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Adiabatic limits and spectral geometry of foliations

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1. Introduction

Let $(M,\mathcal{F})$ be a closed foliated manifold, $\dim M = n$, $\dim\mathcal{F} = p$, $p+q = n$, equipped with a Riemannian metric g_M. We assume that the foliation $\mathcal{F}$ is Riemannian, and the metric g_M is bundle-like. Let $F = T\mathcal{F}$ be the integrable distribution of tangent p-planes to $\mathcal{F}$, and $H = F^{\perp}$ be the orthogonal complement to F. The decomposition of TM into a direct sum, $TM = F \bigoplus H$, induces a decomposition of the metric g_M: $g_M = g_F + g_H$. For any $h > 0$, let $\Delta_h, h > 0$, be the Laplace operator on differential forms defined by a metric g_h on M, given by the formula $g_h = g_F + h^{-2} g_H$. The operator Δ_h is an elliptic differential operator with the positive definite, scalar principal symbol, which is self-adjoint and has discrete spectrum in the Hilbert space $L^2(M, \Lambda T^*M, g_h)$. The main result of the paper is an asymptotical formula for the eigenvalue distribution function $N_h(\lambda)$ of the operator Δ_h:

$$N_h(\lambda) = \sharp\{\lambda_i(h) \in \text{spec}\ \Delta_h : \lambda_i(h) \le \lambda\}.$$

Theorem 1. *Let $(M,\mathcal{F})$ be a compact Riemannian foliated manifold, equipped with a bundle-like Riemannian metric g_M. Then the asymptotical formula for $N_h(\lambda)$ has the following form:*

$$N_h(\lambda) = h^{-q} \frac{(4\pi)^{-q/2}}{\Gamma((q/2)+1)} \int_{-\infty}^{\lambda} (\lambda-\tau)^{q/2} d_\tau N_{\mathcal{F}}(\tau) + o(h^{-q}), h \to 0, \quad (1)$$

where $N_{\mathcal{F}}(\lambda)$ is the spectrum distribution function of the tangential Laplace operator Δ_F (see (7) and (21) for the definitions).

We refer the reader to Sect. 5 for a detailed formulation of this Theorem. We state also the asymptotical formula for the trace of the operator $f(\Delta_h)$ for any function $f \in C_c(\mathbb{R})$ (see Theorems 2 and 3 below).

The study of the asymptotical behaviour of geometric objects (like as harmonic forms, eta-invariants etc.) associated with a family of Riemannian metrics on fibrations as the metrics become singular was stimulated by Witten's work on adiabatic limits [28]. For further developments see, for instance, [23, 9, 11, 12] and references there.

In the spectral theory of differential operators, problems in question are related with the Born-Oppenheimer approximation which consists in that the Schrödinger operator for a polyatomic molecule is considered in the semiclassical limit where the mass ratio of electronic to nuclear mass tends to zero (see, for instance, [16] and references there). In particular, the result on semiclassical asymptotics for the spectrum distribution function in the fibration case is, essentially, due to [3].

The investigation of semiclassical spectral asymptotics for foliations was started by the author in [17, 18, 20]. There we considered the problem in the operator setting, that is, we studied spectral asymptotics for a self-adjoint hypoelliptic operator A_h of the form $A_h = A + h^m B$, where A is a tangentially elliptic operator of order $\mu > 0$ with the positive tangential principal symbol, and B is a differential operator of order m on M with the positive, holonomy invariant transversal principal symbol, and obtained an asymptotical formula for the spectrum distribution function of this operator when h tends to zero.

The goal of this work is to adapt our results on semiclassical spectral asymptotics to the geometric setting of adiabatic limits on foliations.

An interesting observation related with the asymptotical formula (1) is that its right-hand side depends only on leafwise spectral data of the tangential Laplace operator Δ_F. So, in the case when the leafwise spectrum of Δ_F doesn't coincide with its spectrum in L^2 space on the ambient manifold M (it might be a case if the foliation $\mathcal{F}$ is nonamenable, [19]), there is a $\lambda > 0$ such that $N_h(\lambda) \neq 0$ but

$$\lim_{h\to 0} h^q N_h(\lambda) = 0. \tag{2}$$

The asymptotic behaviour of $N_h(\lambda)$ allows us to introduce spectral characteristics $r_k(\lambda)$ of the leafwise Laplacian related with adiabatic limits. We might expect that some invariants of the function $r_k(\lambda)$ near $\lambda = 0$ are independent of the choice of metric on M, and, moreover, be topological or homotopic invariants of foliated manifolds (just as in the case of Novikov-Shubin invariants [13]). We discuss these questions and their relationships with the spectral theory of leafwise Laplacian in Sect. 7 and with noncommutative spectral geometry of foliations in [21].

The organization of the paper is as follows.

In Sect. 2, we recall necessary facts, concerning to differential operators on foliated manifolds.

In the Sects. 3 and 4, we formulate and prove the asymptotical formula for tr $f(\Delta_h)$ when h tends to zero for any function $f \in C_0(\mathbb{R})$.

In Sect. 5, we rewrite the asymptotical formula of Sect. 3 in terms of spectral characteristics of the operator Δ_F. In particular, this enables us to complete a proof of Theorem 1 on the asymptotic behaviour of the eigenvalue distribution function.

Finally, in Sect. 6 we discuss some results and examples related with the asymptotical behaviour of individual eigenvalues of the operator Δ_h when h tends to zero.

2. Differential operators on foliated manifolds

Let $(M, \mathcal{F})$ be a compact foliated manifold, F be the tangential distribution to $\mathcal{F}$. The embedding $F \subset TM$ induces an embedding of algebras of differential operators $D^\mu(\mathcal{F}) \subset D^\mu(M)$, and differential operators on M obtained in such a way are called tangential differential operators.

More generally, let E be an Hermitian vector bundle on M. We say that a linear differential operator A of order μ acting on $C^\infty(M, E)$ is a tangential operator, if, in any foliated chart $\kappa\ :\ I^p \times I^q \to M$ ($I = (0,1)$ is the open interval) and any trivialization of the bundle E over it, A is of the form

$$A = \sum_{|\alpha|\leq\mu} a_\alpha(x,y) D_x^\alpha, (x,y) \in I^p \times I^q,$$

with a_α, being matrix valued functions on $I^p \times I^q$. Let $D^\mu(\mathcal{F}, E)$ denote the set of all tangential differential operators of order μ acting in $C^\infty(M, E)$.

We say that $A \in D^{m,\mu}(M, \mathcal{F}, E)$ if A is of the form $A = \sum_\alpha B_\alpha C_\alpha$, where $B_\alpha \in D^m(M, E)$, $C_\alpha \in D^\mu(\mathcal{F}, E)$. From symbolic calculus, it can be easily seen that:

(1) if $A_1 \in D^{m_1,\mu_1}(M, \mathcal{F}, E)$, $A_2 \in D^{m_2,\mu_2}(M, \mathcal{F}, E)$, then $A_1 \circ A_2 \in D^{m_1+m_2,\mu_1+\mu_2}(M, \mathcal{F}, E)$;

(2) if $A \in D^{m,\mu}(M, \mathcal{F}, E)$, then the adjoint $A^* \in D^{m,\mu}(M, \mathcal{F}, E)$.

Remark 1. It should be noted that classes $D^{m,\mu}(M, \mathcal{F}, E)$ can be extended to classes of pseudodifferential operators $\Psi^{m,\mu}(M, \mathcal{F}, E)$, which contain, for instance, parametrices for elliptic operators from the classes $D^{m,\mu}(M, \mathcal{F}, E)$. We don't use them here and refer the interested reader to [19] (see also [20]) for details.

Next, recall the definition of the scale of Sobolev type spaces $H^{s,k}(M,\mathcal{F},E)$, $s\in\mathbb{R}, k\in\mathbb{R}$, corresponding to classes $D^{m,\mu}(M,\mathcal{F},E)$ [19,20].

The space $H^{s,k}(\mathbb{R}^n,\mathbb{R}^p,\mathbb{C}^r)$ consists of all $\mathbb{C}^r$-valued tempered distributions $u\in S'(\mathbb{R}^n,\mathbb{C}^r)$ such that $\tilde{u}\in L^2_{\text{loc}}(\mathbb{R}^n,\mathbb{C}^r)$ ($\tilde{u}$ the Fourier transform) and

$$\|u\|^2_{s,k}=\int\int|\tilde{u}(\xi,\eta)|^2(1+|\xi|^2+|\eta|^2)^s(1+|\xi|^2)^k d\xi d\eta<\infty. \quad (3)$$

The identity (3) serves as a definition of a norm $\| \ \|_{s,k}$ in $H^{s,k}(\mathbb{R}^n,\mathbb{R}^p,\mathbb{C}^r)$.

The space $H^{s,k}(M,\mathcal{F},E)$ consists of all $u\in\mathcal{D}'(M,E)$ such that, for any foliated coordinate chart $\kappa: I^p\times I^q\to U=\kappa(I^p\times I^q)\subset M$, any trivialization of the bundle E over it, and for any $\phi\in C^\infty_c(U)$, the function $\kappa^*(\phi u)$ belongs to the space $H^{s,k}(\mathbb{R}^n,\mathbb{R}^p,\mathbb{C}^r)$ ($r=\operatorname{rank} E$). Fix some finite covering $\{U_i : i=1,\dots,d\}$ of M by foliated coordinate patches with foliated coordinate charts $\kappa_i : I^p\times I^q\to U_i=\kappa_i(I^p\times I^q)$ and trivializations of the bundle E over them, and a partition of unity $\{\phi_i\in C^\infty(M) : i=1,\dots,d\}$ subordinate to this covering. A scalar product in $H^{s,k}(M,\mathcal{F},E)$ is defined by the formula

$$(u,v)_{s,k}=\sum_{i=1}^d(\kappa^*(\phi_i u),\kappa^*(\phi_i v))_{s,k}, u,v\in H^{s,k}(M,\mathcal{F},E).$$

Operators of classes $D^{m,\mu}(M,\mathcal{F},E)$ acts in the spaces $H^{s,k}(M,\mathcal{F},E)$ in the following way (see [19,20] for a proof in the scalar case):

Proposition 1. *An operator $A\in D^{m,\mu}(M,\mathcal{F},E)$ defines a linear bounded operator from $H^{s,k}(M,\mathcal{F},E)$ to $H^{s-m,k-\mu}(M,\mathcal{F},E)$ for any $s\in\mathbb{R}$, $k\in\mathbb{R}$.*

Now let us turn to properties of geometric operators on the foliated manifold $(M,\mathcal{F})$. Let H be the orthogonal complement to F, so

$$F\bigoplus H=TM. \quad (4)$$

The decomposition (4) induces a bigrading on ΛT^*M by the formula

$$\Lambda^kT^*M=\bigoplus_{i=0}^k\Lambda^{i,k-i}T^*M, \quad (5)$$

where $\Lambda^{i,j}T^*M=\Lambda^iF^*\bigotimes\Lambda^jH^*, i=1,\dots,p, j=1,\dots,q.$

The de Rham differential d inherits the bigrading (5) in the form

$$d=d_F+d_H+\theta.$$

Here the tangential de Rham differential d_F and the transversal de Rham differential d_H are first order differential operators, and θ is zeroth order. Moreover, the operator d_F doesn't depend on a choice of the bundle-like metric g_M (see, for instance, [26]).

Recall that $\Delta_h, h > 0$, is the Laplace operator on differential forms defined by the metric $g_h = g_F + h^{-2}g_H$. We transfer the family Δ_h to the fixed Hilbert space $L^2(M, \Lambda T^*M, g)$. For this goal, we introduce an isometry $\Theta_h : L^2(M, \Lambda T^*M, g_h) \to L^2(M, \Lambda T^*M, g)$, defined, for $u \in L^2(M, \Lambda^{i,j}T^*M, g_h)$, as $\Theta_h u = h^j u$. The operator Δ_h in the Hilbert space $L^2(M, \Lambda T^*M, g_h)$ corresponds under the isometry Θ_h to the operator $L_h = \Theta_h \Delta_h \Theta_h^{-1}$ in the Hilbert space $L^2(M, \Lambda T^*M) = L^2(M, \Lambda T^*M, g)$.

Lemma 1 ([11]). *We have*

$$L_h = d_h\delta_h + \delta_h d_h,$$

where $d_h = d_F + hd_H + h^2\theta$, *and* $\delta_h = \delta_F + h\delta_H + h^2\theta^*$ *is the adjoint to* d_h *with* δ_F, δ_H *and* θ^*, *being the adjoints to* d_F, d_H *and* θ *respectively. Here we consider the adjoints taken in the Hilbert space* $L^2(M, \Lambda T^*M)$.

By Lemma 1, the operator L_h is of the following form:

$$L_h = \Delta_F + h^2\Delta_H + h^4\Delta_{-1,2} + hK_1 + h^2K_2 + h^3K_3, \tag{6}$$

where

– The operator

$$\Delta_F = d_F\delta_F + \delta_F d_F \in D^{0,2}(M, \mathcal{F}, \Lambda T^*M) \tag{7}$$

is the tangential Laplacian in the space $C^\infty(M, \Lambda T^*M)$.

– The operator

$$\Delta_H = d_H\delta_H + \delta_H d_H \in D^{2,0}(M, \mathcal{F}, \Lambda T^*M)$$

is the transversal Laplacian in the space $C^\infty(M, \Lambda T^*M)$.

– $\Delta_{-1,2} = \theta\theta^* + \theta^*\theta \in D^{0,0}(M, \mathcal{F}, \Lambda T^*M))$.
– $K_1 = d_F\delta_H + \delta_H d_F + \delta_F d_H + d_H\delta_F \in D^{1,0}(M, \mathcal{F}, \Lambda T^*M)$.
– $K_2 = d_F\theta^* + \theta^* d_F + \delta_F\theta + \theta\delta_F \in D^{0,0}(M, \mathcal{F}, \Lambda T^*M)$.
– $K_3 = d_H\theta^* + \theta^* d_H + \delta_H\theta + \theta\delta_H \in D^{1,0}(M, \mathcal{F}, \Lambda T^*M)$.

From now on, we will assume that $(M, \mathcal{F})$ is a Riemannian foliation with the metric g_M, being bundle-like (see, for instance, [26]). The crucial point, concerning to geometrical operators on a Riemannian foliated manifold, is that, in this case, the operators $d_F\delta_H + \delta_H d_F$ and $\delta_F d_H + d_H\delta_F$ belong to $D^{0,1}(M, \mathcal{F}, \Lambda T^*M)$. In particular, this implies

$$K_1 \in D^{0,1}(M, \mathcal{F}, \Lambda T^*M).$$

For any $h > 0$, the operator L_h is a formally self-adjoint, elliptic operator in $L^2(M, \Lambda T^*M)$ with the positive principal symbol. The following Proposition is a refinement of the classical Gårding inequality for the operator L_h.

Proposition 2. *Under current hypotheses, there exist constants $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that for any $h > 0$ small enough and for any $u \in C^\infty(M, \Lambda T^*M)$ we have*

$$(L_h u, u) \geq (1 - C_1 h^2)(\Delta_F u, u) + C_2 h^2 \|u\|^2_{1,0} - C_3 \|u\|^2. \tag{8}$$

Proof. Let $u \in C^\infty(M, \Lambda T^*M)$. By (6), we have

$$\begin{aligned}(L_h u.u) = (\Delta_F u, u) + h^2(\Delta_H u, u) + h^4(\Delta_{-1,2} u, u)\\ +h(K_1 u, u) + h^2(K_2 u, u) + h^3(K_3 u, u).\end{aligned}$$

It clear that $(\Delta_{-1,2} u, u) \geq 0$. By Proposition 1, we have

$$(K_2 u, u) \geq -C_4 \|u\|^2, (K_3 u, u) \geq -C_5 \|u\|^2_{1,0}. \tag{9}$$

So we obtain

$$\begin{aligned}(L_h u.u) \geq (\Delta_F u, u) + h^2(\Delta_H u, u) + h(K_1 u, u) - C_4 h^2 \|u\|^2\\ -C_5 h^3 \|u\|^2_{1,0}.\end{aligned} \tag{10}$$

The operator $\Delta_F + \Delta_H$ is a second order elliptic operator with the positive principal symbol, so, by the standard Gårding inequality, we have

$$((\Delta_F + \Delta_H) u, u) \geq C_6 \|u\|^2_{1,0} - C_7 \|u\|^2, \tag{11}$$

that implies the estimate

$$(L_h u.u) \geq (1 - h^2)(\Delta_F u, u) + C_7 h^2 \|u\|^2_{1,0} + h(K_1 u, u) - C_8 \|u\|^2. \tag{12}$$

Finally, we make use of the inequality

$$|(K_1 u, u)| \leq C_7 \|u\|_{0,1} \|u\| \leq C_8 (h \|u\|^2_{0,1} + h^{-1} \|u\|^2) \tag{13}$$

and the tangential Gårding inequality (see [20])

$$\|u\|^2_{0,1} \leq C_9((\Delta_F u, u) + \|u\|^2),$$

that completes immediately the proof.

Remark 2. In some cases, it is sufficient to use more rough estimate

$$(L_h u, u) \geq C_1 \|u\|^2_{0,1} + C_2 h^2 \|u\|^2_{1,0} - C_3 \|u\|^2, u \in C^\infty(M, \Lambda T^*M), \tag{14}$$

which follows from (8), if we apply the standard Sobolev norm estimate to the term $(\Delta_F u, u)$.

Let $H_h(t) = \exp(-tL_h), t \geq 0$, be the parabolic semigroup of bounded operators in $L^2(M, \Lambda T^*M)$, generated by L_h. For any $t > 0$, the operator $H_h(t)$ is an operator with a smooth kernel. Proposition 2 implies the following norm estimates for operators of this semigroup in $H^{s,k}(M, \mathcal{F}, \Lambda T^*M)$ (see also [20]).

Proposition 3. *We have the following estimates:*

$$\|H_h(t)u\|_{r,k} \leq C_{rsk} t^{(s-k-r)/2} h^{s-r} \|u\|_s, u \in C^\infty(M, \Lambda T^*M),$$

if $r > s, h \in (0,1], 0 < t \leq 1$*, and*

$$\|H_h(t)u\|_{s,k} \leq C_{sk} t^{-k/2} \|u\|_s, u \in C^\infty(M, \Lambda T^*M).$$

if $r = s, h \in [0,1], 0 < t \leq 1$*, where the constants don't depend on* t *and* h*.*

3. Asymptotical formula for functions of the Laplace operator

From now on, we will assume that $(M, \mathcal{F})$ is a Riemannian foliation, equipped with a bundle-like Riemannian metric g_M. In this Section, we state the asymptotical formula for $\mathrm{tr}\, f(\Delta_h)$ when h tends to zero for any function $f \in C_0(\mathbb{R})$.

We will denote by $G_\mathcal{F}$ the holonomy groupoid of $(M, \mathcal{F})$. Let us briefly recall its definition. Let $\sim_h$ be an equivalence relation on the set of leafwise paths $\gamma : [0,1] \to M$, setting $\gamma_1 \sim_h \gamma_2$ if γ_1 and γ_2 have the same initial and final points and the same holonomy maps. The holonomy groupoid $G_\mathcal{F}$ is the set of $\sim_h$ equivalence classes of leafwise paths. $G_\mathcal{F}$ is equipped with the source and the range maps $s, r : G_\mathcal{F} \to M$ defined by $s(\gamma) = \gamma(0)$ and $r(\gamma) = \gamma(1)$. We will identify a point $x \in M$ with an element of $G_\mathcal{F}$ given by the corresponding constant path: $\gamma(t) = x, t \in [0,1]$. Recall also that, for any $x \in M$, the set $G^x_\mathcal{F} = \{\gamma \in G_\mathcal{F} : r(\gamma) = x\}$ is the covering of the leaf through the point x, associated with the holonomy group of the leaf. We will denote by λ_L the Riemannian volume form on each leaf L of $\mathcal{F}$ and by λ^x its lift to a measure on the holonomy covering $G^x_\mathcal{F}, x \in M$.

For any vector bundle E on M, we denote by $C^\infty_c(G_\mathcal{F}, E)$ the space of all smooth, compactly supported Sects. of the vector bundle $r^*E \bigotimes s^*E^*$ over $G_\mathcal{F}$. In other words, for any $k \in C^\infty_c(G_\mathcal{F}, E)$, its value at a point $\gamma \in G_\mathcal{F}$ is a linear map $k(\gamma) : E_{s(\gamma)} \to E_{r(\gamma)}$. We will use a correspondence between tangential kernels $k \in C^\infty_c(G_\mathcal{F}, E)$ and tangential operators $K : C^\infty(M, E) \to C^\infty(M, E)$ via the formula

$$Ku(x) = \int_{G^x_\mathcal{F}} k(\gamma) u(s(\gamma)) d\lambda^x(\gamma), u \in C^\infty(M, E).$$

Now we introduce a notion of the principal h-symbol of the operator Δ_h. It is well-known (see, for instance, [24,26]) that the conormal bundle H^* to the foliation $\mathcal{F}$ has a partial (Bott) connection, which is flat along leaves of $\mathcal{F}$. So we can lift the foliation $\mathcal{F}$ to a foliation $\mathcal{F}_H$ in the conormal bundle H^*. The leaf $\tilde{L}_\nu$ of the foliation $\mathcal{F}_H$ through a point $\nu \in H^*$ is diffeomorphic to the holonomy covering $G^x_{\mathcal{F}}$ of the leaf $L_x, x = \pi(\nu)$, of the foliation $\mathcal{F}$ through the point x (here $\pi : H^* \to M$ is the bundle map) and has trivial holonomy.

Denote by

$$\Delta_{\mathcal{F}_H} : C^\infty(H^*, \pi^* \Lambda T^* M) \to C^\infty(H^*, \pi^* \Lambda T^* M)$$

the lift of the leafwise Laplacian Δ_F to a tangentially elliptic operator on H^* with respect to $\mathcal{F}_H$.

Definition 1. *The principal h-symbol of the operator Δ_h is a tangentially elliptic operator on H^* with respect to the foliation $\mathcal{F}_H$, given by the formula*

$$\sigma_h(\Delta_h) = \Delta_{\mathcal{F}_H} + g_H,$$

where g_H is the multiplication operator by the function $g_H(\nu), \nu \in H^*$.

The holonomy groupoid $G_{\mathcal{F}_H}$ of the lifted foliation $\mathcal{F}_H$ consists of all triples $(\gamma, \nu) \in G_{\mathcal{F}} \times H^*$ such that $r(\gamma) = \pi(\nu)$, and $s(\gamma) = dh^*_\gamma(\nu)$, where dh^*_γ is the codifferential of the holonomy map with the source map $s : G_{\mathcal{F}_H} \to H^*, s(\gamma, \nu) = dh^*_\gamma(\nu)$, and the range map $r : G_{\mathcal{F}_H} \to H^*, r(\gamma, \nu) = \nu$. We have the map $\pi_G : G_{\mathcal{F}_H} \to G_{\mathcal{F}}$, given by $\pi_G(\gamma, \nu) = \gamma$. Denote by $\mathrm{tr}_{\mathcal{F}_H}$ the trace on the von Neumann algebra $W^*(G_{\mathcal{F}_H}, \pi^* \Lambda T^* M)$ of all tangential operators on H^* with respect to the foliation $\mathcal{F}_H$, given by the holonomy invariant measure $dx\, d\nu$ on H^* [6]. For any tangential operator K on $(H^*, \mathcal{F}_H)$, given by a tangential kernel $k \in C^\infty_c(G_{\mathcal{F}_H}, \pi^* \Lambda T^* M)$, $k = k(\gamma, \nu)$, we have

$$\mathrm{tr}_{\mathcal{F}_H}(K) = \int_{H^*} \mathrm{Tr}_{\pi^* \Lambda T^* M}\, k(x, \nu) dx d\nu.$$

Theorem 2. *For any function $f \in C_0(\mathbb{R})$, we have the asymptotical formula*

$$\mathrm{tr}\, f(\Delta_h) = (2\pi)^{-q} h^{-q} \mathrm{tr}_{\mathcal{F}_H}\, f(\sigma_h(\Delta_h)) + o(h^{-q}), h \to 0. \qquad (15)$$

We will prove this theorem in the next section, and now we conclude this section by a remark.

Remark 3. Let us compare the formula (15) with what we have in the case of a Schrödinger operator. Let $H_h = -h^2 \Delta + V(x), x \in M$, be a Schrödinger operator on a compact manifold M with an operator-valued potential $V \in$

$\mathcal{L}(H)$ (H is a Hilbert space) such that $V(x)^* = V(x)$ (the fibration case). Then the corresponding asymptotical formula has the following form:

$$\operatorname{tr} f(\Delta_h) = (2\pi)^{-n} h^{-n} \int_{T^*M} \operatorname{Tr} f(\sigma_h(H_h)(x,\xi)) dx d\xi + o(h^{-n}),\\ h \to 0+, \quad (16)$$

where $\sigma_h(H_h)(x,\xi)$ is the operator-valued principal h-symbol, $\sigma_h(H_h)(x,\xi) = |\xi|^2 + V(x), (x,\xi) \in T^*M$. So we see that the formula (15) has the same form as (16) with the difference that here $f(\sigma_h(H_h))$ is an element of the foliation C^*-algebra $C^*(G_{\mathcal{F}_H}, \pi^* \Lambda T^*M)$, which is a noncommutative analoque of the algebra of continuous functions on $H^*/\mathcal{F}_H$, and the integration over T^*M and the fibrewise trace in (16) are replaced by the integration in a sense of the noncommutative integration theory [6].

4. Proof of Theorem 2

This section is devoted to a proof of Theorem 2. Without loss of generality, we will consider the asymptotical behaviour of $\operatorname{tr} f(L_h)$. The proof of Theorem 2 relies on the comparison of the operator L_h with some operator $\bar{L}_h$ of the almost product structure as in [20] with a subsequent use of results of [20] (see also [17,18]) on semiclassical spectral asymptotics for elliptic operators on foliated manifolds.

So let an operator $\bar{L}_h \in D^{2,0}(M,\mathcal{F},\Lambda T^*M)$ be given by the formula

$$\bar{L}_h = \Delta_F + h^2 \Delta_H.$$

The operator $\bar{L}_h$ satisfies the conditions of [17,18,20], that is, it is of the form $\bar{L}_h = A + h^2 B$, where $A = \Delta_F$ is a second order tangentially elliptic operator with the scalar, positive tangential principal symbol, and $B = \Delta_H$ be a second order differential operator on M with the scalar, positive, holonomy invariant transversal principal symbol. Indeed, it is easy to see that the transversal principal symbol of operator Δ_H, which is the restriction of its principal symbol from T^*M to the conormal bundle H^*, is given by the formula $\sigma(\nu) = g_{H^*}(\nu) I, \nu \in H^*$, and its holonomy invariance is equivalent to the assumption on the metric g_M to be bundle-like.

Remark 4. The only fact which we need from the holonomy invariance conditition is that the commutator $[A,B]$, which, by general symbolic calculus, belongs to the class $D^{2,1}(M,\mathcal{F},\Lambda T^*M)$, is an operator of the class $D^{1,2}(M,\mathcal{F},\Lambda T^*M)$, that can be checked by a straightforward calculation.

The operator $\bar{L}_h$ generates the parabolic semigroup $\bar{H}_h(t) = e^{-t\bar{L}_h}, t \geq 0$, in the space $L^2(M,\Lambda T^*M)$. It is clear that these operators are smoothing operators when $t > 0$. By [20], the operators of the parabolic semigroup $\bar{H}_h(t)$ satisfy the same estimates as in Proposition 3:

$$\|\bar{H}_h(t)u\|_{r,k} \leq C_{r,s,k} t^{(s-k-r)/2} h^{s-r} \|u\|_s, u \in C^\infty(M, \Lambda T^*M), \quad (17)$$

if $r > s, h \in (0,1], 0 < t \leq 1$, and

$$\|\bar{H}_h(t)u\|_{s,k} \leq C_{sk} t^{-k/2} \|u\|_s, u \in C^\infty(M, \Lambda T^*M). \quad (18)$$

if $r = s, h \in [0,1], 0 < t \leq 1$, where the constants don't depend on t and h.

First, we state the norm estimates for the difference $H_h(t) - \bar{H}_h(t)$.

Proposition 4. *We have the estimate*

$$\|(H_h(t) - \bar{H}_h(t))u\|_{r,k} \leq C_{r,s,k} t^{(s-k-r)/2} h^{s-r+1} \|u\|_s,$$
$$u \in C^\infty(M, \Lambda T^*M),$$

if $r > s, h \in (0,1], 0 < t \leq 1$, *and the estimate*

$$\|(H_h(t) - \bar{H}_h(t))u\|_{s,k} \leq C_{sk} t^{-k/2} \|u\|_s, u \in C^\infty(M, \Lambda T^*M).$$

if $r = s, h \in [0,1], 0 < t \leq 1$, *where the constants don't depend on* t *and* h.

Proof. For the proof, we make use of the Duhamel formula

$$H_h(t) - \bar{H}_h(t) = \int_0^t H_h(\tau)(\bar{L}_h - L_h)\bar{H}_h(t-\tau)\, d\tau.$$

We know norm estimates for operators $H_h(t)$ and $\bar{H}_h(t)$ (see Proposition 3 and (17)-(18)) and the explicit formula for the difference $\bar{L}_h - L_h$:

$$L_h - \bar{L}_h = h^4 \Delta_{-1,2} + hK_1 + h^2 K_2 + h^3 K_3.$$

from where Proposition 4 is proved in a usual way.

Next, we pass from the Sobolev estimates for the operator $H_h(t) - \bar{H}_h(t)$ to pointwise and trace estimates.

Proposition 5. *Under current hypotheses, we have the estimate*

$$|\mathrm{tr}(H_h(t) - \bar{H}_h(t))| \leq Ch^{1-q}.$$

Proof. For the proof, we make use the following proposition (see [20] for a scalar case):

Proposition 6. *Let $(M, \mathcal{F})$ be a compact foliated manifold, E be an Hermitian vector bundle on M. For any $s > p/2$ and $k > q/2$, there is a continuous embedding*

$$H^{s,k}(M, \mathcal{F}, E) \subset C(M, E).$$

Moreover, for any $s > p/2$ and $k > q/2$, there is a constant $C_{s,k} > 0$ such that, for each $\lambda \geq 1$,

$$\sup_{x \in M} |u(x)| \leq C_{s,l} \lambda^{q/2} (\lambda^{-s} \|u\|_{s,k} + \|u\|_{0,k+s}), u \in H^{s,k}(M, \mathcal{F}, E).$$

Let $H_h(t, x, y)$ ($\bar{H}_h(t, x, y)$) be the integral kernels of operators $H_h(t)$ ($\bar{H}_h(t)$) respectively. Then, by Propositions 4 and 6 (with $\lambda = h^{-1}$), we obtain:

$$\sup_{x \in M} |H_h(t, x, x) - \bar{H}_h(t, x, x)| \leq C h^{1-q},$$

that immediately completes the proof.

Denote by $h_{\mathcal{F}}(t, \gamma) \in C^\infty(G_{\mathcal{F}}, \Lambda T^* M)$ the tangential kernel of the smoothing tangential operator $\exp(-t\Delta_F)$.

Proposition 7. *For any $t > 0$, we have the asymptotical formula (as $h \to 0$)*

$$\operatorname{tr} e^{-tL_h} = (2\pi)^{-q} h^{-q} \int_M \left(\int_{H_x^*} e^{-t g_H(\nu)} d\nu \right) \operatorname{Tr}_{\Lambda T^* M} h_{\mathcal{F}}(t, x) dx + O(h^{1-q}). \tag{19}$$

Proof. By Propositions 3 and 6, we have the estimate

$$\operatorname{tr} e^{-tL_h} \leq C h^{-q}, h \to 0.$$

Moreover, by Proposition 5, asymptotics of traces of the operators $H_h(t)$ and $\bar{H}_h(t)$ when h tends to zero have the same leading terms (of order h^{-q}), and we can apply the asymptotical formula of [17, 18, 20] to complete the proof.

Remark 5. Since

$$\int_{H_x^*} e^{-t g_H(\nu)} d\nu = \pi^{q/2} t^{-q/2},$$

the formula (19) can be rewritten in a simpler form:

$$\operatorname{tr} e^{-tL_h} = (4\pi t)^{-q/2} h^{-q} \int_M \operatorname{Tr}_{\Lambda T^* M} h_{\mathcal{F}}(t, x) dx + O(h^{1-q}), h \to 0. \tag{20}$$

From (20), we can also obtain an asymptotical formula for the spectrum distribution function, but it is more convenient for us to use the formula in the form (19).

Remark 6. For any $x \in M$, the restriction $h_{\mathcal{F}}(t,\gamma) \in C^\infty(G^x_{\mathcal{F}}, \Lambda T^*M)$ of $h_{\mathcal{F}}$ on $G^x_{\mathcal{F}}$ is the kernel of the operator $\exp(-t\Delta_x)$, where Δ_x the restriction of Δ_F on $G^x_{\mathcal{F}}$ (see also Sect. 5). This fact cannot be extended to more general functions $f(\Delta_F)$ (see [19]), that is closely related with so-called spectrum coincidence theorems and with the nonstandard asymptotical formula (2).

Proof of Theorem 2. The tangential kernel $h_{\mathcal{F}_H}(t) \in C^\infty(G_{\mathcal{F}_H}, \pi^*\Lambda T^*M)$ of the operator $\exp(-t\Delta_{\mathcal{F}_H})$ is related with the tangential kernel $h_{\mathcal{F}}(t) \in C^\infty(G_{\mathcal{F}}, \Lambda T^*M)$ of the operator $\exp(-t\Delta_F)$ by the formula

$$h_{\mathcal{F}_H}(t,\gamma,\nu) = h_{\mathcal{F}}(t,\gamma).$$

The crucial point is that, since $\mathcal{F}$ is a Riemannian foliation, the operators $\Delta_{\mathcal{F}_H}$ and g_H commutes as operators on H^*. In particular, we have

$$e^{-t\sigma_h(\Delta_h)} = e^{-tg_H(\nu)}e^{-t\Delta_{\mathcal{F}_H}}, t > 0.$$

So the formula (19) can be rewritten in terms of the notation of this section as follows:

$$\mathrm{tr}\, e^{-tL_h} = h^{-q}\mathrm{tr}_{\mathcal{F}_H}\, e^{-t\sigma_h(\Delta_h)} + O(h^{1-q}), h \to 0.$$

from where, using standard approximation arguments, the theorem follows immediately.

Remark 7. The passage from the operator L_h to the operator $\bar{L}_h$ resembles the passage from the Riemannian connection on M to the almost product connection as in [1,26].

5. Formulation in terms of leafwise spectral characteristics

Here we rewrite the asymptotical formula (15) in terms of spectral characteristics of the operator Δ_F. In particular, it allows us to complete the proof of Theorem 1 on the asymptotic behaviour of the eigenvalue distribution function.

Recall that Δ_F denotes the tangential Laplacian in the space $C^\infty(M, \Lambda T^*M)$ (see (7)). Let us restrict the operator Δ_F to the leaves of the foliation $\mathcal{F}$ and lift the restrictions to holonomy coverings $G^x_{\mathcal{F}}$ via the map s. We obtain the family

$$\Delta_x : C^\infty_c(G^x_{\mathcal{F}}, s^*\Lambda T^*M) \to C^\infty_c(G^x_{\mathcal{F}}, s^*\Lambda T^*M)$$

of Laplacians on holonomy coverings of leaves. Since the foliation $\mathcal{F}$ is Riemannian, it can be checked that the operator Δ_x is formally self-adjoint in $L^2(G^x_{\mathcal{F}}, s^*\Lambda T^*M)$, that, in turn, implies its essential self-adjointness in this Hilbert space (with initial domain $C^\infty_c(G^x_{\mathcal{F}}, s^*\Lambda T^*M)$) for any $x \in M$.

For each $\lambda \in \mathbb{R}$, the kernel $e(\gamma,\lambda), \gamma \in G_{\mathcal{F}}$ of the spectral projections of the operators Δ_x, corresponding to the semiaxis $(-\infty,\lambda]$, defines an element of the von Neumann algebra $W^*(G_{\mathcal{F}}, \Lambda T^*M)$. The section $e(\gamma,\lambda)$ is a leafwise smooth section of the bundle $(s^*\Lambda T^*M)^* \bigotimes r^*\Lambda T^*M$ over $G_{\mathcal{F}}$.

We introduce the spectrum distribution function $N_{\mathcal{F}}(\lambda)$ of the operator Δ_F by the formula

$$N_{\mathcal{F}}(\lambda) = \int_M \mathrm{Tr}_{\Lambda T^*M}\, e(x,\lambda)dx, \lambda \in \mathbb{R}. \tag{21}$$

By [19], for any $\lambda \in \mathbb{R}$, the function $\mathrm{Tr}_{\Lambda T^*M}\, e(x,\lambda)$ is a bounded measurable function on M, therefore, the spectrum distribution function $N_{\mathcal{F}}(\lambda)$ is well-defined and takes finite values.

Theorem 3. *For any function $f \in C_0^\infty(\mathbb{R})$, we have the following asymptotic formula (as $h \to 0$):*

$$\mathrm{tr}\, f(L_h) = h^{-q}\frac{(4\pi)^{-q/2}}{\Gamma(q/2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma^{q/2-1}f(\tau+\sigma)\,d\sigma\,dN_{\mathcal{F}}(\tau)+o(h^{-q}).$$

Proof. Let $E_{g_H}(\tau)$ and $E_\Delta(\sigma)$ denote the spectral projections of the operators g_H and $\Delta_{\mathcal{F}_H}$ in $L^2(H^*, \pi^*\Lambda T^*M)$ respectively. Since these operators commute, we have

$$f(\sigma_h(\Delta_h)) = f(\Delta_F + g_H) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tau+\sigma)\, dE_{g_H}(\tau)\, dE_\Delta(\sigma)$$

is a tangential operator on H^* with respect to $\mathcal{F}_H$ with the tangential kernel

$$k_{f(\sigma_h(\Delta_h))}(\gamma,\nu) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tau+\sigma)\, dE_{g_H}(\tau)(\nu)\, dE_\Delta(\gamma,\sigma).$$

So we obtain

$$\begin{aligned}\mathrm{tr}_{\mathcal{F}_H} f(\sigma_h(\Delta_h)) &= \int_M\int_{H_x^*}\mathrm{Tr}_{\Lambda T^*M}\, k_{f(\sigma_h(\Delta_h))}(x,\nu)dxd\nu \\ &= \int_M\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tau+\sigma)\,(\int_{H_x^*} dE_{g_H}(\tau)(\nu)\, d\nu) \\ &\quad d_\sigma(\mathrm{Tr}_{\Lambda T^*M}\, E_\Delta(x,\sigma))\, d\tau\, dx,\end{aligned}$$

from where, taking into account that $E_{g_H}(\tau)(\nu) = \chi_{\{g_H(\nu)\le\tau\}} I_{\pi^*\Lambda T^*M}$ and

$$\int_{H_x^*} E_{g_H}(\tau)(\nu)\, d\nu = volume\{\nu \in H_x^* : g_H(\nu) \le \tau\} = \omega_q\tau^{q/2},$$

where $\omega_q = \frac{\pi^{q/2}}{\Gamma((q/2)+1)}$ is the volume of the unit ball in $\mathbb{R}^q$, we immediately obtain the desired formula.

In a particular case when f is the characteristic function of the semi-axis $(-\infty, \lambda)$, Theorem 3 implies the following theorem on the asymptotic behaviour of the spectrum distribution function $N_h(\lambda)$, which is exactly Theorem 1 formulated in terms of the operator L_h.

Theorem 4. *Under current hypothesis, we have*

$$N_h(\lambda) = h^{-q}\frac{(4\pi)^{-q/2}}{\Gamma((q/2)+1)}\int_{-\infty}^{\lambda}(\lambda-\tau)^{q/2}\,dN_{\mathcal{F}}(\tau)+o(h^{-q}), h\to 0$$

for any $\lambda \in \mathbb{R}$.

6. Eigenvalue limits

Here we discuss the asymptotical behaviour of individual eigenvalues of the operator Δ_h when h tends to zero. As usual, we will, equivalently, consider the operator L_h instead of Δ_h. Moreover, we will consider eigenvalues of this operator on differential k-forms with a fixed k. Therefore, we will write L_h^k for the restriction of the operator L_h on $C^\infty(M, \Lambda^k T^*M)$ $k = 1, \dots, n$, omitting k where it is not essential.

For any $h > 0$, L_h is an analytic family of type (B) of self-adjoint operators in sense of [15]. Therefore, for $h > 0$, the eigenvalues of L_h depends analytically on h. Thus there are (countably many) analytic functions $\lambda_i(h)$ such that

$$\operatorname{spec} L_h = \{\lambda_i(h) : i = 1, 2, \dots\}, h > 0.$$

Proposition 8. *Under current hypotheses, for any* i*, there exists a limit*

$$\lim_{h\to 0+}\lambda_i(h) = \lambda_{\lim,i}. \tag{22}$$

Moreover, if v_h *is a normalized eigenform associated with the eigenvalue* $\lambda_i(h)$, $L_h v_h = \lambda_i(h) v_h$, $\|v_h\| = 1$, *then we have the estimates*

$$\|v_h\|_{0,1} < C_1,\ h\|v_h\|_{1,0} < C_2, \tag{23}$$

with the constants C_1 *and* C_2 *independent of* $h \in (0, 1]$.

Proof. By [15], the functions $\lambda_i(h)$ satisfy the following equality

$$\begin{aligned}\lambda_i'(h) &= ((dL_h/dh)v_h, v_h)\\ &= ((2h\Delta_H + 4h^3\Delta_{-1,2} + K_1 + 2hK_2 + 3h^2K_3)v_h, v_h),\end{aligned}$$

from where, using the positivity of operators Δ_H and $\Delta_{-1,2}$ in $L^2(M, \Lambda T^* M)$, and the estimates (9) and (13) (with $h = 1$), we obtain

$$\lambda_i'(h) \geq -C_1\|v_h\|_{0,1}^2 - C_2h^2\|v_h\|_{1,0}^2 - C_3. \tag{24}$$

The estimate (14) implies

$$C_1\|v_h\|^2_{0,1}+C_2h^2\|v_h\|^2_{1,0}\leq C_3\lambda_i(h)+C_4, h\in(0,1]. \tag{25}$$

By (24) and (25), we conclude that

$$\lambda'_i(h)\geq -C_5\lambda_i(h)-C_6.$$

This estimate can be rewritten in the following way:

$$\frac{d}{dh}((\lambda_i(h)+\frac{C_6}{C_5})e^{C_5h})\geq 0,$$

that means that the function $(\lambda_i(h)+\frac{C_6}{C_5})e^{C_5h}$ is not decreasing in h for h small enough. By the positivity of the operator L_h in $L^2(M,\Lambda T^*M)$, each eigenvalue $\lambda_i(h)$ is positive, so the function $(\lambda_i(h)+\frac{C_6}{C_5})e^{C_5h}$ semibounded from below near zero. Therefore, this function has a limit when h tends to zero, that, clearly, implies the existence of the limit for the function λ_i.

The second assertion of this proposition is an immediate consequence of the first one and the estimate (25).

Proposition 8 allows us to introduce the limiting spectrum of the operator Δ^k_h as the set of all limiting values $\lambda^k_{\lim,i}$, given by (22):

$$\sigma_{\lim}(\Delta^k_h)=\{\lambda^k_{\lim,i}:i=0,1,\dots\}.$$

By analogy with semiclassical asymptotics for a Schrödinger operator, we may assume that the structure of the limiting spectrum $\sigma_{\lim}(\Delta^k_h)$ is defined in a big extent by a limiting value of the bottoms of spectra of operators Δ^k_h. So let

$$\lambda^k_0(h)=\min_{u\in C^\infty(M,\Lambda^kT^*M)}\frac{(\Delta^k_hu,u)}{\|u\|^2},$$

and

$$\lambda^k_{\lim,0}=\lim_{h\to 0}\lambda^k_0(h).$$

There are two other quantities: the bottom $\lambda^k_{F,0}$ of the spectrum of the operator Δ^k_F in $L^2(M,\Lambda^kT^*M)$:

$$\lambda^k_{F,0}=\min_{u\in C^\infty(M,\Lambda^kT^*M)}\frac{(\Delta^k_Fu,u)}{\|u\|^2},$$

and the bottom $\lambda^k_{\mathcal{F},0}$ of the leafwise spectrum of the operator Δ^k_F:

$$\lambda^k_{\mathcal{F},0}=\inf\{\lambda^k_{L,0}:L\in V/\mathcal{F}\},$$

where

$$\lambda^k_{L,0} = \min_{u \in C^\infty_c(L,\Lambda^k T^*M)} \frac{(\Delta^k_L u, u)}{\|u\|^2},$$

the operator Δ^k_L is the restriction of the operator Δ^k_F on the leaf L.

Proposition 9. *Under current hypotheses, we have the following relations:*

$$\lambda^k_{F,0} \le \lambda^k_{\lim,0} \le \lambda^k_{\mathcal{F},0}, \ k = 1, \dots, n. \tag{26}$$

Proof. Let v_h be the normalized eigenform associated with the bottom eigenvalue $\lambda^k_0(h)$: $L^k_h v_h = \lambda^k_0(h) v_h, \ \|v_h\| = 1$. By the definition of $\lambda^k_{F,0}$, we have the estimate

$$(\Delta^k_F v_h, v_h) \ge \lambda^k_{F,0}.$$

By (12), we obtain

$$\lambda^k_0(h) \ge (1-h^2)\lambda^k_{F,0} + C_1 h^2 \|v_h\|^2_{1,0} + h(K_1 v_h, v_h) - C_2 h^2, \tag{27}$$

where C_1 and C_2 are positive constants. By (23), we have

$$\lim_{h\to 0} h(K_1 v_h, v_h) = 0,$$

that, by (27), immediately completes the proof of the first inequality in (26).

By Theorem 1, $N^k_h(\lambda) > 0$ for any $\lambda > \lambda^k_{\mathcal{F},0}$ and h small enough, from where the second inequality in (26) follows immediately.

We conclude this section with some remarks and examples, concerning the quantities $\lambda^k_{F,0}$, $\lambda^k_{\lim,0}$ and $\lambda^k_{\mathcal{F},0}$.

Remark 8. When the foliation $\mathcal{F}$ is a fibration or, more general, is amenable in a sense of [19], the relations (26) turns out to be identities [19].

Remark 9. We don't know if the equality $\lambda^k_{F,0} = \lambda^k_{\lim,0}$ is always true. It is, clearly, true for $k = 0$: $\lambda^0_{F,0} = \lambda^0_{\lim,0} = 0$. More general, if the Betti number $b_k(M)$ is not zero, then $\lambda^k_0(h) = 0$ for all h, that also implies $\lambda^k_{F,0} = \lambda^k_{\lim,0} = 0$.

Remark 10. Here we give an example of the foliation such that the bottom $\lambda^0_{F,0} = 0$ of the operator Δ^0_F in $L^2(M)$ is a point of discrete spectrum.

Let Γ be a discrete, finitely generated group such that

(a) Γ has property (T) of Kazhdan;

(b) Γ is embedded in a compact Lie group G as a dense subgroup.

For definitions and examples of such groups, see, for instance, [14,22].

Let us take a compact manifold X such that $\pi_1(X) = \Gamma$. Let $\tilde{X}$ be the universal covering of X equipped with a left action of Γ by deck transformations. We will assume that Γ acts on G by left translations. Let us consider the suspension foliation $\mathcal{F}$ on a compact manifold $M = \tilde{X} \times_\Gamma G$ (see, for instance, [5]). A choice of a left invariant metric on G provides a bundle-like metric on M, so $\mathcal{F}$ is a Riemannian foliation. We may assume that leafwise metric is chosen in such a way that any leaf of the foliation $\mathcal{F}$ is isometric to $\tilde{X}$.

There is defined a natural action of Γ on M and the operator Δ^0_F is invariant under this action. Let $E(0,\lambda)$, $\lambda > 0$, denote the spectral projection of the operator Δ^0_F in $L^2(M)$, corresponding to the interval $(0,\Lambda)$, and $E(0,\lambda)L^2(M)$ be the corresponding Γ-invariant spectral subspace.

Claim. In this example, $\lambda^0_{F,0} = 0$ is a nondegenerate point of discrete spectrum of Δ^0_F, that is, an isolated eigenvalue of multitplicity 1.

From the contrary, let us assume that zero lies in the essential spectrum of the operator Δ^0_F in $L^2(M)$. Then, for any $\varepsilon > 0$ and $\lambda > 0$, there is a function $u_\varepsilon \in C^\infty(M)$ such that u_ε belongs to the space $E(0,\lambda)L^2(M)$, $\|u_\varepsilon\| = 1$ and

$$(\Delta_F u_\varepsilon, u_\varepsilon) = \|\nabla_F u_\varepsilon\| \leq \varepsilon, \tag{28}$$

where ∇_F denotes the leafwise gradient. From (28), we can easily derive that the representation of the group Γ in $E(0,\lambda)L^2(M)$ has an almost invariant vector, that, by the property (T), implies the existence of an invariant vector $v_0 \in E(0,\lambda)L^2(M)$.

Since Γ is dense in G, Γ-invariance of v_0 implies its G-invariance, that, in turn, implies that v_0 is a lift of some non-zero element $v \in C^\infty(X)$ via the natural projection $M \to X$. It can be easily checked that v belongs to the corresponding spectral space $E(0,\lambda)L^2(X)$ of the Laplace operator Δ_X in $L^2(X)$. From other hand, the operator Δ_X has a discrete spectrum, so zero is an isolated point in the spectrum of Δ_X, and the space $E(0,\lambda)L^2(X)$ is trivial if $\lambda > 0$ is small enough. So we get a contradiction, which implies that zero lies in the discrete spectrum of the operator Δ^0_F in $L^2(M)$.

Remark 11. If $\mathcal{F}$ is given by a fibration, zero is also an isolated point in the spectrum of the operator Δ^0_F in $L^2(M)$, but, in that case, it is an eigenvalue of infinite multitplicity, and, therefore, lies in the essential spectrum of Δ^0_F in $L^2(M)$.

Remark 12. Unlike the scalar case, it is not always the case that all of the semiaxis $[\lambda_{\lim,0}, +\infty)$ is contained in $\sigma_{lim}(\Delta_h)$. Indeed, let, as in the example of Remark 10, $\lambda^0_{F,0} = 0$ is a nondegenerate point of discrete spectrum of Δ^0_F. Then, by means of the perturbation theory of the discrete spectrum (see, for instance, [15]), we can state that, for $h > 0$ small enough, $\lambda^0(h) = 0$ is the only eigenvalue of Δ^0_h near zero. So we conclude that $\lambda_{lim,0} = 0$ but there exists a $\lambda_1 > 0$ such that $\sigma_{lim}(\Delta_h) \bigcap [0,\lambda_1] = \{0\}$.

7. Concluding remarks

In this section, we discuss some aspects of the main asymptotical formula (1), and, especially, of the nonstandard formula (2). We will make use of the notation of previous sections.

The whole picture which we observe in the foliation case is the following. Generally, for any $k = 0, 1, \dots, n$, we have only that $\lambda^k_{F,0} \le \lambda^k_{\mathrm{lim},0} \le \lambda^k_{\mathcal{F},0}$, and these relations turn into identities, if the foliation $\mathcal{F}$ is a fibration or, more general, is amenable (see Remark 8).

By (1), the function $N^k_h(\lambda)$ behaves in a usual way when λ is greater than the bottom of the leafwise spectrum of Δ^k_F:

$$N^k_h(\lambda) \sim Ch^{-q}, \lambda \ge \lambda^k_{\mathcal{F},0},$$

but, if $\lambda^k_{F,0} < \lambda^k_{\mathcal{F},0}$, there might be limiting values for eigenvalues $\lambda^k_i(h)$ of the operator Δ^k_h, lying in the interval $(\lambda^k_{F,0}, \lambda^k_{\mathcal{F},0})$. So the function $N^k_h(\lambda)$ is nontrivial on the interval $(\lambda^k_{\mathrm{lim},0}, \lambda^k_{\mathcal{F},0})$, but, since the right-hand side of (1) depends only on leafwise spectral data of the operator Δ^k_F, we have

$$\lim_{h\to 0+} h^q N^k_h(\lambda) = 0,\ \lambda < \lambda^k_{\mathcal{F},0}. \tag{29}$$

It means that the set of eigenvalues of Δ^k_h in the interval $(\lambda^k_{\mathrm{lim},0}, \lambda^k_{\mathcal{F},0})$ is "thin" in the whole set of eigenvalues of Δ_h. By analogy with [27], (29) in the case $k = 0$ may be called as a weak foliated version of "Riemann hypothesis".

This is quite different from what we have in the case of a Schrödinger operator or in the fibration case. For instance, if H_h is a Schrödinger operator on a compact manifold M (we may consider M, being equipped with a trivial foliation $\mathcal{F}$ which leaves are points): $H_h = -h^2\Delta + V(x), x \in M$, we have $\lambda_{F,0} = \lambda_{\mathrm{lim},0} = \lambda_{\mathcal{F},0} = \inf V_-$, where $V_-(x) = \min(V(x), 0), x \in M$, and the following asymptotical formula for the spectrum distribution function $N_h(\lambda)$ in the semiclassical limit:

$$N_h(\lambda) = (2\pi)^{-n} h^{-n} \int_{\{(x,\xi):\xi^2+V(x)\le\lambda\}} dx d\xi + o(h^{-n}), h \to 0+ .$$

So we have only two possibilities: $N_h(\lambda) \sim Ch^{-n}$, if $\lambda > \inf V_-$ ($n = \dim M$), and $N_h(\lambda) = 0$, if $\lambda \le \inf V_-$.

We can point out facts in spectral theory of coverings, which are very similar to ones in spectral theory of foliations mentioned above. For simplicity, consider only the Laplace-Beltrami operator on functions.

Let $\tilde{M} \to M$ be a normal covering with a covering group Γ. Recall that a tower of coverings is a set $\{M_i\}^\infty_{i=1}$ of finite-fold subcoverings of this covering with the corresponding covering groups Γ_i such that:

(1) for each i, Γ_i is a normal subgroup of finite index in Γ;
(2) for each i, Γ_{i+1} is contained in Γ_i;
(3) $\bigcap_i \Gamma_i = \{e\}$.

Let $\sigma(\Delta_{M_i})$ be the set of eigenvalues of the Laplacian Δ_{M_i} on M_i. For any i, we have an embedding $\sigma(\Delta_{M_i}) \subset \sigma(\Delta_{M_{i+1}})$, and when i tends to infinity the spectrum $\sigma(\Delta_{M_i})$ of the finite covering M_i approaches to the limit $\sigma_{lim}(\Delta) = \bigcup_i \sigma(\Delta_{M_i})$. Then, the bottom $\lambda_{\mathrm{lim},0}$ of the limiting spectrum $\sigma_{\mathrm{lim}}(\Delta)$ and the bottom $\lambda_{M,0}$ of the spectrum $\sigma(\Delta_M)$ of the manifold M are, clearly, equal to 0. In general, the bottom $\lambda_{\tilde{M},0}$ of the spectrum $\sigma(\Delta_{\tilde{M}})$ of the covering manifold $\tilde{M}$ is is not less than $\lambda_{M,0} = 0$, and, by [4], the identity $\lambda_{\tilde{M},0} = \lambda_{M,0}$ holds if and only if the group Γ is amenable.

Moreover, by [10], for any function $f \in C_c^\infty(\mathbb{R})$, we have

$$\lim_{i\to\infty} (vol\ M_i)^{-1} \mathrm{tr}\ f(\Delta_{M_i}) = \mathrm{tr}_\Gamma\ f(\Delta_{\tilde{M}}),$$

where tr_Γ is the von Neumann trace on the algebra of Γ-invariant operators on $\tilde{M}$ [2]. In particular, if $N_i(\lambda)$ is the eigenvalue distribution function of the Laplace-Beltrami operator Δ_{M_i}, then

$$\lim_{i\to\infty} (vol\ M_i)^{-1} N_i(\lambda) = N_\Gamma(\lambda), \lambda \in \mathbb{R},$$
$$\lim_{i\to\infty} (vol\ M_i)^{-1} N_i(\lambda) = 0, \lambda < \lambda_{\tilde{M},0},$$

where $N_\Gamma(\lambda)$ is the spectrum distribution function of the operator $\Delta_{\tilde{M}}$ constructed by means of the Γ-trace tr_Γ, $\lambda_{\tilde{M},0} = \inf \sigma(\Delta_{\tilde{M}})$.

A little bit more general possibility to arrange a finite-dimensional approximation of the spectrum of a covering, making use of sequences of finite-dimensional representations of the covering group Γ, converging to the left regular representations of Γ, is considered in [27]. Analogues of (1) and (29) can be also found in [27].

Actually, both of these two problems – the spectral problem for the Laplacian on a covering and the spectral problem for the leafwise Laplacian on a foliated manifold – can be considered as type II spectral problems in a sense of theory of operator algebras, and asymptotical spectral problems mentioned above can be treated as finite-dimensional (of type I) approximations to these spectral problems. This gives some explanation to analogies, which we observed above. In above considerations, we also meet notions connected with such approximations: amenability and Kazhdan's property (T).

Let us introduce quantitative spectral characteristics of the tangential Laplacian Δ_F^k related with adiabatic limits. For any λ, let $r_k(\lambda)$ be given as

$$r_k(\lambda) = -\limsup_{h\to 0} \ln N_h^k(\lambda)/\ln h.$$

Otherwise speaking, $r_k(\lambda)$ equals the least bound of all r such that $N_h^k(\lambda) \leq Ch^{-r}, h \to 0$. If $\lambda < \lambda^k_{\lim,0}$, we put $r_k(\lambda) = -\infty$.

One can easily state the following properties of $r_k(\lambda)$:

1. $0 \leq r_k(\lambda) \leq q$ for any $\lambda \geq \lambda^k_{\lim,0}$;
2. $r_k(\lambda)$ is not decreasing in λ;
3. $r_k(\lambda) = q$ if $\lambda > \lambda^k_{\mathcal{F},0}$.
4. if the foliation $\mathcal{F}$ is amenable, then:

$$r_k(\lambda) = q, \lambda > \lambda^k_{\mathcal{F},0}, r_k(\lambda) = -\infty, \lambda \leq \lambda^k_{\mathcal{F},0}.$$

5. $r_k(\lambda) = 0$ iff the interval $[0, \lambda]$ lies in the discrete spectrum of the operator Δ_F^k in $L^2(M, \Lambda^k T^*M)$ (the property (T) case; see Remark 10).

We might expect that some invariants of the function $r_k(\lambda)$ introduced above near $\lambda = 0$ are independent of the choice of metric on M (otherwise speaking, are coarse invariants), and, moreover, are topological or homotopic invariants of foliated manifolds.

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References

1. Alvarez Lopez, J.A., Tondeur, T.: Hodge decomposition along the leaves of a Riemannian foliation. J. Funct. Anal. **99**, 443–458 (1991)
2. Atiyah, M.F.: Elliptic operators, discrete groups and von Neumann algebras. Asterisque **32–33**, 43–72 (1976)
3. Balazard-Konlein, A.: Asymptotique semi-classique du spectre pour des operateurs a symbole operatoriel. C.R.Acad. Sc. Paris, Serie I, **301**, 903–906 (1985)
4. Brooks, R.: The spectral geometry of tower of coverings. J.Diff.Geom. **23**, 97–107 (1986)
5. Camacho, C., Lins Neto, A.: Geometric Theory of Foliations. Boston, Basel, Stuttgart: Birkhauser 1985
6. Connes, A.: Sur la theorie non-commutative de l'integration. In: Algebres d'operateurs, pp. 19-143, Lecture Notes Math., v.725. Berlin, Heidelberg, New York: Springer 1979
7. Connes, A.: Noncommutative geometry. London: Academic Press 1994
8. Connes, A., Moscovici, H.: The local index formula in noncommutative geometry. Geom. and Funct. Anal. **5**, 174–243 (1995)
9. Dai, X.: Adiabatic limits, non-multiplicity of signature and the Leray spectral sequence. J. Amer. Math. Soc. **4**, 265–321 (1991)
10. Donnelli, H.: On the spectrum of towers. Proc. Amer. Math. Soc. **87**, 322–329 (1983)
11. Forman, R.: Spectral Sequences and Adiabatic Limits. Commun. Math. Phys. **168**, 57–116 (1995)
12. Ge, Z.: On adiabatic limits and Rumin's complex. (Preprint, 1994)
13. Gromov, M., Shubin, M.: Von Neumann spectra near zero. Geom. and Funct. Anal. **1**, 375–404 (1991)

14. de la Harpe, P., Valette, A.: La Propiete (T) de Kazhdan pour les Groupes Localement Compacts. Asterisque 175. Soc. Math. France 1989
15. Kato, T.: Perturbation theory for linear operators. Berlin: Springer 1980
16. Klein, M., Martinez, A., Seiler, R., Wang, X.P.: On the Born-Oppenheimer expansion for polyatomic molecules. Comm. Math. Phys. **143**, 607–639 (1992)
17. Kordyukov, Yu.A.: On semiclassical asymptotics of spectrum of elliptic operators on a foliated manifold. Math. Zametki **53**, no. 1, 157–159 (1993)
18. Kordyukov, Yu.A.: On semiclassical asymptotics of spectrum of hypoelliptic operators on a foliated manifold, Funkts. analiz i ego prilozh. **29**, no. 3, 80–82 (1995)
19. Kordyukov, Yu.A.: Functional calculus for tangentially elliptic operators on foliated manifolds. In: Analysis and Geometry in Foliated Manifolds, Proceedings of the VII International Colloquium on Differential Geometry, Santiago de Compostela, 1994. - Singapore: World Scientific 1995 113-136.
20. Kordyukov, Yu.A.: Semiclassical spectral asymptotics on a foliated manifold (to appear)
21. Kordyukov, Yu.A.: Noncommutative spectral geometry of Riemannian foliations (Preprint IHES/M/96/2, 1996; to appear in Manuscr.Math.)
22. Lubotzky, A.: Discrete groups, Expanding Graphs and Invariant Measures. Basel: Birkhauser, 1994
23. Melrose, R., Mazzeo, R.: The adiabatic limit, Hodge cohomology and Leray's spectral sequence for a fibration. J. Diff. Geom. **31**, 185–213 (1991)
24. Molino, P.: Riemannian foliations. Progr. in Math., vol. 73. Boston: Birkhauser 1988
25. Moore, C.C., Schochet, C.: Global Analysis of Foliated Spaces. Math. Sci. Res. Inst. Pub., vol. 9. New York: Springer 1988
26. Reinhart, B.L.: Differential Geometry of Foliations. Berlin, Heidelberg, New York: Springer 1983
27. Sunada, T., Nishio, M.: Trace formulae in spectral geometry. In: Proc. of the Intern. Congress of Math., Japan, 1991. - P. 577-587.
28. Witten, E.: Global gravitational anomalies. Comm. Math. Phys. **100**, 197–229 (1985)