M is compact, it easily follows that there exists a finite dimensional subspace $W \subset \mathfrak{X}(\mathcal{F})$ such that

$$T_x\mathcal{F} = \{Z(x) \mid Z \in W\} .$$

Then $V = W \oplus \mathbb{R}X \subset \mathfrak{X}(M, \mathcal{F})$ is a finite dimensional transitive subspace. Fix an Euclidean metric on V so that X has norm one and is orthogonal to W. Then the following result was shown in [22].

Lemma 2.2 (Sarkaria). *For any $f \in C^\infty_c(V)$, the operator*

$$P = \int_V Z_1^* \cdot f(Z)\, dZ$$

on $\Omega(\mathcal{F})$ is smoothing, and its smoothing kernel depends continuously on f (with respect to the C^∞ topologies).

2.3. **Proof of Theorem 1.1.** For each $Z \in W$ and $t \in \mathbb{R}$, the maps

$$((1-s)X)_t \circ (Z + stX)_1 : (M, \mathcal{F}) \to (M, \mathcal{F}) , \quad s \in I ,$$

define a leafwise homotopy between $X_t \circ Z_1 = (tX)_1 \circ Z_1$ and $(Z + tX)_1$. So

$$\Pi \circ Z_1^* \circ X_t^* = \Pi \circ (Z + tX)_1^* : \Omega(\mathcal{F}) \to \Omega(\mathcal{F})$$

by (4).

Now take any $f \in C^\infty_c(\mathbb{R})$ and any $g \in C^\infty_c(W)$ with $\int_W g(Z)\, dZ = 1$, and let

$$B_f = \int_V Z_1^* \cdot h(Z)\, dZ : \Omega(\mathcal{F}) \to \Omega(\mathcal{F}) ,$$

where $h \in C^\infty_c(V)$ is given by $h(Z + tX) = f(t) \cdot g(Z)$ for $Z \in W$ and $t \in \mathbb{R}$. By Lemma 2.2, such a B_f is a smoothing operator whose smoothing kernel depends continuously on h, and thus on f. We also have

$$A_f = \Pi \circ \int_W \int_{\mathbb{R}} Z_1^* \circ X_t^* \cdot f(t) \cdot g(Z)\, dt\, dZ \circ \Pi = \Pi \circ B_f \circ \Pi .$$

On the other hand, it was proved by the authors in [3] that $\Pi : \Omega(\mathcal{F}) \to \Omega(\mathcal{F})$ is continuous, and has an extension to a bounded linear operator on every Sobolev space of leafwise differential forms, and thus to $\Omega(\mathcal{F})'$. So, by Lemma 2.1, the operator $A_f = \Pi \circ B_f \circ \Pi$ is smoothing. Moreover its smoothing kernel depends continuously on A_f, and thus on B_f. In turn, B_f depends continuously on its smoothing kernel, and thus on f. So the smoothing kernel of A_f depends continuously on f. It follows that A_f is a trace class operator, as well as each $A_f^{(i)}$, and their traces depend continuously on f by (5). Therefore each β^i_{dis} is a distribution.

2.4. **The dependence of the distributional Betti numbers.**

Lemma 2.3. *The distributional Betti numbers depend only on $\mathcal{F}$ and the class of X in $\mathfrak{X}(M,\mathcal{F})/\mathfrak{X}(\mathcal{F})$.*

Proof. Suppose that $Y \in \mathfrak{X}(M,\mathcal{F})$ defines the same class as X in $\mathfrak{X}(M,\mathcal{F})/\mathfrak{X}(\mathcal{F})$. Then, for all t,

$$(X + s(Y-X))_t : (M,\mathcal{F}) \to (M,\mathcal{F}) \;, \quad s \in I \;,$$

is an leafwise homotopy between X_t and Y_t. So

$$\Pi \circ \int_{\mathbb{R}} X_s^* \cdot f(s)\, ds = \Pi \circ \int_{\mathbb{R}} Y_s^* \cdot f(s)\, ds \tag{6}$$

by (4).

Take another bundle-like metric on M so that Y is of norm one and orthogonal to the leaves, and let $\Pi' : \Omega(\mathcal{F}) \to \mathcal{H}'(\mathcal{F})$ be the corresponding orthogonal projection onto the corresponding leafwise harmonic forms. Then

$$\Pi' : \mathcal{H}(\mathcal{F}) \overset{\cong}{\to} \mathcal{H}'(\mathcal{F}) \tag{7}$$

by (2).

For any $f \in C_c^\infty(\mathbb{R})$, let

$$B_f = \Pi' \circ \int_{\mathbb{R}} Y_s^* \cdot f(s)\, ds \circ \Pi' : \Omega(\mathcal{F}) \to \Omega(\mathcal{F}) \;,$$

and let $B_f^{(i)}$ denote its restriction to $\Omega^i(\mathcal{F})$. Then the distributional Betti numbers β'^i_{dis}, determined by $\mathcal{F}, Y$ and the new bundle-like metric, are given by $\langle \beta'^i_{\mathrm{dis}}, f\rangle = \mathrm{Tr}\left(B_f^{(i)}\right)$.

By (6), (7), and since Π, Π' are projections, it follows that

$$\begin{aligned}
\mathrm{Tr}\left(A_f^{(i)}\right) &= \mathrm{Tr}\left(\Pi \circ \int_{\mathbb{R}} X_s^* \cdot f(s)\, ds : \Omega^i(\mathcal{F}) \to \Omega^i(\mathcal{F})\right) \\
&= \mathrm{Tr}\left(\Pi \circ \int_{\mathbb{R}} Y_s^* \cdot f(s)\, ds : \Omega^i(\mathcal{F}) \to \Omega^i(\mathcal{F})\right) \\
&= \mathrm{Tr}\left(\Pi' \circ \int_{\mathbb{R}} Y_s^* \cdot f(s)\, ds : \Omega^i(\mathcal{F}) \to \Omega^i(\mathcal{F})\right) \\
&= \mathrm{Tr}\left(B_f^{(i)}\right) \;.
\end{aligned}$$

Therefore $\beta^i_{\mathrm{dis}} = \beta'^i_{\mathrm{dis}}$ as desired.

3. The distributional Euler characteristic and functions of the leafwise Laplacian

3.1. **A family of smoothing operators.** Let $UB(\mathbb{R})$ be the space of uniformly bounded Borel functions on $\mathbb{R}$. Since the operator $D_{\mathcal{F}} = d_{\mathcal{F}} + \delta_{\mathcal{F}}$ is essentially self-adjoint in $L^2\Omega(\mathcal{F})$, the Spectral Theorem defines a "functional calculus map"

$$UB(\mathbb{R}) \to \operatorname{End}(L^2\Omega(\mathcal{F}))\ , \quad \phi \mapsto \phi(D_{\mathcal{F}})\ ,$$

where $\operatorname{End}(L^2\Omega(\mathcal{F}))$ denotes the bounded linear endomorphisms of $L^2\Omega(\mathcal{F})$.

Let $\mathcal{A}$ be the set of functions $\phi : \mathbb{R} \to \mathbb{C}$ that extend to entire functions on $\mathbb{C}$ so that, for each compact subset $K \subset \mathbb{R}$, the set of functions $x \mapsto \phi(x + iy)$, $y \in K$, is bounded in the Schwartz space $\mathcal{S}(\mathbb{R})$. Such an $\mathcal{A}$ is a Fréchet algebra, and, in fact, a module over $\mathbb{C}[z]$. This algebra contains all functions with compactly supported Fourier transform, and functions $x \mapsto e^{-tx^2}$ with $t > 0$. By [21], the above functional calculus map, given by the Spectral Theorem, restricts to a "functional calculus map" $\mathcal{A} \to \operatorname{End}(\Omega(\mathcal{F}))$, which is a continuous homomorphism of $\mathbb{C}[z]$-modules and of algebras.

Let $x_1, \dots, x_p, y_1, \dots, y_q$ be foliation coordinates on a foliation patch U; i.e.,

$$(x_1, \dots, x_p, y_1, \dots, y_q) : U \to \mathbb{R}^p \times \mathbb{R}^q$$

is a diffeomorphism so that the slices $\mathbb{R}^p \times \{*\}$ correspond to the plaques of $\mathcal{F}$ in U. Then the differential forms

$$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_r}\ ,$$

for multiindixes $I = (i_1, \dots, i_r)$ with $1 \le i_1 < \dots < i_r \le p$, form a base of $\Omega\left(\mathcal{F}|_U\right)$ as $C^\infty(U)$-module. An operator B in $\Omega(\mathcal{F})$ is local when, for any $\alpha \in \Omega(\mathcal{F})$ and any $x \in M$, the value $(B\alpha)(x)$ depends only on the germ of α at x, and thus B defines an operator $B|_U$ in $\Omega\left(\mathcal{F}|_U\right)$ for any open subset $U \subset M$. Recall that a leafwise differential operator B in $\Omega(\mathcal{F})$ is a local operator in $\Omega(\mathcal{F})$ such that, for arbitrary foliation coordinates $x_1, \dots, x_p, y_1, \dots, y_q$ on any foliation patch U, with respect to the $C^\infty(U)$-base dx_I of $\Omega\left(\mathcal{F}|_U\right)$, the restriction $B|_U$ is given by a matrix whose entries are linear combinations, with coefficients in $C^\infty(U)$, of the leafwise partial derivatives

$$\frac{\partial^k}{\partial x^K} = \frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_p^{k_p}}\ ,$$

for multiindixes $K = (k_1, \dots, k_p) \in \mathbb{N}^p$, where $k = k_1 + \dots + k_p$. Now, a family $\{B_t \mid t \in \mathbb{R}\}$ of leafwise differential operators is called smooth when, for any foliation patch U with foliation coordinates $x_1, \dots, x_p, y_1, \dots, y_q$, in the corresponding expression of $B_t|_U$, the above coefficients of the partial derivatives $\partial^k/\partial x^K$ depend smoothly on t (they are C^∞ functions on $U \times \mathbb{R}$). The support of such a family is the closure in $\mathbb{R}$ of the set of points t with $B_t \neq 0$.

Proposition 3.1. *Let $\phi \in \mathcal{A}$ and let $\{B_t \mid t \in \mathbb{R}\}$ be a smooth compactly supported family of leafwise differential operators in $\Omega(\mathcal{F})$. Then*

$$B = \left(\int_{\mathbb{R}} X_t^* \circ B_t\, dt\right) \circ \phi(D_{\mathcal{F}})$$

is a smoothing operator in $\Omega(\mathcal{F})$ whose smoothing kernel depends continuously on ϕ and $\{B_t \mid t \in \mathbb{R}\}$.

Proposition 3.1 follows from Lemma 2.1 by showing that, for any k, l, there are some N and C such that

$$\left\| B : W^k\Omega(\mathcal{F}) \to W^l\Omega(\mathcal{F}) \right\| \leq \int_{\mathbb{R}} \left| (I - \partial_\xi^2)^N \hat{\phi}(\xi) \right| e^{C|\xi|} \, d\xi \; . \tag{8}$$

The proof of (8) is omitted here because it is rather technical. It is given in our work [3] with more generality.

Observe that Theorem 1.1 is a direct consequence of Proposition 3.1. In fact, by Proposition 3.1, for $f \in C_c^\infty(\mathbb{R})$ and $t > 0$, the operator

$$B_{t,f} = \int_{\mathbb{R}} X_s^* \cdot f(s) \, ds \circ e^{-t\Delta_{\mathcal{F}}}$$

on $\Omega(\mathcal{F})$ is smoothing and its smoothing kernel depends continuously on f. Hence $A_f = \Pi \circ B_{t,f} \circ \Pi$ satisfies the same properties. This also shows that the operator

$$A'_f = \int_{\mathbb{R}} X_t^* \cdot f(t) \, dt \circ \Pi = B_{t,f} \circ \Pi$$

is smoothing with smoothing kernel depending continuously on f, and we have $\mathrm{Tr}(A'_f) = \mathrm{Tr}(A_f)$ since Π is a projection.

3.2. **The distributional Euler characteristic and the leafwise heat operator.** Let $B_{t,f}^{(i)}$ denote the restriction of $B_{t,f}$ to $\Omega^i(\mathcal{F})$ for each degree i.

Lemma 3.2. *For any $f \in C_c^\infty(\mathbb{R})$, we have $\mathrm{Tr} B_{t,f}^{(i)} \to \mathrm{Tr} A_f^{(i)}$ as $t \to \infty$.*

Proof. Note that

$$\begin{aligned} B_{t,f} - A'_f &= \int_{\mathbb{R}} X_s^* \cdot f(s) \, ds \circ \left(e^{-t\Delta_{\mathcal{F}}} - \Pi \right) \\ &= \int_{\mathbb{R}} X_t^* \cdot f(t) \, dt \circ e^{-\Delta_{\mathcal{F}}} \circ \left(e^{-(t-1)\Delta_{\mathcal{F}}} - \Pi \right) \; . \end{aligned}$$

On the other hand, by [3], the operator $e^{-(t-1)\Delta_{\mathcal{F}}} - \Pi$ has a continuous extension $\Omega(\mathcal{F})' \to \Omega(\mathcal{F})'$, and converges to zero in $\mathcal{L}(\Omega(\mathcal{F})', \Omega(\mathcal{F})')$ (equipped with the topology of bounded convergence) as $t \to \infty$. Therefore $B_{t,f} - A'_f$ converges to zero in $\mathcal{L}(\Omega(\mathcal{F})', \Omega(\mathcal{F}))$ as $t \to \infty$, and thus its smoothing kernel converges to zero by Lemma 2.1, and the result follows.

For the sake of simplicity, it is worthwhile to use the supertrace notation. Consider $\Omega(\mathcal{F})$ as a $\mathbb{Z}_2$-graded space:

$$\Omega(\mathcal{F}) = \Omega^+(\mathcal{F}) \oplus \Omega^-(\mathcal{F}) \; ,$$

where $\Omega^+(\mathcal{F}) = \Omega^{\mathrm{even}}(\mathcal{F})$ and $\Omega^-(\mathcal{F}) = \Omega^{\mathrm{odd}}(\mathcal{F})$. For any $\mathbb{Z}_2$-homogeneous operator P on $\Omega(\mathcal{F})$, let $P^\pm$ denote its restrictions to $\Omega^\pm(\mathcal{F})$. If moreover P is of trace class and $\mathbb{Z}_2$-degree zero, its *supertrace* is defined as

$$\mathrm{Tr}^{\mathrm{s}}(P) = \mathrm{Tr}(P^+) - \mathrm{Tr}(P^-) \; .$$

In particular,

$$\langle \chi_{\mathrm{dis}}(\mathcal{F}), f \rangle = \mathrm{Tr}^{\mathrm{s}}(A_f)$$

for all $f \in C_c^\infty(\mathbb{R})$.

Choose an even function in $\mathcal{A}$, which can be written as $x \mapsto \psi(x^2)$. Then, for $t > 0$ and $f \in C_c^\infty(\mathbb{R})$, let

$$C_{t,\psi,f} = \int_{\mathbb{R}} X_s^* \cdot f(s)\, ds \circ \psi(t\Delta_{\mathcal{F}})^2 : \Omega(\mathcal{F}) \to \Omega(\mathcal{F}) .$$

In particular, $B_{t,f} = C_{t,\psi,f}$ when $\psi(x^2) = e^{-x^2/2}$.

Lemma 3.3. *$\mathrm{Tr}^{\mathrm{s}} C_{t,\psi,f}$ is independent of t.*

Proof. It is similar to the proof of the corresponding result in the heat equation proof of the usual Lefschetz trace formula [7, 13]. We have

$$\begin{aligned}
\frac{d}{dt}\mathrm{Tr}^{\mathrm{s}} C_{t,\psi,f} &= 2\,\mathrm{Tr}^{\mathrm{s}}\left(\int_{\mathbb{R}} X_s^* \cdot f(s)\, ds \circ \Delta_{\mathcal{F}} \circ \psi'(t\Delta_{\mathcal{F}}) \circ \psi(t\Delta_{\mathcal{F}})\right)\\
&= 2\,\mathrm{Tr}\left(\int_{\mathbb{R}} X_s^* \cdot f(s)\, ds \circ d_{\mathcal{F}}^- \circ \delta_{\mathcal{F}}^+ \circ \psi'\left(t\Delta_{\mathcal{F}}^+\right) \circ \psi\left(t\Delta_{\mathcal{F}}^+\right)\right)\\
&\quad - 2\,\mathrm{Tr}\left(\int_{\mathbb{R}} X_s^* \cdot f(s)\, ds \circ d_{\mathcal{F}}^+ \circ \delta_{\mathcal{F}}^- \circ \psi'\left(t\Delta_{\mathcal{F}}^-\right) \circ \psi\left(t\Delta_{\mathcal{F}}^-\right)\right)\\
&\quad + 2\,\mathrm{Tr}\left(\int_{\mathbb{R}} X_s^* \cdot f(s)\, ds \circ \delta_{\mathcal{F}}^- \circ d_{\mathcal{F}}^+ \circ \psi'\left(t\Delta_{\mathcal{F}}^+\right) \circ \psi\left(t\Delta_{\mathcal{F}}^+\right)\right)\\
&\quad - 2\,\mathrm{Tr}\left(\int_{\mathbb{R}} X_s^* \cdot f(s)\, ds \circ \delta_{\mathcal{F}}^+ \circ d_{\mathcal{F}}^- \circ \psi'\left(t\Delta_{\mathcal{F}}^-\right) \circ \psi\left(t\Delta_{\mathcal{F}}^-\right)\right) .
\end{aligned}$$

On the other hand, since the function $x \mapsto \psi'(x^2)$ is in $\mathcal{A}$, we have

$$\begin{aligned}
&\mathrm{Tr}\left(\int_{\mathbb{R}} X_s^* \cdot f(s)\, ds \circ d_{\mathcal{F}}^\mp \circ \delta_{\mathcal{F}}^\pm \circ \psi'\left(t\Delta_{\mathcal{F}}^\pm\right) \circ \psi\left(t\Delta_{\mathcal{F}}^\pm\right)\right)\\
&\quad= \mathrm{Tr}\left(d_{\mathcal{F}}^\mp \circ \int_{\mathbb{R}} X_s^* \cdot f(s)\, ds \circ \psi'\left(t\Delta_{\mathcal{F}}^\pm\right) \circ \psi\left(t\Delta_{\mathcal{F}}^\pm\right) \circ \delta_{\mathcal{F}}^\pm\right)\\
&\quad= \mathrm{Tr}\left(\psi\left(t\Delta_{\mathcal{F}}^\pm\right) \circ \delta_{\mathcal{F}}^\pm \circ d_{\mathcal{F}}^\mp \circ \int_{\mathbb{R}} X_s^* \cdot f(s)\, ds \circ \psi'\left(t\Delta_{\mathcal{F}}^\pm\right)\right)\\
&\quad= \mathrm{Tr}\left(\int_{\mathbb{R}} X_s^* \cdot f(s)\, ds \circ \psi'\left(t\Delta_{\mathcal{F}}^\pm\right) \circ \psi\left(t\Delta_{\mathcal{F}}^\pm\right) \circ \delta_{\mathcal{F}}^\pm \circ d_{\mathcal{F}}^\mp\right)\\
&\quad= \mathrm{Tr}\left(\int_{\mathbb{R}} X_s^* \cdot f(s)\, ds \circ \delta_{\mathcal{F}}^\pm \circ d_{\mathcal{F}}^\mp \circ \psi'\left(t\Delta_{\mathcal{F}}^\pm\right) \circ \psi\left(t\Delta_{\mathcal{F}}^\pm\right)\right) ,
\end{aligned}$$

where we have used the well known fact that, if A is a trace class operator and B is bounded, then AB and BA are trace class operators with the same trace. Therefore $\frac{d}{dt}\mathrm{Tr}^{\mathrm{s}} C_{t,\psi,f} = 0$ as desired.

The following result follows directly from Lemmas 3.2 and 3.3.

Corollary 3.4. *We have*

$$\mathrm{Tr}^{\mathrm{s}} B_{t,f} = \langle \chi_{\mathrm{dis}}(\mathcal{F}), f\rangle$$

for any $t > 0$ and $f \in C_c^\infty(\mathbb{R})$.

Like in [21, p. 463], choose a sequence of smooth even functions $\phi_m \in \mathcal{A}$, which we write as $\phi_m(x) = \psi_m(x^2)$, with $\phi_m(0) = 1$, and whose Fourier transforms $\widehat{\phi_m}$ are compactly supported and tend to the function $\phi(x) = e^{-x^2/2}$ as $m \to \infty$ (in the topology of the Schwartz space $\mathcal{S}(\mathbb{R})$). Without loss of generality, we can also assume that, for each N and C,

$$\int \left|(\mathrm{id}-\partial_\xi^2)^N\left(\widehat{\phi_m}(\xi)-\hat{\phi}(\xi)\right)\right| e^{C|\xi|}\,d\xi \to 0 \tag{9}$$

as $m\to\infty$. Consider the operator

$$C_{t,m,f}=C_{t,\psi_m,f}=\int_G X_g^*\cdot f(g)\,dg\circ\psi_m(t\Delta_{\mathcal{F}})^2$$

on $\Omega(\mathcal{F})$.

Lemma 3.5. *For any $t>0$ and $f\in C_c^\infty(\mathbb{R})$, we have*

$$\mathrm{Tr}^{\mathrm{s}}C_{t,m,f}\to\mathrm{Tr}^{\mathrm{s}}B_{t,f}=\langle\chi_{\mathrm{dis}}(\mathcal{F}),f\rangle$$

as $m\to\infty$.

Proof. Combining (9) and (8), we get that

$$C_{t,m,f}-B_{t,f}\to 0$$

in $\mathcal{L}(\Omega(\mathcal{F})',\Omega(\mathcal{F}))$ as $t\to\infty$. By Lemma 2.1, it follows that the smoothing kernel of $C_{t,m,f}$ converges uniformly to the smoothing kernel of $B_{t,f}$, and the result follows.

3.3. **Description of the smoothing kernels.** According to the structure of Lie foliations [12, 19], the foliation $\mathcal{F}$ can be described as follows. There is a finitely generated subgroup $\Gamma\subset\mathbb{R}$ that acts on the right in some manifold L such that:

- M is diffeomorphic to the orbit space $L\times_\Gamma\mathbb{R}$ of the right Γ-action on $\widetilde{M}=L\times\mathbb{R}$ given by
$$(x,s)\cdot\gamma=(x\cdot\gamma,s+\gamma)\;,\quad (x,s)\in L\times\mathbb{R}\;,\quad \gamma\in\Gamma\;;$$
say $M=L\times_\Gamma\mathbb{R}$. Thus the canonical projection $\pi:\widetilde{M}\to M$ is a covering map.
- The leaves of the lifting $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ to $\widetilde{M}$ are the fibers $L\times\{t\}$, $t\in\mathbb{R}$, of the second factor projection $D:\widetilde{M}\to\mathbb{R}$.
- The flow of the lifting $\widetilde{X}$ of X to $\widetilde{M}$ is given by $\widetilde{X}_t(x,s)=(x,t+s)$, $t\in\mathbb{R}$.

Let $\mathcal{G},\widetilde{\mathcal{G}}$ denote the holonomy groupoids of $\mathcal{F},\widetilde{\mathcal{F}}$ respectively. Since the leaves of $\mathcal{F},\widetilde{\mathcal{F}}$ have trivial holonomy groups, we have

$$\mathcal{G}\equiv\{(x,y)\in M\mid x,y\text{ lie in the same leaf of }\mathcal{F}\}\;,$$
$$\widetilde{\mathcal{G}}\equiv\left\{(\tilde{x},\tilde{y})\in\widetilde{M}\;\middle|\;\tilde{x},\tilde{y}\text{ lie in the same leaf of }\widetilde{\mathcal{F}}\text{ (fiber of }D)\right\}\;.$$

Thus $\mathcal{G},\widetilde{\mathcal{G}}$ are C^∞ submanifolds of $M\times M$ and $\widetilde{M}\times\widetilde{M}$, respectively. Moreover $\pi\times\pi:\widetilde{\mathcal{G}}\to\mathcal{G}$ is a covering map whose group of deck transformations is

$$\mathrm{Aut}\left(\widetilde{\mathcal{G}}\to\mathcal{G}\right)\equiv\mathrm{Aut}(\pi)\equiv\Gamma\;,$$

where $\sigma\in\mathrm{Aut}(\pi)$ corresponds to $\sigma\times\sigma\in\mathrm{Aut}\left(\widetilde{\mathcal{G}}\to\mathcal{G}\right)$. Let $s,r:\mathcal{G}\to M$ the source and the range projections, which are the restrictions of the factor projections $M\times M\to M$.

Recall the definition of the global action of the convolution algebra

$$C_c^\infty\left(\mathcal{G}, r^*\bigwedge T\mathcal{F}^* \otimes s^*\bigwedge T\mathcal{F}\right)$$

in $\Omega(\mathcal{F})$. For any

$$k \in C_c^\infty\left(\mathcal{G}, r^*\bigwedge T\mathcal{F}^* \otimes s^*\bigwedge T\mathcal{F}\right)\ , \quad \alpha \in \Omega(\mathcal{F})\ ,$$

the element $k \cdot \alpha \in \Omega(\mathcal{F})$ is given by

$$(k\cdot\alpha)(x) = \int_{L_x} k(x,y)\,\alpha(y)\,\omega_{\mathcal{F}}(y)\ , \quad x \in M\ ,$$

where L_x is the leaf of $\mathcal{F}$ through $x \in M$.

Consider the lifting of the fixed bundle-like metric on M to $\widetilde{M}$ and its restriction to the leaves of $\mathcal{F}$. Let $\omega_{\widetilde{M}}$ denote the Riemannian volume element of $\widetilde{M}$, and $\omega_{\widetilde{\mathcal{F}}}$ the Riemannian volume element of $\widetilde{\mathcal{F}}$. We also have a global action of any

$$\tilde{k} \in C^\infty\left(\widetilde{\mathcal{G}}, r^*\bigwedge T\widetilde{\mathcal{F}}^* \otimes s^*\bigwedge T\widetilde{\mathcal{F}}\right)\ ,$$

supported in an R-neighborhood of the diagonal $\widetilde{M} \equiv \widetilde{\Delta} \subset \widetilde{\mathcal{G}}$ for some $R > 0$, on the space $U\Omega(\widetilde{\mathcal{F}})$ of uniformly bounded differential forms in $\Omega(\widetilde{\mathcal{F}})$: For any $\alpha \in U\Omega(\widetilde{\mathcal{F}})$, the element $\tilde{k}\cdot\alpha \in U\Omega(\widetilde{\mathcal{F}})$ is given by

$$\left(\tilde{k}\cdot\alpha\right)(\tilde{x}) = \int_{\tilde{L}_{\tilde{x}}} \tilde{k}(\tilde{x},\tilde{y})\,\alpha(\tilde{y})\,\omega_{\widetilde{\mathcal{F}}}(\tilde{y})\ , \quad \tilde{x} \in \widetilde{M}\ .$$

By [21], if h is a bounded Borel function on $\mathbb{R}$ such that its Fourier transform $\hat{h} \in C_c^\infty(\mathbb{R})$, then the operator $h(D_{\mathcal{F}})$ on $\Omega(\mathcal{F})$ is represented by some element of

$$C_c^\infty\left(\mathcal{G}, r^*\bigwedge T\mathcal{F}^* \otimes s^*\bigwedge T\mathcal{F}\right)\ .$$

Moreover, it follows from the proof of Assertion 1 in [21, p.461] that, for any function h in the Schwartz space $\mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\hat{h} \subset [-R, R]$, the operator $h(D_{\mathcal{F}})$ on $\Omega(\mathcal{F})$ is represented by a leafwise smoothing kernel on $\mathcal{G}$ supported in the R-neighborhood of the diagonal $M \equiv \Delta \subset \mathcal{G}$.

The map $\pi\times\pi : \widetilde{\mathcal{G}} \to \mathcal{G}$ restricts to a diffeomorphism $\pi\times\pi : \widetilde{L}\times\widetilde{L} \to L\times L$ for any leaf $\widetilde{L}$ of $\widetilde{\mathcal{F}}$ ($L = \pi(\tilde{L})$). Hence, the lift of the leafwise smoothing kernel of $h(D_{\mathcal{F}})$ to $\widetilde{\mathcal{G}}$ is supported in the R-neighborhood of the diagonal $\widetilde{M} \equiv \widetilde{\Delta} \subset \widetilde{\mathcal{G}}$, and thus defines an operator $h\left(D_{\widetilde{\mathcal{F}}}\right)$ on $U\Omega\left(\widetilde{\mathcal{F}}\right)$. It is clear that the diagram

$$\begin{array}{ccc} U\Omega\left(\widetilde{\mathcal{F}}\right) & \xrightarrow{h\left(D_{\widetilde{\mathcal{F}}}\right)} & U\Omega\left(\widetilde{\mathcal{F}}\right) \\ {\scriptstyle (\pi\times\pi)^*}\uparrow & & \uparrow{\scriptstyle (\pi\times\pi)^*} \\ \Omega(\mathcal{F}) & \xrightarrow{h(D_{\mathcal{F}})} & \Omega(\mathcal{F}) \end{array}$$

commutes.

Since $\widehat{\phi_m}$ is compactly supported, the operator $\psi_m(t\Delta_{\mathcal{F}})^2$ is represented by a leafwise smoothing kernel

$$k_{m,t} \in C_c^\infty\left(\mathcal{G}, r^*\bigwedge T\mathcal{F}^* \otimes s^*\bigwedge T\mathcal{F}\right)\ .$$

The action of $k_{m,t}$ on $\Omega(\mathcal{F})$ defines the operator $\psi_m(t\Delta_{\mathcal{F}})^2$ in $\Omega(\mathcal{F})$: For any $\alpha \in \Omega(\mathcal{F})$, we have

$$\left(\psi_m(t\Delta_{\mathcal{F}})^2\alpha\right)(x) = (k_{m,t}\cdot\alpha)(x) = \int_{L_x} k_{m,t}(x,y)\,\alpha(y)\,\omega_{\mathcal{F}}(y)\ , \quad x \in M\ .$$

This operator is equal to the operator $\psi_m(t\Delta_{\mathcal{F}})^2$ in $\Omega(\mathcal{F})$ defined by the Spectral Theorem.

Let $\tilde{k}_{m,t}$ be the lift of $k_{m,t}$ to $\widetilde{\mathcal{G}}$, which is also supported in the R-neighborhood of the diagonal $\widetilde{M} \equiv \widetilde{\Delta} \subset \widetilde{\mathcal{G}}$ for some $R > 0$. For $\beta \in U\Omega\left(\widetilde{\mathcal{F}}\right)$, we have

$$\left(\psi_m\left(t\Delta_{\widetilde{\mathcal{F}}}\right)^2\beta\right)(\tilde{x}) = \int_{\widetilde{L}_{\tilde{x}}} \tilde{k}_{m,t}(\tilde{x},\tilde{y})\,\beta(\tilde{y})\,\omega_{\widetilde{\mathcal{F}}}(\tilde{y})\ , \quad \tilde{x} \in \widetilde{M}\ .$$

For any $f \in C_c^\infty(\mathbb{R})$, consider the operator

$$\widetilde{C}_{t,m,f} = \int_{\mathbb{R}} \widetilde{X}_s^*\cdot f(s)\,ds \circ \psi_m(t\Delta_{\widetilde{\mathcal{F}}})^2 : U\Omega\left(\widetilde{\mathcal{F}}\right) \to U\Omega\left(\widetilde{\mathcal{F}}\right)\ .$$

Obviously, $\widetilde{C}_{t,m,f}\circ\pi^* = \pi^*\circ C_{t,m,f}$ on $\Omega(\mathcal{F})$.

Lemma 3.6. *For any $f \in C_c^\infty(\mathbb{R})$, $t > 0$ and m, the operator $\widetilde{C}_{t,m,f}$ is a smoothing operator whose smoothing kernel $\tilde{c}_{t,m,f}$ is supported in an R-neighborhood of the diagonal in $\widetilde{M}\times\widetilde{M}$ for some $R > 0$, and*

$$\tilde{c}_{t,m,f}((x,u),(y,v)) = \widetilde{X}_{v-u}^*\circ\tilde{k}_{m,t}((x,v),(y,v))\cdot f(v-u) \tag{10}$$

for $(x,u),(y,v) \in L\times\mathbb{R} = \widetilde{M}$.

Proof. For $\alpha \in U\Omega\left(\widetilde{\mathcal{F}}\right)$, we have

$$\begin{aligned}
&\left(\widetilde{C}_{t,m,f}\alpha\right)(x,u)\\
&= \left(\int_{\mathbb{R}}\widetilde{X}_s^*\cdot f(s)\,ds\circ\psi_m\left(t\Delta_{\widetilde{\mathcal{F}}}\right)^2\alpha\right)(x,u)\\
&= \int_{\mathbb{R}}\widetilde{X}_s^*\left(\psi_m\left(t\Delta_{\widetilde{\mathcal{F}}}\right)^2\alpha\right)(x,u+s)\cdot f(s)\,ds\\
&= \int_{\mathbb{R}}\widetilde{X}_{v-u}^*\left(\int_{L\times\{v\}}\tilde{k}_{m,t}((x,v),(y,v))\alpha(y,v)\,\omega_{\widetilde{\mathcal{F}}}(y,v)\right)\cdot f(v-u)\,dv\\
&= \int_{\widetilde{M}}\widetilde{X}_{v-u}^*\circ\tilde{k}_{m,t}((x,v),(y,v))\alpha(y,v)\cdot f(v-u)\,\omega_{\widetilde{M}}(y,v)\ ,
\end{aligned}$$

by using the change of variable $s = v-u$. Hence $\widetilde{C}_{t,m,f}$ is defined by the smoothing kernel given in (10), and the result follows.

For each $(x,u) \in \widetilde{M}$, let $[x,u] = \pi(x,v)$. It is easy to see that the kernels $\tilde{c}_{t,m,f}$ and $c_{t,m,f}$ are related by the formula

$$c_{t,m,f}([x,u],[y,v]) \equiv \sum_{\gamma\in\Gamma}\tilde{c}_{t,m,f}((x,u),(y\cdot\gamma,v+\gamma))\ , \tag{11}$$

where we use the identity

$$\bigwedge T_{[x,u]}\widetilde{\mathcal{F}}^*\otimes\bigwedge T_{[y,v]}\widetilde{\mathcal{F}} \equiv \bigwedge T_{(x,u)}\mathcal{F}^*\otimes\bigwedge T_{(y\cdot\gamma,v+\gamma)}\mathcal{F}$$

via the map π. The sum in (11) is finite because $\tilde{c}_{t,m,f}$ is supported in an R-neighborhood of the diagonal in $\widetilde{M} \times \widetilde{M}$ for some $R > 0$.

4. Distributional Euler characteristic and Connes' Euler characteristic

The goal of this section is to prove Theorem 1.2.

Lemma 4.1. *Given $R > 0$, there is some neighborhood U of 0 in $\mathbb{R}$ so that*

$$\pi : \left\{(y,v) \mid (y,u) \in B_{\widetilde{\mathcal{F}}}((x,u),R)\ ,\ v-u \in U\right\} \to M$$

is injective for any $(x,u) \in \widetilde{M}$, where $B_{\widetilde{\mathcal{F}}}((x,u),R)$ denotes the ball of radius R and centered at (x,u) in the leaf $L \times \{u\}$.

Proof. Since M is compact, there exists a compact subset $K \subset \widetilde{M}$ with $\pi(K) = M$. Note that, if the statement holds for $(x,u) \in K$, then it holds for all $(x,u) \in \widetilde{M}$.

Assume the result is false. Then there exist sequences $(x_i,u_i),(y_i,v_i) \in \widetilde{M}$, and a sequence $\gamma_i \in \Gamma$ with $(x_i,u_i) \in K$, $\gamma_i \neq 0$, and such that (y_i,v_i) and $(y_i\cdot\gamma_i, v_i+\gamma_i)$ approach $B_{\widetilde{\mathcal{F}}}((x_i,u_i),R)$ in the sense that the distance between the terms of this sequences to this set converges to zero.

Since K is compact, we can assume that there exists $\lim_i(x_i,u_i)_i = (x,u) \in \widetilde{M}$. Hence, (y_i,v_i) and $(y_i\cdot\gamma_i, v_i+\gamma_i)$ approach the relatively compact set $B_{\widetilde{\mathcal{F}}}((x,u),R)$. It follows that, for infinitely many i, the points (y_i,v_i) and $(y_i\cdot\gamma_i, v_i+\gamma_i)$ lie in some compact neighborhood Q of $B_{\widetilde{\mathcal{F}}}((x,u),R)$; thus $Q\cdot\gamma_i \bigcap Q \neq \emptyset$ for infinitely many i. This implies that there exists some $\gamma \in \Gamma$ such that $\gamma_i = \gamma$ for infinitely many i. In particular, $\gamma \neq 0$.

On the other hand, since (y_i,v_i) and $(y_i\cdot\gamma_i, v_i+\gamma_i)$ approach $B_{\widetilde{\mathcal{F}}}((x,u),R)$, which is relatively compact, we can assume that there exist

$$\lim_i (y_i,v_i)\ ,\quad \lim_i (y_i\cdot\gamma_i, v_i+\gamma_i)$$

in $\overline{B_{\widetilde{\mathcal{F}}}((x,u),R)}$. Therefore, if $\lim_i(y_i,v_i) = (y,v)$, then

$$(y,v) \in L \times \{u\}\ ,\quad (y\cdot\gamma, v+\gamma) = \lim_i (y_i\cdot\gamma_i, v_i+\gamma_i) \in L \times \{u\}\ ,$$

yielding $v = u = v+\gamma$, and thus $\gamma = 0$. The result follows from this contradiction.

Lemma 4.2. *For each m, there is a neighborhood U of 0 in $\mathbb{R}$ such that the map π is injective on the support of $\tilde{c}_{t,m,f}((x,u),\cdot)$ for all $(x,u) \in \widetilde{M}$ if t is small enough and the support of f is contained in U.*

Proof. For any fixed $R > 0$, choose some neighborhood U of 0 in $\mathbb{R}$ satisfying the statement of Lemma 4.1. For any m, we have

$$\operatorname{supp}\left(F\left(x \mapsto \psi_m\left(tx^2\right)\right)\right) = \sqrt{t}\cdot\operatorname{supp}\left(F\left(x \mapsto \psi_m\left(x^2\right)\right)\right)\ ,$$

where F denotes the Fourier transform. So, since $\operatorname{supp}\left(F\left(x \mapsto \phi_m\left(x^2\right)\right)\right)$ is compact, it follows that

$$\operatorname{supp}\left(F\left(x \mapsto \psi_m\left(tx^2\right)\right)\right) \subset \left[-\frac{R}{2},\frac{R}{2}\right]$$

if t is small enough, for each m. Thus the leafwise smoothing kernel of $\psi_m\left(t\Delta_{\widetilde{\mathcal{F}}}\right)$ is supported in the $\frac{R}{2}$-neighborhood of the diagonal $\widetilde{\Delta}\subset\widetilde{\mathcal{G}}$, and the leafwise smoothing kernel of $\psi_m\left(t\Delta_{\widetilde{\mathcal{F}}}\right)^2$ is supported in the R-neighborhood of the diagonal $\widetilde{\Delta}\subset\widetilde{\mathcal{G}}$. ¿From Lemma 3.6, we get

$$\operatorname{supp}(\tilde{c}_{t,m,f}((x,u),\cdot))\subset\left\{(y,v)\mid(y,u)\in B_{\widetilde{\mathcal{F}}}((x,u),R)\ ,\ v-u\in U\right\}$$

for any $(x,u)\in\widetilde{M}$ if t is small enough, and the result follows by Lemma 4.1.

By noncommutative integration theory [8], the holonomy invariant transverse measure Λ defines a trace on the von Neumann algebra of $\mathcal{F}$, which can be shortly described as follows. The twisted convolution algebra

$$C_c^\infty\left(\mathcal{G},r^*\bigwedge T\mathcal{F}^*\otimes s^*\bigwedge T\mathcal{F}\right)$$

is contained in the (twisted) von Neumann algebra $W^*\left(\mathcal{F},\bigwedge T\mathcal{F}^*\right)$, and, for any

$$k\in C_c^\infty(\mathcal{G},r^*\bigwedge T\mathcal{F}^*\otimes s^*\bigwedge T\mathcal{F})\ ,$$

the trace $\operatorname{Tr}_\Lambda(k)$ is finite and given by the formula

$$\operatorname{Tr}_\Lambda(k)=\int_M\operatorname{tr}k([x,u],[x,u])\,\omega_M([x,u])\ . \tag{12}$$

Now fix U as in Lemma 4.2, and let $f\in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}f\subset U$.

Proposition 4.3. *For all $t>0$, we have*

$$\operatorname{Tr}^{\mathrm{s}}C_{t,m,f}=f(0)\cdot\operatorname{Tr}^{\mathrm{s}}_\Lambda\psi_m(t\Delta_{\mathcal{F}})^2\ . \tag{13}$$

Proof. Recall that $\operatorname{Tr}^{\mathrm{s}}C_{t,m,f}$ is independent of t by Lemma 3.3, and $\operatorname{Tr}^{\mathrm{s}}_\Lambda\psi_m(t\Delta_{\mathcal{F}})^2$ is also independent of t by [21]. So we need only prove this statement for a single t.

If t is small enough, we have

$$\begin{aligned}\operatorname{Tr}^{\mathrm{s}}C_{t,m,f}&=\int_M\operatorname{tr}^{\mathrm{s}}(c_{t,m,f}([x,u],[x,u])\,\omega_M([x,u])\\&=\int_{\widetilde{M}}\operatorname{tr}^{\mathrm{s}}(\tilde{c}_{t,m,f}((x,u),(x,u)))\,\omega_{\widetilde{M}}(x,u)\\&=f(0)\cdot\int_{\widetilde{M}}\operatorname{tr}^{\mathrm{s}}(\tilde{k}_{m,t}((x,u),(x,u)))\,\omega_{\widetilde{M}}(x,u)\\&=f(0)\cdot\int_M\operatorname{tr}^{\mathrm{s}}(k_{m,t}([x,u],[x,u]))\,\omega_M([x,u])\\&=f(0)\cdot\operatorname{Tr}^{\mathrm{s}}_\Lambda\psi_m\left(t\Delta_{\mathcal{F}}\right)^2\end{aligned}$$

by (11), Lemma 4.2, Lemma 3.6 and (12) since

$$k_{m,t}\in C_c^\infty\left(\mathcal{G},s^*\bigwedge T\mathcal{F}^*\otimes r^*\bigwedge T\mathcal{F}\right)\ .$$

Now we recall some facts on Connes' Betti numbers. The family

$$\{P_{i,L}\mid L\text{ is a leaf of }\mathcal{F}\}\ ,$$

where each $P_{i,L}$ is the orthogonal projection onto the space of square integrable harmonic i-forms on L, defines a projection P_i in the twisted foliation von Neumann algebra $W^*\left(\mathcal{F},\bigwedge T\mathcal{F}^*\right)$. As in [8], one can define Λ-Betti numbers $\beta^i_\Lambda(\mathcal{F})$ as

$$\beta^i_\Lambda(\mathcal{F})=\operatorname{Tr}_\Lambda P_i\ .$$

Then the Λ-Euler characteristic of $\mathcal{F}$ is

$$\chi_\Lambda(\mathcal{F}) = \sum_i (-1)^i \beta^i_\Lambda(\mathcal{F}) \ .$$

Using the corresponding supertrace notion, this formula can be rewritten as

$$\chi_\Lambda(\mathcal{F}) = \mathrm{Tr}^{\mathrm{s}}_\Lambda P \ ,$$

where $P = \sum_i P_i$. By [21], we have that $\mathrm{Tr}^{\mathrm{s}}_\Lambda \left(\psi_m \left(t\Delta_{\mathcal{F}}\right)^2\right)$ is independent of t, and

$$\mathrm{Tr}^{\mathrm{s}}_\Lambda \left(\psi_m \left(t\Delta_{\mathcal{F}}\right)^2\right) \to \chi_\Lambda(\mathcal{F}) \tag{14}$$

as $m \to \infty$ for all $t > 0$ (independently of m!).

Proof of Theorem 1.2. Fix a neighborhood U of 0 in $\mathbb{R}$ as in Lemma 4.2. Let $f \in C^\infty_c(\mathbb{R})$ with supp$f \in U$. Combining Proposition 4.3, Lemma 3.3 and (14), we have

$$\mathrm{Tr}^{\mathrm{s}} C_{t,m,f} = f(0) \cdot \mathrm{Tr}^{\mathrm{s}}_\Lambda \left(\psi_m \left(t\Delta_{\mathcal{F}}\right)^2\right) \to f(0) \cdot \chi_\Lambda(\mathcal{F}) \tag{15}$$

as $m \to \infty$ for any $t > 0$.

Fix any $\varepsilon > 0$. From Lemmas 3.5 and 3.3, it follows that

$$|\mathrm{Tr}^{\mathrm{s}} C_{t,m,f} - \mathrm{Tr}^{\mathrm{s}} B_{t,f}| < \varepsilon$$

for any $t > 0$ if m is large enough. But

$$\mathrm{Tr}^{\mathrm{s}} B_{t,f} = \langle \chi_{\mathrm{dis}}(\mathcal{F}), f\rangle$$

for any $t > 0$ by Corollary 3.4. So

$$|\mathrm{Tr}^{\mathrm{s}} C_{t,m,f} - \langle \chi_{\mathrm{dis}}(\mathcal{F}), f\rangle| < \varepsilon \tag{16}$$

for any $t > 0$ if m is large enough.

¿From (15) and (16), it follows that

$$|\langle \chi_{\mathrm{dis}}(\mathcal{F}), f\rangle - f(0) \cdot \chi_\Lambda(\mathcal{F})| < \varepsilon$$

for any $\varepsilon > 0$, yielding $\chi_{\mathrm{dis}}(\mathcal{F}) = \chi_\Lambda(\mathcal{F}) \cdot \delta_0$ on U since $\varepsilon > 0$ and $f \in C^\infty_c(U)$ are arbitrary.

5. Localization theorem

Theorem 5.1. *The distribution $\chi_{\mathrm{dis}}(\mathcal{F})$ is supported in the set of all $s \in \mathbb{R}$ such that X_s has a fixed point in M.*

Proof. Let V be an open subset in $\mathbb{R}$ such that X_s has no fixed points for all $s \in V$. We have to prove that $\chi_{\mathrm{dis}}(\mathcal{F}) = 0$ on V. Note that the fact that X_s has no fixed points for all $s \in V$ is equivalent to

$$\widetilde{X}_s(x, u) \neq (x \cdot \gamma, u + \gamma)$$

for any $s \in V$, $\gamma \in \Gamma$ and $(x, u) \in \widetilde{M}$.

One can prove an analogue of Lemma 4.2, asserting that there exists a neighborhood U of 0 in $\mathbb{R}$ such that, for any $s_0 \in \mathbb{R}$ and for each m, if $f \in C^\infty_c(U + s_0)$, then π is injective on the support of $\tilde{c}_{t,m,f}((x, u), \cdot)$ for $(x, u) \in \widetilde{M}$ if t is small enough. ¿From this, it follows that, for each m, if t is small enough, then, for any $[x, u] \in M$, either

$$c_{t,m,f}([x, u], [x, u]) = 0$$

or

$$c_{t,m,f}([x,u],[x,u]) = \tilde{c}_{t,m,f}((x,u),(x\cdot\gamma,u+\gamma))$$

for some $\gamma \in \Gamma \cap (U + s_0)$. In the latter case, such a γ is uniquely determined by (x,u).

Take $s_0 \in V$ and some neighborhood U of 0 as above, satisfying also $U + s_0 \subset V$. Since s_0 is an arbitrary point of V, it is enough to show that $\langle \chi_{\rm dis}, f\rangle = 0$ for any $f \in C^\infty(U+s_0)$. Then, by Lemma 3.5 and Corollary 3.4, it is enough to show that $\mathrm{Tr}^{\mathrm s} C_{t,m,f} = 0$ for each m and t small enough (depending on m).

We have

$$\begin{aligned}
\mathrm{Tr}^{\mathrm s} C_{t,m,f} &= \int_M \mathrm{tr}^{\mathrm s}(c_{t,m,f}([x,u],[x,u]))\,\omega_M([x,u]) \\
&= \int_{\widetilde M} \mathrm{tr}^{\mathrm s}\,(\tilde c_{t,m,f}((x,u),(x\cdot\gamma,u+\gamma)))\ \omega_{\widetilde M}(x,u) \\
&= \int_{\widetilde M} \mathrm{tr}^{\mathrm s}\left(\widetilde X^*_\gamma \circ \tilde k_{m,t}((x,u+\gamma),(x\cdot\gamma,u+\gamma))\right)\cdot f(\gamma)\,\omega_{\widetilde M}(x,u)\ ,
\end{aligned}$$

for the appropriate choice of $\gamma \in \Gamma$, where we use the identity

$$\bigwedge T_{(y,v)}\widetilde{\mathcal F}^* \equiv \bigwedge T_{(y\cdot\gamma,v+\gamma)}\widetilde{\mathcal F}^*$$

given by the diagonal action of γ on $\widetilde M$.

Since $\mathrm{supp} f \subset U + s_0$, we can consider only those $[x,u] \in M$ with $\gamma \in U + s_0$ for some choice of $\gamma \in \Gamma$ with $(x\cdot\gamma, u+\gamma)$ in the support of $\tilde c_{t,m,f}$. Thus $\gamma \in V$, yielding

$$(x\cdot\gamma,u+\gamma) \neq (x,u+\gamma)$$

by assumption. It follows that

$$\tilde k_{m,t}((x,u+\gamma),(x\cdot\gamma,u+\gamma)) \to 0$$

as $t \to 0$ uniformly on $(x,u) \in M$. Since $\mathrm{Tr}^{\mathrm s} C_{t,m,f}$ is independent of t, we get $\mathrm{Tr}^{\mathrm s} C_{t,m,f} = 0$ for each m, as desired.

6. The Lefschetz trace formula

The goal of this section is to prove Theorem 1.3.

By Theorem 5.1, in order to evaluate

$$\mathrm{Tr}^{\mathrm s} C_{t,m,f} = \int_M \mathrm{tr}^{\mathrm s} c_{t,m,f}([x,u],[x,u])\,\omega_M([x,u])$$

asymptotically as $t \to 0$, it is enough to integrate over small neighborhoods of closed orbits.

As in the proof of Theorem 5.1, take a neighborhood U of 0 in $\mathbb R$ such that, for any $s_0 \in \mathbb R$ and for each m, if $f \in C^\infty_c(U+s_0)$, then π is injective in the support of $\tilde c_{t,m,f}((x,u),\cdot)$ for all $(x,u) \in \widetilde M$ if t is small enough.

Let s_0 be the period of some closed orbit of X. There exist finitely many closed orbits with the period in $U + s_0$. Hence, the neighborhood U can be chosen so that s_0 is the only period that belongs to $s_0 + U$, and thus only this period may be in $\mathrm{supp} f$. Take a closed orbit of period s_0, and let c be the corresponding primitive closed orbit with length $l = l(c)$; thus $s_0 = kl$ for some integer $k > 0$. We also get that $l \in \Gamma$, and

$$(x,u+l) = (x\cdot l, u+l)\ , \quad (x,u+kl) = (x\cdot kl, u+kl)$$

if $[x, u]$ is in c. So x is a fixed point of the action of l on L, and there are no other fixed points of elements of $\Gamma \cap (s_0 + U)$ in some open neighborhood W of x in L because all X-orbits are simple. Note also that

$$\pi(\{x\} \times [0, l]) = c$$

and

$$\pi : \{x\} \times (0, l) \to c$$

is a C^∞ embedding. Moreover $\pi(W \times [0, l])$ is an open neighborhood of c where there are no other orbits of period in $s_0 + U$, and

$$\pi : W \times (0, l) \to M$$

is a C^∞ embedding.

Denote by

$$\tilde{k}_t \in C^\infty \left(\widetilde{\mathcal{G}}, r^* \bigwedge T\widetilde{\mathcal{F}}^* \otimes s^* \bigwedge T\widetilde{\mathcal{F}}\right)$$

the leafwise smoothing kernel of the leafwise heat operator $e^{-t\Delta_{\widetilde{\mathcal{F}}}}$. Then, since $\tilde{k}_{m,t}(x, \cdot)$ converges to $\tilde{k}_t(x, \cdot)$ as $m \to \infty$, by Lemma 3.6, it is enough to compute the asymptotics as $t \downarrow 0$ of

$$\begin{aligned} &\int_{W \times [0,l]} \operatorname{tr}^{\mathrm{s}} \left(\widetilde{X}_\gamma^* \circ \tilde{k}_t\left((y, v + kl), (y \cdot kl, v + kl)\right)\right) \cdot f(kl)\, \omega_{\widetilde{M}}(y, v) \\ &\quad = \quad f(kl) \cdot \int_0^l \int_W \operatorname{tr}^{\mathrm{s}} \left(\widetilde{X}_{kl}^* \circ \tilde{k}_t\left((y, v + kl), (y \cdot kl, v + kl)\right)\right) \omega_{W,v}(y)\, dv\ , \end{aligned}$$

where $\omega_{W,v}$ is the restriction of $\omega_{\widetilde{\mathcal{F}}}$ to $W \times \{v\} \equiv W$, and we use the identity

$$\bigwedge T_{(y,v)} \widetilde{\mathcal{F}}^* \equiv \bigwedge T_{(y \cdot kl, v + kl)} \widetilde{\mathcal{F}}^*$$

given by the diagonal action of kl on $\widetilde{M}$. But, by [7, 13], the integral

$$\int_W \operatorname{tr}^{\mathrm{s}} \left(\widetilde{X}_{kl}^* \circ \tilde{k}_t\left((y, v + kl), (y \cdot kl, v + kl)\right)\right) \omega_{W,v}(y)$$

converges as $t \downarrow 0$ to

$$\operatorname{sign} \det \left(\mathrm{id} - X_{kl*} : T_{(x,v)}(W \times \{v\}) \to T_{(x,v)}(W \times \{v\})\right)\ ,$$

which is independent of v, and the proof is finished.

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