Functional calculus for tangentially elliptic operators on foliated manifolds *

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1 Introduction

A differential operator A on a compact foliated manifold $(M, \mathcal{F})$ is called tangentially elliptic, if A can be restricted to the leaves of the foliation with the restrictions, being elliptic operators on the leaves. The study of tangentially elliptic operators was initiated by A.Connes [3] in the context of the noncommutative integration theory and developed extensively up to now (see, for instance, [16, 4, 5] and references there).

In this paper we are, mainly, interested in the spectral theory of tangentially elliptic operators in the global representation, which deals with spectral properties of tangentially elliptic operators considered as differential operators on the ambient foliated manifold M and studies relations of

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various spectral characteristics of such operators with their leafwise counterparts and with geometric and dynamic invariants of the foliation $\mathcal{F}$.

One of well-known problems of such kind, originated in the spectral theory of differential operators with almost-periodic and random coefficients, is a spectrum coincidence problem, which asks for relationships between the global spectrum and the leafwise spectrum of tangentially elliptic operators. A slightly more general problem is the problem on tangentiality of operators $f(A)$ for formally self-adjoint tangentially elliptic operator A, that is, the problem about relations between the operator $f(A)$ and the corresponding leafwise operators $f(A_L)$. Spectral problems for tangentially elliptic operators in the global representation also arise in a study of the smooth leafwise cohomology $H^*(M, \mathcal{F})$.

One of the main results of this paper is a construction of classes $\tilde{\Psi}^{m,l}(M, \mathcal{F})$ of pseudodifferential operators on the compact foliated manifold $(M, \mathcal{F})$, having different orders in tangential and transversal directions, and of the corresponding scale of anisotropic Sobolev spaces $H^{s,k}(M, \mathcal{F})$, and development of a pseudodifferential functional calculus for tangentially elliptic operators in the global representation. Namely, we give a construction of operators $f(A)$ for a tangentially elliptic operator A, study continuity of the operators $f(A)$ in the spaces $H^{s,k}(M, \mathcal{F})$ and provide a description of these operators as pseudodifferential operators of classes $\tilde{\Psi}^{m,l}(M, \mathcal{F})$.

We consider other properties of operators $f(A)$: C^*-algebraic functional calculus, relationships with the corresponding leafwise operators $f(A_L)$, global continuity and measurability of their leafwise kernels. We also obtain a result on the spectrum coincidence problem for amenable foliations and state existence and some simple properties of the spectrum distribution function for any tangentially elliptic operator.

Note that some results of this work are extensions of results obtained earlier (see, for instance, [17, 5, 8, 1, 12] and references there). For example, our methods don't use the finite propagation speed arguments and, therefore, can be applied to a tangentially elliptic operator of an arbitrary order.

The results and technique developed here have been used by the author in a study of the following problems, which can be viewed as a type I spectrum regularization of a (type II) leafwise spectrum problem for tangentially elliptic operators.

Let A be a self-adjoint tangentially elliptic operator of order m with the positive tangential principal symbol. Consider a self-adjoint elliptic differential operator A_h of order m, depending on $h > 0$, of the form

$$A_h = A + h^m B,$$

where B is an elliptic operator of order m with the positive principal symbol. Then the operator A_h is a semibounded from below operator in $L^2(M)$, having the discrete spectrum. The problem in question is to derive the asymptotical formula for the eigenvalue distribution function $N_h(\lambda)$ of A_h when h tends to 0, or, more generally, to study the asymptotical behaviour of $\operatorname{tr} f(A_h)$ for any function $f \in S(\mathbb{R})$. Such an asymptotic formula for $N_h(\lambda)$ was obtained in [14].

It is clear that the spectrum of the operator A in the global representation is a limit of the spectra of operators A_h, when h tends to zero, therefore, the investigation of these problems necessarily involves some results, concerning to the spectral theory of tangentially elliptic operators in the global representation.

Now we say some words about the organization of this paper. Section 2 contains necessary facts, concerning pseudodifferential operators on foliated manifolds. In particular, we introduce the classes of pseudodifferential operators and the scale of anisotropic Sobolev spaces mentioned above. Section 3 is devoted to the pseudodifferential functional calculus for tangentially elliptic operators. Section 4 contains various results, concerning to relations of operators $f(A)$ with their leafwise analogues and global behaviour of tangential kernels of these operators. In Subsection 4.1, we describe an approach to the functional calculus for tangentially elliptic operator, based on the theory of C^*-algebras. In Subsection 4.2, using C^*-algebraic calculus, we obtain some results on the spectrum coincidence problem. In Subsection 4.3, we discuss the problem of tangentiality of operators $f(A)$. Subsection 4.4 contains results on mesurability and continuity of tangential kernels of the operators $f(A)$, and, finally, Subsection 4.5 is devoted to the existence and some properties of the spectrum distribution function of tangentially elliptic operators.

2 Pseudodifferential calculus on foliated manifolds

2.1 Preliminaries

Here we collect together necessary facts and notations, concerning operators on foliated manifolds. This is based on the presentations in [3, 4, 16].

In the sequel, $(M,\mathcal{F})$ is a (connected) compact foliated manifold, $n=\dim M$, $p=\dim\mathcal{F}$. We denote by G the holonomy groupoid of the foliation $\mathcal{F}$ and will assume that G is Hausdorff. As usual, we use notation $s:G\to M$ and $r:G\to M$ for the source and range mappings respectively, $G^x=\{\gamma\in G: r(\gamma)=x\}$.

We fix a Riemannian metric g on M with the corresponding distance function ρ and the volume form dx. Let ν^x be a fixed smooth density on G^x for any $x\in M$, which is the Riemannian volume of the leafwise Riemannian metric induced by g.

Let I be the open interval $(-1,1)$, $p_2: I^n\to I^q$ the projection onto the second factor in the product $I^n=I^p\times I^q$.

A foliation chart for $\mathcal{F}$ is a coordinate chart $\phi: U\to I^n$ with $U\subset M$ open such that the level sets $p(y)=(p_2\circ\phi)^{-1}(y), y\in I^q$, are the connected components of the restrictions of the leaves of $\mathcal{F}$ to U.

A covering of M by foliation charts $\{(U_j,\phi_j)\}$ is said to be *good* if the covering is locally finite and any non-empty intersection of coordinates patches is a contractible space.

A foliation chart $\phi: U\to I^n$ is *regular* if ϕ has an extension $\tilde\phi:\tilde U\to(-1-\varepsilon,1+\varepsilon)^n, \varepsilon>0$, where $U\subset\tilde U, \tilde\phi|U=\phi$, and the level sets $\tilde p(y)$ are connected subsets of leaves of $\mathcal{F}$. Without loss of generality, we will only consider regular foliation charts $\phi: U\to I^n$ such that the r_0-neighborhood of U is contained in a foliation coordinate patch $\tilde U$ with some (fixed) $r_0>0$.

A cover $\{U_j\}$ of M has a *Lebesque number* $c_1>0$ if, for any set $X\subset M$ of diameter less than c_1, there is j, for which $X\subset U_j$.

We say that a covering $\{(U_j,\phi_j\}$ of M by foliated charts is c_1*-regular* if the covering is good, each chart ϕ_j is regular, and there is a Lebesque number $c_1>0$ for this covering.

Usually we will denote the local coordinates in $\mathbb{R}^n=\mathbb{R}^p\times\mathbb{R}^q$ by $(x,y), x\in\mathbb{R}^p, y\in\mathbb{R}^q$, and the dual coordinates by $(\xi,\eta),\xi\in\mathbb{R}^p,\eta\in\mathbb{R}^q$. We will also write multi-indices $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ as $\alpha=(\alpha',\alpha'')$, where $\alpha'=(\alpha_1,\alpha_2,\dots,\alpha_p),\alpha''=(\alpha_{p+1},\dots,\alpha_n)$.

A differential operator A acting on $C^\infty(M)$ is called *a tangential operator*, if, in any foliation chart, it takes the form

$$A=\sum_{|\alpha|\le m} a_\alpha(x,y)D^\alpha_x.$$

Given a tangential differential operator A, define *the tangential (complete) symbol* of A by

$$\sigma(x,y,\xi)=\sum_{|\alpha|\le m} a_\alpha(x,y)\xi^\alpha.$$

where $a_\alpha\in C^\infty(I^n)$, and define *the tangential principal symbol* of A by

$$\sigma_m(x,y,\xi)=\sum_{|\alpha|=m} a_\alpha(x,y)\xi^\alpha.$$

If σ_m is invertible for $\xi\ne0$, then A is said to be *tangentially elliptic*.

Recall the definition of classes $\Psi^m(\mathcal{F})$ of tangential pseudodifferential operators (cf. for details, [3, 16]). An element $A \in \Psi^m(\mathcal{F})$ is determined by its restrictions to the leaves of $\mathcal{F}$. For each leaf L, the restriction of P on L is a pseudodifferential operator P_L on L of order m with the distributional kernel supported in the δ-neighborhood of the diagonal $\Delta_L \subset L \times L$ for some $\delta > 0$ independent of L. Moreover, the restrictions of P to foliated coordinate patches are given by the formula

$$Pu(x,y) = (2\pi)^{-p} \int e^{i(x-x_1)\xi} p(x,y,\xi) u(x_1,y)\, dx_1\, d\xi,$$

where $u \in C_c^\infty(I^n), x \in I^p, y \in I^q$, and the tangential symbol p is given by a smooth family of leafwise symbols $\{p(x,y,\xi) \in S^m(I^p \times \mathbb{R}^p) : y \in I^q\}$.

We will write $\Psi^m(\mathcal{F},\delta)$ for the set of all operators $A \in \Psi^m(\mathcal{F})$ with fixed δ.

2.2 Classes of pseudodifferential operators

In this section, we construct classes $\tilde{\Psi}^{m,k}(M,\mathcal{F})$, which contain both usual pseudodifferential operators from Hormander's classes $\Psi^m(M)$ and tangential pseudodifferential operators and are closed under the composition of operators. Due to these properties, they are convenient for investigation of the action of tangential pseudodifferential operators in the Sobolev spaces on the ambient manifold M. Classes $\Psi^{m,k}(M,\mathcal{F},\delta)$, which we introduce now, provide a description of kernels of operators from $\tilde{\Psi}^{m,k}(M,\mathcal{F})$ near the diagonal in $M \times M$. A global description of the kernels is given in terms of an action of operators in the corresponding Sobolev spaces scale, therefore, we, at first, introduce this scale and then complete the construction of classes $\tilde{\Psi}^{m,k}(M,\mathcal{F})$ (see the next Subsection).

We say that a function $a \in C^\infty(I^n \times \mathbb{R}^n)$ belongs to the class $S^{m,k}(I^n \times \mathbb{R}^n, \mathbb{R}^p)$, if, for any multiindices α and β, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$|\partial^\alpha_{(\xi,\eta)} \partial^\beta_{(x,y)} a(x,y,\xi,\eta)| \leq C_{\alpha,\beta}(1+|\xi|+|\eta|)^{m-|\alpha''|}(1+|\xi|)^{k-|\alpha'|}, (x,y) \in I^n, (\xi,\eta) \in \mathbb{R}^n.$$

The class $\Psi^{m,k}(M,\mathcal{F},\delta), \delta < r_0$, consists of operators A, acting in $C^\infty(M)$, such that:

a) its distributional kernel $K_A(x,y)$ vanishes outside of the δ-neighborhood of the diagonal in $M \times M$;

b) in any regular foliation chart, A is given by a symbol $a \in S^{m,k}(I^n \times \mathbb{R}^n, \mathbb{R}^p)$ via the usual formula

$$Au(x,y) = (2\pi)^{-n} \int e^{i((x-x_1)\xi+(y-y_1)\eta)} a(x,y,\xi,\eta) u(x_1,y_1)\, dx_1\, dy_1\, d\xi\, d\eta,$$

where $u \in C_c^\infty(I^n), x \in I^p, y \in I^q$.

It is clear that $\Psi^m(\mathcal{F},\delta)$ is contained in $\Psi^{0,m}(M,\mathcal{F},\delta)$. The classes $S^{m,k}(I^n \times \mathbb{R}^n, \mathbb{R}^p)$ can be obtained as particular cases of Hormander's classes $S(m,g)$ [9] (cf. also [10]), if we take

$$m(x,y,\xi,\eta) = (1+|\xi|^2+|\eta|^2)^{s/2}(1+|\xi|^2)^{k/2},$$
$$g_{(x,y,\xi,\eta)}(x_1,y_1,\xi_1,\eta_1) = |x_1|^2+|y_1|^2+(1+|\xi|^2)^{-1}|\xi_1|^2+(1+|\xi|^2+|\eta|^2)^{-1}|\eta_1|^2,$$

therefore, their basic properties, concerning symbolic calculus and L^2-estimates, can be easily derived from results of [9, 10]. In particular, we have the following assertions:

Proposition 2.1. *If $A \in \Psi^{m_1,k_1}(M,\mathcal{F},\delta_1)$ and $B \in \Psi^{m_2,k_2}(M,\mathcal{F},\delta_2)(\delta_1+\delta_2 < r_0)$, then $C = AB \in \Psi^{m_1+m_2,k_1+k_2}(M,\mathcal{F},\delta_1+\delta_2)$. Moreover, if $a \in S^{m_1,k_1}(I^n \times \mathbb{R}^n, \mathbb{R}^p)$ and $b \in S^{m_2,k_2}(I^n \times \mathbb{R}^n, \mathbb{R}^p)$ are the complete symbols of the operators A and B in a foliated chart, then the complete symbol c of the composition $C = AB$ belongs to $S^{m_1+m_2,k_1+k_2}(I^n \times \mathbb{R}^n, \mathbb{R}^p)$, and, for any natural N,*

$$c(x,y.\xi,\eta) - \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial^\alpha_{(\xi,\eta)} a(x,y,\xi,\eta) D^\alpha_{(x,y)} b(x,y,\xi,\eta) \in S^{m_1+m_2,k_1+k_2-N}(I^n \times \mathbb{R}^n, \mathbb{R}^p).$$

Proposition 2.2. *An operator $P \in \Psi^{m,k}(M, \mathcal{F}, \delta)$, $m \le 0$, $k \le 0$, defines a bounded operator from $L^2(M)$ to $L^2(M)$.*

For any operator $P \in \Psi^{m,k}(M, \mathcal{F}, \delta)$, given by the complete symbol $p \in S^{m,k}(I^n \times \mathbb{R}^n, \mathbb{R}^p)$ in some foliation chart, its principal symbol is the class in the quotient

$$S^{m,k}(I^n \times \mathbb{R}^n, \mathbb{R}^p)/S^{m,k-1}(I^n \times \mathbb{R}^n, \mathbb{R}^p)$$

defined by p. It can be easily seen that the principal symbol is invariant under foliated coordinate changes, and the principal symbol of the composition of operators A and B is equal to the product of the principal symbols of the operators A and B. Taking this into account, we will give the definition of ellipticity as follows.

A symbol $a \in S^{m,k}(I^n \times \mathbb{R}^n, \mathbb{R}^p)$ is said to be *elliptic* if there exists $r \in S^{m,k-1}(I^n \times \mathbb{R}^n, \mathbb{R}^p)$ such that a symbol $b = a + r$ satisfies the estimate

$$|b(x, y, \xi, \eta)| \ge C(1 + |\xi| + |\eta|)^m (1 + |\xi|)^k, \tag{2.1}$$

for any $(x, y) \in I^n$, $(\xi, \eta) \in \mathbb{R}^n \backslash \{0\}$ with some constants $C > 0, R > 0$.

An operator $A \in \Psi^{m,k}(M, \mathcal{F}, \delta)$ is said to be *elliptic*, if, in any foliation chart, its complete symbol is elliptic. It is easy to see that any tangentially elliptic differential operator of order k is an elliptic operator of class $\Psi^{0,k}(M, \mathcal{F}, \delta)$ for any $\delta > 0$.

Proposition 2.3. *For any elliptic operator $A \in \Psi^{m,k}(M, \mathcal{F}, \delta)$, there exists a parametrix, that is, an operator $P \in \Psi^{-m,-k}(M, \mathcal{F}, \delta)$ such that*

$$AP = I - R_1, \quad PA = I - R_2, \quad R_1, R_2 \in \Psi^{0,-\infty}(M, \mathcal{F}, 2\delta).$$

2.3 Sobolev spaces and self-adjointness

Using the classes $\Psi^{m,k}(M, \mathcal{F}, \delta)$, we can introduce the corresponding scale of Sobolev type spaces $H^{s,k}(M, \mathcal{F})$, $s \in \mathbb{R}$, $k \in \mathbb{R}$, and state their basic properties in the usual manner (cf., for instance, [20]).

For any $s \in \mathbb{R}$, $k \in \mathbb{R}$, fix an elliptic operator $\Lambda_{s,k} \in \Psi^{s,k}(M, \mathcal{F}, \delta)$. Without loss of generality, we may assume that the operator $\Lambda_{s,k}$ is formally self-adjoint and $\Lambda_{-s,-k}$ is a parametrix for $\Lambda_{s,k}$. The space $H^{s,-\infty}(M, \mathcal{F})$, $s \in \mathbb{R}$, consists of all $u \in \mathcal{D}'(M)$, which can be represented in the form $u = \sum_\alpha A_\alpha \phi_\alpha$ with some $A_\alpha \in \Psi^{0,+\infty}(M, \mathcal{F}, \delta), \phi_\alpha \in H^s(M)$. Finally, the space $H^{s,k}(M, \mathcal{F})$ consists of all $u \in H^{s,-\infty}(M, \mathcal{F})$ such that $\Lambda_{s,k} u \in L^2(M)$. The spaces $H^{s,k}(M, \mathcal{F})$ can be easily turned into Hilbert spaces. For instance, if $s \ge 0, k \ge 0$, an inner product in $H^{s,k}(M, \mathcal{F})$ can be defined by the formula

$$(u, v)_{s,k} = (\Lambda_{s,k} u, \Lambda_{s,k} u) + (u, v), \quad u, v \in H^{s,k}(M, \mathcal{F}).$$

Now we will give an equivalent definition of the spaces $H^{s,k}(M, \mathcal{F})$ in terms of local coordinates. The space $H^{s,k}(\mathbb{R}^n, \mathbb{R}^p)$ consists of all $u \in S'(\mathbb{R}^n)$ such that $\tilde{u} \in L^2_{\text{loc}}(\mathbb{R}^n)$ ($\tilde{u}$ the Fourier transform) and

$$\|u\|^2_{s,k} = \iint |\tilde{u}(\xi, \eta)|^2 (1 + |\xi|^2 + |\eta|^2)^s (1 + |\xi|^2)^k d\xi d\eta < \infty.$$

The last identity may serve as a definition of a Hilbert norm $\|u\|_{s,k}$ in the space $H^{s,k}(\mathbb{R}^n, \mathbb{R}^p)$.

Now the space $H^{s,k}(M, \mathcal{F})$ can be defined as the set of all $u \in \mathcal{D}'(M)$ such that, for any foliated coordinate chart $\kappa : I^p \times I^q \to U = \kappa(I^p \times I^q) \subset M$ and for any $\phi \in C^\infty_c(U)$, the function $\kappa^*(\phi u)$ belongs to the space $H^{s,k}(\mathbb{R}^n, \mathbb{R}^p)$. Fix some regular covering $\{U_i : i = 1, \dots, d\}$ of M by foliated coordinate patches with the foliated coordinate charts $\kappa_i : I^p \times I^q \to U_i = \kappa_i(I^p \times I^q)$, and a partition of unity $\{\phi_i \in C^\infty(M) : i = 1, ..., d\}$ subordinate to this covering. An inner product in $H^{s,k}(M,\mathcal{F})$ is defined by the formula

$$(u,v)_{s,k} = \sum_{i=1}^{d} (\kappa^*(\phi_i u), \kappa^*(\phi_i v))_{s,k}, \quad u, v \in H^{s,k}(M,\mathcal{F}).$$

Proofs of the following two Propositions can easily be given by a slight modification of proofs of the corresponding facts from the standard theory of pseudodifferential operators (cf., for instance, [20]).

Proposition 2.4. *(1)*(continuity in Sobolev spaces) *An operator $P \in \Psi^{m,l}(M,\mathcal{F})$ defines a linear bounded operator from $H^{s,k}(M,\mathcal{F})$ to $H^{s-m,k-l}(M,\mathcal{F})$ for any $s \in \mathbb{R}$, $k \in \mathbb{R}$.*

(2)(elliptic regularity) *If $P \in \Psi^{m,l}(M,\mathcal{F})$ is elliptic and $u \in H^{s,-\infty}(M,\mathcal{F})$, $Pu \in H^{s-m,k}(M,\mathcal{F})$, then $u \in H^{s,k+l}(M,\mathcal{F})$ and*

$$\|u\|_{s,k+l} \leq C(\|Pu\|_{s-m,k} + \|u\|_{s,-\infty})$$

with the constant $C > 0$, not depending on u.

Proposition 2.5. *A formally self-adjoint elliptic operator $P \in \tilde{\Psi}^{0,k}(M,\mathcal{F})$, $k > 0$, defines a self-adjoint operator in $L^2(M)$ with a domain $H^{0,k}(M,\mathcal{F})$.*

Now we complete classes $\Psi^{m,k}(M,\mathcal{F},\delta)$ so that we obtain algebras of pseudodifferential operators.

The class $\tilde{\Psi}^{m,-\infty}(M,\mathcal{F})$ consists of all operators $K : \mathcal{D}'(M) \to \mathcal{D}'(M)$, which define a continuous map

$$K : H^{s,k}(M,\mathcal{F}) \to H^{s-m,+\infty}(M,\mathcal{F})$$

for any s and k.

It is easy to check the following property (pseudolocality):

if the distributional kernel of an operator $A \in \Psi^{m,k}(M,\mathcal{F},\delta)$ vanishes in some neighborhood of the diagonal in $M \times M$, then A belongs to $\tilde{\Psi}^{m,-\infty}(M,\mathcal{F})$.

Finally, the class $\tilde{\Psi}^{m,k}(M,\mathcal{F})$ consists of all operators $A : \mathcal{D}'(M) \to \mathcal{D}'(M)$, which can be represented as

$$A = A_1 + K, \tag{2.2}$$

where $A_1 \in \Psi^{m,k}(M,\mathcal{F},\delta)$ and $K \in \tilde{\Psi}^{m,-\infty}(M,\mathcal{F})$.

The pseudolocality property mentioned above implies that, if $A \in \tilde{\Psi}^{m,k}(M,\mathcal{F})$, then, for any $\delta > 0$, there exists a representation of the form (2.2) with $A_1 \in \Psi^{m,k}(M,\mathcal{F},\delta)$. We also denote by $\tilde{\Psi}^k(\mathcal{F})$ the set of all operators $A \in \tilde{\Psi}^{0,k}(M,\mathcal{F})$, which can be represented in the form (2.2) with $A_1 \in \Psi^k(\mathcal{F})$.

Throughout in this paper, we will consider only classical pseudodifferential operators, which can be defined as follows.

Let $m \in \mathbb{C}$. Recall that a function $a \in S^{\operatorname{Re} m}(I^n \times \mathbb{R}^n)$ is called *a classical* (or *polyhomogeneous*) symbol, if a can be represented as an asymptotic sum $a \sim \sum_{j=0}^\infty a_j$, where $a_j \in C^\infty(I^n \times \mathbb{R}^n)$ is a homogeneous function of degree $m - j$ in ξ for $|\xi| > 1$. An operator $A \in \Psi^m(M)$ is called *a classical* operator, if, in any coordinate system, it is given by a classical symbol modulo smoothing operators.

Let $m \in \mathbb{C}, k \in \mathbb{C}$. A function $a \in S^{\operatorname{Re} m, \operatorname{Re} k}(I^n \times \mathbb{R}^n, \mathbb{R}^p)$ is called *a classical* (or *polyhomogeneous*) symbol, if a can be represented as an asymptotic sum (in the scale $S^{\operatorname{Re} m, s}(I^n \times \mathbb{R}^n, \mathbb{R}^p)$, $s \in \mathbb{R}$)

$$a \sim \sum_{j=0}^{\infty} a_j, \tag{2.3}$$

where a_j is of the form

$$a_j = b_j c_j, \tag{2.4}$$

b_j is a classical symbol from $S^{\mathrm{Re}\, m}(I^n \times \mathbb{R}^n)$ and $c_j = c_j(x, y, \xi) \in C^\infty(I^n \times \mathbb{R}^p)$ is a homogeneous function of degree $k - j$ in ξ for $|\xi| > 1$.

An operator $A \in \tilde{\Psi}^{m,k}(M)$ is called *a classical* operator, if, in any foliation chart, it is given by a classical symbol modulo smoothing operators.

3 Pseudodifferential functional calculus

Let D be a formally self-adjoint tangentially elliptic operator of order $m > 0$ on a closed foliated manifold $(M, \mathcal{F})$. By Proposition 2.5, D is essentially self-adjoint, and we have via the spectral theorem a bounded linear operator $f(D)$ in the Hilbert space $L^2(M)$ for any bounded Borel function f on $\mathbb{R}$.

When D is not self-adjoint, we can try to define operators $f(D)$ as a bounded linear operator in some Hilbert space (like as $L^2(M)$ or Sobolev spaces) for functions, holomorphic in some neighborhood of the spectrum of D, using the holomorphic functional calculus via the Cauchy integral formula

$$f(D) = \frac{i}{2\pi} \int_\Gamma f(\lambda)(D - \lambda)^{-1}\, d\lambda, \tag{3.1}$$

where Γ is a contour in the complex plane, containing the spectrum of D. Well-known examples of such functional calculus can be found, for instance, in [19, 7], where the constructions of the complex powers and of the heat semigroup for elliptic operators with the positive principal symbol are given.

Finally, it is possible to give a description of the operator $f(D)$ as a pseudodifferential operator on M for functions f from symbol classes. There are many approaches to the pseudodifferential functional calculus (see, for instance, [19, 18, 22] and the bibliography there).

In this Section, we construct functions $f(A)$ of tangentially elliptic operator A with the positive tangential principal symbol, study the continuity of operators $f(A)$ in the Sobolev spaces $H^{m,k}(M, \mathcal{F})$ and give a description of these operators in terms of the classes $\tilde{\Psi}^{m,k}(M, \mathcal{F})$.

We will use an approach to the functional calculus due to M. Taylor (cf., for instance, [22]), based on the Fourier inversion formula. The construction of the operators $f(A)$ consists of several steps. At first, we develop the machinery of complex powers by the standard scheme of Seeley (cf. [19, 20]) and reduce our considerations to functions of the first order operator $P = A^{1/m}$. Then we prove existence of the wave semigroup e^{itP} and study its properties, using energy estimates for t large and geometric optics constructions for all t around zero. Finally, for a general function f, we use the following formula

$$f(P) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(t) e^{itP}\, dt. \tag{3.2}$$

Remark that all our considerations, except for geometrical optics constructions, are valid under more general assumptions than mentioned above. Namely, at the beginning of this Section, we assume that A is an elliptic classical pseudodifferential operator of class $\tilde{\Psi}^{0,m}(M, \mathcal{F})$ with the positive principal symbol.

3.1 Complex powers

Throughout in this Subsection, we assume that A is an elliptic classical pseudodifferential operator of class $\tilde{\Psi}^{0,m}(M, \mathcal{F})$ with the positive principal symbol. We start a construction of complex powers A^z with construction of a parametrix $P(\lambda)$ for the operator $A - \lambda$ as an operator with a parameter, that is, of a parametrix, which has a right behaviour when λ tends to infinity.

Denote by Λ_ε the angle in the complex plane:

$$\Lambda_\varepsilon = \{\lambda \in \mathbb{C} : |\arg \lambda| > \varepsilon\}.$$

Let $a \sim \sum_{j=0}^{\infty} a_j$ be an asymptotic expansion of the complete symbol of the operator A in some foliation chart (as in (2.3)). For any $\varepsilon > 0$, define the functions $p_{-m-l}(\lambda), \lambda \in \Lambda_\varepsilon, l = 0, 1, \dots,$ by the following system

$$(a_m - \lambda)p_{-m} = 1, \tag{3.3}$$

$$(a_m - \lambda)p_{-m-l} + \frac{1}{\alpha!} \sum_{j<l, j+k+|\alpha|=l} \partial_\xi^\alpha b_{-m-j} D_x^\alpha a_{m-k} = 0, \quad l > 0. \tag{3.4}$$

The functions $p_{-m-l}(\lambda)$ are not of the form (2.4), but it can be easily checked that they satisfy the following estimate

$$|D^\beta_{(x,y)} D^\alpha_{(\xi,\eta)} p_{-m-l}(x, y, \xi, \eta, \lambda)| \leq C_{\alpha\beta}(1 + |\xi| + |\lambda|^{1/m})^{-m}(1 + |\xi|)^{-l-|\alpha'|}(1 + |\xi| + |\eta|)^{-|\alpha''|},$$
$$(x, y) \in I^n, (\xi, \eta) \in \mathbb{R}^n, \lambda \in \Lambda_\varepsilon,$$

where α and β are any multi-indices. For any natural N, put $p_{(N)} = p_{-m} + p_{-m-1} + \dots + p_{-m-N+1}$. Then $p_{(N)}$ satisfies the estimates

$$|D^\beta_{(x,y)} D^\alpha_{(\xi,\eta)} p_{(N)}(x, y, \xi, \eta, \lambda)| \leq C_{\alpha\beta}(1 + |\xi| + |\lambda|^{1/m})^{-m}(1 + |\xi|)^{-|\alpha'|}(1 + |\xi| + |\eta|)^{-|\alpha''|},$$
$$(x, y) \in I^n, (\xi, \eta) \in \mathbb{R}^n, \lambda \in \Lambda_\varepsilon. \tag{3.5}$$

Now we take a covering of M by foliation charts, construct an operator with the complete symbol $p_{(N)}(\lambda)$ in any foliation patch of this covering, and glue these local operators together in the global operator $P_{(N)}(\lambda) \in \Psi^{0,-m}(M, \mathcal{F}, \delta), \lambda \in \Lambda_\varepsilon$, with some $\delta < r_0$. Decompose the operator A in the sum $A = A_1 + K$, where $A_1 \in \Psi^{0,m}(M, \mathcal{F}, \delta)$ and $K \in \tilde{\Psi}^{0,-\infty}(M, \mathcal{F})$ (see (2.2)). By (3.3), it can be easily seen that

$$P_{(N)}(\lambda)(A_1 - \lambda) = I - R_{(N)}(\lambda), \quad \lambda \in \Lambda_\varepsilon,$$

where $R_{(N)}(\lambda) \in \Psi^{0,-N}(M, \mathcal{F}, 2\delta)$ has the complete symbol $r_{(N)}(\lambda)$, satisfying the following estimates:

$$|D^\beta_{(x,y)} D^\alpha_{(\xi,\eta)} r_{(N)}(x, y, \xi, \eta, \lambda)| \leq C_{\alpha\beta}(1 + |\xi| + |\lambda|^{1/m})^{-m}(1 + |\xi|)^{m-N-|\alpha'|}(1 + |\xi| + |\eta|)^{-|\alpha''|},$$
$$(x, y) \in I^n, (\xi, \eta) \in \mathbb{R}^n, \lambda \in \Lambda_\varepsilon. \tag{3.6}$$

From (3.5) and (3.6), we can easily obtain the norm estimates:

$$\|P_{(N)}(\lambda) : H^{s,k}(M, \mathcal{F}) \to H^{s,k+q}(M, \mathcal{F})\| \leq C_{s,k}(1 + |\lambda|^{1/m})^{q-m}, \lambda \in \Lambda_\varepsilon, \tag{3.7}$$

$$\|R_{(N)}(\lambda) : H^{s,k}(M, \mathcal{F}) \to H^{s,k+q+N-m}(M, \mathcal{F})\| \leq C_{s,k,N}(1 + |\lambda|^{1/m})^{q-m}, \lambda \in \Lambda_\varepsilon, \tag{3.8}$$

for any real s and k and for any q, $0 \leq q \leq m$.

Is it clear that

$$P_{(N)}(\lambda)(A - \lambda) = I - R'_{(N)}(\lambda), \quad \lambda \in \Lambda_\varepsilon,$$

where $R'_{(N)}(\lambda) \in \tilde{\Psi}^{0,-N}(M, \mathcal{F})$ is given by the formula $R'_{(N)}(\lambda) = R_{(N)}(\lambda) - P_{(N)}(\lambda)K$. By (3.7) and (3.8), the operator $R'_{(N)}(\lambda)$ satisfies the estimate

$$\|R'_{(N)}(\lambda) : H^{s,k}(M, \mathcal{F}) \to H^{s,k+q+N-m}(M, \mathcal{F})\| \leq C_{s,k,N}(1 + |\lambda|^{1/m})^{q-m}, \lambda \in \Lambda_\varepsilon,$$

for any real s and k and for any q, $0 \le q \le m$. Therefore, for all $\lambda \in \Lambda_\varepsilon$ large enough, the operator $A-\lambda$ is invertible as an unbounded operator in $H^{s,k}(M)$ and the inverse operator can be represented as

$$(A-\lambda)^{-1} = P_{(N)}(\lambda) + R'_{(N)}(\lambda)(A-\lambda)^{-1}.$$

From this, we immediately obtain the following proposition:

Proposition 3.1. *Let $A \in \tilde{\Psi}^{0,m}(M,\mathcal{F})$ be as above. Then, for any $\varepsilon > 0$, there exists a constant $R > 0$ such that, for any $\lambda \in \Lambda_\varepsilon$, $|\lambda| > R$, the operator $A-\lambda$ is invertible as an unbounded operator in $L^2(M)$, the inverse operator $(A-\lambda)^{-1}$ belongs to $\tilde{\Psi}^{0,-m}(M,\mathcal{F})$, and for any $s \in \mathbb{R}$ and $k \in \mathbb{R}$, satisfies the following norm estimates:*

$$\|(A-\lambda)^{-1} : H^{s,k}(M,\mathcal{F}) \to H^{s,k}(M,\mathcal{F})\| \le C_{s,k}(1+|\lambda|)^{-1}, \lambda \in \Lambda_\varepsilon, |\lambda| > R,$$
$$\|(A-\lambda)^{-1} : H^{s,k}(M,\mathcal{F}) \to H^{s,k+m}(M,\mathcal{F})\| \le C_{s,k}, \lambda \in \Lambda_\varepsilon, |\lambda| > R.$$

When A is a tangentially elliptic differential operator of order m with the positive principal symbol, then, using a leafwise parametrix for $A-\lambda$ as an operator with parameter (see also Subsection 4.1), we can prove that $(A-\lambda)^{-1} \in \tilde{\Psi}^{-m}(\mathcal{F})$. Since we can only state H^s-estimates for the operator $(A-\lambda)^{-1}R'(\lambda)$, the operator $(A-\lambda)^{-1}$ is not, in general, a tangential operator, that is, it is not given by a family of leafwise operators (see Subsection 4.3 for a detailed discussion).

Let us note the following corollary of Proposition 3.1, which is used in [14] and is an immediate consequence of the previous results and semigroup theory (see, for instance, [7]).

Proposition 3.2. *Let $A \in \tilde{\Psi}^{0,m}(M,\mathcal{F})$ be an elliptic operator with the positive principal symbol.*

(1)(Gårding inequality) *For any $s \in \mathbb{R}$, $k \in \mathbb{R}$, there exist constants $C_1 > 0$ and C_2 such that*

$$\operatorname{Re}(Au,u)_{s,k} \ge C_1\|u\|^2_{s,k+m/2} - C_2\|u\|^2_{s,-\infty}, \quad u \in C^\infty(M).$$

(2)(Heat semigroup) *The operator A generates a strongly continuous semigroup e^{-tA} of bounded linear operators in $H^{s,k}(M,\mathcal{F})$, satisfying the estimate*

$$\|e^{-tA}\|_{s,k} \le Ct^{-(k-l)/m}\|u\|_{s,l}, \quad u \in C^\infty(M), \quad 0 < t < T,$$

for any $s \in \mathbb{R}, k > l, T > 0$, with the constant $C > 0$, not depending on t.

Now we turn to the construction of the complex powers A^z for an elliptic operator $A \in \tilde{\Psi}^{0,m}(M,\mathcal{F})$, satisfying the above conditions. First of all, replacing A by $A + cI$, we may assume that the spectrum $\sigma(A)$ of the operator A does not contain the semi-axis $(-\infty, 0]$. This implies the existence of a constant $\rho > 0$ such that the disk of the radius ρ, centered at the origin, is not contained in $\sigma(A)$.

Let Γ be a contour in the complex plane of the form $\Gamma = \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3$, where $\lambda = re^{i\alpha}, +\infty > r > \rho$, on Γ_1, $\lambda = \rho e^{i\phi}, \alpha > \phi > -\alpha$, on Γ_2, $\lambda = re^{-i\alpha}, \rho < r < +\infty$, on Γ_3, ($\alpha \in (0,\pi)$ is arbitrary).

The bounded operator A^z, $\operatorname{Re} z < 0$, in $L^2(M)$ is defined by the formula

$$A^z = \frac{i}{2\pi}\int_\Gamma \lambda^z (A-\lambda)^{-1} d\lambda,$$

where a branch of the analytic function λ^z is chosen so that $\lambda^z = e^{z\ln\lambda}$ for $\lambda > 0$.

This definition is extended to all z, putting $A^z = A^{z-k}A^k$ for any z, $\operatorname{Re} z < k$, where k is natural and A^k is the usual power of the operator A. Proposition 3.1 implies that the operator A^z is an operator of class $\Psi^{0,m\operatorname{Re} z}(M,\mathcal{F})$ with the principal symbol, being equal to a^z.

3.2 Action in Sobolev spaces

Now we prove existence of the wave group e^{itP} generated by an elliptic operator $P \in \tilde{\Psi}^{0,1}(M,\mathcal{F})$ with the real principal symbol and state norm estimates for the operators of this group.

Lemma 3.3. *Let $P \in \tilde{\Psi}^{0,1}(M,\mathcal{F})$ be an elliptic operator with the real principal symbol. Then, for any $s \in \mathbb{R}$ and $k \in \mathbb{R}$, there exist constants α and $C > 0$ such that*

$$\|e^{itP} : H^{s,k}(M) \to H^{s,k}(M)\| \leq C\, e^{\alpha|t|}, \quad t \in \mathbb{R},$$

Proof. Proposition 3.2 implies existence of a holomorphic semigroup e^{-zP}, $\operatorname{Re} z > 0$, of linear bounded operators in $L^2(M)$, generated by the operator P, and boundedness of the operators of this semigroup in $H^{s,k}(M,\mathcal{F})$ for all s and k. Denote by D a subspace of $C^\infty(M)$, consisting of all $u \in C^\infty(M)$ of the form $u = e^{-\tau P}v$ with some $v \in C^\infty(M)$ and $\tau > 0$. It is clear that D is dense in the space $H^s(M)$ for any $s \in \mathbb{R}$ and $e^{itA}(D) \subseteq C^\infty(M)$ for any $t \in \mathbb{R}$. In our further calculations, we will assume that $u \in D$.

We have

$$\begin{aligned}\frac{d}{dt}\|\Lambda_{s,k}e^{itP}\|^2 &= -2\mathrm{Im}(\Lambda_{s,k}Pe^{itP}u, \Lambda_{s,k}e^{itP}u)\\ &\leq 2|([\Lambda_{s,k},P]e^{itP}u, \Lambda_{s,k}e^{itP}u)| + 2|\mathrm{Im}(P\Lambda_{s,k}e^{itP}u, \Lambda_{s,k}e^{itP}u)| \end{aligned} \tag{3.9}$$

By Proposition 2.1, the operator $[\Lambda_{s,k}, iP]$ is an operator of class $\Psi^{s,k}(M,\mathcal{F})$, that, by Proposition 2.4, implies the estimate

$$\|[\Lambda_{s,k}, iP]e^{itP}u\| \leq C\|e^{itP}u\|_{s,k}. \tag{3.10}$$

The operator P has the real principal symbol, therefore, $P - P^* \in \tilde{\Psi}^{0,0}(M,\mathcal{F})$, and we obtain the estimate

$$|\mathrm{Im}(P\Lambda_{s,k}e^{itP}u, \Lambda_{s,k}e^{itP}u)| \leq C\|\Lambda_{s,k}e^{itP}u\|^2. \tag{3.11}$$

So, from (3.9), (3.10) and (3.11), we obtain the estimate

$$\frac{d}{dt}\|e^{itP}u\|^2_{s,k} \leq C\|e^{itP}u\|^2_{s,k}.$$

The Gronwall lemma implies the estimate

$$\|e^{itP}u\|_{s,k} \leq C\, e^{\alpha|t|}\|u\|_{s,k}, \quad t \in \mathbb{R}, \quad u \in D,$$

that immediately completes the proof. □

Finally, we will obtain $H^{s,k}$-estimates for operators $f(P)$ under the current hypotheses on P. We say that a function $f \in S(\mathbb{R})$ belongs to the space $S(\mathbb{R},\alpha)$, if it extends to a holomorphic function $f(z)$, defined in the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < \alpha\}$, such that, for any $\eta \in \mathbb{R}$, $|\eta| < \alpha$, the function $f(\cdot + i\eta)$ belongs to $S(\mathbb{R})$ with the seminorms, uniformly bounded on η, $|\eta| < \beta$, for any $\beta < \alpha$. We put also

$$S(\mathbb{R},+\infty) = \bigcap_{\alpha \geq 0} S(\mathbb{R},\alpha)$$

Proposition 3.4. *Let $P \in \tilde{\Psi}^{0,1}(M,\mathcal{F})$ be an elliptic operator with the real principal symbol. Then, for any $s \in \mathbb{R}$, there exists a constant $\alpha > 0$ such that, for any function $f \in S(\mathbb{R},\alpha)$, the operator $f(P)$ defines a continuous mapping from $H^s(M)$ to $H^{s,+\infty}(M,\mathcal{F})$ with the following estimate for its norm*

$$\|f(P) : H^s(M) \to H^{s,k}(M,\mathcal{F})\| \leq C\int |F((1+|\lambda|^2)^{k/2}f)(t)|e^{\alpha|t|}dt$$

Proof. Proposition 3.4 is an immediate consequence of the formula (3.2), Lemma 3.3 and the Paley-Wiener theorem. □

3.3 $f(A)$ as pseudodifferential operators

Here we consider the case when A is a tangentially elliptic operator of order m with the positive tangential principal symbol. Following the lines of [22] (see also [14]) and using the norm estimates obtained above and geometrical optics construction, we will give a description of the operators $f(A)$ as pseudodifferential operators on M and, as a corollary, results on continuity of these operators in the Sobolev spaces $H^{s,k}(M,\mathcal{F})$ for functions f from symbol classes.

As mentioned in Subsection 3.1, under the current hypotheses on A, the operator A^z belongs to the class $\tilde{\Psi}^{m\,\mathrm{Re}\,z}(\mathcal{F})$ and, therefore, we can only consider the case of operators $f(P)$, where $P = A^{1/m}$ is an elliptic operator of class $\tilde{\Psi}^1(\mathcal{F})$ with the positive principal symbol.

We recall that a function f on $\mathbb{R}$ belongs to the class $S^q(\mathbb{R})$, $q \in \mathbb{R}$, if, for any natural j, there exists a constant $C_j > 0$ such that

$$|f^{(j)}(t)| \leq C_j(1+|t|)^{q-j}, \quad t \in \mathbb{R}.$$

Further, we say that a function f on $\mathbb{R}$ belongs to the class $S^q(\mathbb{R}, W)$, $q \in \mathbb{R}$, $W > 0$, if f extends to a holomorphic function in the strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < W\}$ such that, for any $\eta \in \mathbb{R}$ with $|\eta| < W$, the function $f(\cdot + i\eta)$ belongs to $S^q(\mathbb{R})$ with all the seminorms, bounded on compacts in the interval $|\eta| < W$.

Proposition 3.5. *Let P be an elliptic operator of class $\tilde{\Psi}^1(\mathcal{F})$ with the real principal symbol. If $f \in S^q(\mathbb{R}, \infty)$, then $f(P) \in \tilde{\Psi}^q(\mathcal{F})$.*

Proof. Using a partition of unity, we decompose the function f into a sum $f = f_1 + f_2$, where $\tilde{f}_1$ is supported in an appropriate neighborhood of zero, and $\tilde{f}_2$ vanishes in some neighborhood of zero.

The desired assertion for the function f_2 follows from the following lemma.

Lemma 3.6. *Let P be an elliptic operator of class $\tilde{\Psi}^1(\mathcal{F})$ with positive principal symbol. If $f \in S^q(\mathbb{R}, \infty)$, and $\tilde{f}$ vanishes in some neighborhood of zero, then $f(P) \in \tilde{\Psi}^{0,-\infty}(\mathcal{F})$.*

Proof. The lemma is a simple sequence of Lemma 3.3 and the fact that, for any function $f \in S^q(\mathbb{R})$, the function $\tilde{f}(t)$ and all its derivatives are rapidly decreasing on $\mathbb{R}\backslash(-\varepsilon, \varepsilon)$ for any $\varepsilon > 0$ (see [22]). □

To prove Proposition for the function f_1, we replace the operator e^{itP} for small t by its leafwise geometrical optics approximation $W(t)$.

At first, decompose P into a sum $P = P_1 + P_2$, where $P_1 \in B\Psi^1(\mathcal{F}, \delta)$ with δ small enough, $P_2 \in B\tilde{\Psi}^{-\infty}(\mathcal{F})$. Let $p(x, y, \xi)$ be the complete symbol of P_1 in some fixed foliation chart. Then we can assume the parametrix $W(t)$ for e^{itP_1} is of the following form

$$(W(t)u)(x, y) = \iint a(t, x, y, \xi) e^{i(\phi(t,x,y,\xi) - x_1\xi)} u(x_1, y)\, dx_1\, d\xi.$$

Here ϕ is the solution of the Cauchy problem of the eikonal equation

$$\frac{\partial \phi}{\partial t} = p(x, y, \nabla_x \phi), \quad \phi(0, x, y, \xi) = x\xi$$

for $|t| \leq r$, $x \in B$; $a(t, x, y, \xi)$ is determined by the usual transport equations of geometrical optics (see, for instance, [22, Chapter VIII]) with $a(0, x, \xi) = 1$.

Further, using a regular covering of M by foliated coordinate patches and a partition of unity subordinate to this covering, we can glue these local parametrices together into a global one and obtain a Fourier integral operator $W(t) : C^\infty(M) \to C^\infty(M)$ such that

$$(D_t - P_1)W(t) = K(t), \quad |t| < T, \quad W(0) = I + K_0,$$

where $K_0 \in B\Psi^{-\infty}(\mathcal{F},\delta)$, $\{K(t),\ |t|<T\}$ is a smooth family of operators from $B\Psi^{-\infty}(\mathcal{F},\delta)$ ($\delta < r_0$, T depends on δ). To estimate the difference between the operator e^{itP} and its geometrical optics approximation, we make use of the following well-known formula

$$W(t) - e^{itP} = e^{itP}K_0 + i\int_0^t e^{i(t-\tau)P}(K(\tau) - P_2W(\tau))d\tau.$$

This formula and Lemma 3.3 immediately imply that $W(t)-e^{itP}$ belongs to $\tilde{\Psi}^{0,-\infty}(M,\mathcal{F})$ with the following estimate for its norm

$$\|W(t) - e^{itP} : H^s(M) \to H^{s,k}(M,\mathcal{F})\| \leq C_{s,k}e^{\alpha|t|}, \quad |t| < r,$$

for any real s and k with the constant $C_{s,k} > 0$, not depending on t. After that, repeating word by word the arguments of [22], we can prove existence of a constant $r > 0$ such that, for any function $f \in S^q(\mathbb{R})$ with the Fourier transform supported in the interval $(-r,r)$, the operator $f(P)$ is an operator of class $\tilde{\Psi}^q(\mathcal{F})$ with the principal symbol, being equal to $f(p)$, that completes the proof. □

Now, combining the results, obtained in this Section, we can formulate the final result on the pseudodifferential functional calculus.

Theorem 3.7. *Let A be a tangentially elliptic differential operator on M of order m with the positive tangential principal symbol. Then, for any $s \in \mathbb{R}$, there exist constants $W > 0$ and $c > 0$ such that, for any function f on the real line such that the function $g(t) = f(t^m - c)$, $t \in \mathbb{R}$, belongs to the space $S^q(\mathbb{R}, W)$, the operator $f(A)$ defines a continuous mapping*

$$f(A) : H^{s,k}(M,\mathcal{F}) \to H^{s,k-q}(M,\mathcal{F})$$

for any $k \in \mathbb{R}$.

In particular, there exists a constant $c > 0$ such that, for any function f on the real line such that the function $g(t) = f(t^m - c)$, $t \in \mathbb{R}$, belongs to the space $S^q(\mathbb{R}, +\infty)$, the operator $f(A)$ belongs to the class $\tilde{\Psi}^q(\mathcal{F})$.

3.4 The case of Finsler foliations

All the facts on functional calculus for tangentially elliptic operators stated above can be essentially improved in the case of Riemannian foliations or in a slightly more general case of Finsler foliations.

Recall that *the transversal principal symbol* σ_P of an operator $P \in \Psi^m(M)$ is the restriction of the principal symbol of P to $N^*\mathcal{F}\backslash 0$, where $N^*\mathcal{F}$ denotes the conormal bundle to $\mathcal{F}$. For any smooth leafwise path γ from $x \in M$ to $y \in M$, it is defined the map $(dh_\gamma)^* : N^*_y\mathcal{F} \to N^*_x\mathcal{F}$, being the codifferential of the holonomy map, corresponding to γ (cf., for instance, [16]). We say that the transversal principal symbol of the operator P is *holonomy invariant*, if, for any smooth leafwise path γ from x to y, the following equality holds:

$$\sigma_P((dh_\gamma)^*\xi) = \sigma_P(\xi), \quad \xi \in N^*_y\mathcal{F}.$$

If $p_m \in S^m(I^n \times \mathbb{R}^n)$ is the principal symbol of the operator $P \in \Psi^m(M)$ in some foliated chart, then its transversal principal symbol σ_P is given by the formula $\sigma_p(x,y,\eta) = p_m(x,y,0,\eta)$, $(x,y) \in I^n$, $\eta \in \mathbb{R}^n$, and the holonomy invariance of σ_p means its independence of x.

We say that the foliation $\mathcal{F}$ is *Finsler*, if there exists a positive homogeneous (of degree 1) holonomy invariant function on $N^*\mathcal{F}\backslash 0$.

It is clear that this condition is equivalent to the existence of an operator $\Lambda_1 \in \Psi^1(M)$ with the positive holonomy invariant transversal principal symbol. Any Riemannian foliation is Finsler, since in this case an operator with the positive holonomy invariant transversal principal symbol is given by the operator $(I+\Delta)^{1/2}$, where Δ is the Laplacian of a bundle-like metric on M.

The main fact, concerning to pseudodifferential operators on Finsler foliations, is contained in the following lemma.

Lemma 3.8. *If the foliation $\mathcal{F}$ is Finsler, and $P \in \tilde{\Psi}^{0,1}(M,\mathcal{F})$ is a self-adjoint elliptic operator, then, for any $s \in \mathbb{R}$ and $k \in \mathbb{R}$, we have the estimate*

$$\|e^{itP} : H^{s,k}(M,\mathcal{F}) \to H^{s,k}(M,\mathcal{F})\| \leq C_{s,k}(1+|t|)^{|s|}, \quad t \in \mathbb{R},$$

with the constant $C_{s,k} > 0$, not depending on t.

Proof. We make use of the following well-known identity

$$\Lambda_s e^{itP} = e^{itP}\Lambda_s + \int_0^t e^{i(t-\tau)P}[\Lambda_s, iP]e^{i\tau P}d\tau. \tag{3.12}$$

We may assume that the operator $\Lambda_s \in \Psi^s(M)$, defining the Sobolev norm, has the positive holonomy invariant transversal principal symbol. Then, using symbolic calculus, it can be easily checked (see also [14]) that $[\Lambda_s, P] \in \Psi^{s-1,1}(M,\mathcal{F})$, that, by Proposition 2.4, implies the estimate

$$\|[\Lambda_s, P]u\|_{0,k} \leq C\|u\|_{s-1,k+1}, \quad u \in C^\infty(M).$$

Since the norm in the space $H^{0,k}(M,\mathcal{F})$ can be equivalently defined by means of the operators P^k, and e^{itP} is a unitary group of bounded operators in $L^2(M)$, the operators e^{itP} are bounded in $H^{0,k}(M,\mathcal{F})$ uniformly on $t \in \mathbb{R}$. From (3.12), we obtain that

$$\|e^{itP}u\|_{s,k} \leq \|u\|_{s,k} + C\int_0^t \|e^{i\tau P}u\|_{s-1,k+1}d\tau, u \in C^\infty(M). \tag{3.13}$$

The lemma can be immediately derived from (3.13) by induction arguments. □

Now, repeating the arguments of previous Subsections, we obtain the following results on the functional calculus for tangentially elliptic operators on Finsler foliations: the first one is an analogue of Proposition 3.4 and the second one is an analogue of Proposition 3.7.

Proposition 3.9. *Let the foliation $\mathcal{F}$ be Finsler and $P \in \tilde{\Psi}^{0,1}(M,\mathcal{F})$ a formally self-adjoint elliptic operator. Then, for any $s \in \mathbb{R}$ and for any function $f \in S(\mathbb{R})$, the operator $f(P)$ defines a continuous mapping from $H^s(M)$ to $H^{s,+\infty}(M,\mathcal{F})$ with the following estimate for its norm*

$$\|f(P) : H^s(M) \to H^{s,k}(M,\mathcal{F})\| \leq C\int |F((1+|\lambda|^2)^{k/2}f)(t)|(1+|t|)^{|s|}dt$$

Theorem 3.10. *Let the foliation $\mathcal{F}$ be Finsler and A a formally self-adjoint tangentially elliptic differential operator on M of order m with the positive tangential principal symbol. Then, for any $s \in \mathbb{R}$, there exists a constant $c > 0$ such that, for any function f on the real line such that the function $g(t) = f(t^m - c)$, $t \in \mathbb{R}$, belongs to the space $S^q(\mathbb{R})$, the operator $f(A)$ belongs to the class $\tilde{\Psi}^q(\mathcal{F})$ and, in particular, defines a continuous mapping*

$$f(A) : H^{s,k}(M,\mathcal{F}) \to H^{s,k-q}(M,\mathcal{F})$$

for any $s \in \mathbb{R}$ and $k \in \mathbb{R}$.

4 Global aspects of the functional calculus

4.1 C^*-algebraic functional calculus

Here we review an approach to the global functional calculus for tangentially elliptic operators from the point of the theory of C^*-algebras. Throughout in this Section, we will assume that the measure dx is holonomy invariant. At first, recall the definitions of operator algebras, associated with foliated manifolds (see, for instance, [16]). One defines a multiplication and an involution on the space $C_c^\infty(G)$ by the formulas

$$(k_1 * k_2)(\gamma) = \int k_1(\gamma')k_2({\gamma'}^{-1}\gamma)d\nu^x(\gamma'), \quad \gamma \in G,$$
$$k^*(\gamma) = \overline{k(\gamma^{-1})}, \quad \gamma \in G,$$

converting $C_c^\infty(G)$ into an associative algebra with involution. For any $x \in M$, there is a natural homomorphism R_x of $C_c^\infty(G)$ in the algebra $\mathcal{L}(L^2(G^x,\nu^x))$ of linear bounded operators in $L^2(G^x,\nu^x)$, defined by the formula

$$(R_x(k)u)(\gamma) = \int k(\gamma^{-1}\gamma')u(\gamma')d\nu^x(\gamma'), \quad \gamma \in G.$$

The reduced C^*-algebra $C_r^*(G)$ of the groupoid G is the completion of $C_c^\infty(G)$ with respect to the norm

$$\|k\|_{C_r^*(G)} = \sup_{x\in M}\|R_x(k)\|.$$

and *the von Neumann algebra $W^*(M,\mathcal{F})$ of the foliation $\mathcal{F}$* can be defined as the closure of the image of the representation $R = \{R_x; x \in M\}$ of the algebra $C_c^\infty(G)$ on the measurable field of the Hilbert spaces $\mathcal{H} = \{L^2(G^x,\nu^x)\}$ in the weak topology on $\mathcal{L}(\mathcal{H})$.

Define a norm on $C_c^\infty(G)$ by the formula

$$\|k\|_1 = \max\left(\sup_{x\in M}\int |k(\gamma)|d\nu^x(\gamma), \sup_{x\in M}\int |k(\gamma^{-1})|d\nu^x(\gamma)\right).$$

Then $C_c^\infty(G)$ becomes a normed $*$-algebra. The Banach algebra $L^1(G)$ is the completion of $C_c^\infty(G)$ with respect to the norm $\|k\|_1$.

The formula

$$R_M(k)u(x) = \int k(\gamma)u(s(\gamma))d\nu^x(\gamma)$$

defines a bounded representation of $C_c^\infty(G)$ in the space $L^2(M)$. One can define the C^*-algebra $C_M^*(G)$ as the uniform closure of $R_M(C_c^\infty(G))$ in $\mathcal{L}(L^2(M))$. By Theorem 2.1 of [6], we have the estimate

$$\|k\|_{C_r^*(G)} \le \|R_M(k)\|, \quad k \in C_c^\infty(G), \tag{4.1}$$

therefore, the reduced C^*-algebra $C_r^*(G)$ is a quotient of $C_M^*(G)$, so we have the natural projection

$$\pi : C_M^*(G) \to C_r^*(G).$$

Let A be a formally self-adjoint tangentially elliptic operator with the positive tangential principal symbol. Denote by $\{A_x : x \in M\}$ be the corresponding family of uniformly elliptic differential operators along leaves. Then (cf., for instance, [3]), A_x is essentially self-adjoint in $L^2(G^x,\nu^x)$, and, for any bounded Borel function f on $\mathbb{R}$, we have an essentially bounded, measurable, holonomy invariant field $\{f(A_x) : x \in M\}$ of bounded operators in $\mathcal{H} = \{L^2(G^x,\nu^x)\}$. Denote by $k_{f(A)} \in \mathcal{D}'(G)$ the distributional kernel of this field.

Proposition 4.1. *Let A be a formally self-adjoint tangentially elliptic differential operator with the positive tangential principal symbol. For any function $f \in S(\mathbb{R},+\infty)$, the tangential kernel $k_{f(A)}$ belongs to $L^1(G)$, and the operator $f(A)$ has the form $f(A) = R_M(k_{f(A)})$. In particular, $f(A)$ belongs to the C^*-algebra $C^*_M(G)$.*

Proof. For a proof, we make use of the leafwise construction parametrix with parameter. Namely, in any foliation chart, the operator A is given by a family $\{A_y : y \in I^q\}$ of elliptic operators on $I^q \subset \mathbb{R}^q$ with the positive principal symbol, and we obtain a (local) leafwise parametrix $P(\lambda)$ for $A-\lambda$, gathering together usual parametrices with parameter for the operators $A_y - \lambda$.

So, for any $\varepsilon > 0$ and $\delta > 0$, we can construct an operator $P(\lambda) \in \Psi^{-m}(\mathcal{F},\delta)$, $\lambda \in \Lambda_\varepsilon$, such that

$$(A-\lambda)P(\lambda) = I - R(\lambda), \quad R(\lambda) \in \Psi^{-\infty}(\mathcal{F}, 2\delta).$$

Moreover, from standard estimates for parametrices with parameter ([19, 16, 20], see also Subsection 3.1), it can be easily seen that the tangential kernels $p(\lambda)$ and $r(\lambda)$ of the operators $P(\lambda)$ and $R(\lambda)$ respectively belong to the space $C^\infty_c(G)$ with the following estimate for their L^1-norms:

$$\|p(\lambda)\|_1 \leq C_1(1+|\lambda|^{1/m})^{-m}, \quad \|r(\lambda)\|_1 \leq C_2(1+|\lambda|^{1/m})^{-m}, \quad \lambda \in \Lambda_\varepsilon, \tag{4.2}$$

with the constants $C_1 > 0$ and $C_2 > 0$, not depending on λ. By (4.2), there exists a constant $R > 0$ such that $1 - r(\lambda)$ is invertible as an element of the Banach algebra $L^1(G)$ for $\lambda \in \Lambda_\varepsilon, |\lambda| > R$. Therefore, for such a λ, the operator $A - \lambda$ is invertible in the space $L^2(M)$, and the inverse operator $(A-\lambda)^{-1}$ is of the form $(A-\lambda)^{-1} = R_M(k(\lambda))$, where $k(\lambda) = p(\lambda) * (1-r(\lambda))^{-1}$ is an element of the algebra $L^1(G)$ with the following estimate for its norm:

$$\|k(\lambda)\|_1 \leq C(1+|\lambda|)^{-1}, \quad \lambda \in \Lambda_\varepsilon, \quad |\lambda| > R, \tag{4.3}$$

where $C > 0$ and $R > 0$ do not depend on λ .

Using the Cauchy integral formula (see (3.1)) and (4.3), we immediately complete the proof of Proposition 4.1. ☐

Since the space $S(\mathbb{R},+\infty)$ is dense in the space $C(\mathbb{R})$ in the uniform norm, Proposition 4.1 immediately implies the following assertion.

Corollary 4.2. *Let A be a formally self-adjoint tangentially elliptic operator of order $m > 0$ with the positive tangential symbol. Then the operator $f(A)$ belongs to $C^*_M(G)$ for any function $f \in C(\mathbb{R})$, and, for any $x \in M$, we have*

$$R_x(\pi(f(A))) = f(A_x).$$

4.2 Spectral coincidence for amenable foliations

Here we will show how the C^*-algebraic calculus implies a result on the spectrum coincidence problem. The approach to the spectrum coincidence problem, based on the theory of C^*-algebras, was used in [2] for C^*-dynamical systems with the group $\mathbb{R}^n$. We refer the reader to [11] and the bibliography there for other approaches to the spectrum coincidence problem.

We say that the foliation $\mathcal{F}$ is *amenable*, if the natural projection $\pi : C^*_M(G) \to C^*_r(G)$ is injective (and, hence, is an isomorphism). Some sufficient conditions for the amenability of the foliation $\mathcal{F}$ can be found in [6].

Proposition 4.3. *Let A be a tangentially elliptic operator of order $m > 0$.*

(1) The spectrum $\sigma_M(A)$ of A in the L^2 space on the ambient manifold M contains its leafwise spectrum

$$\sigma_{\mathcal{F}}(A) = \overline{\bigcup\{\sigma(A_x) : x \in M\}},$$

where $\sigma(A_x)$ is the spectrum of the operator A_x in $L^2(G^x, \nu^x)$.

(2) If the foliated manifold $(M,\mathcal{F})$ is amenable, then $\sigma_M(A)$ coincides with $\sigma_{\mathcal{F}}(A)$.

Proof. It can be esily seen that the operator A is invertible iff the operators A^*A and AA^* are invertible, therefore, we may assume that A is a formally self-adjoint tangentially elliptic operator with the positive tangential principal symbol.

By Corollary 4.2 (see also (4.1)), we have

$$\sup_{x\in M} \|f(A_x)\| \leq \|f(A)\|, \tag{4.4}$$

therefore, the identity $f(A) = 0$ for some function $f \in C(\mathbb{R})$ implies the identity $f(A_x) = 0$ for all x, that completes the proof of the first assertion of Proposition.

If the foliation is amenable, then we have the equality in (4.4), from where the second assertion of Proposition is immediate. □

4.3 Tangentiality of the operator $f(A)$

Let A be a formally self-adjoint tangentially elliptic operator of order m with the positive tangential symbol. Here we will consider conditions on a function $f \in C(\mathbb{R})$, under which the operator $f(A)$ is a tangential operator. At first, let us recall the precise definition of a tangential operator (cf. [16]).

An operator $P : C^\infty(M) \to C^\infty(M)$ is *a tangential operator*, if there is a holonomy invariant family of operators $\{P_x : C^\infty_b(G^x) \to C^\infty_b(G^x)\}$, such that the following diagram commutes:

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{P} & C^\infty(M) \\ {\scriptstyle s_x^*}\downarrow & & {\scriptstyle s_x^*}\downarrow \\ C^\infty_b(G^x) & \xrightarrow{P_x} & C^\infty_b(G^x), \end{array}$$

where the map $s_x^* : C^\infty(M) \to C^\infty_b(G^x)$ is induced by the source map $s : G^x \to M$ ($C^\infty_b(G^x)$ denotes the Frechet space of C^∞-bounded smooth functions on G^x as a manifold of bounded geometry [13]), or, equivalently,

$$P_x s_x^* u = s_x^* P u, \quad u \in C^\infty(M).$$

Remark that, for any $k \in C^\infty_c(G)$, the operator $K = R_M(k)$ is a tangential operator, given by the tangential kernel k, and its restrictions to the leaves are equal to $R_x(k)$.

Theorem 4.4. *Let A be a tangentially elliptic differential operator on M of order m with the positive tangential principal symbol. Then there exist constants $W > 0$ and $c > 0$ such that, for any function f on the real line such that the function $g(t) = f(t^m - c)$, $t \in \mathbb{R}$, belongs to the space $S^q(\mathbb{R}, W)$, the operator $f(A)$ is a tangential operator on M with restrictions to the leaves, being equal to $f(A_x)$.*

Proof. For a proof, we apply Theorem 3.7 and the following theorem, concerning to operators on manifolds of bounded geometry:

Theorem 4.5. *([15]) Let $A \in BD^m(V)$ be a uniformly elliptic differential operator with the positive principal symbol on a manifold of bounded geometry V. Then, there exist constants $W > 0$ and $c > 0$ such that, for any function f on the real line such that the function $g(t) = f(t^m - c)$, $t \in \mathbb{R}$, belongs to the space $S^q(\mathbb{R}, W)$, the operator $f(A)$ defines a continuous mapping*

$$f(A) : C^\infty_b(V) \to C^\infty_b(V).$$

From the proofs of these Theorems, it can be seen that, for any real s, k, there exist constants W and c such that, for any function f, satisfying the hypotheses of Theorem 4.4 with these W and c, there exists a sequence $f_n \in S(\mathbb{R}, +\infty)$ such that $f_n(A)$ converges to $f(A)$ in the uniform topology of $\mathcal{L}(H^{s,k}(M, \mathcal{F}), H^{s,k-q}(M, \mathcal{F}))$ and $f_n(A_x)$ converges to $f(A_x)$ in the uniform topology of $\mathcal{L}(H^l(G^x), H^{l-q}(G^x))$. By Lemma 4.1, we have

$$f_n(A_x) s_x^* u = s_x^* f_n(A) u, \quad u \in C^\infty(M) \tag{4.5}$$

for any n and $x \in M$. Choosing W and c in an appropriate way, we can provide the convergence of both sides of the identity (4.5) in the uniform norm of $C(M)$, that completes the proof of Theorem. □

4.4 Global regularity of tangential kernels

Let A be a formally self-adjoint tangentially elliptic operator with the positive tangential principal symbol, and $\{A_x : x \in M\}$ the corresponding family of uniformly elliptic differential operators along the leaves. Here we state sufficient conditions on a function $f \in B(\mathbb{R})$, under which the tangential kernel $k_{f(A)}$ of the operator field $\{f(A_x) : x \in M\}$ is a continuous (or measurable) function on G.

Theorem 4.6. *Let f be a bounded Borel function on $\mathbb{R}$, satisfying the estimate*

$$|f(\lambda)| \leq C_s(1+|\lambda|)^{-s/m}, \lambda \in \mathbb{R},$$

for some $s > p/2$ with the constant $C_s > 0$, not depending on λ. Then the tangential kernel $k_{f(A)}$ is a bounded, leafwise smooth, measurable function on G, satisfying the estimate

$$\sup_{\gamma \in G} |k_{f(A)}(\gamma)| \leq C \sup_{\lambda \in \mathbb{R}} |(1+|\lambda|)^{s/m} f(\lambda)|,$$

where $C > 0$ does not depend on f.

Proof. The tangential kernel k of a tangential operator K can be obtained as $k(\gamma) = K_x[\delta_\gamma], \gamma \in G^x, x \in M$, where $\delta_\gamma \in \mathcal{D}'(G^x)$ is the delta function at the point $\gamma \in G^x$. Moreover, $\delta_\gamma \in H^{-s}(G^x)$ for any $s > p/2$ with the H^{-s}-norm, uniformly bounded on x. The leafwise Sobolev embedding theorem implies that the tangential kernel k of a tangential operator K, which can be extended to an uniformly bounded field of bounded operators $K_x : H^{-s}(G^x) \to H^s(G^x)$, is a bounded, leafwise smooth function on G, satisfying the estimate

$$\sup_{\gamma \in G^x} |k(\gamma)| \leq C_s \|K_x(\delta_\gamma)\|_s \tag{4.6}$$

for any $s > p/2, x \in M$, with the constant $C_s > 0$, not depending on x. Since the right-hand side of (4.6) can be written as $\|(I + A_x^2)^{s/2m} K_x (I + A_x^2)^{s/2m} u_x\|$ with some uniformly bounded family $u_x \in L^2(G^x, \nu^x)$, any tangential operator K such that, for some $s > p/2$, the family $\{(I + A_x^2)^{s/2m} K_x (I + A_x^2)^{s/2m}\}$ defines an element of the von Neumann algebra $W^*(M, \mathcal{F})$ of the foliation $\mathcal{F}$ (that is, it can be represented as a weak limit of a sequence of tangential operators with tangential kernels from $C_c^\infty(G)$) has the tangential kernel which is a measurable function on G. Finally, if we rewrite the inequality (4.6) in the form

$$\begin{aligned}\sup_{\gamma \in G^x} |k(\gamma)| &\leq C_s \|K_x : H^{-s}(G^x) \to H^s(G^x)\| \\ &\leq C_s \|(I + A_x^2)^{s/2m} K_x (I + A_x^2)^{s/2m} : L^2(G^x, \nu^x) \to L^2(G^x, \nu^x)\|,\end{aligned} \tag{4.7}$$

we will see that a tangential operator K such that the family $\{(I + A_x^2)^{s/2m} K_x (I + A_x^2)^{s/2m}\}$ defines an element of the reduced foliation C^*-algebra $C^*(G)$ for some $s > p/2$, has the tangential kernel, being a continuous, leafwise smooth function on G. Using these facts in the particular case $K = f(A)$ under the current hypotheses on A and f, we immediately complete the proof of Theorem 4.6. □

4.5 Spectrum distribution function for tangentially elliptic operators

Here, using the results of Subsection 4.4, we state the existence and some properties of the spectrum distribution function for any formally self-adjoint tangentially elliptic operator A with the positive tangential principal symbol.

The holonomy invariant measure dx defines the normal semi-finite faithful trace $\mathrm{tr}_{\mathcal{F}}$ on the von Neumann algebra $W^*(M,\mathcal{F})$ of the foliation $\mathcal{F}$. For any tangential operator K with the bounded measurable kernel k, $\mathrm{tr}_{\mathcal{F}}(K)$ is finite and is given by the formula

$$\mathrm{tr}_{\mathcal{F}}(K) = \int_M k(x)dx.$$

For any $\lambda \in \mathbb{R}$ we denote by $E(\lambda)_x$ the spectral projection of the operator A_x, corresponding to the interval $[-\infty, \lambda)$, given by the spectral theorem. Then the family $E(\lambda) = \{E(\lambda)_x : x \in M\}$ defines an element of the algebra $W^*(M,\mathcal{F})$, and we can define *the spectrum distribution function* of the operator A (with values in $[0, +\infty]$) by the formula

$$N(\lambda) = \mathrm{tr}_{\mathcal{F}}(E(\lambda)), \quad \lambda \in \mathbb{R}.$$

By Theorem 4.6, for any $\lambda \in \mathbb{R}$, the tangential kernel $e(\lambda, \gamma)$, $\gamma \in G$, of the operator $E(\lambda)$ is a measurable bounded function on G. Therefore, the function $N(\lambda)$ takes finite values for any $\lambda \in \mathbb{R}$, and we have the following formula:

$$N(\lambda) = \int_M e(\lambda, x)dx, \quad \lambda \in \mathbb{R}.$$

From general properties of the trace $\mathrm{tr}_{\mathcal{F}}$, it can be easily derived the following proposition (see, for instance, [21]):

Proposition 4.7. *The spectrum distribution function $N(\lambda)$ is non-decreasing in λ. The set of points of increase of this function (that is, the set of all $\lambda \in \mathbb{R}$ such that $N(\lambda+\varepsilon) - N(\lambda-\varepsilon) > 0$ for any $\varepsilon > 0$) coincides with the leafwise spectrum $\overline{\bigcup\{\sigma(A_x) : x \in M\}}$ of the operator A.*

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