THE ALGEBRAIC THEORY OF THE FUNDAMENTAL GERM

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Dedicated to Dennis Sullivan on the occasion of his 60th birthday

ABSTRACT. This paper introduces a notion of fundamental group appropriate for laminations.

INTRODUCTION

Let L be a lamination, that is, a space modeled on a "deck of cards" $\mathbb{R}^n \times \mathsf{T}$, where T is a topological space (usually 2^{nd} countable and metrizable), and overlap homeomorphisms take cards to cards continuously in the deck direction T. One thinks of L as a family of manifolds (the leaves, formed from maximal continuations of overlapping cards) bound by a transversal topology prescribed locally by T. Using this picture, many structures and constructions familiar to the theory of manifolds can be extended to laminations via the ansatz:

> Replace manifold object A by a family of manifold objects $\{A_L\}$ existing on the leaves of L and respecting the transverse topology.

For example, one defines a smooth structure to be a family of smooth structures on the leaves in which the card gluing homeomorphisms occurring in a pair of overlapping decks vary transversally in the smooth topology. Continuing in this way, many familiar constructions over $\mathbb{R}$, such as tensors, de Rham cohomology groups, *etc.* may be defined.

Identifying those constructions classically defined over $\mathbb{Z}$ is not as straightforward, especially if we wish to follow tradition and define them geometrically. To see why this is true, let us consider the case of an exceptionally well-behaved lamination: that occurring as an inverse limit

$$\widehat{M} \;=\; \lim_{\longleftarrow} M_\alpha$$

of manifolds by covering maps. Such a system induces a direct limit of de Rham cohomology groups, and there is a canonical map from this limit into $H^*(\widehat{M};\,\mathbb{R})$ with dense image. In fact, here we can use the system to define – by completion of limits – the homology groups $H_*(\widehat{M};\,\mathbb{R})$ as well. If we want to use this point of view to define the groups π_1, $H_*(\cdot\;\mathbb{Z})$, $H^*(\cdot\;\mathbb{Z})$, we immediately run into difficulty since the systems they induce have trivial limits. The purpose of this paper is to introduce for certain classes of weakly-minimal[1] laminations L a construction $[\![\pi]\!]_1(L,x)$ called the fundamental germ, a generalization of π_1 which intends to address this omission in the theory of laminations.

The intuition which guides our construction is that of the lamination as irrational manifold. Recall that for a pointed manifold (M,x), the deck group of the universal cover $(\widetilde{M},\tilde{x}) \to (M,x)$ may be identified with $\pi_1(M,x)$. In particular, $\pi_1(M,x)$ tells us how to make identifications within $(\widetilde{M},\tilde{x})$ so as to recover (M,x) by quotient. We complicate this

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[1] A lamination is weakly-minimal if it has a dense leaf.

picture by imagining that we have disturbed the process of identifying π_1 orbits, so that instead, points in an orbit merely approximate one another through some auxiliary transversal space T. The result is that $(\widetilde{M},\tilde{x})$ does not produce a manifold but rather a coiled-up version of itself: which forms a dense leaf (L,x) of a weakly-minimal lamination L. The germ of the transversal T about x may be interpreted as the failed attempt of (L,x) to form an identification topology at x. The fundamental germ $[\![\pi]\!]_1(L,x)$ is then a device which records algebraically the dynamics of (L,x) as it approaches x through the topology of T. We have indicated the idea in Figure 1.

Naively, one might define an element of $[\![\pi]\!]_1(L,x)$ as a tail equivalence class of a sequence of approaches $\{x_\alpha\}$, where $L \ni x_\alpha \to x$ through T. In this paper, the laminations under consideration have the property that there is a group G acting on L in such a way that every approach is of the form $\{g_\alpha x\}$, for $g_\alpha \in G$. We shall then take $[\![\pi]\!]_1(L,x)$ to consist of tail equivalence classes of sequences of the form[2] $\{g_\alpha h_\alpha^{-1}\}$, where $g_\alpha x,\ h_\alpha x \to x$ in T. The group structure of G is used to define a groupoid structure on $[\![\pi]\!]_1(L,x)$ through component-wise multiplication of sequences: with respect to this structure, $\pi_1(L,x)$ is contained in $[\![\pi]\!]_1(L,x)$ as a subgroup. In practice, $[\![\pi]\!]_1(L,x)$ is only a groupoid; but for many well-behaved laminations such as inverse limit solenoids, Sullivan solenoids and linear foliations of torii, it is a group.

When $L = M$ is a manifold (a lamination with one leaf), $[\![\pi]\!]_1(M,x)$ is equal to ${}^*\pi_1(M,x)$, the non-standard version of $\pi_1(M,x)$: the group of tail equivalence classes of *all* sequences in $\pi_1(M,x)$. When L is a weakly-minimal lamination contained in a manifold M, there is a map $[\![\pi]\!]_1(L,x) \to {}^*\pi_1(M,x)$ whose image consists of those classes of sequences in $\pi_1(M,x)$ that correspond to the holonomy of L. Thus, in expanding π_1 to its non-standard counterpart, it is possible to detect algebraically sublaminations invisible to π_1.

One can profitably think of $[\![\pi]\!]_1(L,x)$ as consisting of equivalence classes of sequences of "G-diophantine approximations". In the case of an irrational foliation F_r of the torus $\mathbb{T}^2$ by lines of slope $r \in \mathbb{R}\setminus\mathbb{Q}$, this is literally true: the elements of $[\![\pi]\!]_1(F_r,x)$ are precisely equivalence classes of convergent sequences of diophantine approximations of r. More generally, in $[\![\pi]\!]_1$ we find an algebraic-topological tool which enables us to systematically translate the geometry of laminations into the analysis of non-linear diophantine approximation.

There is also an analogue of covering space theory for $[\![\pi]\!]_1$. In particular, there is a lamination $[\![\overline{L}]\!]$ which plays the role of a universal cover, on which $[\![\pi]\!]_1(L,x)$ acts with quotient L. Identifying classes of well-behaved subgroupoids C of $[\![\pi]\!]_1(L,x)$, we obtain intermediate quotient laminations L_{C} which are laminated coverings of L. When $[\![\pi]\!]_1(L,x)$ is a group and C is a normal subgroup, the quotient $[\![\pi]\!]_1(L,x)/\mathsf{C}$ may be identified with the deck group of $L_{\mathsf{C}} \to L$. These considerations give rise to a Galois theory of laminations.

For a lamination L which is not necessarily weakly-minimal but has a non-trivial minimal set M (such as a compact foliation, in particular, a foliation of S^3), there is a counterpart of the fundamental germ which describes how a given leaf (L,x) accumulates at a fixed point $\hat{x} \in M$ (rather than how (L,x) accumulates upon itself). The groupoid $[\![\overset{\infty}{\pi}]\!]_1(L;\, x,\hat{x})$ constructed from the associated sequence classes is called the fundamental germ at infinity. We expect that, together with the topological invariants of the leaves, the germs $[\![\pi]\!]_1$ and $[\![\overset{\infty}{\pi}]\!]_1$ will play an important role in the classification of laminations.

[2]Taking sequences of this form has the advantage of guaranteeing a groupoid structure. This would not be true if we used sequences of the form $\{g_\alpha\}$.

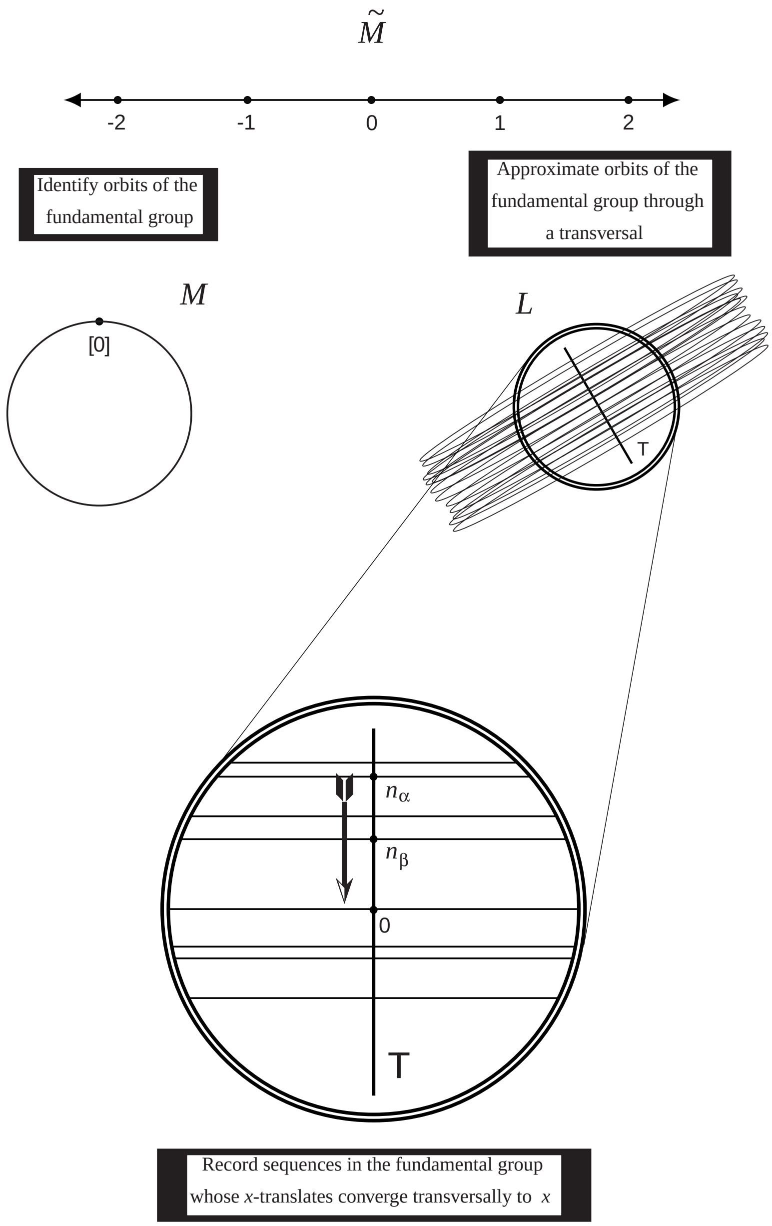

FIGURE 1. The Lamination as Irrational Manifold

We have broken up our study into two parts, each of which will be carried out in a separate paper. In this first installment, *The Algebraic Theory of the Fundamental Germ*, the focus is on laminations which arise through group actions: suspensions, quasi-suspensions [3], double coset foliations and locally-free Lie group actions. In the second installment, *The*

[3] A quasi-suspension is a quotient of a suspension.

Geometric Theory of the Fundamental Germ (to appear later) this construction is studied in the presence of Riemannian geometry along the leaves. There, the fundamental germ is represented using isometries of a fixed dense leaf. In the case of an hyperbolic Riemann surface lamination L, this amounts to an algebraic uniformization theorem [Ge1]: a subgroupoid $[\![\Gamma]\!] \subset PSL(2, {}^*\mathbb{R})$ of non-standard $PSL(2,\mathbb{R})$ acting by isometries on a laminated space $[\![\mathbb{H}]\!]$ with quotient L. In general, the geometric construction will also allow us to extend the definition of $[\![\pi]\!]_1$ to laminations which do not arise algebraically but have nice leaf-wise geometry.

The following contains a summary of the contents of the present paper.

§**1: The Germs of a Group** §1.1 discusses ultraproducts of sequences of sets, and introduces a more general construction called the ultrascope which will give us greater flexibility later on. In §1.2, the ultrapower ${}^*\mathbb{R}$ of $\mathbb{R}$ is considered. Also known as non-standard $\mathbb{R}$, ${}^*\mathbb{R}$ is the ground field of the germ perspective. §1.3 extends a few of the results found in §1.2 to arbitrary non-standard topological groups. In §1.4, given a nested sequence of subsets $G = \{G_i\}$ about 1 in a group G, four *germs* are defined. They are obtained by applying the ultraproduct and the ultrascope construction to each of G and $G \cdot G^{-1}$. The ultrascope of the nested set $G \cdot G^{-1}$ will occupy most of our attention: it is denoted $[\![G]\!]$, and is called simply the germ of G.

§**2: Examples of Germs I** §2.1, §2.2 and §2.3 describe germs arising from, respectively, nested sequences of subgroups, neighborhood bases in a topological group and actions of a group on a space.

§**3: Examples of Germs II** §3.1 and §3.2 describe germs arising from, respectively, double coset topologies and locally free actions of Lie groups. The examples in this section differ from those in §2 in that the group $\mathfrak{G}$ is a topological group, and the nested set G is contained in a lower dimensional "transversal" subspace $T \subset \mathfrak{G}$. Here, we must broaden our understanding of multiplication in $[\![\mathfrak{G}]\!]$ in order to include products which are asymptotically contained in T.

§**4: The Fundamental Germ of an Algebraic Lamination** §4.1 reviews the definition of a lamination and all of the attendant terminology. §4.2 contains the definition of the fundamental germ of a suspension. In §4.3, §4.4 and §4.5, we calculate, respectively, the fundamental germs of the following suspensions: inverse limit solenoids, linear foliations of torii and the Anosov foliation of the unit tangent bundle of a hyperbolic surface. In §4.6, the fundamental germ of a quasi-suspension is defined, and the fundamental germ of a Sullivan solenoid is calculated. In §4.7, given a triple (G, H, Γ) consisting of a Lie group, a closed Lie subgroup and a discrete subgroup, the fundamental germ is defined for the foliation of $\Gamma\backslash\mathfrak{G}$ by the images of cosets $g\mathfrak{H}$. We consider in this section the case of the geodesic and horocyclic flows on the unit tangent bundle of a hyperbolic surface. In §4.8, the definition is given of the the fundamental germ of a lamination arising through a locally-free action of a Lie group on a space. The example of a geodesic lamination in a hyperbolic surface is briefly discussed.

§**5: Non-standard Completions I** In certain cases, it is possible to translate by elements of G the nested set G to obtain a topology on G which has a well-defined completion $\widehat{G}$. The non-standard completion $[\![\overline{G}]\!]$ is a space on which $[\![G]\!]$ acts with quotient $\widehat{G}$. In §5.1, this construction is motivated by the case of $G = \mathbb{Q}$. In §5.2 and §5.3, $[\![\overline{G}]\!]$ is defined when G comes from, respectively, a nested sequence of subgroups and a neighborhood basis about 1 in a topological group. In these cases, the germ $[\![G]\!]$ is a group which acts by homeomorphisms on $[\![\overline{G}]\!]$.

§**6: Non-standard Completions II** This section is a continuation of §5, where in §6.1, §6.2 and §6.3 the cases of G coming from, respectively, an action topology, a double coset topology and a locally free Lie group action are treated. The translates of G need not yield a completable topology: when they do, G is called topologically tame. For topologically tame G, there is a non-standard completion $[\![\overline{G}]\!]$ on which $[\![G]\!]$ acts as a groupoid of homeomorphisms, with quotient a completion $\widehat{G}$ of G (or a completion $\widehat{T}$ of the relevant transversal subspace).

§**7: The Complete Groupoid Structure** In this section it is assumed that $[\![G]\!]$ arises from a topologically tame nested set. The goal here is to expand the groupoid structure of $[\![G]\!]$ to include its action on $[\![\overline{G}]\!]$. In §7.1 this groupoid structure is defined and in §7.2, it is shown to be invariant with respect to the operation of replacing G by a cofinal sub nested set.

§**8: The Fundamental Germ at Infinity** In this section the fundamental germ at infinity of a compact but not necessarily weakly-minimal lamination is defined. In §8.1 the notion of minimal set is recalled. In §8.2, §8.3 and §8.4, we define $[\![\overset{\infty}{\pi}]\!]_1(L;\, x,\hat{x})$ for a L a suspension, a quasi-suspension and a double coset lamination. For L an inverse limit solenoid or a Sullivan solenoid, it is shown that $[\![\overset{\infty}{\pi}]\!]_1(L;\, x,\hat{x}) \cong [\![\pi]\!]_1(L,x)$. In §8.5, $[\![\overset{\infty}{\pi}]\!]_1$ is calculated for the Reeb foliation of S^3 (a foliation which is not weakly-minimal).

§**9: The Higher Order Fundamental Germ** The higher order fundamental germ is obtained by replacing equivalence classes of sequences $\{g_\alpha\}$ by equivalences classes of certain sequences of sets $\{A_\alpha\}$. In §9.1, given G a nested set about 1 in G, the germ of the power set group 2^G is studied. In §9.2, the higher order fundamental germ $\{\pi\}_1(L,x)$ is defined as the subset of the appropriate power set germ consisting of elements represented by *full* sequences $\{A_\alpha\}$ *i.e.* A_α translates x to a dense subset of an open set in the model transversal T. $\{\pi\}_1$ has the advantage of being natural with respect to change of base leaf (this is not a property shared by $[\![\pi]\!]_1$).

§**10: Dependence on Data** To define the fundamental germ $[\![\pi]\!]_1(L,x)$, three types of data are fixed: a point x, a nested set G (which comes from a neighborhood basis T about a transversal T through x), and an ultrafilter $\mathfrak{U}$. In §10.1, it is shown that $[\![\pi]\!]_1(L,x)$ depends, up to isomorphism, only on the germ of T. In §10.2, it is shown that a change of base point *within a fixed leaf* L induces an isomorphism of germs. In §10.3, a change of base point *and* a change of leaf are considered. Here, it is not in general possible to define a transformation map of fundamental germs. However, passing to the higher order germ $\{\pi\}_1$, it is possible to assert the existence of such a map. Finally, in §10.4, assuming the continuum hypothesis, it is shown that $[\![\pi]\!]_1$ is independent of $\mathfrak{U}$.

§**11: Functoriality** In §11.1, the definition of the map $[\![F]\!]$ induced by a lamination map $F: L \to L'$ is given. It is not known at this point if $[\![F]\!]$ is a groupoid homomorphism. In §11.2, we discuss the class of trained lamination maps, which have the property that the induced map $[\![F]\!]$ is homomorphic.

§**12: Germ Covering Space Theory** In §12.1 the germ universal cover $[\![\overline{L}]\!]$ of a weakly minimal lamination L is defined, using the non-standard completion $[\![\overline{G}]\!]$. An action of $[\![\pi]\!]_1(L,x)$ on $[\![\overline{L}]\!]$ is defined, and it is shown that the quotient $[\![\pi]\!]_1(L,x)\backslash[\![\overline{L}]\!]$ may be identified with L. In §12.2 the class of regular subgroupoids C of $[\![\pi]\!]_1(L,x)$ is defined, and it is shown how they may be used to construct laminated coverings spaces L_{C} over L. In §12.3, we show that when $[\![\pi]\!]_1(L,x)$ is a group and C is a normal subgroup, then the quotient $[\![\pi]\!]_1(L,x)/\mathsf{C}$ may be identified with the group of deck homeomorphisms of the covering $L_{\mathsf{C}} \to L$.

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1. The Germs of a Group

Consider a weakly-minimal lamination $\mathcal{L}$ which has the following property: for every pointed dense leaf $(L,x) \in \mathcal{L}$, there is a transversal T through x and a group $G \subset \mathsf{Homeo}_+(L)$ such that

$$\mathsf{T} \cap L \ \subset \ G \cdot x.$$

In this case, the building blocks which we use to define elements of $[\![\pi]\!]_1(\mathcal{L},x)$ consist of equivalence classes of sequences $\{g_\alpha\}$, where $g_\alpha x \in \mathsf{T}$ and $g_\alpha x \to x$ in the topology of T.

In this section we will consider such classes of sequences in the setting of an abstract group G. Specifically, we will study various sets of classes of sequences which converge to $1 \in G$ with respect to some nested sequence of subsets $\{G_i\}$ about 1. These sets are collectively called germs: after giving the definitions of four types of germs, we examine carefully a number of examples which appear later as fundamental germs.

1.1. **Ultraproducts and Ultrascopes.** The study of tail equivalence classes of sequences in families of structures first appeared in model theory [Hew], [Sk], and later played an important role in the invention of non-standard analysis [Ro]. The task of identifying sequences so as to conserve algebraic structure is accomplished through the use of ultrafilters.

Let $\mathbb{N} = \{0,1,2,\dots\}$ denote the natural numbers. An *ultrafilter* on $\mathbb{N}$ is a subset $\mathfrak{U} \subset 2^{\mathbb{N}}$ satisfying the four conditions:

(1) $\emptyset \notin \mathfrak{U}$.
(2) If $X \in \mathfrak{U}$ and $Y \supset X$ then $Y \in \mathfrak{U}$.
(3) If $X, Y \in \mathfrak{U}$ then $X \bigcap Y \in \mathfrak{U}$.
(4) For any $X \in 2^{\mathbb{N}}$, either $X \in \mathfrak{U}$ or $X' = \mathbb{N} \setminus X \in \mathfrak{U}$.

Note 1. The dual $\mathfrak{U}' = \{X \mid X' \in \mathfrak{U}\}$ of an ultrafilter defines a maximal ideal in the Boolean algebra $2^{\mathbb{N}}$, and conversely, the dual of every maximal ideal of $2^{\mathbb{N}}$ defines an ultrafilter. Zorn's lemma is thus required to assert the existence of ultrafilters. The dual $\mathfrak{U}'$ of an ultrafilter is principal if and only if $\mathfrak{U}$ contains a finite set.

Property (3) implies that an ultrafilter is a directed set with respect to set inclusion, and principal ultrafilters do not yield interesting directed sets, so

Assumption: *All ultrafilters considered in this paper are non-principal.*

Let $S = \{S_i\}$ be a sequence of sets, $\mathfrak{U}$ an ultrafilter on $\mathbb{N}$. For each $X \in \mathfrak{U}$, write

$$S_X = \prod_{j \in X} S_j.$$

Since $\mathfrak{U}$ is directed, the $\{S_X\}$ form a direct system with respect to the cartesian projections:

$$S_{X_2} \to S_{X_1} \quad \text{if} \quad X_1 \subset X_2.$$

The *ultraproduct* of the S_i with respect to $\mathfrak{U}$ is the limit

$$\prod_{\mathfrak{U}} S_i \;:=\; \varinjlim S_X.$$

Elements of the ultraproduct are denoted $[x_i]$ or $[x]$ if the indexing is understood. If $S_i = S$ for all i, the ultraproduct is called the *ultrapower* of S, denoted ${}^*S_{\mathfrak{U}}$ or ${}^*S$ if no confusion arises; its elements are denoted ${}^*x$.

If S consists of nested sets, $\prod_{\mathfrak{U}} S_i$ may be regarded as the set of tails of sequences $\{x_i\}$ which converge with respect to S in a lock-step fashion: that is $x_i \in S_i$ for all i. This suggests a more general construction which takes into account *all* convergent sequences. Thus, denote by ${}^\bullet S$ the set of sequences which converge with respect to S: sequences $\{x_\alpha\}$ whose elements are eventually contained in any fixed S_j. For each $X \in \mathfrak{U}$, define a map $P_X : {}^\bullet S \to {}^\bullet S$ by restriction of indices:

$$P_X(\{x_\alpha\}) = \{x_\alpha\}|_{\alpha \in X}.$$

The *ultrascope* is the direct limit

$$\bigodot_{\mathfrak{U}} S_i \;:=\; \varinjlim_{P_X} {}^\bullet S.$$

Elements of the ultrascope are denoted $[\![x_\alpha]\!]$ or $[\![x]\!]$.

Note 2. There is a canonical inclusion

$$\prod_{\mathfrak{U}} S_i \longrightarrow \bigodot_{\mathfrak{U}} S_i.$$

When $S_i = S$ for all i, the ultrascope coincides with the ultrapower.

Proposition 1. *For any nested set S,*

$$\bigodot_{\mathfrak{U}} S_i \;=\; \bigcap {}^*(S_i)_{\mathfrak{U}} \;\supseteq\; {}^*\Big(\bigcap S_i\Big)_{\mathfrak{U}}.$$

The inclusion $\supseteq$ is an equality if and only if for i large, S_i is equal to a fixed set S_{i_0}.

Proof. Every sequence $\{x_\alpha\} \in {}^\bullet S$ is by definition eventually in any S_i hence defines an element of ${}^*(S_i)_{\mathfrak{U}}$. Conversely, any element of $\bigcap {}^*(S_i)_{\mathfrak{U}}$ may be represented by an element of $\{x_\alpha\} \in {}^\bullet S$, hence defines an element of $\bigodot_{\mathfrak{U}} S_i$. If S_i does not stabilize, then a sequence of the form $\{x_\alpha\}$, $x_\alpha \in S_i \setminus S_j$, $i < j$ and $i \to \infty$, does not define an element of ${}^*(\bigcap S_i)_{\mathfrak{U}}$. If $S_i = S_{i_0}$ eventually, then $\bigodot_{\mathfrak{U}} S_i = {}^*(S_{i_0})_{\mathfrak{U}} = {}^*(\bigcap S_i)_{\mathfrak{U}}$. □

Note 3. Given S a nested set, let $N = \{n_0, n_1, \dots\} \subset \mathbb{N}$ be infinite, and let $S' = \{S'_i\}$ be the nested set $S'_i = S_{n_i}$. Then the natural inclusion ${}^\bullet S' \hookrightarrow {}^\bullet S$ yields a bijection

$$\bigodot_{\mathfrak{U}} S'_i \longleftrightarrow \bigodot_{\mathfrak{U}} S_i.$$

Note 4. If S is a (nested) sequence of subgroups or subrings, the induced component-wise operations on the S_X descend to similar operations making the ultraproduct (the ultrascope) a group or ring.

Proposition 2. *If S_i consists of a (nested) sequence of subfields, the ultraproduct (ultrascope) is a field.*

Proof. That the ultraproduct (ultrascope) is a division ring is clear. All that needs to be shown is that there are no zero-divisors. Suppose that for $[x_i], [y_i] \in \prod_{\mathfrak{U}} S_i$, we have $[x_i][y_i] = [0]$. Represent $[x_i], [y_i]$ by sequences $\{x_i\}, \{y_i\}$ such that $x_i y_i = 0$ for all i. Let

$$X = \{i \in \mathbb{N} |\ x_i = 0\} \quad \text{and} \quad Y = \{i \in \mathbb{N} |\ y_i = 0\}.$$

Note that $X' \subset Y$. If $X \in \mathfrak{U}$, then $[x_i] = [0]$. Otherwise, by property (4) in the definition of ultrafilters, $X' \in \mathfrak{U}$, hence so is Y by property (2). In this case $[y_i] = [0]$. The proof for the ultrascope is identical. □

Theorem 1. *Let G and $G = \{G_i\}$ be respectively, a group (or ring or field) and a sequence of groups (or rings or fields), each of cardinality at most the continuum. Let $\mathfrak{U}$, $\mathfrak{U}'$ be two ultrafilters on $\mathbb{N}$. Then assuming the continuum hypothesis,*

$$(1) \qquad {}^*G_{\mathfrak{U}} \;\cong\; {}^*G_{\mathfrak{U}'} \quad \text{and} \quad \prod_{\mathfrak{U}} G_i \;\cong\; \prod_{\mathfrak{U}'} G_i.$$

If G is nested, then in addition

$$\bigodot_{\mathfrak{U}} G_i \;\cong\; \bigodot_{\mathfrak{U}'} G_i.$$

Proof. (1) follows immediately from Corollary 6.1.2 of [Ch-Ke]. If G is nested, then the isomorphism ${}^*(G_0)_{\mathfrak{U}} \to {}^*(G_0)_{\mathfrak{U}'}$ may be chosen so that for every G_i, ${}^*(G_i)_{\mathfrak{U}} \mapsto {}^*(G_i)_{\mathfrak{U}'}$. Indeed, this may be accomplished by adding to the language the names of the sets G_i. Then by Proposition 1, the second statement follows. □

1.2. **Non-standard $\mathbb{R}$.** The field ${}^*\mathbb{R}$ is called *non-standard* $\mathbb{R}$. There is a canonical embedding $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ given by the constant sequences, and we will not distinguish between $\mathbb{R}$ and its image in ${}^*\mathbb{R}$.

For ${}^*x, {}^*y \in {}^*\mathbb{R}$, we say that ${}^*x < {}^*y$ if there exists $X \in \mathfrak{U}$ and representative sequences $\{x_i\}$, $\{y_i\}$ such that $x_i < y_i$ for all $i \in X$. The non-negative nonstandard reals are defined

$${}^*\mathbb{R}_+ \;=\; \{{}^*x \in {}^*\mathbb{R} \mid\ {}^*x \geq 0\}.$$

The Euclidean norm $|\cdot|$ on $\mathbb{R}$ extends to a ${}^*\mathbb{R}_+$-valued norm on ${}^*\mathbb{R}$.

An element ${}^*x$ of ${}^*\mathbb{R}$ is called *infinite* if for all $r \in \mathbb{R}$, $|{}^*x| > r$.

Proposition 3. *${}^*\mathbb{R}$ is a totally-ordered, non-Archimedean field.*

Proof. The quality of being totally-ordered follows from an argument similar to that used to rule out zero divisors in Proposition 2. If ${}^*x$ is infinite, then for any $r \in \mathbb{R}$ we have $|nr| < |{}^*x|$ for all $n \in \mathbb{N}$: hence ${}^*\mathbb{R}$ is non-Archimedian. □

There are two obvious topologies that we may give ${}^*\mathbb{R}$:

- The *enlargement topology* ${}^*\tau$, generated by sets of the form ${}^*A$, where $A \subset \mathbb{R}$ is open. This topology is convenient when regarding $\mathbb{R}$ as a quotient of a subring of ${}^*\mathbb{R}$ (see Proposition 7 below).
- The *internal topology* $[\tau]$, generated by sets of the form
$$[A_i] \;=\; \prod_{\mathfrak{U}} A_i,$$
where $A_i \subset \mathbb{R}$ is open for all i. This topology is most natural when regarding ${}^*\mathbb{R}$ as a space in its own right.

We have ${}^*\tau \subset [\tau]$, the inclusion being strict.

Note 5. Given ${}^*x < {}^*y$, let
$$({}^*x, {}^*y) \;=\; \{{}^*z \mid {}^*x < {}^*z < {}^*y\}.$$
Then if $\{x_i\}$, $\{y_i\}$ represent ${}^*x$, ${}^*y$, we have
$$({}^*x, {}^*y) \;=\; \prod_{\mathfrak{U}} (x_i, y_i).$$
It follows that $[\tau]$ is the order topology.

Proposition 4. *The enlargement topology* ${}^*\tau$ *is* 2^{nd}*-countable but non Hausdorff. The internal topology* $[\tau]$ *is Hausdorff but not* 2^{nd}*-countable.*

Proof. ${}^*\tau$ is not Hausdorff since it is impossible to separate points ${}^*x$ and ${}^*y$ represented by asymptotic sequences. It is 2^{nd}-countable since it is generated by ${}^*A$, where the A run over a countable basis of $\mathbb{R}$.

Given ${}^*x$, ${}^*y \in {}^*\mathbb{R}$ represented by sequences $\{x_i\}$, $\{y_i\}$, let $A_i, B_i \subset \mathbb{R}$ be disjoint opens containing x_i, y_i respectively. Then $[A_i]$, $[B_i]$ are disjoint $[\tau]$-opens containing ${}^*x$ and ${}^*y$ respectively. Thus $[\tau]$ is Hausdorff.

Note that there exists an uncountable set $\mathcal{A} \subset {}^*\mathbb{R}$ such that given ${}^*x$, ${}^*y \in \mathcal{A}$, the distance $|x_i - y_i|$ between elements of representative sequences $\to \infty$. We may then find $[\tau]$-opens about every ${}^*x \in \mathcal{A}$ which are pair-wise disjoint. Thus $[\tau]$ is not 2^{nd}-countable. □

Proposition 5. $({}^*\mathbb{R}, [\tau])$ *is a real, infinite dimensional topological vector space.*

Proof. It is clear that scalar multiplication by $\mathbb{R}$ is a continuous operation with respect to $[\tau]$. Given ${}^*x \in {}^*\mathbb{R}$ represented by $\{x_i\}$, we have ${}^*x + [A_i] = [x_i + A_i] \in [\tau]$ for all $[A_i] \in [\tau]$. It follows that addition is $[\tau]$-continuous. Since $[\tau]$ is not second countable, ${}^*\mathbb{R}$ must be infinite dimensional. □

Note 6. ${}^*\mathbb{R}$ is not a topological group with respect to ${}^*\tau$. Every open set ${}^*A$ contains bounded points (*i.e.* elements represented by bounded sequences). If ${}^*x > 0$ is unbounded and ${}^*A > 0$, then ${}^*x + {}^*A$ contains only unbounded elements.

Let
$${}^*\mathbb{R}_{\text{fin}} \;=\; \{{}^*x \in {}^*\mathbb{R} \mid \exists\, M \in \mathbb{R} \text{ such that } |{}^*x| < M\}.$$

Proposition 6. ${}^*\mathbb{R}_{\text{fin}}$ *is a topological subring of* ${}^*\mathbb{R}$ *with respect to both the* ${}^*\tau$ *and* $[\tau]$ *topologies.*

Proof. That ${}^*\mathbb{R}_{\text{fin}}$ is a $[\tau]$-topological subring of ${}^*\mathbb{R}$ is clear. Given ${}^*x \in {}^*\mathbb{R}_{\text{fin}}$, let

$$M_{{}^*x} \;=\; \inf_{{}^*x<M} M.$$

Then for every enlargement open ${}^*A$ we have

$${}^*x + {}^*A \;=\; {}^*(M_{{}^*x} + A) \quad\text{and}\quad {}^*x \cdot {}^*A \;=\; {}^*(M_{{}^*x} \cdot A).$$

□

The additive subgroup of *infinitesimals* of ${}^*\mathbb{R}$ is defined

$${}^*\mathbb{R}_\varepsilon \;\;=\;\; \{{}^*\varepsilon \mid \; |{}^*\varepsilon\,| < M \text{ for all } M \in \mathbb{R}_+\}.$$

For ${}^*x \in {}^*\mathbb{R}$, the coset

$$\mu({}^*x) \;=\; {}^*x + {}^*\mathbb{R}_\varepsilon$$

is called the *galaxy* of ${}^*x$. If ${}^*y \in \mu({}^*x)$, we write

$${}^*x \;\simeq\; {}^*y$$

and say that ${}^*x$ is *infinitesimal* to ${}^*y$.

Proposition 7. *${}^*\mathbb{R}_{\text{fin}}$ is a local ring with maximal ideal ${}^*\mathbb{R}_\varepsilon$. The quotient ${}^*\mathbb{R}_{\text{fin}}/{}^*\mathbb{R}_\varepsilon$ is isomorphic to $\mathbb{R}$, homeomorphic with respect to the quotient ${}^*\tau$-topology.*

Proof. Given ${}^*x \in {}^*\mathbb{R}_{\text{fin}}$ and ${}^*\varepsilon \in {}^*\mathbb{R}_\varepsilon$, $|{}^*x \cdot {}^*\varepsilon| < M$ for any positive real M, so ${}^*\mathbb{R}_\varepsilon$ is an ideal. If ${}^*\mathbb{R}_\varepsilon$ is contained in an ideal $\mathfrak{m}$, then there exists $r \in \mathfrak{m}$, with $0 \neq r \in \mathbb{R}$. Then $\mathbb{R} \subset \mathfrak{m}$ and since ${}^*\mathbb{R}_\varepsilon \subset \mathfrak{m}$, $\mathfrak{m} = {}^*\mathbb{R}_{\text{fin}}$. Thus ${}^*\mathbb{R}_\varepsilon$ is maximal.

On the other hand, let $\mathfrak{a} \subset {}^*\mathbb{R}_{\text{fin}}$ be a non zero ideal not contained in ${}^*\mathbb{R}_\varepsilon$, and let ${}^*r \in \mathfrak{a} \setminus (\mathfrak{a} \cap {}^*\mathbb{R}_\varepsilon)$. Then for any ${}^*\varepsilon \in {}^*\mathbb{R}_\varepsilon$, the equation

$${}^*x \cdot {}^*r = {}^*\varepsilon$$

has a solution in ${}^*\mathbb{R}_{\text{fin}}$. Thus ${}^*\mathbb{R}_\varepsilon \subset \mathfrak{a}$ and by maximality of ${}^*\mathbb{R}_\varepsilon$, ${}^*\mathbb{R}_\varepsilon = \mathfrak{a}$. It follows that ${}^*\mathbb{R}_{\text{fin}}$ is local.

Given ${}^*x \in {}^*\mathbb{R}_{\text{fin}}$, let $M_{{}^*x}$ be as in the proof of Proposition 6. The map ${}^*x \mapsto M_{{}^*x}$ descends to an isomorphism ${}^*\mathbb{R}_{\text{fin}}/{}^*\mathbb{R}_\varepsilon \to \mathbb{R}$ which takes every quotient class of ${}^*\tau$-open ${}^*A$ to its counterpart $A \subset \mathbb{R}$. □

Note 7. ${}^*\mathbb{R}_\varepsilon$ is clopen in the $[\tau]$-topology; the $[\tau]$-quotient topology on ${}^*\mathbb{R}_{\text{fin}}/{}^*\mathbb{R}_\varepsilon$ is discrete.

Note 8. ${}^*\mathbb{R}_\varepsilon$ is not an ideal in ${}^*\mathbb{R}$. The abelian group ${}^\star\mathbb{R} := {}^*\mathbb{R}/{}^*\mathbb{R}_\varepsilon$ (with the quotient ${}^*\tau$-topology) is called the *extended reals*. By Proposition 7, ${}^\star\mathbb{R}$ contains a subfield isomorphic to $\mathbb{R}$.

1.3. **Non-standard Topological Groups.** Let $\mathfrak{G}$ be a complete topological group. Some of the properties satisfied by ${}^*\mathbb{R}$ also hold for ${}^*\mathfrak{G}$: in this section, we briefly discuss those that shall need later.

If τ denotes the topology of $\mathfrak{G}$, then the topologies ${}^*\tau$ and $[\tau]$ are defined exactly as in §1.2. ${}^*\mathfrak{G}$ is a topological group in the $[\tau]$ topology, but not in the ${}^*\tau$ topology.

Denote by ${}^*\mathfrak{G}_\varepsilon$ the classes of sequences converging to the unit element 1.

Note 9. ${}^*\mathfrak{G}_\varepsilon$ is a group since a product of sequences converging to 1 in a topological group is again a sequence converging to 1.

Let ${}^*\mathfrak{G}_{\text{fin}}$ be the subset of ${}^*\mathfrak{G}$ all of whose elements are represented by sequences which converge to an element of $\mathfrak{G}$. We have the following analogue of Proposition 7, whose proof we leave to the reader.

Proposition 8. *$^*\mathfrak{G}_\varepsilon$ is a normal subgroup of $^*\mathfrak{G}_{\text{fin}}$ and $^*\mathfrak{G}_{\text{fin}}/^*\mathfrak{G}_\varepsilon$ is isomorphic to $\mathfrak{G}$, homeomorphic with respect to the quotient $^*\tau$-topology.*

We call the left coset space $^\star\mathfrak{G} := {}^*\mathfrak{G}/{}^*\mathfrak{G}_\varepsilon$ (with the quotient $^*\tau$-topology) the *extended* $\mathfrak{G}$. It contains $\mathfrak{G}$ as a topological subgroup.

Note 10. If $\mathfrak{G}$ is compact or abelian, then $^\star\mathfrak{G}$ is a group.

1.4. **The Germs of a Group.** Given G an arbitrary group, let $G = \{G_i\}$ be a nested sequence of subsets about 1. The ultraproduct

$$[G]^{\mathsf{pre}}_G \;:=\; \prod_{\mathfrak{U}} G_i$$

is called the *lock-step pregerm* of G with respect to G; the ultrascope

$$\llbracket G\rrbracket^{\mathsf{pre}}_G \;:=\; \bigodot_{\mathfrak{U}} G_i$$

is called the *pregerm*.

The collection $G\cdot G^{-1} = \{G_i\cdot G_i^{-1}\}$ is also a nested set about 1. The ultraproduct

$$[G]_G \;:=\; [G]^{\mathsf{pre}}_{G\cdot G^{-1}}$$

is called the *lock-step germ* of G with respect to G; the ultrascope

$$\llbracket G\rrbracket_G \;:=\; \llbracket G\rrbracket^{\mathsf{pre}}_{G\cdot G^{-1}}$$

is called the *germ*.

We will often omit the G-subscript in the interest of clear notation, though it is obvious that these constructions depend heavily on our choice of nested set.

Note 11. The pregerms enjoy no special algebraic structure. The point of using the associated nested set $G\cdot G^{-1}$ is that both of $[G]$ and $\llbracket G\rrbracket$ are groupoids, since by definition all of their elements are invertible. Moreover, the associated nested set pays additional dividends: sometimes producing nested groups from nested sets which are not groups (*c.f.* Example 3 below).

The relationship between the four constructions is given by the commutative diagram:

$$\begin{array}{ccc} [G]^{\mathsf{pre}} & \subset & [G] \\ \cap & & \cap \\ \llbracket G\rrbracket^{\mathsf{pre}} & \subset & \llbracket G\rrbracket \end{array},$$

where the horizontal inclusions are induced by $G_i = G_i\cdot 1 \subset G_i\cdot G_i^{-1}$.

2. Examples of Germs I

2.1. **Nested Subgroups.** The following family of examples have the property that either the G_i or the $G_iG_i^{-1}$ are groups.

Germ Example 1. $G_i = G$ for all i.

Here $[G]^{\mathsf{pre}} = [G] = \llbracket G\rrbracket^{\mathsf{pre}} = \llbracket G\rrbracket = {}^*G$, which is a group.

Germ Example 2. $G_i = H_i$ is a normal subgroup of G for all i.

Proposition 9. *$[G]^{\mathsf{pre}} = [G]$ and $\llbracket G\rrbracket^{\mathsf{pre}} = \llbracket G\rrbracket$ are groups.*

Proof. This follows immediately from the fact that $H_iH_i^{-1} = H_i$. □

If G is a sequence of finite-index normal subgroups cofinal in the family of all finite-index normal subgroups (of all p^n-index subgroups, p a prime, $n \in \mathbb{N}$) , we obtain the *pro-finite germs*, denoted $[G]_{\hat{\varepsilon}}$ and $[\![G]\!]_{\hat{\varepsilon}}$ ($[G]_{\hat{\varepsilon},p}$ and $[\![G]\!]_{\hat{\varepsilon},p}$).

Note 12. Consider $G = \mathbb{Z}$. Since every subgroup of $\mathbb{Z}$ is an ideal, the germs $[\mathbb{Z}]_{\hat{\varepsilon}} \subset [\![\mathbb{Z}]\!]_{\hat{\varepsilon}} \subset {}^*\mathbb{Z}$ are ideals, being respectively an ultraproduct and an ultrascope of ideals. However, they are not principal ideals (principal ideals are countable). Thus, unlike $\mathbb{Z}$, ${}^*\mathbb{Z}$ is not a principal ideal domain.

Germ Example 3. For $d \in \mathbb{N}$, let

$$G_{\mathsf{BS}} = G_{\mathsf{BS}}(d) = \langle f, x : \; fxf^{-1} = x^d \rangle$$

be the Baumslag-Solitar group, and take

$$G_i = \left\{ f^m x^{rd^i} \;\middle|\; m, r \in \mathbb{Z} \right\}.$$

Theorem 2. *$[G_{\mathsf{BS}}]^{\mathsf{pre}}$ and $[\![G_{\mathsf{BS}}]\!]^{\mathsf{pre}}$ are not groupoids. $[G_{\mathsf{BS}}]$ and $[\![G_{\mathsf{BS}}]\!]$ are groups.*

Proof. We first observe by induction that in G_{BS},

$$x^{-d^i} f = f x^{-d^{i-1}} \tag{2}$$

for all $i > 0$. Now consider the sequence

$$\{g_i\} = \{ f^{-m_i} x^{d^i} \},$$

where $m_i > i > 0$. Note that $\{g_i\}$ defines an element in both $[G_{\mathsf{BS}}]^{\mathsf{pre}}$ and $[\![G_{\mathsf{BS}}]\!]^{\mathsf{pre}}$. Using (2), we may write the inverse sequence

$$\{g_i^{-1}\} = \{x^{-d^i} f^{m_i}\} = \{ f^i x^{-1} f^{m_i - i} \}.$$

Since $m_i > i$, we cannot use the defining relation of G_{BS} to move the remaining $f^{m_i - i}$ to the left of the x-term. It follows that $\{g_i^{-1}\}$ defines neither an element of $[G_{\mathsf{BS}}]^{\mathsf{pre}}$ nor of $[\![G_{\mathsf{BS}}]\!]^{\mathsf{pre}}$. In particular, neither $[G_{\mathsf{BS}}]^{\mathsf{pre}}$ nor $[\![G_{\mathsf{BS}}]\!]^{\mathsf{pre}}$ have the structure of a groupoid.

To see that $[G_{\mathsf{BS}}]$ and $[\![G_{\mathsf{BS}}]\!]$ are groups, it suffices to see that $G_i \cdot G_i^{-1}$ is a group for all i. Write a generic element $g \in G_i \cdot G_i^{-1}$ in the form

$$g = f^l x^{rd^i} f^m$$

for $l, m, r \in \mathbb{Z}$. Then an element gh^{-1}, $g, h \in G_i \cdot G_i^{-1}$ may be written

$$gh^{-1} = f^l x^{rd^i} f^m x^{sd^i} f^n = \begin{cases} f^l x^{(r+sd^m)d^i} f^{m+n} & \text{if } m > 0 \\ \\ f^{l+m} x^{(rd^m + s)d^i} f^n & \text{if } m \leq 0 \end{cases},$$

where $l, m, n, r, s \in \mathbb{Z}$. It follows that $gh^{-1} \in G_i \cdot G_i^{-1}$. □

2.2. **(Representations in) Topological Groups.** In the next family of examples, the G_i arise from a neighborhood basis about 1 of a topological group. In this case we may not assume that G_i or $G_i \cdot G_i^{-1}$ are groups.

Germ Example 4. $G = \mathfrak{G}$ a topological group, $G_i = U_i$ is a neighborhood basis about 1.

Note 13. By replacing G_i by $G_i \bigcup G_i^{-1}$ we get a basis of groupoids: in this case we can then assert at least that $[\mathfrak{G}]^{\mathsf{pre}}$ and $[\mathfrak{G}]$ are groupoids.

Proposition 10. *$[\![\mathfrak{G}]\!]^{\mathsf{pre}} = [\![\mathfrak{G}]\!] = {}^*\mathfrak{G}_{\varepsilon}$ is a group. If G_i is a group for all i, then $[\mathfrak{G}]^{\mathsf{pre}} = [\mathfrak{G}]$ are groups. If $G_i \cdot G_i^{-1}$ is a group for all i, then $[\mathfrak{G}]$ is a group.*

Proof. If $\{g_i\}$, $\{h_i\}$ are sequences converging to 1, then so is $\{g_i h_i^{-1}\}$. Passing to direct limits gives the first statement. The second and third statements are obvious. □

Germ Example 5. Let $\rho : G \to \mathfrak{G}$ be a representation into a topological group $\mathfrak{G}$ with U_i a neighborhood basis about 1. G is defined:

$$G_i = \{g \mid \rho(g) \in U_i\}.$$

The discussion here is the pull-back of that of *Germ Example* 4. In particular, we have the analogue of Proposition 10:

Proposition 11. *Let G be defined as in* Germ Example *5. Then $[\![G]\!]^{\mathsf{pre}} = [\![G]\!]$ is a group.*

An important instance of *Germ Example* 5 comes when $G = \mathbb{Z}^p$ and $\mathfrak{G}$ = the q-torus $\mathbb{T}^q = \mathbb{R}^q/\mathbb{Z}^q$.

Germ Example 6. Fix $\mathbf{r}_1, \dots, \mathbf{r}_p \in \mathbb{R}^q$ a set of $\mathbb{Q}$-independent column vectors. Let $\mathbf{R}$ be the $q \times p$ matrix whose kth column is $\mathbf{r}_k$. Define $\rho : \mathbb{Z}^p \to \mathbb{T}^q$ by $\rho(\mathbf{n}) = \overline{\mathbf{Rn}}$, where $\overline{\mathbf{m}}$ means the image of $\mathbf{m}$ in $\mathbb{T}^q$. Then G is defined

$$G_i = \left\{\mathbf{n} \in \mathbb{Z}^p \mid d(\rho(\mathbf{n}), \overline{\mathbf{0}}) < 1/i \right\},$$

where d is the Euclidean distance function on $\mathbb{T}^q$.

Denote by ${}^*\mathbb{Z}^p_{\mathbf{R}}$ the corresponding germ: by Proposition 11, it is a subgroup of ${}^*\mathbb{Z}^p$. When $p = q = 1$, then $\mathbf{R} = r \in \mathbb{R}$ and we write ${}^*\mathbb{Z}_r$.

Theorem 3. *${}^*\mathbb{Z}^p_{\boldsymbol{R}}$ is an ideal if and only if $\boldsymbol{R} \in M_{q,p}(\mathbb{Q})$.*

Proof. Suppose that $\mathbf{R} \in M_{q,p}(\mathbb{Q})$ and let a_k = the l.c.d. of the entries of $\mathbf{r}_k$. Write

$$\mathfrak{a} = (a_1) \oplus \cdots \oplus (a_p)$$

where (a_k) is the ideal generated by a_k. Note that ${}^*\mathfrak{a} \subset {}^*\mathbb{Z}^p_{\mathbf{R}}$. On the other hand, rationality of the entries of the $\mathbf{r}_k$ implies that a sequence $\{\mathbf{n}_i\} \subset \mathbb{Z}^p$ defines an element of ${}^*\mathbb{Z}^p_{\mathbf{R}}$ if and only if there exists $X \in \mathfrak{U}$ such that $\rho(\mathbf{n}_i) = \overline{\mathbf{0}}$ for all $i \in X$. This is equivalent to $\mathbf{n}_i \in \mathfrak{a}$ for all $i \in X$. Thus ${}^*\mathbb{Z}^p_{\mathbf{R}} = {}^*\mathfrak{a}$ which is an ideal in ${}^*\mathbb{Z}^p$.

Suppose now that $\mathbf{r} = \mathbf{r}_k \notin \mathbb{Q}^q$ for some k, $1 \geq k \leq p$. Let $\{\mathbf{n}_i\}$ represent an element of ${}^*\mathbb{Z}_{\mathbf{R}}$, and denote by $\{n_i\}$ the sequence of k-th coordinates of the $\mathbf{n}_i$. Note that $\overline{n_i\mathbf{r}} \neq \overline{\mathbf{0}}$ for all i since $\mathbf{r}$ is not rational. In fact, for any j_0 we may find a sequence of integers $\{m_i\}$ such that $\overline{m_i n_i \mathbf{r}}$ is *not* within $1/j_0$ of $\overline{\mathbf{0}}$. Let $\mathbf{m}_i \in \mathbb{Z}^p$ be the vector whose kth coordinate is m_i and whose other coordinates are 0. Then the sequence $\{\mathbf{m}_i \cdot \mathbf{n}_i\}$ does not converge with respect to G. It follows that ${}^*\mathbb{Z}^p_{\mathbf{R}}$ is not an ideal in ${}^*\mathbb{Z}^p$. □

Note 14. Theorem 3 draws another sharp distinction between $\mathbb{Z}$ and ${}^*\mathbb{Z}$: every subgroup of the former is an ideal, while this is false for the latter.

The group ${}^*\mathbb{Z}^p_{\mathbf{R}}$ has the following alternate description:

$$(3)\qquad {}^*\mathbb{Z}^p_{\mathbf{R}} = \left\{ {}^*\mathbf{n} \in {}^*\mathbb{Z}^p \;\middle|\; \exists\, {}^*\mathbf{n}^{\perp} \in {}^*\mathbb{Z}^q \text{ such that } \mathbf{R}({}^*\mathbf{n}) - {}^*\mathbf{n}^{\perp} \in {}^*\mathbb{R}^q_{\varepsilon} \right\}.$$

Given ${}^*\mathbf{n} \in {}^*\mathbb{Z}^p_{\mathbf{R}}$, the corresponding element ${}^*\mathbf{n}^{\perp} \in {}^*\mathbb{Z}^q$ is called the *dual* of ${}^*\mathbf{n}$; it is uniquely determined. From (3), it is clear that the set

$$({}^*\mathbb{Z}^p_{\mathbf{R}})^{\perp} = \left\{ {}^*\mathbf{n}^{\perp} \;\middle|\; {}^*\mathbf{n}^{\perp} \text{ is the dual of } {}^*\mathbf{n} \in {}^*\mathbb{Z}^p_{\mathbf{R}} \right\}$$

is a subgroup of ${}^*\mathbb{Z}^q$, called the *dual* of ${}^*\mathbb{Z}^p_{\mathbf{R}}$.

Note 15. When $\mathbf{R} \in M_{q,p}(\mathbb{R}\setminus\mathbb{Q})$ has a left-inverse $\mathbf{S}$, we have $({}^*\mathbb{Z}^p_{\mathbf{R}})^{\perp} = {}^*\mathbb{Z}^q_{\mathbf{S}}$.

Similarly, the set

$$ {}^*\mathbb{R}^q_{\mathbf{R},\varepsilon} = \left\{ {}^*\varepsilon \in {}^*\mathbb{R}^q_{\varepsilon} \;\middle|\; \exists {}^*\mathbf{n} \in {}^*\mathbb{Z}^p_{\mathbf{R}} \text{ such that } \mathbf{R}({}^*\mathbf{n}) - {}^*\mathbf{n}^{\perp} = {}^*\varepsilon \right\} $$

is a subgroup of ${}^*\mathbb{R}^q_{\varepsilon}$, called the *group of rates* of $\mathbf{R}$.

The following proposition is an immediate consequence of (3).

Proposition 12. *The maps ${}^*\boldsymbol{n} \mapsto {}^*\boldsymbol{n}^{\perp}$ and ${}^*\boldsymbol{n} \mapsto {}^*\varepsilon$ define isomorphisms*

$$ {}^*\mathbb{Z}^p_{\boldsymbol{R}} \;\cong\; ({}^*\mathbb{Z}^p_{\boldsymbol{R}})^{\perp} \qquad \textit{and} \qquad {}^*\mathbb{Z}^p_{\boldsymbol{R}} \;\cong\; {}^*\mathbb{R}^q_{\boldsymbol{R},\varepsilon}. $$

Note 16 (A.Verjovsky). Using formulation (3) of ${}^*\mathbb{Z}^p_{\mathbf{R}}$, it follows that every triple

$$ ({}^*\mathbf{n}, {}^*\mathbf{n}^{\perp}, {}^*\varepsilon) $$

represents a convergent sequence of diophantine approximations of $\mathbf{R}$. Thus we may regard ${}^*\mathbb{Z}^p_{\mathbf{R}}$ as the *group of diophantine approximations* of $\mathbf{R}$.

For example, when $m = n = 1$ and $r \in \mathbb{R}\setminus\mathbb{Q}$, ${}^*n$ and ${}^*n^{\perp}$ are equivalence classes of sequences $\{x_i\}$ and $\{y_i\} \subset \mathbb{Z}$, and ${}^*\varepsilon$ an equivalence class of sequence $\{\varepsilon_i\} \subset \mathbb{R}$, $\varepsilon_i \to 0$, such that

$$ \left| r - \frac{y_i}{x_i} \right| \;=\; \left| \frac{\varepsilon_i}{x_i} \right| \longrightarrow 0. $$

Conversely, every convergent sequence of diophantine approximations of r defines uniquely a triple $({}^*n, {}^*n^{\perp}, {}^*\varepsilon)$.

Recall that two irrational numbers $r, s \in \mathbb{R}\setminus\mathbb{Q}$ are equivalent if there exists $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ such that $s = (ar+b)/(cr+d)$.

Proposition 13. *If r and s are equivalent irrational numbers, then ${}^*\mathbb{Z}_r \cong {}^*\mathbb{Z}_s$.*

Proof. Given ${}^*n \in {}^*\mathbb{Z}_r$,

$$ (cr+d){}^*n \;\simeq\; -c{}^*n^{\perp} + d{}^*n \;\in\; {}^*\mathbb{Z}. $$

Write ${}^*m = -c{}^*n^{\perp} + d{}^*n$. Then ${}^*m \in {}^*\mathbb{Z}_s$, since

$$ s{}^*m \;\simeq\; (ar+b){}^*n \;\simeq\; -a{}^*n^{\perp} + b{}^*n \;\in\; {}^*\mathbb{Z}. $$

The association ${}^*n \mapsto {}^*m$ defines an injective homomorphism

$$ \psi : {}^*\mathbb{Z}_r \to {}^*\mathbb{Z}_s, $$

with inverse defined

$$ \psi^{-1}({}^*m) \;\simeq\; (-cs+a){}^*m. $$

□

Note 17. Two irrational numbers r, s are called *virtually equivalent* if there exists $A \in SL(2,\mathbb{Q})$ (or equivalently, $A \in GL(2,\mathbb{Z})$) such that $A(r) = s$. In this case, there exists a pair of injective homomorphisms

$$ \psi_1 : {}^*\mathbb{Z}_r \hookrightarrow {}^*\mathbb{Z}_s \quad \text{and} \quad \psi_2 : {}^*\mathbb{Z}_s \hookrightarrow {}^*\mathbb{Z}_r, $$

defined as in Proposition 13. These maps are mutually inverse to each other if and only if $A \in SL(2,\mathbb{Z})$. In other words, ${}^*\mathbb{Z}_r$ and ${}^*\mathbb{Z}_s$ are *virtually isomorphic*.

We are led to make the following conjecture.

Conjecture 1. *If ${}^*\mathbb{Z}_r \cong {}^*\mathbb{Z}_s$ for irrational numbers r, s, then r and s are equivalent.*

2.3. **Action Topologies.** In the next example, the nested set G comes to us through an action of G on a topological space.

Let G be a group, F a (2nd countable metrizable) topological space. Let $\rho : G \to \mathsf{Homeo}(F)$ be a representation. We say that $t \in F$ is *minimal* if the orbit $\rho(G)(t)$ is dense in F.

Germ Example 7. Fix a minimal $t \in F$ and a neighborhood basis $\{U_i\}$ about t. G is defined

$$G_i = \{g \in G \mid \rho_g(t) \in U_i\}.$$

We denote the corresponding pregerm and germ by $[\![G]\!]_t^{\mathsf{pre}}$ and $[\![G]\!]_t$, respectively. In this example, we are using the topology not of the topological group $\mathsf{Homeo}(F)$ but of its local action on F at t. Since this is generally a finer topology, we cannot expect to reap the strong algebraic qualities of *Germ Examples* 5 and 6.

Note 18. One can replace F by a a smooth manifold M, a Riemannian manifold (M,γ), an algebraic variety V, a topological group $\mathfrak{G}$, a vector space V, *etc.*, and consider representations in $\mathsf{Diff}(M)$, $\mathsf{Isom}(M,\gamma)$, $\mathsf{Rat}(V)$, $\mathsf{Iso}(\mathfrak{G})$, $GL(V)$ and so forth.

Note 19. We observe that *Germ Examples* 2 and 6 may be put in this form. In *Germ Example* 2, the canonical map $\iota : G \to \widehat{G}$ yields the right-multiplication representation $\rho : G \to Homeo(\widehat{G})$,

$$\rho_g(\hat{g}) = \hat{g} \cdot \iota(g)^{-1}.$$

The germs so induced agree with those constructed in *Germ Example* 2 since the nested set G induced by the action topology is the same as that induced by ι. The same is true of *Germ Example* 6, where ρ has image in the subgroup $\mathbb{T}^q < \mathsf{Homeo}(\mathbb{T}^q)$.

Notation 1. From here on, we write

$$g(t) = \rho_g(t).$$

The following sub-case of *Germ Example* 7 corresponds to a foliation defined on the unit tangent bundle of a hyperbolic surface.

Germ Example 8. Let $G = \Gamma$ be a Fuchsian group, $\rho : \Gamma \to Homeo(S^1)$ the representation defined by extending the action of Γ on $\mathbb{H}$ to the boundary.

Let us consider the modular group $\Gamma = PSL(2,\mathbb{Z})$. Let $t = 0$; its orbit is dense in S^1. Notice then that for every element in $PSL(2,\mathbb{Z})$ represented by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

we have $A(0) = b/d$.

Lemma 1. *If* ${}^*A = \begin{pmatrix} {}^*a & {}^*b \\ {}^*c & {}^*d \end{pmatrix} \in [\![PSL(2,\mathbb{Z})]\!]_0^{\mathsf{pre}}$, *then*

(1) ${}^*b/{}^*d \in {}^*\mathbb{R}_\varepsilon$.
(2) ${}^*c$, ${}^*d \in {}^*\mathbb{Z} \setminus \mathbb{Z}$.

Proof. By definition of G, we have ${}^*A(0) = {}^*b/{}^*d \in {}^*\mathbb{R}_\varepsilon$ for any ${}^*A \in [\![PSL(2,\mathbb{Z})]\!]_0^{\mathsf{pre}}$.

Since $|{}^*b| \geq 1$, (1) implies that ${}^*d \in {}^*\mathbb{Z} \setminus \mathbb{Z}$. On the other hand, since ${}^*a{}^*d - {}^*b{}^*c = 1$, it follows that ${}^*c \in {}^*\mathbb{Z} \setminus \mathbb{Z}$ as well. $\square$

Proposition 14. $[\![PSL(2,\mathbb{Z})]\!]_0^{\mathsf{pre}}$ *is neither a groupoid nor a monoid.*

Proof. Consider the sequence $\{A_i\}$ where

$$A_i = \begin{pmatrix} 1 & 1 \\ i & i+1 \end{pmatrix}. \tag{4}$$

Clearly $\{A_i\}$ defines an element of $[\![PSL(2,\mathbb{Z})]\!]_0^{\mathsf{pre}}$ but the inverse sequence does not, since $A_i^{-1}(0) = -1$ for all i. Thus, $[\![PSL(2,\mathbb{Z})]\!]_0^{\mathsf{pre}}$ is not a groupoid.

Consider now the sequence $\{A_i\}$ given by

$$A_i = \begin{pmatrix} i & -1 \\ 1-i^2 & i \end{pmatrix},$$

defining an element of $[\![PSL(2,\mathbb{Z})]\!]_0^{\mathsf{pre}}$. Since the orbit of 0 by $PSL(2,\mathbb{Z})$ is dense in S^1, we may find a sequence $\{B_i\}$, $B_n = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL(2,\mathbb{Z})$, so that the ratio $R_i = d_i/b_i \to +\infty$ and satisfies

$$1-i^2+iR_i \;\approx\; i^{-1},$$

to whatever degree of accuracy we desire. (An error term on the order of $o(i^{-1})$ will do.) Such a sequence defines an element of $[\![PSL(2,\mathbb{Z})]\!]_0^{\mathsf{pre}}$. We find that

$$A_iB_i(0) \;=\; A_i(R_i^{-1}) \;=\; \frac{i-R_i}{1-i^2+iR_i} \;\approx\; \frac{i^{-1}(1-i^{-1})}{i^{-1}} \;\longrightarrow\; 1.$$

It follows that the product $\{A_iB_i\}$ does not define an element of $[\![PSL(2,\mathbb{Z})]\!]_0^{\mathsf{pre}}$, so the latter is not a monoid. ☐

Note 20. Note that the isotropy subgroup I_0 of $PSL(2,\mathbb{Z})$ at 0 consists of elements represented by matrices of the form

$$\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix},$$

where $n \in \mathbb{Z}$. In particular, $I_0 \cong \mathbb{Z}$.

Recall that we may regard $PSL(2,\mathbb{Z}) < {}^*PSL(2,\mathbb{Z})$ via the constant sequences.

Lemma 2. $PSL(2,\mathbb{Z}) \bigcap [\![PSL(2,\mathbb{Z})]\!]_0 = I_0$.

Proof. Clearly $I_0 \subset PSL(2,\mathbb{Z}) \bigcap [\![PSL(2,\mathbb{Z})]\!]_0$. If $\{C = A_iB_i^{-1}\}$ represents an element of $PSL(2,\mathbb{Z}) \bigcap [\![PSL(2,\mathbb{Z})]\!]_0$ then $\{CB_i\}$ represents an element of $[\![PSL(2,\mathbb{Z})]\!]_0^{\mathsf{pre}}$. But this is only the case if $C(0) = 0$, *i.e.* $C \in I_0$. ☐

Theorem 4. $[\![PSL(2,\mathbb{Z})]\!]_0$ *is not a group.*

Proof. Consider the sequences $\{X_i\}$, $\{Y_i\}$ defined

$$X_i \;=\; \begin{pmatrix} i & 1 \\ i^2+i-1 & i+1 \end{pmatrix}$$

and

$$Y_i \;=\; \begin{pmatrix} 2i+3 & -i-2 \\ -2i^2-3i+2 & i^2+2i-1 \end{pmatrix}.$$

Each defines an element of $[\![PSL(2,\mathbb{Z})]\!]_0^{\mathsf{pre}} \subset [\![PSL(2,\mathbb{Z})]\!]_0$. But

$$X_iY_i \;=\; \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

for all i. By Lemma 2, it follows that the product does not define an element of $[\![PSL(2,\mathbb{Z})]\!]_0$. □

By Lemma 2, ${}^*I_0 \cong {}^*\mathbb{Z}$ is a subgroup of $[\![PSL(2,\mathbb{Z})]\!]_0$. At present, we know of no others that are not contained in ${}^*I_0$.

Note 21. In analogy with *Note* 16 in *Germ Example 6*, the elements of $[\![PSL(2,\mathbb{Z})]\!]_0$ may be interpreted as equivalence classes of convergent sequences of $PSL(2,\mathbb{Z})$-diophantine approximations (see [Be-Do] for a brief survey). Here, one selects a point $t \in \mathbb{R}\cup\{\infty\} \approx \partial\mathbb{H}$ in which $I_t \neq 1$. Then given $\xi \in \mathbb{R}\cup\{\infty\}$ in the limit set of $PSL(2,\mathbb{Z})$, a diophantine approximation of ξ based at t is a sequence $A_i \in PSL(2,\mathbb{Z})$ such that $|\xi - A_i(t)| \to 0$. In the example considered above, $t = \xi = 0$.

3. Examples of Germs II

The examples in this section differ from those of §2 in that G is a topological group, and the nested set G is contained in a lower dimensional subspace. Here we broaden our understanding of multiplication in $[\![G]\!]$ in order to accommodate products which are infinitesimal to elements of $[\![G]\!]$ (rather than contained in $[\![G]\!]$).

3.1. **Double Coset Topologies.** Let $\mathfrak{G}$ be a Lie group, $\mathfrak{H} < \mathfrak{G}$ a closed Lie subgroup and $\Gamma < \mathfrak{G}$ a discrete subgroup. Denote by $p : \widetilde{\mathfrak{H}} \to \mathfrak{H}$ the universal cover of $\mathfrak{H}$.

An element $g \in \mathfrak{G}$ is called *minimal* (with respect to $\mathfrak{H},\Gamma$) if the coset $g\mathfrak{H}$ projects to a dense subset of $\Gamma\backslash\mathfrak{G}$. A subset $T^g \subset \mathfrak{G}$ is called a *local section* (at g) of the quotient map $\mathfrak{G} \to \mathfrak{G}/\mathfrak{H}$ if T^g maps homeomorphically onto an open subset containing $\bar{g} = g\mathfrak{H}$. A nested set $\mathcal{T}^g = \{T_i^g\}$ is called a *basis of local sections* (about g) if each T_i^g is a local section, and the image of $\mathcal{T}^g$ in $\mathfrak{G}/\mathfrak{H}$ is a basis about $\bar{g}$.

Note 22. If $\mathcal{T} = \{T_i\}$ is a basis of local sections about 1, then for any $g \in \mathfrak{G}$, there exists $M > 0$ such that

(1) $g \cdot \mathcal{T} \cdot g^{-1} = \{g \cdot T_i \cdot g^{-1}\}_{i>M}$ is a basis of local sections about 1.
(2) $\mathcal{T} \cdot g = \{T_i \cdot g\}_{i>M}$ is a basis of local sections about g.

For $g \in \mathfrak{G}$, denote by $\sigma_g(h) = ghg^{-1}$ the conjugation map.

Germ Example 9. Let $G = \mathfrak{H}$ and $\mathcal{T} = \{T_i\}$ be a basis of local sections about 1 which maps homeomorphically onto its image in $\Gamma\backslash\mathfrak{G}$. For $g \in \mathfrak{G}$ minimal, G is defined:

$$G_i = \left\{ \tilde{h} \in \widetilde{\mathfrak{H}} \;\middle|\; \sigma_g(p(\tilde{h})) \;\in\; \Gamma \cdot T_i \right\}.$$

We shall denote this germ $[\![\mathfrak{H}]\!]_{\Gamma,g}$.

Note 23. Since $p^{-1}(e) \cong \pi_1\mathfrak{H}$, we have ${}^*\pi_1\mathfrak{H} < [\![\mathfrak{H}]\!]_{\Gamma,g}$.

Note 24. We may put *Germ Example* 6 into this form: by taking $\mathfrak{G} = \mathbb{R}^{p+q}$, $\mathfrak{H} = \mathbf{R}(\mathbb{R}^p)$ and $\Gamma = \mathbb{Z}^{p+q}$.

Note 25. If we use another basis $\mathcal{T}'$ of local sections about 1, the germ obtained is different. However, $[\![\mathfrak{H}]\!]_{\Gamma,g} \subset {}^*\widetilde{\mathfrak{H}}$ maps injectively into ${}^\star\widetilde{\mathfrak{H}}$ (see §1.3); its image, denoted

$${}^\star\mathfrak{H}_{\Gamma,g},$$

is independent of the choice of $\mathcal{T}$.

Denote

$$[\![T]\!] \ := \ \bigodot_{\mathfrak{U}} T_i \ \subset \ {}^*\mathfrak{G}_{\varepsilon}.$$

An alternate definition of $[\![\mathfrak{H}]\!]_{\Gamma,g}$ is

$$(5) \qquad [\![\mathfrak{H}]\!]_{\Gamma,g} \ = \ \left\{ {}^*\tilde{h} \in {}^*\widetilde{\mathfrak{H}} \ \middle| \ \exists\, {}^*\gamma \in {}^*\Gamma \text{ such that } {}^*\mathfrak{e} \ = \ {}^*\gamma \cdot \sigma_g(p({}^*\tilde{h})) \ \in \ [\![T]\!] \right\}.$$

The set

$$[\![\mathfrak{H}]\!]^{\perp}_{\Gamma,g} \ = \ \{{}^*\gamma\}$$

of ${}^*\gamma$ occurring in (5) is called the *dual* of $[\![\mathfrak{H}]\!]_{\Gamma,g}$. The set

$${}^*\mathfrak{G}^{\Gamma,g}_{\varepsilon} \ = \ \{{}^*\mathfrak{e}\}$$

of ${}^*\mathfrak{e}$ occurring in (5) is called the *set of rates*.

Since $\mathfrak{H}$ is not discrete, we will define multiplication in $[\![\mathfrak{H}]\!]_{\Gamma,g}$ up to ${}^\star\mathfrak{H}_{\Gamma,g}$. Specifically, the product

$$[\![\tilde{h}]\!][\![\tilde{h}']\!] = [\![\tilde{h}'']\!]$$

is defined if the images of $[\![\tilde{h}\tilde{h}']\!]$ and $[\![\tilde{h}'']\!]$ in ${}^\star\widetilde{\mathfrak{H}}$ are equal. By *Note* 25, the algebra of $[\![\mathfrak{H}]\!]_{\Gamma,g}$ is independent of our choice of T.

Proposition 15. *If ${}^\star\widetilde{\mathfrak{H}}$ is a group, then the product in $[\![\mathfrak{H}]\!]_{\Gamma,g}$ is associative.*

Proof. We assume that $\pi_1\mathfrak{H} = 1$: the proof in the non simply connected case is identical. Let $[\![h]\!]$, $[\![h]\!]'$ and $[\![h]\!]''$ be such that

- $[\![h]\!][\![h']\!]$ and $[\![h']\!][\![h'']\!]$ are defined.
- $[\![h]\!]([\![h']\!][\![h'']\!])$ is defined.

The first item means that there exists $[\![h_1]\!], [\![h_2]\!] \in [\![\mathfrak{H}]\!]_{\Gamma,g}$ and ${}^*\varepsilon_1, {}^*\varepsilon_2 \in {}^*\mathfrak{H}_{\varepsilon}$ such that

$$(6) \qquad [\![h]\!][\![h']\!] = [\![h_1]\!]{}^*\varepsilon_1 \quad \text{and} \quad [\![h']\!][\![h'']\!] = [\![h_2]\!]{}^*\varepsilon_2.$$

In particular, if we let $({}^*\gamma, {}^*\mathfrak{e})$, $({}^*\gamma', {}^*\mathfrak{e}')$, $({}^*\gamma'', {}^*\mathfrak{e}')$, $({}^*\gamma_1, {}^*\mathfrak{e}_1)$ and $({}^*\gamma_2,, {}^*\mathfrak{e}_2)$ be the duals and rates of the aforementioned, then equation (6) implies that

$$(7) \qquad ({}^*\gamma^{-1} \cdot {}^*\mathfrak{e})(({}^*\gamma')^{-1} \cdot {}^*\mathfrak{e}') \ = \ (({}^*\gamma_1)^{-1} \cdot {}^*\mathfrak{e}_1)\,\sigma_g({}^*\varepsilon_1)$$

and

$$(8) \qquad (({}^*\gamma')^{-1} \cdot {}^*\mathfrak{e}')(({}^*\gamma'')^{-1} \cdot {}^*\mathfrak{e}'') \ = \ (({}^*\gamma_2)^{-1} \cdot {}^*\mathfrak{e}_2)\,\sigma_g({}^*\varepsilon_2).$$

Now the second item above implies that there exists $[\![\check{h}]\!]$, ${}^*\check{\varepsilon}$, ${}^*\check{\gamma}$ and ${}^*\check{\mathfrak{e}}$ so that

$$[\![h]\!][\![h_2]\!] \ = \ [\![\check{h}]\!]{}^*\check{\varepsilon}$$

or using (8)

$$({}^*\gamma^{-1} \cdot {}^*\mathfrak{e})(({}^*\gamma')^{-1} \cdot {}^*\mathfrak{e}')(({}^*\gamma'')^{-1} \cdot {}^*\mathfrak{e}'')\,\sigma_g({}^*\varepsilon_2^{-1}) \ = \ (({}^*\check{\gamma})^{-1} \cdot {}^*\check{\mathfrak{e}})\,\sigma_g({}^*\check{\varepsilon}).$$

By (7), we may write this equation as

$$(({}^*\gamma_1)^{-1} \cdot {}^*\mathfrak{e}_2)\,\sigma_g({}^*\varepsilon_1)(({}^*\gamma'')^{-1} \cdot {}^*\mathfrak{e}'')\,\sigma_g({}^*\varepsilon_2^{-1}) \ = \ (({}^*\check{\gamma})^{-1} \cdot {}^*\check{\mathfrak{e}})\,\sigma_g({}^*\check{\varepsilon})$$

or

$$\sigma_g\big([\![h_1]\!]{}^*\varepsilon_1[\![h'']\!]{}^*\varepsilon_2^{-1}\big) \ = \ \sigma_g\big([\![\check{h}]\!]{}^*\check{\varepsilon}\big).$$

Since ${}^\star\mathfrak{H}$ is a group, ${}^*\mathfrak{H}_{\varepsilon}$ is a normal subgroup of ${}^*\mathfrak{H}$. This implies that

$$[\![h_1 h'']\!] \ \in \ [\![\check{h}]\!] \cdot {}^*\mathfrak{H}_{\varepsilon},$$

that is, the product is associative. □

Corollary 1. *If ${}^\star\widetilde{\mathfrak{H}}$ is a group, then $[\![\mathfrak{H}]\!]_{\Gamma,g}$ is a groupoid.*

Let us consider the case $\mathfrak{G} = SL(2,\mathbb{R})$, $\Gamma = SL(2,\mathbb{Z})$ and $\mathfrak{H}$ = the 1-parameter subgroup $H^+ = \{A_r^+\}$, ($H^- = \{A_r^-\}$)where

$$A_r^+ = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \quad \left(A_r^- = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \right)$$

for $r \in \mathbb{R}$.

Note 26. The Lie algebra of $SL(2,\mathbb{R})$ is generated by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

and the vector field generated by X (by Y) integrates to H^+ (H^-). The calculations we shall carry out for $\mathfrak{H} = H^+$ are identical to those for $\mathfrak{H} = H^-$. Accordingly, we shall consider only H^+; for the remainder of this section, we write $H = H^+$.

Note 27. In this case, it is possible to choose local sections T_i to be open neighborhoods about 1 in the Lie subgroup $\mathfrak{B}$ whose Lie algebra is generated by Y and Z. It follows that the ultrascope $[\![T]\!]$ is the group ${}^*\mathfrak{B}_\varepsilon$.

Note 28. *Germ Example* 8 also arises from a double coset topology, by taking $\mathfrak{H} = \mathfrak{B}$ and $\Gamma = SL(2,\mathbb{Z})$.

The germ $[\![H]\!]_{SL(2,\mathbb{Z}),g}$ appears to be quite complicated: in fact, we offer no theorems and only some conjectures which we state shortly. To get a feel for the subtlety of $[\![H]\!]_{SL(2,\mathbb{Z}),g}$, we walk through a sample calculation using the relatively simple choice

$$g = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

The coset of g is

$$gH = \begin{pmatrix} \sqrt{2} & \sqrt{2}r+1 \\ 1 & r+\sqrt{2} \end{pmatrix};$$

it is dense in $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ (it does not define a cycle, and by a theorem of Hedlund [He], must be dense). Thus g is minimal. The conjugate of H by g is

$$\sigma_g(H) = \begin{pmatrix} 1-\sqrt{2}r & 2r \\ -r & 1+\sqrt{2}r \end{pmatrix}.$$

Proposition 16. *For every* ${}^*\gamma = \begin{pmatrix} {}^*a & {}^*b \\ {}^*c & {}^*d \end{pmatrix} \in [\![H]\!]^{\perp}_{SL(2,\mathbb{Z}),g}$, *we have*

- ${}^*a-1,\ {}^*c \in {}^*\mathbb{Z}_{\sqrt{2}}$ *and* ${}^*b,\ {}^*d-1 \in {}^*\mathbb{Z}_{\sqrt{2}} \cap ({}^*\mathbb{Z}_{\sqrt{2}})^{\perp}$.
- ${}^*b = ({}^*a-1)^{\perp}$ *and* ${}^*c = ({}^*d-1)^{\perp}$.

Proof. Let ${}^*\gamma \in {}^*SL(2,\mathbb{Z})$, $A_{{}^*r} \subset {}^*H$ satisfy (5). Then there exists ${}^*\varepsilon_1,\dots,{}^*\varepsilon_4 \in {}^*\mathbb{R}_\varepsilon$ such that

$$(9)\qquad {}^*\gamma\cdot\sigma_g(A_{{}^*r}) = \begin{pmatrix} 1+{}^*\varepsilon_1 & {}^*\varepsilon_2 \\ {}^*\varepsilon_3 & 1+{}^*\varepsilon_4 \end{pmatrix}.$$

Carrying out the product on the left hand side gives the equations in ${}^*a, {}^*b$:

$$\begin{aligned} {}^*a-\sqrt{2}{}^*a{}^*r-{}^*b{}^*r &= 1+{}^*\varepsilon_1 \\ 2{}^*a{}^*r+{}^*b+\sqrt{2}{}^*b{}^*r &= {}^*\varepsilon_2 \end{aligned}$$

which in turn yield

$$\sqrt{2}^*a + {}^*b = \sqrt{2}(1 + {}^*\varepsilon_1) + {}^*\varepsilon_2, \tag{10}$$

or

$$^*a + \frac{^*b}{\sqrt{2}} = 1 + ({}^*\varepsilon_1 + \frac{^*\varepsilon_2}{\sqrt{2}}).$$

It follows that $^*a - 1 \in {}^*\mathbb{Z}_{\sqrt{2}}$ and $^*b = ({}^*a-1)^\perp \in {}^*\mathbb{Z}_{1/\sqrt{2}} = ({}^*\mathbb{Z}_{\sqrt{2}})^\perp$. Equation (10) also shows that $^*b \in {}^*\mathbb{Z}_{\sqrt{2}}$. The calculation is qualitatively the same for $^*c, {}^*d$; the details are left to the reader. □

Conjecture 2. $[\![H]\!]^\perp_{SL(2,\mathbb{Z}),g}$ *is not a group.*

Note 29. The evidence for this conjecture, we believe, comes from the fact that the germ $[\![H]\!]_{SL(2,\mathbb{Z})}$ is the fundamental germ (based at g) of the horocyclic flow on the unit tangent bundle $\mathrm{T}^1_*(PSL(2,\mathbb{Z})\backslash\mathbb{H}^2)$ of the modular orbifold (see §4.7). This horocyclic flow is dual to an Anosov flow on $\mathrm{T}^1_*(PSL(2,\mathbb{Z})\backslash\mathbb{H}^2)$, whose fundamental germ (at x) is $[\![PSL(2,\mathbb{Z})]\!]_x$ (see *Note* 50, §4.5). As such, the dual $[\![H]\!]^\perp_{SL(2,\mathbb{Z}),g}$ should be related to some $[\![PSL(2,\mathbb{Z})]\!]_x$, $x \in S^1$, which we have seen in the case of $x = 0$, is not a group.

For $X \subset {}^*\mathbb{R}$, the galaxy of X is defined $\mu(X) = \{\mu({}^*x) \mid {}^*x \in X\}$ (see §1.2 for the definition of $\mu(x)$). If X, Y are two sets, $X/Y = \{x/y \mid x \in X,\ y \in Y\}$.

Proposition 17. *If* $A_{^*r} \in [\![H]\!]_{SL(2,\mathbb{Z}),g}$, *then*

$$^*r \ \in\ \frac{\mu({}^*\mathbb{Z}_{\sqrt{2}})}{\mu(0)} \ \bigcap\ \frac{\mu(({}^*\mathbb{Z}_{\sqrt{2}})^\perp)}{\mu(2)}.$$

Proof. We have

$$^*r \ =\ -\frac{^*a - 1 - {}^*\varepsilon_1}{\sqrt{2}^*a + {}^*b} \ =\ -\frac{\sqrt{2}({}^*a - 1 - \varepsilon_1)}{2^*a + \sqrt{2}^*b}.$$

But (10) says that $2a + \sqrt{2}b \equiv 2 \mod {}^*\mathbb{R}_\varepsilon$. Thus

$$^*r \ \in\ \frac{^*b + \sqrt{2}^*\varepsilon_1}{\mu(2)} \ \in\ \frac{\mu(({}^*\mathbb{Z}_{\sqrt{2}})^\perp)}{\mu(2)}.$$

Using the equations arising from (9) that involve $^*c$, $^*d$ one obtains the second inclusion $^*r \in \mu({}^*\mathbb{Z}_{\sqrt{2}})/\mu(0)$. □

Note 30. The ease with which we were able to eliminate $^*r$ in obtaining (10) is not possible for more general g. In particular, we do not obtain a nice general description of the elements of $[\![H]\!]_g$ and $[\![H]\!]^\perp_g$ using the groups $^*\mathbb{Z}_r$.

Let Γ be a co-finite volume Fuchsian group (*i.e.* $\Gamma\backslash\mathbb{H}^2$ is of finite volume). The horocyclic flow on $\mathrm{T}^1_*(\Gamma\backslash\mathbb{H}^2)$ is dynamically well-behaved in many respects. For example, [Da-Sm], all of the dense orbits are uniformly distributed with respect to the unique $SL(2,\mathbb{R})$ invariant measure. When Γ is co-compact, the horocyclic flow is uniquely ergodic [Fu]. Although the germ $[\![H]\!]_{\Gamma,g}$ contains much finer (metrical) information than that comprised by the aforementioned ergodicity (measure theoretic) results, we nevertheless state the following, very optimistic conjecture.

Conjecture 3. *Let* Γ *be a co-finite volume Fuchsian group. Then for any* $g \in SL(2,\mathbb{R})$ *minimal with respect to* Γ, $[\![H]\!]_{\Gamma,g}$ *is a group.*

Now consider the subgroup $G = \{B_r\}$ of $SL(2,\mathbb{R})$ consisting of matrices of the form

$$B_r = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix},$$

$r \in \mathbb{R}$.

Let Γ be a co-finite volume Fuchsian group. The germ $[\![G]\!]_{\Gamma,g}$ is a groupoid by Corollary 1; it is the fundamental germ of the geodesic flow on $\mathrm{T}^1_*(\Gamma\backslash\mathbb{H}^2)$ (see §4.7). The geodesic flow is in contrast rather badly behaved from the point of view of ergodic theory, so here we make the following conjecture.

Conjecture 4. *Let Γ be a co-finite volume Fuchsian group. For any $g \in SL(2,\mathbb{R})$ minimal with respect to Γ, $[\![G]\!]_g$ is* not *a group.*

3.2. **Locally Free Lie Group Actions.** Let $\mathfrak{B}$ be a k-dimensional Lie group, $X \subset M^n$ a subspace of an n-dimensional smooth manifold, $n > k$, and $\theta : \mathfrak{B} \to \mathsf{Homeo}(X)$ a continuous homomorphism. We say that θ is locally free if for all $x \in X$, the isotropy subgroup $I_x < \mathfrak{B}$ is discrete.

A *transversal* to θ at x is a submanifold T of M of dimension $n-k$ such that $x \in T$, and the intersection of the orbit $Orb(x) = \theta(\mathfrak{B})\cdot x$ with T is totally-disconnected. Let $\mathcal{T} = \{T_i\}$ be a neighborhood basis about x in T, and denote by $p : \widetilde{\mathfrak{B}} \to \mathfrak{B}$ the universal cover.

Germ Example 10. $\mathcal{G}$ is defined by

$$G_i \;=\; \{\tilde{g} \in \widetilde{\mathfrak{B}} \mid \; \theta_{p(\tilde{g})}(x) \;\in\; T_i\}.$$

We denote this germ $[\![\mathfrak{B}]\!]_{X,x}$.

Note 31. We may put *Germ Example* 9 in this form by taking $\mathfrak{B} = \mathfrak{H}$, $X = M^n = \Gamma\backslash\mathfrak{G}$ and the action θ is that induced $\Gamma\backslash\mathfrak{G}$ by right-multiplication in $\mathfrak{G}$.

Note 32. We must define multiplication in $[\![\mathfrak{B}]\!]_{X,x}$ up to ${}^\star\mathfrak{B}$, as in §3.1.

Let M^n be a Riemannian manifold. Fix a point $x \in M^n$ and a tangent vector $v \in \mathrm{T}_xM$. Let l be the complete geodesic determined by v, X its closure (itself a union of geodesics). Then there is a locally free action of $\mathbb{R}$ given by geodesic flow along X. Denote the resulting germ ${}^*\mathbb{R}_{M,v} \subset {}^*\mathbb{R}$. If l is a closed geodesic, then ${}^*\mathbb{R}_{M,v}$ is isomorphic to ${}^*\mathbb{Z}$ and is hence a group. The following is a relative of Conjecture 4:

Conjecture 5. *Let $M^2 = \Sigma$ be a hyperbolic surface, $v \in \mathrm{T}_*\Sigma$ a vector which is not tangent to a closed geodesic. Then ${}^*\mathbb{R}_{\Sigma,v}$ is not a group.*

4. The Fundamental Germ of an Algebraic Lamination

This section contains the definition of the fundamental germ for four types of laminations: suspensions, quasi-suspensions, double-coset foliations and laminations arising from the action of a Lie Group. We shall refer to these types of laminations as *algebraic laminations* for the duration of this paper.

4.1. **Laminations.** A deck of cards is a product $\mathbb{R}^n \times \mathsf{T}$, where T is a topological space. A card is a subset of the form $C = O \times \{\mathsf{t}\}$, where $O \subset \mathbb{R}^n$ is open and $\mathsf{t} \in \mathsf{T}$.

A lamination of dimension n is a space $\mathcal{L}$ equipped with a maximal atlas $\mathcal{A} = \{\phi_\alpha\}$ consisting of charts with range in a fixed deck of cards $\mathbb{R}^n \times \mathsf{T}$, such that each transition homeomorphism $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ satisfies the following conditions:

(1) For every card $C \in \mathsf{Dom}(\phi_{\alpha\beta})$, $\phi_{\alpha\beta}(C)$ is a card.

(2) The family of homeomorphisms $\{\phi_{\alpha\beta}(\cdot,\mathsf{t})\}$ is continuous in t.

An open (closed) transversal in $\mathcal{L}$ is a subset of the form $\phi_\alpha^{-1}(\{x\}\times\mathsf{T}')$ where T' is open (closed) in T. An open (closed) flow box is a subset of the form $\phi_\alpha^{-1}(O\times\mathsf{T}')$, where O is open and $\mathsf{T}'\subset\mathsf{T}$ is open (closed).

A card in $\mathcal{L}$ is a subset of the form $\phi_\alpha^{-1}(C)$ for C a card in the deck $\mathbb{R}^n\times\mathsf{T}$. A leaf $L\subset\mathcal{L}$ is a maximal continuation of overlapping cards in $\mathcal{L}$. A lamination is weakly minimal if it has a dense leaf; it is minimal if all of its leaves are dense.

A lamination map $f:\mathcal{L}\to\mathcal{L}'$ is a continuous map which respects the leaf structure: for every leaf $L\subset\mathcal{L}$, $f(L)$ is contained in a leaf of $\mathcal{L}'$.

If the transitions $\phi_{\alpha\beta}$ are smooth and $\{\phi_{\alpha\beta}(\cdot,\mathsf{t})\}$ is a normal family of smooth homeomorphisms, we say that $\mathcal{L}$ is a smooth lamination. A Lie group lamination is a smooth lamination $\mathcal{L}$ which has the structure of a group, in which the group operations yield smooth lamination homeomorphisms.

Theorem 5. $(({}^*\mathbb{R},+),[\tau])$ *has the structure of a Lie group lamination.*

Proof. Let $\mathsf{T}={}^*\mathbb{R}/\mathbb{R}$, equipped with the $[\tau]$-quotient topology. T is a real infinite dimensional vector space. The quotient map ${}^*\mathbb{R}\to\mathsf{T}$ gives ${}^*\mathbb{R}$ the structure of a smooth lamination with leaves the cosets ${}^*x+\mathbb{R}$. It is clear that addition and inversion define lamination maps. ☐

Note 33. Note that

$$\mathsf{T}_{\mathrm{fin}} \;:=\; {}^*\mathbb{R}_{\mathrm{fin}}/\mathbb{R} \;\cong\; {}^*\mathbb{R}_{\varepsilon},$$

the isomorphism being a homeomorphism with respect to the $[\tau]$-topology. In this case, the projection ${}^*\mathbb{R}_{\mathrm{fin}}\to\mathsf{T}_{\mathrm{fin}}$ has a canonical section $\mathsf{T}_{\mathrm{fin}}\to{}^*\mathbb{R}_{\varepsilon}$. Hence ${}^*\mathbb{R}_{\mathrm{fin}}$ is diffeomorphic to the deck of cards $\mathbb{R}\times{}^*\mathbb{R}_{\varepsilon}$.

4.2. **The Fundamental Germ of a Suspension.** Let B be a manifold, F a 2^{nd} countable metrizable space and $\rho:\pi_1B\to\mathsf{Homeo}(F)$ a representation. The suspension of ρ is the space

$$\mathcal{L} \;=\; \widetilde{B}\times_\rho F$$

defined by quotienting the product $\widetilde{B}\times F$ by the diagonal action of π_1B:

$$\alpha\cdot(\tilde{x},\,t) \;=\; (\alpha\cdot\tilde{x},\,\rho_\alpha(t)).$$

The diagonal action is properly discontinuous and leaf preserving, hence $\mathcal{L}$ is a lamination.

Note 34. $\mathcal{L}$ is a fiber bundle over B with model fiber F. If $K=\ker(\rho)$ and $(L,x)\subset\mathcal{L}$ is a pointed leaf, we have $K\trianglelefteq\pi_1(L,x)$. $\mathcal{L}$ is weakly-minimal if and only if $\rho(\pi_1B)$ has a dense orbit.

Note 35. The restriction $p|_L$ of the projection $p:\mathcal{L}\to B$ to a leaf L is a covering map. Suppose that p_L is a normal covering (we say that L is *normal*). The deck group D_L of $p|_L$ has the property that

$$D_L\cdot x \;=\; L\cap F_x,$$

where F_x is the fiber of p through x. In particular, if we give $(L\cap F_x)\subset F_x$ the subspace topology, we have an inclusion

$$D_L\hookrightarrow\mathsf{Homeo}(L\cap F_x).$$

Let L be a weakly-minimal suspension with (L,x) a normal, pointed dense leaf. In keeping with the intuition outlined in the introduction, we want to construct the fundamental germ from tail-equivalence classes of sequences in $\pi_1 B$ which translate x to sequences in $L\bigcap F_x$ converging in to x. If L is not simply connected, then these sequences should take into account the topology of L *i.e.* we want ${}^*\pi_1(L,x)$ to be a subgroup of the germ. The following definition accomplishes these goals.

Let

$$\rho_0 : \pi_1 B \longrightarrow \mathsf{Homeo}(L\cap F_x)$$

be the composition

$$\pi_1 B \longrightarrow \pi_1 B\big/(p|_L)_*(\pi_1 L) \;\;\cong\;\; D_L \hookrightarrow \mathsf{Homeo}(L\cap F_x).$$

Definition 1. The *fundamental germ* of L at x is

$$[\![\pi]\!]_1(L,x) \;\;:=\;\; [\![\pi_1 B]\!]_x,$$

where the germ $[\![\pi_1 B]\!]_x$ is obtained by pulling back a basis $\{U_i\}\subset L\cap F_x$ about x along ρ_0.

Note 36. If $G=\{G_i\}$ is the induced nested set, we have

$$(p|_L)_*(\pi_1 L) \;\;\subset\;\; \bigcap G_i.$$

Hence $[\![\pi]\!]_1(L,x)$ contains a subgroup isomorphic to ${}^*\pi_1(L,x)$.

On the other hand, we saw in § 2.3, *Germ Example* 7, that given $t\in F$, the representation ρ used to define L defines a germ

$$[\![\pi_1 B]\!]_t.$$

Theorem 6. *Let $(\tilde{x},t)\in\widetilde{B}\times F$ be any point lying above $x\in L$. Then*

$$[\![\pi]\!]_1(L,x) \;\;=\;\; [\![\pi_1 B]\!]_t.$$

Proof. We have $\pi_1 L\cong I_t$ = the isotropy subgroup of $\rho(\pi_1 B)$ at t. The action of $\rho(\pi_1 B)/I_t$ on the orbit of t is the same as the action of D_L on $F_x\cap L$. It follows that the germs so obtained are equal. ☐

The next three sections are devoted to calculating fundamental germs of some well-known suspensions, using the results of §2.

4.3. **Inverse Limit Solenoids.** A *solenoid* is a lamination in which F is a totally-disconnected space.

Let $C=\{\rho_\alpha : M_\alpha\to M\}$ be an inverse system of pointed manifolds and normal covering maps with initial object M; denote by

$$\widehat{M} \;\;=\;\; \widehat{M}_C \;\;:=\;\; \varprojlim M_\alpha$$

the limit. By definition, $\widehat{M}\subset\prod M_\alpha$; thus elements of $\widehat{M}$ are denoted (x_α), where $x_\alpha\in M_\alpha$.

Note 37. We may identify the universal covers $\widetilde{M}_\alpha$ with $\widetilde{M}$ and choose the universal covering maps $\widetilde{M}\to M_\alpha$ to be compatible with the system C. By universality, there exists a canonical map

$$i:\widetilde{M}\to\widehat{M}.$$

Let $H_\alpha=(\rho_\alpha)_*(\pi_1 M_\alpha)$. Associated to C is the inverse limit of deck groups

$$\hat{\pi}_1 M \;\;:=\;\; \varprojlim \pi_1 M/H_\alpha.$$

Note 38. By universality of inverse limits, the projections $\pi_1 M \to \pi_1 M/H_\alpha$ yield a canonical homomorphism

$$\iota : \pi_1 M \longrightarrow \hat{\pi}_1 M$$

with dense image.

Note 39. $\hat{\pi}_1 M$ is a Cantor group. The closures of the images $\iota(H_\alpha)$ are clopen, and give a neighborhood basis about 1.

Note 40. An element $\hat{g} = (g_\alpha) \in \hat{\pi}_1 M$ acts on $\hat{x} = (x_\alpha) \in \widehat{M}$ by the rule

$$\hat{g} \cdot \hat{x} = (g_\alpha \cdot x_\alpha).$$

This action preserves the fiber $F_{\hat{x}}$ through $\hat{x}$. Conversely, any two elements of F_x may are related by some $\hat{g} \in \hat{\pi}_1 M$.

Let $\rho : \pi_1 M \to \mathsf{Homeo}(\hat{\pi}_1 M)$ be the right multiplication representation, defined

$$\rho_\alpha(\hat{g}) = \hat{g} \cdot (\iota(\alpha))^{-1}.$$

Proposition 18. *$\widehat{M}$ is homeomorphic to the suspension*

$$\widetilde{M} \times_\rho \hat{\pi}_1 M.$$

In particular, $\widehat{M}$ is a solenoid.

Proof. Let

$$\Upsilon : \widetilde{M} \times \hat{\pi}_1 M \longrightarrow \widehat{M}$$

be the map defined $(\tilde{x}, \hat{g}) \mapsto \hat{g} \cdot i(\tilde{x})$. Υ is invariant with respect to the diagonal action of $\pi_1 M$, and descends to a homeomorphism $\widetilde{M} \times_\rho \hat{\pi}_1 M \to \widehat{M}$. □

By *Note* 39, it follows that we may take $G = \{H_{\alpha_i}\}$ to be a nested sequence cofinal in $\{H_\alpha\}$. Then for any $x \in \widehat{M}$, $[\![\pi]\!]_1(\widehat{M}, x)$ is equal to the germ $[\![\pi_1 M]\!]_G$ (see *Germ Example* 2) . If G is cofinal in the lattice of normal finite index subgroups of $\pi_1 M$, we obtain the pro-finite germ $[\![\pi_1 M]\!]_{\hat{\varepsilon}}$.

Note 41. A manifold M is an inverse limit solenoid with respect to the trivial inverse system $\{id : M \to M\}$. Here $\hat{\pi}_1 = 1$. The representation ρ is therefore trivial and induces the nested set $G_i = \pi_1(M, x)$ for all i. It follows that

$$[\![\pi]\!]_1(M, x) = {}^*\pi_1(M, x).$$

4.4. **Linear Foliations of Torii.** Let V be a p-dimensional subspace of $\mathbb{R}^{p+q}$. Denote by $\widetilde{F}_V$ the foliation of $\mathbb{R}^{p+q}$ by cosets $\mathbf{v} + V$. The image F_V of $\widetilde{F}_V$ in the torus $\mathbb{T}^{p+q} = \mathbb{R}^{p+q}/\mathbb{Z}^{p+q}$ gives a foliation of the latter by Euclidean manifolds.

V may be regarded as the graph of a $q \times p$ matrix map

$$\mathbf{R} : \mathbb{R}^p \to \mathbb{R}^q$$

whose columns are $\mathbb{Q}$-linearly independent. Let $\rho : \mathbb{Z}^p \to \mathbb{T}^q$ be the representation obtained by composing $\mathbf{R}$ with the quotient $\mathbb{R}^q \to \mathbb{T}^q$ (see §2.2, *Germ Example* 6). Here, we view $\mathbb{T}^q \subset \mathsf{Homeo}_+(\mathbb{T}^q)$.

Proposition 19. *F_V is homeomorphic to the suspension $\mathbb{R}^p \times_\rho \mathbb{T}^q$.*

Proof. Let $P_0 : \mathbb{R}^{p+q} \to \mathbb{R}^p \times \mathbb{T}^q$ be the map defined

$$\mathbb{R}^p \times \mathbb{R}^q \ni (\mathbf{x}, \mathbf{y}) \longmapsto \left(\mathbf{x}, \overline{\mathbf{y}} - \overline{\mathbf{Rx}}\right),$$

where $\overline{\mathbf{w}}$ is the image of $\mathbf{w} \in \mathbb{R}^q$ in $\mathbb{T}^q$. Let P be the composition of P_0 with the projection $\xi : \mathbb{R}^p \times \mathbb{T}^q \to \mathbb{R}^p \times_\rho \mathbb{T}^q$. Then P is a covering homomorphism with kernel $\mathbb{Z}^{p+q}$, hence $\mathbb{R}^p \times_\rho \mathbb{T}^q \cong \mathbb{T}^{p+q}$. Since $V = (\mathbf{x}, \mathbf{Rx})$,

$$P(V) = \xi(\mathbb{R}^p \times \overline{\mathbf{0}});$$

thus $P(V)$ is a leaf of the suspension. It follows that P defines a lamination map $\widetilde{\mathcal{F}}_V \to \mathbb{R}^p \times_\rho \mathbb{T}^q$ which descends to the desired homeomorphism. □

Note 42. Let $\mathbf{r}_i$ be the ith column vector of $\mathbf{R}$. If $\mathbf{r}_i \in \mathbb{Q}^q$ for all i, the leaves of $\mathcal{F}_V$ are homeomorphic to $\mathbb{T}^p$ and are not dense. If at least one of the $\mathbf{r}_i$ has an irrational coordinate, then the leaves of $\mathcal{F}_V$ are non-compact and dense, homeomorphic to the quotient of $\mathbb{R}^p$ by a lattice with as many generators as rational $\mathbf{r}_i$.

Let ${}^*\mathbb{Z}^p_{\mathbf{R}}$ be the germ defined in *Germ Example* 6. The following is an immediate consequence of Proposition 19:

Corollary 2. $[\![\pi]\!]_1(\mathcal{F}_V, x) = {}^*\mathbb{Z}^p_{\mathbf{R}}$.

4.5. **Anosov Flows.** Let $\Sigma = \mathbb{H}^2/\Gamma$ be a finite volume hyperbolic surface and let $\rho : \Gamma \to \mathsf{Homeo}(S^1)$ be defined by extending the action of Γ on $\mathbb{H}^2$ to $\partial\mathbb{H}^2 \approx S^1$. The suspension

$$\mathcal{F}_\Gamma = \mathbb{H}^2 \times_\rho S^1$$

is called the *Anosov flow*.

Proposition 20. *The underlying space of $\mathcal{F}_\Gamma$ is homeomorphic to the unit tangent bundle* $\mathrm{T}^1_*\Sigma$.

Proof. Let $v \in \mathrm{T}^1_*\mathbb{H}^2$ be based at $z \in \mathbb{H}^2$. Let $x \in \partial\mathbb{H}^2 \approx S^1$ be the terminus of the oriented geodesic determined by v. The map $\mathrm{T}^1_*\mathbb{H}^2 \to \mathbb{H}^2 \times S^1$ defined $v \mapsto (z, x)$ descends to the desired homeomorphism. □

If z is any point in the leaf which is the image of $\mathbb{H}^2 \times \{x\}$, $x \in S^1$. Then

$$[\![\pi]\!]_1(\mathcal{F}_\Gamma, z) = [\![\Gamma]\!]_x,$$

where $[\![\Gamma]\!]_x$ is the germ considered in *Germ Example* 7 of §2.3. For $\Gamma = PSL(2,\mathbb{Z})$ and $x = 0$, we have seen that $[\![\pi]\!]_1(\mathcal{F}_{PSL(2,\mathbb{Z})}, z) = [\![PSL(2,\mathbb{Z})]\!]_0$ is not a group.

4.6. **The Fundamental Germ of a Quasi-Suspension.** Let $\mathcal{L} = \widetilde{B} \times_\rho F$ be a suspension over a base B. We say that $\mathcal{L}$ is *normal* if the restriction p_L of the projection $p : \mathcal{L} \to B$ to any leaf L is a normal covering.

Assumption. *Throughout this section, $\mathcal{L}$ will be a normal, weakly minimal suspension.*

We define an action of $\pi_1 B$ on $\mathcal{L}$ by

$$x \mapsto \overline{\gamma} \cdot x,$$

where $x \in L$ and $\overline{\gamma}$ is the image of $\gamma \in \pi_1 B$ in $\pi_1 B/(p_L)_*(\pi_1 L) \cong D_L$ = the deck group of p_L.

A lamination homeomorphism $f : \mathcal{L} \to \mathcal{L}$ is *weakly fiber-preserving* if for every fiber F_x,

$$f(F_x) = \bigcup_{i=1}^{n} E_{x_i}, \tag{11}$$

where $E_x \subset F_x$ denotes a subset of the fiber F_x. The collection $\mathsf{Homeo}_{\omega-\mathsf{fib}}(\mathcal{L})$ of weakly fiber-preserving homeomorphisms is clearly a group.

Note 43. Since the fibers are disjoint, each E_x occurring in (11) must be open. In particular, if F is connected, a weakly fiber-preserving homeomorphism is fiber-preserving. Thus, the concept of a weakly fiber-preserving homeomorphism is most interesting when the fibers are totally disconnected: that is, when $\mathcal{L}$ is a solenoid.

Suppose $H < \mathsf{Homeo}_{\omega-\mathsf{fib}}(\mathcal{L})$ is a subgroup acting properly discontinuously on $\mathcal{L}$. The quotient

$$\mathcal{L}^\flat = H\backslash\mathcal{L}$$

is called a *quasi-suspension* (over B).

Let (L,x) be a pointed dense leaf of $\mathcal{L}$, F the fiber through x. Denote by $(\mathcal{L}^\flat, x^\flat)$ the image of (L,x) in $\mathcal{L}^\flat$, and by $F^\flat$ the image of an open subset $F' \subset F$ which is evenly covered by the union $H\cdot F'$. Let G be the group generated by the actions of $\pi_1 B$ and H on $\mathcal{L}$.

Note 44. A sequence $\{x^\flat_\alpha\}$ of points converging to $x^\flat$ in $F^\flat$ lifts to a set of sequences in $\mathcal{L}$ indexed by H: each sequence is a translate by some $h \in H$ of a lift $\{x_\alpha\} \subset F'$ which converges to x. On the other hand, any sequence of the form $\{h_\alpha x_\alpha\}$, where $h_\alpha \in H$, projects to $\{x^\flat_\alpha\}$ as well.

Let U_i be a neighborhood basis about $x \in F$. Define a nested set $\mathcal{G} = \{G_i\}$ in G by

$$G_i \;=\; \{h\gamma \mid\; h \in H,\, \gamma \in \pi_1 B \text{ and } \overline{\gamma}\cdot x \in U_i\}.$$

Denote by $[\![\mathcal{G}]\!]_{x^\flat}$ the corresponding germ. In view of *Note* 44, the following definition is in keeping with our over-riding intuition.

Definition 2. The fundamental germ of the quasi-suspension $\mathcal{L}^\flat$ at $x^\flat$ is defined

$$[\![\pi]\!]_1(\mathcal{L}^\flat, x^\flat) \;=\; [\![\mathcal{G}]\!]_{x^\flat}.$$

Note 45. By definition of $\mathcal{G}$, both ${}^*\pi_1(L,x)$ and ${}^*H$ are subgroups of $[\![\pi]\!]_1(\mathcal{L}^\flat, x^\flat)$. In addition, the quotient map $\mathcal{L} \to \mathcal{L}^\natural$ induces an inclusion

$$[\![\pi]\!]_1(\mathcal{L}, x) \hookrightarrow [\![\pi]\!]_1(\mathcal{L}^\flat, x^\flat).$$

The following is an important example which comes from holomorphic dynamics.

Let $U, V \subset \mathbb{C}$ be regions conformal to the unit disc, with $\overline{U} \subset V$. A proper conformal map $f: U \to V$ is called a polynomial-like map. The conjugacy class of f is uniquely determined by a pair

$$(p, \partial f),$$

where p is a complex polynomial of degree d and $\partial f: S^1 \to S^1$ is a smooth, expanding degree d map of S^1 [Do-Hu].

The limit of the inverse system

$$\widehat{S}^1 \;=\; \varprojlim (S^1 \xleftarrow{\partial f} S^1 \xleftarrow{\partial f} S^1 \xleftarrow{\partial f} \dots)$$

is an inverse limit solenoid which may be identified with the suspension

$$\mathbb{R} \times_\rho \widehat{\mathbb{Z}}_d,$$

where $\widehat{\mathbb{Z}}_d$ is the group of d-adic integers and $\rho: \mathbb{Z} \to \mathsf{Homeo}(\widehat{\mathbb{Z}}_d)$ is the representation

$$\rho_m(\hat{n}) \mapsto \hat{n} - m.$$

Note 46. Every leaf of $\widehat{S}^1$ is homeomorphic to $\mathbb{R}$. ∂f defines a map of the inverse system $S^1 \leftarrow S^1 \leftarrow S^1 \leftarrow \dots$ to itself, which induces a homeomorphism $\partial \hat{f} : \widehat{S}^1 \to \widehat{S}^1$.

Let $\widehat{A}$ denote the suspension

$$\mathbb{H}^2 \times_\rho \widehat{\mathbb{Z}}_d$$

obtained by extending to $\mathbb{H}^2 \times \widehat{\mathbb{Z}}_d$ the twist identification used to define $\mathbb{R} \times_\rho \widehat{\mathbb{Z}}_d$ *e.g.*

$$(z, \hat{n}) \sim (\gamma^m \cdot z,\ \rho_m(\hat{n}))$$

for $m \in \mathbb{Z}$, where

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is the affine extension of the map $x \mapsto x+1$ to $\mathbb{H}^2$. The base of the suspension $\mathbb{H}^2 \times_\rho \widehat{\mathbb{Z}}_d$ is thus the singular hyperbolic annulus $A = \langle\gamma\rangle \backslash \mathbb{H}^2$.

The map $\partial \hat{f}$ extends to a weakly fiber-preserving homeomorphism $\hat{f} : \widehat{A} \to \widehat{A}$ which acts properly discontinuously on $\widehat{A}$.

Note 47. There exists a hyperbolic structure on L such that $\hat{f}$ is an isometry [Su].

The quotient

$$\widehat{S}_f := \langle \hat{f} \rangle \backslash \widehat{A}$$

is a quasi-suspension called the *Sullivan solenoid* [Gh], [Su].

Note 48. The action of $\pi_1 A \cong \mathbb{Z}$ on $\widehat{A}$ is induced by the map $(z, \hat{n}) \mapsto (z, \hat{n} + m)$ on $\mathbb{H}^2 \times \widehat{\mathbb{Z}}_d$.

Theorem 7. *For any $x^\flat \in \widehat{S}_f$, $[\![\pi]\!]_1(\widehat{S}_f, x^\flat)$ is isomorphic to the Baumslag-Solitar germ $[\![G_{\mathsf{BS}}]\!]$.*

Proof. Let γ denote the generator of $\pi_1 A$, regarded (see *Note* 48) as a homeomorphism of $\widehat{A}$. Then

$$\hat{f} \gamma \hat{f}^{-1} = \gamma^d.$$

It follows that the group G generated by γ and $\hat{f}$ is isomorphic to the Baumslag-Solitar group G_{BS} (see *Germ Example* 3, §2.1). The nested set G considered there corresponds exactly to the nested set used to define $[\![\pi]\!]_1(\widehat{S}_f, x^\flat)$. □

Corollary 3. $[\![\pi]\!]_1(\widehat{S}_f, x^\flat)$ *is a group.*

4.7. **The Fundamental Germ of a Double Coset Foliation.** Let $\mathfrak{G}$ be a Lie group, $\mathfrak{H}$ a closed Lie subgroup, $\Gamma < \mathfrak{G}$ a discrete subgroup. The foliation of $\mathfrak{G}$ by left cosets $g\mathfrak{H}$ descends to a foliation $F_{\mathfrak{H},\Gamma}$ of $\Gamma \backslash \mathfrak{G}$, called a double coset foliation.

Suppose that $F_{\mathfrak{H},\Gamma}$ is weakly minimal and $g \in \mathfrak{G}$ is minimal[4]. Denote by $\bar{g}$ the image of g in $\Gamma \backslash \mathfrak{G}$. Let $T = \{T_i\}$ be a nested sequence of local sections about 1 and take G to be the nested set defined in *Germ Example* 9, §3.1.

Definition 3. The fundamental germ of the double coset foliation $F_{\mathfrak{H},\Gamma}$ at $\bar{g}$ is

$$[\![\pi]\!]_1(F_{\mathfrak{H},\Gamma},\ \bar{g}) = [\![\mathfrak{H}]\!]_{\Gamma,g}.$$

[4]See §3.1 for this and other definitions used in this connection.

Note 49. Let $\{\tilde{h}_\alpha\}$ be a sequence representing an element of $[\![\mathfrak{H}]\!]_{\Gamma,g}$. Note that for any i, eventually

$$g\cdot p(\tilde{h}_\alpha) \;\subset\; \Gamma\cdot T_i\cdot g.$$

Let $\bar{g}_\alpha$ denote the image of $g\cdot p(\tilde{h}_\alpha)$ in $\Gamma\backslash\mathfrak{G}$. Note that there exists i such that $T_i\cdot g$ projects to a transversal $\overline{T}$ through $\bar{g}$. Then eventually $\{\bar{g}_\alpha\}\subset\overline{T}$ and converges to $\bar{g}$. Thus, Definition 3 conforms to our guiding intuition.

Proposition 21. *Suppose that for $g,g'\in\mathfrak{G}$ are minimal and $\bar{g}=\bar{g}'$. Then*

$$[\![\mathfrak{H}]\!]_{\Gamma,g} \;\cong\; [\![\mathfrak{H}]\!]_{\Gamma,g'}.$$

Proof. By hypothesis, there exists $\gamma\in\Gamma$ with $g'=\gamma g$. Let T be the basis of local sections about 1 used to define $[\![\mathfrak{H}]\!]_{\Gamma,g}$, $G\subset\widetilde{\mathfrak{H}}$ the corresponding nested set. Given $\tilde{h}\in G_i$, by definition

$$\sigma_g(p(\tilde{h})) \;\in\; \Gamma\cdot T_i.$$

Since $\sigma_{g'}=\sigma_\gamma\circ\sigma_g$, this implies

$$\sigma_{g'}(p(\tilde{h})) \;\in\; \sigma_\gamma(\Gamma\cdot T_i) \;=\; \Gamma\cdot\sigma_\gamma(T_i).$$

By *Note* 22 (1), $\sigma_\gamma(T)$ is a basis of local sections about 1. By *Note* 25 and the fact that multiplication is defined in ${}^\star\widetilde{\mathfrak{H}}$, the germs calculated at g and g' must be isomorphic. □

Let Γ be a co-finite volume Fuchsian group. Denote by $\Sigma=\Gamma\backslash\mathbb{H}^2$ and by $\mathrm{T}^1_*\Sigma$ the unit tangent bundle of Σ.

Recall that every $v\in\mathrm{T}^1_*\mathbb{H}^2$ determines three oriented, parametrized curves: a unique geodesic γ and two horocycles $\mathfrak{h}_+$, $\mathfrak{h}_-$ tangent to , respectively, $\gamma(\infty)$ and $\gamma(-\infty)$. By parallel translating v along these curves, we obtain three flows on $v\in\mathrm{T}^1_*\mathbb{H}^2$. The three flows are Γ-invariant, and define flows on $\mathrm{T}^1_*\Sigma$. The corresponding foliations are denoted F^γ_Γ, H^+_Γ and H^-_Γ.

Let $\mathfrak{G}=SL(2,\mathbb{R})$ and consider the following three choices for $\mathfrak{H}$: B and $H^\pm$, the 1-parameter subgroups considered in *Germ Example* 9, §3.1.

Proposition 22. *The foliations $F_{G,\Gamma}$ and $F_{H^\pm,\Gamma}$ are homeomorphic to F^γ_Γ and $H^\pm_\Gamma$, respectively.*

Proof. Given $v_0\in\mathrm{T}^1_*\mathbb{H}^2$ and $A\in SL(2,\mathbb{R})$, the map $A\mapsto A_*(v_0)$ descends to a homeomorphism

$$\Gamma\backslash SL(2,\mathbb{R}) \longrightarrow \mathrm{T}^1_*\Sigma \tag{12}$$

taking $F_{G,\Gamma}$ and $F_{H^\pm,\Gamma}$ to F^γ_Γ and $H^\pm_\Gamma$, respectively. □

In particular, if v corresponds to $\bar{g}$ via (12), we find that

$$[\![\pi]\!]_1(F^\gamma_\Gamma,v)=[\![G]\!]_g \quad\text{and}\quad [\![\pi]\!]_1(H^\pm_\Gamma,v)=[\![H^\pm]\!]_g.$$

Note 50. The Anosov foliation F_Γ is homeomorphic to the sum $F^\gamma_\Gamma\oplus H^+_\Gamma$. In particular, H^-_Γ is complimentary to F_Γ. This explains are belief that the duals $[\![H^\pm]\!]^\perp_{\Gamma,g}$ should be related to $[\![\Gamma]\!]_x$ for some $x\in S^1$ (see Conjecture 2).

4.8. **The Fundamental Germ of a Locally Free Lie Group Action.** Let $\mathfrak{B}$ be a Lie group of dimension k, M^n an n-manifold, $n > k$, $X \subset M^n$. Let $\theta : \mathfrak{B} \to \mathsf{Homeo}(X)$ be a locally free representation[5]. X has the structure $L_{\mathfrak{B}}$ of a lamination, whose leaves are the orbits by $\mathfrak{B}$ [Go].

Choose a point $x \in X$, a transversal T to θ at x and a neighborhood basis $\mathcal{T}$ about x in T.

Definition 4. The fundamental germ of $L_{\mathfrak{B}}$ at x is

$$[\![\pi]\!]_1(L_{\mathfrak{B}}, x) = [\![\mathfrak{B}]\!]_x.$$

5. Non-standard Completions I

Let G be a group, $\mathcal{G}$ a nested set about 1, $[\![G]\!]$ the associated germ. In this section, we assume that $\mathcal{G}$ is either a nested sequence of subgroups, or obtained from a neighborhood basis in a topological group. In this case, we may translate $\mathcal{G}$ by elements of G to obtain a metrizable topology. We use this topology to associate to G two topological spaces: a standard completion $\widehat{G}$ and a non-standard completion $[\![\overline{G}]\!]$.

If $\mathcal{G}$ is sufficiently well-behaved, both completions will be groups. In either event, $[\![G]\!]$ acts on the left on $[\![\overline{G}]\!]$ by homeomorphisms, and the quotient gives a homeomorphism

$$[\![G]\!]\backslash[\![\overline{G}]\!] \;\approx\; \widehat{G}$$

(the homeomorphism being an isomorphism when the completions are groups). The principle role $[\![\overline{G}]\!]$ will play is that of a unit space for an enhanced groupoid structure on $[\![G]\!]$ (see §7). These considerations will be used in §12 to construct laminated coverings.

We begin by motivating the non-standard completion with a simple example. Then, in the interest of clarity, we shall define the non-standard completion following the path of examples of §1 and §2. There will be some overlap with [Ro, §4, 8] and [Gol], however we will more or less keep our notation and terminology.

5.1. **A Motivating Example.** Let ${}^*\mathbb{Q} \subset {}^*\mathbb{R}$ be the non-standard rationals equipped with the induced ${}^*\tau$-topology. Consider the subring

$${}^*\mathbb{Q}_{\text{fin}} \;=\; {}^*\mathbb{Q} \cap {}^*\mathbb{R}_{\text{fin}}$$

of bounded elements.

Note 51. ${}^*\mathbb{Q}_{\text{fin}}$ is a local ring with closed maximal ideal

$${}^*\mathbb{Q}_{\varepsilon} \;=\; {}^*\mathbb{Q} \cap {}^*\mathbb{R}_{\varepsilon}.$$

(See Proposition 7.) The quotient ${}^*\mathbb{Q}_{\text{fin}}/{}^*\mathbb{Q}_{\varepsilon}$ is then a topological field with respect to the quotient ${}^*\tau$-topology.

Proposition 23. *${}^*\mathbb{Q}_{\text{fin}}/{}^*\mathbb{Q}_{\varepsilon}$ and $\mathbb{R}$ are isomorphic as topological fields.*

Proof. [6] Let ${}^*q + {}^*\mathbb{Q}_{\varepsilon}$ be a coset of ${}^*\mathbb{Q}_{\text{fin}}/{}^*\mathbb{Q}_{\varepsilon}$. Note that ${}^*q + {}^*\mathbb{Q}_{\varepsilon}$ can contain at most one rational number. Given ${}^*q' \in {}^*q + {}^*\mathbb{Q}_{\varepsilon}$, define $D_1 = \{q \in \mathbb{Q} \mid q \leq {}^*q'\}$ and $D_2 = \{q \in \mathbb{Q} \mid q > {}^*q'\}$. The pair (D_1, D_2) defines a Dedekind cut, hence a real number r. The association ${}^*q + {}^*\mathbb{Q}_{\varepsilon} \mapsto r$ defines the desired isomorphism. $\square$

That is: the quotient ${}^*\mathbb{Q}_{\text{fin}}/{}^*\mathbb{Q}_{\varepsilon}$ is isomorphic to the completion of $\mathbb{Q}$ with respect to the Euclidean topology.

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[5] See §3.2 for definitions.

[6] This argument is due to Robinson [Ro, pp. 56].

5.2. **Nested Subgroups.** Suppose that $G=\{H_i\}$ is a nested sequence of normal subgroups of G. As we have seen in §1.4, the associated germs $[G]$ and $[\![G]\!]$ are groups. There is another group which can naturally be formed from the data given by G: the pro-group $\widehat{G}$, defined

$$\widehat{G} = \varprojlim G/H_i.$$

The relationship is:

Theorem 8. *There exist groups* $[\overline{G}]$, $[\![\overline{G}]\!]$ *containing respectively* $[G]$ *and* $[\![G]\!]$ *as normal subgroups such that*

$$[\overline{G}]/[G] \;\cong\; \widehat{G} \;\cong\; [\![\overline{G}]\!]/[\![G]\!].$$

Proof. Let $\overline{G}$ denote the group of sequences $\{g_i\}$ in G which satisfy

$$g_i g_j^{-1} \in H_i \tag{13}$$

for all i,j with $j>i$. For each $X\in\mathfrak{U}$, define $\overline{G}_X$ by restriction to the X-indices and

$$[\overline{G}] \;=\; \varinjlim \overline{G}_X.$$

Note that $G_X=\prod_{i\in X} H_i$ is a normal subgroup of $\overline{G}_X$, hence $[G]\lhd[\overline{G}]$.

Define

$$Q_X:\overline{G}_X \longrightarrow \prod_{i\in X} G/H_i$$

by $\{g_i\}\mapsto(g_iH_i)$. Since $\{g_i\}$ satisfies (13), whenever $j>i$, g_jH_j maps onto g_iH_i by the projection $G/H_j\to G/H_i$. It follows that $Q_X(\overline{G}_X)=\widehat{G}$ and $\mathsf{Ker}(Q_X)=G_X$. The Q_X are compatible with the direct limit and yield the desired isomorphism $[\overline{G}]/[G]\cong\widehat{G}$.

Let ${}^{\bullet}\overline{G}$ be the set of sequences $\{g_\alpha\}$ which satisfy the condition: for every i, there exists N such that

$$g_\alpha g_\beta^{-1}\in H_i \tag{14}$$

whenever $\alpha,\beta>N$. Given $\{g_\alpha\}$, $\{h_\alpha\}\in{}^{\bullet}\overline{G}$ and i, choose N so that both sequences satisfy (14). Then for $\alpha,\beta>N$,

$$(g_\alpha h_\alpha)(g_\beta h_\beta)^{-1} \;\in\; g_\alpha H_i g_\beta^{-1} \;=\; g_\alpha g_\beta^{-1} H_i \;\subset\; H_i.$$

Thus ${}^{\bullet}\overline{G}$ is a group. Define

$$[\![\overline{G}]\!] \;=\; \varinjlim {}^{\bullet}\overline{G}$$

to be the direct limit group (with respect to the index-restriction homomorphisms P_X of §1.1).

Let $\iota: G\to\widehat{G}$ be the canonical map. A sequence $\{g_\alpha\}$ satisfying (14) is convergent with respect to the pro-topology, hence the image $\{\iota(g_\alpha)\}$ converges to some element $\hat{g}\in\widehat{G}$. This defines a homomorphism $Q:{}^{\bullet}\overline{G}\to\widehat{G}$ with $\mathsf{Ker}(Q)={}^{\bullet}G$. Since $Q\circ P_X=Q$ and

$$[\![G]\!] \;=\; \varinjlim {}^{\bullet}G,$$

we have $[\![\overline{G}]\!]/[\![G]\!]\cong\widehat{G}$. ☐

The group ($[\overline{G}]$) $[\![\overline{G}]\!]$ is called the *(lock-step) non-standard completion*. In analogy with §1.2, there are two obvious ways we may topologize this group. Let $H=\{g_\alpha H_{i_\alpha}\}$ be a nested sequence of cosets in which $i_\alpha\to\infty$. The ultraproduct

$$[H]^{\mathsf{pre}} \;:=\; \prod_{\mathfrak{U}} (g_\alpha H_{i_\alpha})$$

is called an *infinitesimal coset*. The collection of infinitesimal cosets forms a basis for a topology on $[\![\overline{G}]\!]$ which we call the internal topology, denoted $[\tau]$. $[\overline{G}]$ in turn is given the subspace topology through the inclusion $[\overline{G}] \subset [\![\overline{G}]\!]$.

Note 52. $[\![\overline{G}]\!]$ ($[\overline{G}]$) is a topological group in the $[\tau]$-topology, and the germ $[\![G]\!]$ ($[G]$) is an open subgroup. For this reason, the isomorphisms appearing in Theorem 8 are not homeomorphisms with respect to the quotient $[\tau]$-topology.

The enlargement topology ${}^*\tau$ on ${}^*G$ is generated by the non-standard cosets

$$^*(gH),$$

where $g \in G$ and $H = H_i$ for some i. $[\![\overline{G}]\!]$ and $[\overline{G}]$ are ${}^*\tau$-topologized as subsets.

Note 53. $[\![\overline{G}]\!]$ ($[\overline{G}]$) is a (non-Hausdorff) topological group in the ${}^*\tau$-topology, and the germ $[\![G]\!]$ ($[G]$) is a closed subgroup. The isomorphisms appearing in Theorem 8 are homeomorphisms in the quotient ${}^*\tau$-topology.

Proposition 24. *If G consists of finite-index subgroups, then $[\![\overline{G}]\!] = {}^*G$.*

Proof. [7] In this case, $\widehat{G}$ is compact. Let ${}^*g \in {}^*G$. Suppose that given a representative $\{g_i\}$, there exists no $X \in \mathfrak{U}$ such that $\{g_i\}|_X$ converges to a point of $\widehat{G}$. For each $\hat{g} \in \widehat{G}$, let $O_{\hat{g}} \ni \hat{g}$ be an open set in $\widehat{G}$ and $X_{\hat{g}} \in \mathfrak{U}$ such that $g_i \notin O_{\hat{g}}$ for all $i \in X_{\hat{g}}$. The collection of $O_{\hat{g}}$ defines an open cover of $\widehat{G}$; let $O_{\hat{g}_\alpha}$ be a finite subcover, $\alpha = 1, \dots, k$. Then $X = \bigcap X_{\hat{g}_\alpha} \in \mathfrak{U}$ and $\{g_i\}|_X \nsubseteq \widehat{G}$, which is absurd. □

Note 54. Unless G is trivial, not every element of ${}^*G$ is represented by a sequence $\{g_i\}$ satisfying (13). Thus $[\overline{G}] \subsetneqq {}^*G$.

Note 55. If $G = \mathbb{Z}$, then every subgroup is finite-index, so $[\![\overline{\mathbb{Z}}]\!] = {}^*\mathbb{Z}$.

Note 56. If the subgroups H_i are of infinite index, the pro-group $\widehat{G}$ is not compact. Then $[\![\overline{G}]\!] \subsetneqq {}^*G$: any infinite sequence $\{g_\alpha\}$ (*i.e.* not compactly contained) does not converge with respect to G.

5.3. **Topological Groups.** Suppose $\mathfrak{G}$ is a topological group and $G = \{U_i\}$ is a neighborhood basis about 1. Let U_r denote the natural right uniformity on $\mathfrak{G}$: it is generated by sets of the form

$$U_i = \{(g,h) \in \mathfrak{G} \times \mathfrak{G} \mid hg^{-1} \in U_i\}. \tag{15}$$

The left uniformity U_l is defined by replacing hg^{-1} in (15) by $h^{-1}g$.

Note 57. The completion $\widehat{\mathfrak{G}}_r$ ($\widehat{\mathfrak{G}}_l$) of $\mathfrak{G}$ with respect to U_r (U_l) is a topological space, and the right (left) uniformity extends to $\widehat{\mathfrak{G}}_r$ ($\widehat{\mathfrak{G}}_l$) [Kel].

Let ${}^\bullet\overline{G}_r$ denote the set of all U_r-Cauchy sequences in $\mathfrak{G}$, $[\![\overline{\mathfrak{G}}]\!]_r$ the $\mathfrak{U}$-direct limit. Let $[\![\overline{\mathfrak{G}}]\!]_{r-\text{fin}}$ denote the subset consisting of elements represented by sequences converging to elements of $\mathfrak{G}$. We define similarly $[\![\overline{\mathfrak{G}}]\!]_{l-\text{fin}}$.

Lemma 3. $[\![\overline{\mathfrak{G}}]\!]_{r-\text{fin}} = [\![\overline{\mathfrak{G}}]\!]_{l-\text{fin}}$.

Proof. Given $g \in \mathfrak{G}$, the left and right translates of the neighborhood basis about 1, $\{gU_i\}$ and $\{U_i g\}$, give bases about g which are compatible. If $g_i \to g$ with respect to either the left or right uniformity, then it converges with respect to both neighborhood bases. □

[7] This argument is due to [Ro, pg. 93]

We thus drop r,l subscripts and write $[\![\overline{\mathfrak{G}}]\!]_{\mathrm{fin}}$.

The internal topology $[\tau]$ on $[\![\overline{\mathfrak{G}}]\!]_r$ is generated by ultraproducts $[H]^{\mathrm{pre}}$ of nested open sets of the form $H = \{U_{i_\alpha} g_\alpha\}$, $i_\alpha \to \infty$. We use the inclusions $[\![\mathfrak{G}]\!] \subset [\![\overline{\mathfrak{G}}]\!]_{\mathrm{fin}} \subset [\![\overline{\mathfrak{G}}]\!]_r$ to give $[\![\mathfrak{G}]\!]$ and $[\![\overline{\mathfrak{G}}]\!]_{\mathrm{fin}}$ the internal topology.

Note 58. $[\![\overline{\mathfrak{G}}]\!]_{\mathrm{fin}}$ is a topological group in the $[\tau]$-topology and the germ $[\![\mathfrak{G}]\!]$ is an open subgroup.

The enlargement topology ${}^*\tau$ is generated by the non-standard translates ${}^*(Ug)$, where $U = U_i$ for some i and $g \in \mathfrak{G}$.

Theorem 9. *Let $\mathfrak{G}$ be a topological group.*

(1) $[\![\mathfrak{G}]\!] \lhd [\![\overline{\mathfrak{G}}]\!]_{\mathrm{fin}}$ *and*

$$\mathfrak{G} \;\cong\; [\![\overline{\mathfrak{G}}]\!]_{\mathrm{fin}}/[\![\mathfrak{G}]\!],$$

a topological isomorphism with respect to the quotient ${}^*\tau$-topology.

(2) $[\![\mathfrak{G}]\!]$ *acts by left-multiplication on* $[\![\overline{\mathfrak{G}}]\!]_r$ *with*

$$\widehat{\mathfrak{G}}_r \;\approx\; [\![\mathfrak{G}]\!]\backslash[\![\overline{\mathfrak{G}}]\!]_r,$$

a homeomorphism with respect to the quotient ${}^*\tau$-topology.

Proof. (1) Given $\{g_\alpha\}$ a sequence representing an element of $[\![\overline{\mathfrak{G}}]\!]_{\mathrm{fin}}$, let g denote the element of $\mathfrak{G}$ it converges to. For any $\{h_\alpha\}$ representing an element of $[\![\mathfrak{G}]\!]$, we have

$$\lim\, g_\alpha h_\alpha g_\alpha^{-1} \;=\; \lim\, g h_\alpha g^{-1} \;=\; 1.$$

Thus $[\![\mathfrak{G}]\!] \lhd [\![\overline{\mathfrak{G}}]\!]_{\mathrm{fin}}$. The action of $[\![\mathfrak{G}]\!]$ identifies precisely those sequences having the same limit, hence the quotient is $\mathfrak{G}$. The quotient ${}^*\tau$-topology is taken to the topology of $\mathfrak{G}$.

(2) Left multiplication by elements of $[\![\mathfrak{G}]\!]$ preserves the property of being Cauchy. Indeed, let $\{g_\alpha\}$ be Cauchy and let $\{h_\alpha\}$ represent an element of $[\![\mathfrak{G}]\!]$. Then

$$\lim\, h_\beta g_\beta g_\alpha^{-1} h_\alpha^{-1} \;=\; \lim\, h_\beta h_\alpha^{-1} \;\to\; 1$$

as $\alpha, \beta \to \infty$: hence the sequence $\{h_\alpha g_\alpha\}$ is Cauchy.

Equivalent Cauchy sequences are precisely those which differ on the left by a sequence $\{h_\alpha\}$ representing an element of $[\![\mathfrak{G}]\!]$. The quotient of the ${}^*\tau$-topology on $[\![\mathfrak{G}]\!]\backslash[\![\overline{\mathfrak{G}}]\!]_r$ is taken to that of $\widehat{\mathfrak{G}}_r$, and we obtain the desired homeomorphism. □

If $\mathsf{U}_l = \mathsf{U}_r$, we drop subscripts and write $\widehat{\mathfrak{G}}$ and $[\![\overline{\mathfrak{G}}]\!]$ for the standard and non-standard completions of $\mathfrak{G}$.

Theorem 10. *Let $\mathfrak{G}$ be a topological group in which $\mathsf{U}_l = \mathsf{U}_r$. Then $\widehat{\mathfrak{G}}$ and $[\![\overline{\mathfrak{G}}]\!]$ are topological groups, $[\![\mathfrak{G}]\!] \lhd [\![\overline{\mathfrak{G}}]\!]$ and*

$$[\![\overline{\mathfrak{G}}]\!]/[\![\mathfrak{G}]\!] \;\cong\; \widehat{\mathfrak{G}},$$

a topological isomorphism in the ${}^*\tau$-topology.

Proof. Since $\mathsf{U}_l = \mathsf{U}_r$, the family $\{\sigma_g\}_{g\in\mathfrak{G}}$ of conjugation homeomorphisms, $\sigma_g(x) = gxg^{-1}$, is equicontinuous at 1. Thus, given U a neighborhood of 1, there exists V a neighborhood of 1 with $\sigma_g(V) \subset U$ for all $g \in \mathfrak{G}$. If $\{g_\alpha\}, \{h_\alpha\}$ are two Cauchy sequences, choose N so that for $\alpha, \beta > N$,

$$g_\alpha g_\beta^{-1},\; h_\alpha h_\beta^{-1} \;\in\; U_i.$$

Then

$$(g_\alpha h_\alpha)(g_\beta h_\beta)^{-1} \in g_\alpha U_i g_\beta^{-1} = \sigma_{g_\alpha}(U_i)\cdot g_\alpha g_\beta^{-1} \subset U_{n_i}\cdot U_i,$$

where $n_i \to \infty$ as $i \to \infty$ (the last inclusion follows from uniform continuity of conjugation). Thus $[\![\overline{\mathfrak{G}}]\!]$ is a group. We leave the rest of the details of the proof to the reader. □

Note 59. If $\widehat{\mathfrak{G}}$ is compact or abelian, then $\mathsf{U}_l = \mathsf{U}_r$.

We also have the following generalization of Proposition 24 (with identical proof):

Proposition 25. *If $\widehat{\mathfrak{G}}$ is compact then $[\![\overline{\mathfrak{G}}]\!] = {}^*\mathfrak{G}$.*

6. Non-standard Completions II

In the next set of examples, the completion construction is not automatic. In particular, we will have to impose a condition called topological tameness on the nested set G in order to obtain the completions $\widehat{G}$ and $[\![\overline{G}]\!]$ of §5.

6.1. **Action Topologies.** Let G be a group, $\rho : G \to \mathsf{Homeo}(F)$ a representation, where F is a metric space. We fix, as in §2.3, $t \in F$ minimal with respect to the action of $\rho(G)$. Recall that the nested set $G = \{G_i\}$ is obtained through a neighborhood basis $\{U_i\}$ about t: that is, $G_i = \{g \in G \,|\, \rho_g(t) \in U_i\}$. Let

$$\mathsf{B}_r = \{G_i g \mid g \in G \text{ and } i \in \mathbb{N}\}$$

be the collection of right cosets with respect to G.

On the other hand, for every open set $U \subset F$, let

$$G_U = \{g \in G \mid \rho_g(t) \in U\}$$

and $\tau_F = \{G_U\}$. τ_F defines a topology on G.

Definition 5. We say that G (or $[\![G]\!]$) is *topologically tame* if B_r generates τ_F. We say that a suspension L is topologically tame if its fundamental germ is topologically tame.

We will need the condition of topological tameness in order to construct the non-standard completion $[\![\overline{G}]\!]_r$ and obtain an analogue of Theorem 9.

Note 60. It is not a difficult exercise to show that the set of left cosets B_l generates τ_F. Unfortunately, the corresponding germ completion $[\![\overline{G}]\!]_l$ cannot be used in upcoming applications *e.g.* especially in the construction of germ covering spaces (see §12).

Note 61. *Germ Examples* 1,2,4 and 5, viewed as action topologies, are topologically tame.

The following result shows that the condition of topological tameness is non-trivial.

Proposition 26. *$[\![PSL(2,\mathbb{Z})]\!]_0$ is not topologically tame.*

Proof. For any i, j and $B \in PSL(2,\mathbb{Z})$, we have $G_iB \cap G_j \neq \emptyset$. Indeed, let

$$A_\alpha = \begin{pmatrix} 0 & -1 \\ 1 & \alpha \end{pmatrix},$$

$\alpha = 1, 2, \ldots$. Then $\{A_\alpha\}$ represents an element of $[\![PSL(2,\mathbb{Z})]\!]_0^{\mathsf{pre}}$ and

$$\lim A_\alpha B(0) = 0.$$

Thus for α large, $A_\alpha B \in G_i \cap G_j B$. Assuming that B_r forms a basis for a topology, we would not be able to separate I from any other $B \in PSL(2,\mathbb{Z})$. It follows that such a topology could not be equal to τ_F. □

Note 62. There is a way to modify elements of $[\![PSL(2,\mathbb{Z})]\!]$ by replacing sequences $\{A_i\}$ by sequences of the form

$$\{A_i O_i\},$$

where the $O_i \in SO(2,\mathbb{R})$, so as to obtain a *geometric germ* which satisfies an analogue of topological tameness [Ge1], [Ge2].

Assume now that $[\![G]\!]$ is topologically tame. Below, $B_\delta(x)$ means the open d-ball in F of radius δ and center x. A sequence $\{U_i\}$ of nested open sets in F is called regular if for all i, $\overline{U}_{i+1} \subset U_i$.

Let $H = \{G_{i_\alpha} g_\alpha\}$ be a nested sequence of right cosets. Define δ_α by

$$\delta_\alpha = \inf \left\{ \delta > 0 \,\middle|\, B_\delta(g_\alpha(x)) \supset G_{i_\alpha} g_\alpha(x) \right\}.$$

We say that H is regular if

(1) $\{B_{\delta_\alpha}(g_\alpha(x))\}$ is a regular nested sequence of open sets in F.
(2) $\lim \delta_\alpha = 0$ and $\lim i_\alpha = \infty$.

An ultraproduct

$$[H]^{\mathsf{pre}} = \prod_{\mathfrak{U}} G_{i_\alpha} g_\alpha$$

of a regular nested set of cosets is called an infinitesimal coset. We define the non-standard completion

$$[\![\overline{G}]\!] = [\![\overline{G}]\!]_r = \bigcup [H]^{\mathsf{pre}},$$

where the union is over all infinitesimal cosets.

The ${}^*\tau$-topology is generated by sets of the form

$${}^*(G_i g) \cap [\![\overline{G}]\!].$$

For $[\![\overline{g}]\!]_1, [\![\overline{g}]\!]_2 \in [\![\overline{G}]\!]$, write

$$[\![\overline{g}]\!]_1 \sim [\![\overline{g}]\!]_2$$

if there exists an infinitesimal coset $[H]^{\mathsf{pre}}$ such that $[\![\overline{g}]\!]_1, [\![\overline{g}]\!]_2 \in [H]^{\mathsf{pre}}$.

The following Proposition is immediate.

Proposition 27. *If $[\![\overline{g}]\!]_1 \sim [\![\overline{g}]\!]_2$, there exists an element $[\![h]\!] \in [\![G]\!]$ such that*

$$[\![h]\!] \cdot [\![\overline{g}]\!]_1 = [\![\overline{g}]\!]_2.$$

Let $\widehat{F}$ denote the metric space completion of F. We define a map

$$\Theta : [\![\overline{G}]\!] \longrightarrow \widehat{F}$$

by $[\![\overline{g}]\!] \mapsto \hat{x} = \lim g_\alpha(x)$, where $\{g_\alpha\}$ represents $[\![\overline{g}]\!]$.

Theorem 11. Θ *induces a homeomorphism*

$$\Theta_{/\sim} : ([\![\overline{G}]\!]/\sim) \longrightarrow \widehat{F}.$$

Proof. Since $[\![\overline{G}]\!]$ is topologically tame, it follows that Θ is surjective. On the other hand, $\Theta([\![\overline{g}]\!]) = \Theta([\![\overline{g}]\!]')$ $\Leftrightarrow$ there are representatives $\{g_\alpha\}$, $\{g'_\alpha\}$ such that $\{g_\alpha(x)\}$, $\{g'_\alpha(x)\}$ are equivalent Cauchy sequences $\Leftrightarrow$ there exists an infinitesimal coset $[H]^{\mathsf{pre}}$ containing both of $[\![\overline{g}]\!]$, $[\![\overline{g}]\!]'$ $\Leftrightarrow$ $[\![\overline{g}]\!] \sim [\![\overline{g}]\!]'$. Thus $\Theta_{/\sim}$ is a bijection. By definition, the quotient ${}^*\tau$-topology on $[\![\overline{G}]\!]/\sim$ is related bi-continuously with the topology of $\widehat{F}$. $\square$

6.2. **Quasi-suspensions.** Let $L = \widetilde{B} \times_{\rho} F$ be a weakly-minimal, normal, topologically tame suspension. We assume as in §6.1 that F is a metric space. Let $H < \mathsf{Homeo}_{\omega-\mathsf{fib}}(L)$ be a subgroup acting properly discontinuously on L, and $L^{\natural} = H\backslash L$ the associated quasi-suspension. See §4.6 for the relevant notation and terminology.

Fix $x^{\natural} \in L^{\natural}$ and $x \in L$ covering $x^{\natural}$. Let $F^{\natural}$ be a transversal through $x^{\natural}$ evenly covered by the set of translates

$$H \cdot F' = \bigcup_{\gamma \in H} \gamma \cdot F',$$

where $x \in F' \subset F$ is an open subset of the fiber through x. We may give $F^{\natural}$ a metric with respect to the identification $F^{\natural} \approx F'$.

Define a topology $\tau_{F^{\natural}}$ on G corresponding to that of $F^{\natural}$ as follows: if $U^{\natural} \subset F^{\natural}$ is covered by $U \subset F'$, then $\tau_{F^{\natural}}$ consists of sets of the form

$$G_{U^{\natural}} \;=\; \{g \in G \mid \; g(x) \in H \cdot U\}.$$

Let B be the collection of right cosets in G with respect to G.

Proposition 28. *B generates $\tau_{F^{\natural}}$.*

Proof. Let G' denote the nested set used to define the germ of the suspension L. Since L is topologically tame, for every open set $U \subset F'$, there exists a coset $G'_i \gamma \subset G'_U$ for some $\gamma \in \pi_1 B$. But

$$G_i = H \cdot G'_i,$$

hence

$$G_i \gamma \subset H \cdot G'_U \subset G_{U^{\natural}}.$$

□

An infinitesimal coset for the quasi-suspension is one made from H translates of infinitesimal cosets of the suspension L. Specifically, if $[H']^{\mathsf{pre}}$ is an infinitesimal coset for L formed from a regular nested set of cosets $G'_{i_\alpha} \gamma_\alpha$ for which $G'_{i_\alpha} \gamma_\alpha(x) \subset F'$, then

$$[H]^{\mathsf{pre}} \;=\; \prod_{\mathfrak{U}} (H \cdot G'_{i_\alpha} \gamma_\alpha) \;=\; {}^*H \cdot [H']^{\mathsf{pre}}$$

is an infinitesimal coset for $L^{\natural}$.

We define as in §6.1

$$[\![\overline{G}]\!] \;=\; \bigcup [H]^{\mathsf{pre}}$$

and the equivalence relation $\sim$ on $[\![\overline{G}]\!]$. Letting $\widehat{F}^{\natural}$ be the metric space completion of $F^{\natural}$, the map $\Theta : [\![\overline{G}]\!] \to \widehat{F}^{\natural}$ is defined as in §5.4 as well. We then have the following analogue of Theorem 11 (with identical proof).

Theorem 12. *Θ induces a homeomorphism*

$$\Theta_{/\sim} : ([\![\overline{G}]\!]/\sim) \longrightarrow \widehat{F}^{\natural}.$$

6.3. **Double Cosets.** Let $\mathfrak{G}$, $\mathfrak{H}$ and Γ be as in §3.1. Fix $g \in \mathfrak{G}$ minimal.

The nested set $G \subset \widetilde{\mathfrak{H}}$ was defined using a basis $T = \{T_i\}$ of local sections about 1. Let $T = T_0$ be the initial member of T, endowed with a metric inducing its topology. As in the previous sections, we want to define $[\![\overline{\mathfrak{H}}]\!]$ using a right coset topology. There are, however, two main differences in this case:

- We will only consider cosets contained in the initial element G_0 of G. Thus, properly speaking, we will construct completions of G_0 and not of $\widetilde{\mathfrak{H}}$.

- Since G_0 is not a group, we will define cosets up to its asymptotic class: that is, with respect to its image in ${}^{\star}\widetilde{\mathfrak{H}}$.

Let $X = \{X_\alpha\} \subset \widetilde{\mathfrak{H}}$ be a nested sequence of subsets for which $\sigma_g(p(X_\alpha)) \subset \Gamma \cdot T$. X is called T-regular if there is a regular sequence of opens $U = \{U_\alpha\} \subset T$ such that

$$\sigma_g(p(X_\alpha)) \;=\; \big(\Gamma \cdot U_\alpha\big) \cap \mathfrak{H}$$

and $\mathsf{diam}(U_\alpha) \to 0$. For $X = \{X_\alpha\} \subset \widetilde{\mathfrak{H}}$, denote by ${}^{\star}X$ the image of the ultraproduct $[X]^{\mathsf{pre}}$ in ${}^{\star}\widetilde{\mathfrak{H}}$.

Definition 6. Let X be T-regular. The ultraproduct $[X]^{\mathsf{pre}}$ is called an *infinitesimal coset* if there exists a sequence of cosets $H = \{G_{i_\alpha} g_\alpha\}$, $i_\alpha \to \infty$, for which

$${}^{\star}H = {}^{\star}X.$$

We define

$$[\![\overline{\mathfrak{H}}]\!] \;=\; \bigcup [H]^{\mathsf{pre}}$$

where the union ranges over all infinitesimal cosets[8]. On the other hand, let

$$[\![\overline{\mathfrak{H}}]\!]_{T\text{-reg}} \;=\; \bigcup [X]^{\mathsf{pre}},$$

where X ranges over all T-regular nested sets in $\widetilde{\mathfrak{H}}$.

The set of T-regular ultraproducts generates a topology $[\tau_T]$ on $[\![\overline{\mathfrak{H}}]\!]_{T\text{-reg}}$. Denote by $[\mathsf{B}]^{\mathsf{pre}}$ the set of infinitesimal cosets.

Definition 7. We say that G (or $[\![\mathfrak{H}]\!]_{\Gamma,g}$) is *topologically tame* if

(1) $[\![\overline{\mathfrak{H}}]\!] = [\![\overline{\mathfrak{H}}]\!]_{T\text{-reg}}$.
(2) $[\mathsf{B}]^{\mathsf{pre}}$ generates the $[\tau_T]$-topology.

Denote by $\widehat{T}$ the metric completion of T. Every point of $\hat{t} \in \widehat{T}$ is determined by a regular sequence of opens sets $U = \{U_\alpha\} \subset T$ in which $\mathsf{diam}(U_\alpha) \to 0$. For $[\![\mathfrak{H}]\!]$ topologically tame, we define a map

$$\Theta : [\![\overline{\mathfrak{H}}]\!] \longrightarrow \widehat{T}$$

by $[\![\overline{h}]\!] \mapsto \hat{t}$ if for some U which determines $\hat{t}$,

$$\sigma_g(p([\![\overline{h}]\!])) \;\in\; {}^{*}\Gamma \cdot [U]^{\mathsf{pre}}.$$

The ${}^{*}\tau_T$-topology on $[\![\overline{\mathfrak{H}}]\!]$ is defined by pulling back the topology of $\widehat{T}$ along Θ. We write $[\![\overline{h}]\!] \sim [\![\overline{h}]\!]'$ if there exists an infinitesimal coset $[H]^{\mathsf{pre}}$ containing both of $[\![\overline{h}]\!]$, $[\![\overline{h}]\!]'$. Then we have the following

Theorem 13. Θ *induces a homeomorphism*

$$\Theta_{/\sim} : ([\![\overline{\mathfrak{H}}]\!]/\sim) \longrightarrow \widehat{T}.$$

Proof. Since the set $[\mathsf{B}]^{\mathsf{pre}}$ generates the $[\tau_T]$-topology, it follows that $\Theta([\![\overline{h}]\!]) = \Theta([\![\overline{h}]\!]')$ if and only if $[\![\overline{h}]\!] \sim [\![\overline{h}]\!]'$. By definition of ${}^{*}\tau_T$, it follows that Θ descends to a homeomorphism $\Theta_{/\sim}$. $\square$

[8]This is somewhat abusive, since we are only using right cosets $G_i g$ where $g \in G_0 \subsetneq \widetilde{\mathfrak{H}}$.

6.4. **Locally Free Lie Group Actions.** Let M be a manifold of dimension n, $X \subset M$ a subspace, $\mathfrak{B}$ a Lie group of dimension $k < n$, $\theta : \mathfrak{B} \to \mathsf{Homeo}(X)$ a continuous representation. Let T be a transversal of X at x, $T = \{T_i\}$ a neighborhood basis in T about x. Fix a metric on T inducing its topology. Let $p : \widetilde{\mathfrak{B}} \to \mathfrak{B}$ be the universal cover homomorphism. See §3.2 for more notation and terminology.

The construction of a germ completion $[\![\overline{\mathfrak{B}}]\!]_{X,x}$ for $[\![\mathfrak{B}]\!]_{X,x}$ is very similar to that discussed in §6.3. Our discussion here will be somewhat abbreviated.

Let $Y = \{Y_\alpha\}$ be a nested sequence of subsets in $\widetilde{\mathfrak{B}}$ for which $p(Y_\alpha) \subset T$. Then the ultraproduct $[Y]^{\mathsf{pre}}$ is called a T-regular coset if there exists a regular nested set of opens $\{U_\alpha\} \subset T$, $\mathsf{diam}(U_\alpha) \to 0$, so that

$$p(Y_\alpha) \;=\; U_\alpha \cap \left(\mathfrak{B} \cdot x\right).$$

For any nested set Y in $\widetilde{\mathfrak{B}}$, denote by ${}^\star Y$ the image of the ultraproduct $[Y]^{\mathsf{pre}}$ in ${}^\star\widetilde{\mathfrak{B}}$. Suppose $[Y]^{\mathsf{pre}}$ is T-regular. If there exists a sequence of translates $H = \{G_{i_\alpha} g_\alpha\}$, $i_\alpha \to \infty$ and $g_\alpha \in \widetilde{\mathfrak{B}}$, such that

$${}^\star Y = {}^\star H\,,$$

then $[Y]^{\mathsf{pre}}$ is called an infinitesimal coset.

Denote by $[\![\overline{\mathfrak{B}}]\!]$ the union of the infinitesimal cosets, $[\![\overline{\mathfrak{B}}]\!]_{T-\mathrm{reg}}$ the union of the T-regular cosets. Let $[\mathsf{B}]^{\mathsf{pre}}$ be the set of infinitesimal cosets. The set of T-regular cosets forms a topology $[\tau_T]$ on $[\![\overline{\mathfrak{B}}]\!]_{T-\mathrm{reg}}$.

We say that $[\![\mathfrak{B}]\!]_{X,x}$ is topologically tame if $[\![\overline{\mathfrak{B}}]\!] = [\![\overline{\mathfrak{B}}]\!]_{T-\mathrm{reg}}$ and if $[\mathsf{B}]^{\mathsf{pre}}$ generates the topology $[\tau_T]$.

Let $\widehat{T}$ be the metric completion of T. We define $\sim$ and $\Theta : [\![\overline{\mathfrak{B}}]\!] \to \widehat{T}$ exactly as in §6.3. If we topologize $[\![\overline{\mathfrak{B}}]\!]$ by pulling back the topology of $\widehat{T}$ along Θ, then we have

Theorem 14. Θ *induces a homeomorphism*

$$\Theta_{/\sim} : ([\![\overline{\mathfrak{B}}]\!]/\sim) \longrightarrow \widehat{T}.$$

7. The Complete Groupoid Structure

Let $[\![G]\!]$ be a topologically tame germ, $[\![\overline{G}]\!]$ its non-standard completion. In addition, if $[\![G]\!]$ comes from a double coset topology or a Lie group action, we assume that ${}^\star G$ is a group.

Until now, we have understood the groupoid structure of $[\![G]\!]$ exclusively in terms of its action on itself: that is, we say that $[\![g]\!][\![h]\!]$ is defined if $[\![gh]\!] \in [\![G]\!]$. In this section, we will expand the groupoid structure to take into account the action of $[\![G]\!]$ on $[\![\overline{G}]\!]$. This will be necessary, among other things, for the consideration of germ covering spaces.

7.1. **Definition of the Complete Groupoid Structure.** Denote by $\widehat{G}$ any of the spaces $\widehat{G}$, $\widehat{\mathfrak{G}}_r$, $\widehat{F}$, $\widehat{F}^\natural$ or $\widehat{T}$ occurring in Theorems 8, 9, 11 *etc*. Let $\sim$ the equivalence relation giving rise to the homeomorphism $([\![\overline{G}]\!]/\sim) \;\approx\; \widehat{G}$.

We define a groupoid structure on $[\![G]\!]$ using its action on $[\![\overline{G}]\!]$. For any $[\![u]\!] \in [\![G]\!]$, let

$$\mathsf{Dom}[\![u]\!] \;=\; \left\{ [\![\overline{g}]\!] \in [\![\overline{G}]\!] \;\middle|\; [\![u]\!]\cdot[\![\overline{g}]\!] = [\![u\overline{g}]\!] \in [\![\overline{G}]\!] \text{ and is } \sim \text{ to } [\![\overline{g}]\!] \right\}$$

$$\mathsf{Ran}[\![u]\!] \;=\; [\![u]\!]\cdot\mathsf{Dom}[\![u]\!].$$

Note 63. If we write $[\![u]\!] = [\![gh^{-1}]\!] \in [\![G]\!]$, for $[\![g]\!], [\![h]\!] \in [\![G]\!]^{\mathsf{pre}}$, then

$$[\![h]\!] \in \mathsf{Dom}[\![u]\!].$$

Thus $\mathsf{Dom}[\![u]\!] \neq \emptyset$.

We define the composition $[\![g]\!][\![h]\!]$ whenever $[\![gh]\!] \in [\![G]\!]$ and

$$\mathsf{Dom}[\![g]\!] \cap \mathsf{Ran}[\![h]\!] \neq \emptyset.$$

Recall that a groupoid is a category such that every morphism is invertible.

Theorem 15. *$[\![G]\!]$ is a groupoid with respect to its action on $[\![\overline{G}]\!]$.*

Proof. The objects consist of the sets $\mathsf{Dom}[\![u]\!]$, $\mathsf{Ran}[\![u]\!]$, the morphisms the elements $[\![u]\!] \in [\![G]\!]$. The inverse $[\![u]\!]^{-1}$ of an element of $[\![G]\!]$ is in $[\![G]\!]$ by construction. Associativity is a triviality except when $[\![G]\!]$ comes from a double coset topology or a Lie group action: but we are assuming in this section that ${}^{\star}G$ is a group, so by Proposition 15, we are done. □

The groupoid structure on $[\![G]\!]$ obtained from the unit space $[\![\overline{G}]\!]$ in henceforward referred to as the *complete groupoid structure*.

7.2. **Change of Unit Space.** Given a groupoid C, the unit space is defined to be

$$X = \Bigg(\bigcup_{\gamma \in \mathsf{MorC}} \mathsf{Dom}\gamma \Bigg) \bigcup \Bigg(\bigcup_{\gamma \in \mathsf{MorC}} \mathsf{Ran}\gamma \Bigg).$$

Clearly the unit space of $[\![G]\!]$ with respect to the complete groupoid structure is $[\![\overline{G}]\!]$.

Let C, C' be two groupoids. A groupoid homomorphism is a functor $\mathsf{F} : \mathsf{C} \to \mathsf{C}'$. A groupoid isomorphism is an invertible groupoid homomorphism.

By *Note* 3, $[\![G]\!]$ is independent of shift in nested set. Thus, if $\{n_0, n_1, \dots\} \subset \mathbb{N}$ is infinite, and we let $G' = \{G_{n_i}\}$, we have a bijection $[\![G]\!]_G \longleftrightarrow [\![G]\!]_{G'}$. Let $[\![\overline{G}]\!]'$ be the subset of $[\![\overline{G}]\!]$ consisting of elements that are represented by sequences which are eventually in G_{n_0}. It is natural to ask what happens to the complete groupoid structure if we replace the unit space $[\![\overline{G}]\!]$ by $[\![\overline{G}]\!]'$.

Note 64. If $[\![\overline{G}]\!]$ is a group, it is not necessarily true that $[\![\overline{G}]\!]'$ is a group as well.

Let $[\![G]\!]'$ be the groupoid whose morphisms are the elements of $[\![G]\!]$ and whose unit space is $[\![\overline{G}]\!]'$.

Note 65. There is an injective groupoid homomorphism $\mathsf{F} : [\![G]\!]' \to [\![G]\!]$ given by the identity on morphisms, and by the correspondence $\mathsf{Dom}'[\![u]\!] \longmapsto \mathsf{Dom}[\![u]\!]$.

Lemma 4. *For all $[\![\overline{g}]\!] \in [\![\overline{G}]\!]$, there exists $g \in G$ such that $[\![\overline{g}g]\!] \in [\![\overline{G}]\!]'$.*

Proof. Let us consider the case of an action topology *i.e.* G acts on a space F with dense orbit. Let x be the point with respect to which the nested set G is constructed. Since $[\![G]\!]$ is topologically tame, we may right-translate any neighborhood basis U about x to $U \cdot g$, a neighborhood basis about $y = g(x)$. This means that any infinitesimal coset in $[\![\overline{G}]\!]$ can be right-translated to an infinitesimal coset in $[\![\overline{G}]\!]'$. The cases presented by a quasi-suspension, double coset and locally free Lie group action are similar, and are left to the reader. □

Theorem 16. *F is a groupoid isomorphism.*

Proof. We must show that whenever $\llbracket u\rrbracket\llbracket v\rrbracket$ is defined in $\llbracket G\rrbracket$, it is defined as well in $\llbracket G\rrbracket'$. Let $\llbracket \bar{g}\rrbracket \in \mathsf{Dom}\llbracket uv\rrbracket$. If $[\mathcal{H}]^{\mathsf{pre}}$ is an infinitesimal coset containing $\llbracket \bar{g}\rrbracket$, then by Lemma 4 we may find an element $g \in G$ such that $[\mathcal{H}\cdot g]^{\mathsf{pre}}$ is an infinitesimal coset with respect to G'. It follows then that the action of $\llbracket uv\rrbracket$ on $\llbracket \bar{g}g\rrbracket$ is defined and $\sim \llbracket \bar{g}g\rrbracket$. Thus $\mathsf{Dom}'\llbracket uv\rrbracket \neq \emptyset$, and F is an isomorphism. ☐

8. The Fundamental Germ at Infinity

The fundamental germ of a weakly-minimal lamination $\mathcal{L}$ describes the asymptotic accumulation of a dense leaf L onto itself. On the other hand, compact laminations which are not weakly-minimal always have sublaminations on which every leaf accumulates. In §8.2, we define the fundamental germ at infinity $\llbracket \overset{\infty}{\pi}\rrbracket_1$, which records algebraically the accumulation of a leaf on such a sublamination. In addition to extending the germ perspective to non weakly-minimal laminations such as the Reeb foliation, $\llbracket \overset{\infty}{\pi}\rrbracket_1$ also gives an indication of the limiting behavior of $\llbracket \pi\rrbracket_1$ as the base point x approaches transversally a point $\hat{x}$ on a distinct leaf.

8.1. **Minimal Sets.** Let $\mathcal{L}$ be any laminated space, not neccessarily weakly-minimal. A set is said to be saturated if it meets every leaf of $\mathcal{L}$. A minimal set $\mathcal{M}$ is a minimal element in the set of subspaces of $\mathcal{L}$, ordered by inclusion, which are non-empty, closed and saturated in $\mathcal{L}$ [Go].

Note 66. Minimal sets, when they exist, are sublaminations. Compact laminations always have minimal sets; trivial laminations do not.

Note 67. If Γ is a co-finite volume Fuchsian group, then the minimal sets of the horocyclic flows $\mathcal{H}^{\pm}_{\Gamma}$, are the closed orbits. The Reeb foliation of S^3, which is not weakly-minimal, has minimal set a 2-torus.

Let $\mathcal{L}$ be a lamination with minimal set $\mathcal{M}$, $x \in \mathcal{L}$ a point contained in a leaf L, $\hat{x} \in \mathcal{M}$. In order to define the fundamental germ at infinity, we consider sequences of transformations of x which converge to $\hat{x}$. Let us do this following our train of examples.

8.2. **Suspensions.** Suppose $\mathcal{L} = \widetilde{B}\times_{\rho} F$ is a suspension with minimal set $\mathcal{M}$. Given $\hat{x} \in \mathcal{M}$, choose $x \in \mathcal{L}$ so that $\hat{x} \in F = F_x$ = the fiber through x. Let $\{U_i\}$ be a neighborhood basis about $\hat{x}$ in F. Define a nested set $\mathcal{H}$ by

$$H_i \;=\; \{g \in \pi_1 B \mid\; gx \in U_i\}.$$

Definition 8. The *fundamental germ at infinity* (at $(x,\hat{x})$) is

$$\llbracket \overset{\infty}{\pi}\rrbracket_1(\mathcal{L};\, x,\hat{x}) \;:=\; \llbracket \mathcal{H}\rrbracket \;=\; \bigodot_{\mathfrak{U}} H_i H_i^{-1}.$$

Note 68. $[\mathcal{H}]^{\mathsf{pre}} = \prod_{\mathfrak{U}} H_i$ is a basis element of the topology defined on the germ completion $\llbracket \overline{\pi_1 B}\rrbracket$.

Note 69. Viewed as acting on itself on the left, $\llbracket \overset{\infty}{\pi}\rrbracket_1(\mathcal{L};\, x,\hat{x})$ is a groupoid. In contrast, this is false for both of $[\mathcal{H}]^{\mathsf{pre}}$ and $\llbracket \mathcal{H}\rrbracket^{\mathsf{pre}}$, since neither contain 1.

Given a manifold M, let $\widehat{M}$ be an inverse limit solenoid associated to a system $\mathcal{C} = \{M_i \to M\}$ of normal covers over M (see §4.3). Let $\mathcal{G} = \{H_i\} \subset \pi_1 M$ be the associated system of subgroups.

By *Note* 40, any element $\hat{x}$ in the fiber of x is a translate $\hat{\gamma}\cdot x$, where $\hat{\gamma} \in \widehat{\pi}_1 M$. Let $\{\gamma_\alpha\}$ be a sequence in $\pi_1 M$ converging to $\hat{\gamma}$ in the profinite topology. We may choose $\{\gamma_\alpha\}$ and

$i_\alpha \to \infty$ so that $H = \{\gamma_\alpha H_{i_\alpha}\}$ is an infinitesimal coset corresponding to a neighborhood basis about $\hat{x}$. We use this H to define $[\![\overset{\infty}{\pi}]\!]_1(\widehat{M};\, x,\hat{x})$. Denote by $[\![\hat{\gamma}]\!]$ the element of the germ $[\![\overset{\infty}{\pi}]\!]_1(\widehat{M};\, x,\hat{x})$ defined by $\{\gamma_\alpha\}$.

Theorem 17. $[\![\overset{\infty}{\pi}]\!]_1(\widehat{M};\, x,\hat{x})$ *is a group, isomorphic to* $[\![\pi]\!]_1(\widehat{M},x)$.

Proof. Every element of $[\![h]\!] \in [\![H]\!]^{\mathrm{pre}}$ may be written in the form

$$[\![h]\!] = [\![\hat{\gamma}]\!][\![u]\!]$$

for $[\![u]\!] \in [\![\pi_1]\!](\widehat{B},x)$. It follows that

$$[\![\overset{\infty}{\pi}]\!]_1(\widehat{M};\, x,\hat{x}) \;=\; [\![\hat{\gamma}]\!]\cdot[\![\pi]\!]_1(\widehat{M},x)\cdot[\![\hat{\gamma}]\!]^{-1} \;\cong\; [\![\pi]\!]_1(\widehat{M},x). \tag{16}$$

□

8.3. **Quasi-suspensions.** Let $L = \widetilde{B}\times_\rho F$ be a normal, weakly-minimal suspension, $L^\natural = L/H$ a quasi-suspension. Choose $\hat{x}^\natural$ a point in a minimal set of $L^\natural$, $x^\natural$ a point on a dense leaf of $L^\natural$, $F^\natural$ a fiber through $x^\natural$. We assume that $x^\natural$ has been selected so that $\hat{x}^\natural \in F^\natural$. Let x, $\hat{x}$ and F' be two points and an open subset of a fiber F in L covering $x^\natural$, $\hat{x}^\natural$ and $F^\natural$, respectively. Let $U^\natural = \{U_i^\natural\}$ be a neighborhood basis about $\hat{x}^\natural$ in $F^\natural$. We may assume that $U^\natural$ is evenly covered with respect to $L \to L^\natural$: let $U = \{U_i\}$ be the lifted neighborhood basis about $\hat{x}$ in F'. This defines a nested set $H^\natural$ by

$$H_i^\natural \;=\; \{h\gamma \mid\; h\in H,\, \gamma\in\pi_1 B \text{ and } \overline{\gamma}\cdot x\in U_i\}\,.$$

The *fundamental germ at infinity* of $L^\natural$ (at $(x^\natural,\hat{x}^\natural)$) is defined

$$[\![\overset{\infty}{\pi}]\!]_1(L^\natural;\, x^\natural,\hat{x}^\natural) \;=\; [\![H^\natural]\!].$$

Consider in this connection the Sullivan solenoid $\widehat{S}_f$ associated to a degree d polynomial-like map $f: U\to V$. Recall that $\widehat{S}_f$ is a quasi-suspension $\widehat{A}/H$, where $H = \langle \hat{f}\rangle$ and $\hat{f}:\widehat{A}\to\widehat{A}$. (See §4.6 for details.)

Theorem 18. $[\![\overset{\infty}{\pi}]\!]_1(\widehat{S}_f; x^\natural,\hat{x}^\natural) \;=\; [\![\pi]\!]_1(\widehat{S}_f,x^\natural)$.

Proof. Every element of $[\![h^\natural]\!] \in [\![\overset{\infty}{\pi}]\!]_1(\widehat{S}_f; x^\natural,\hat{x}^\natural)$ may be written in the form

$$[\![h^\natural]\!] \;=\; \hat{f}^{*m}\cdot[\![h]\!]\cdot\hat{f}^{*n}$$

where $[\![h]\!] \in [\![\overset{\infty}{\pi}]\!]_1(\widehat{A};\, x,\hat{x})$ and ${}^*m, {}^*n \in {}^*\mathbb{Z}$. In fact, by (16), we may write

$$[\![h^\natural]\!] \;=\; \hat{f}^{*m}\cdot[\![\hat{\gamma}]\!][\![u]\!][\![\hat{\gamma}]\!]^{-1}\cdot\hat{f}^{*n}$$

where $[\![u]\!] \in [\![\pi]\!]_1(\widehat{A},x)$ and $[\![\hat{\gamma}]\!]$ is the element used in Theorem 17. But $\widehat{S}_f$ is a quotient of a suspension over a base B homeomorphic to an annulus. Thus $\pi_1 B$ is abelian, and we have

$$[\![\hat{\gamma}]\!][\![u]\!][\![\hat{\gamma}]\!]^{-1} \;=\; [\![u]\!].$$

□

8.4. **Double Cosets and Locally Free Lie Group Actions.**

8.4.1. *Double Cosets.* Let $F_{\mathfrak{H},\Gamma}$ be the double coset foliation associated to the triple $(\mathfrak{G},\mathfrak{H},\Gamma)$. Let $T=\{T_i\}$ be a basis of local sections about 1. Let $\hat{g}$ be such that $\hat{g}\mathfrak{H}$ projects to $\Gamma\backslash\mathfrak{G}$ as the leaf of a minimal set. Choose minimal $g\in\mathfrak{G}$. We define a nested set H by

$$H_i \;=\; \left\{\tilde{h}\in\widetilde{\mathfrak{H}}\;\middle|\; g\cdot p(h)\cdot\hat{g}^{-1}\;\in\;\Gamma\cdot T_i\right\}.$$

Then the *fundamental germ at infinity* (at $(g,\hat{g})$ is

$$[\![\overset{\infty}{\pi}]\!]_1(F_{\mathfrak{H},\Gamma};\, g,\hat{g}) \;=\; [\![H]\!].$$

8.4.2. *Locally Free Lie Group Actions.* Now let $\mathfrak{B}$, $X\subset M^n$ and $\theta:\mathfrak{B}\to\mathsf{Homeo}(X)$ be as in §3.2; denote by $L_{\mathfrak{B}}$ the associated lamination of X. Let T be a transversal containing points $x,\,\hat{x}\in X$, where x lies in a dense leaf and $\hat{x}$ in a minimal set. Let $T=\{T_i\}$ be a neighborhood basis about $\hat{x}$ in T. We define

$$H_i \;=\; \left\{\tilde{g}\in\widetilde{\mathfrak{B}}\;\middle|\;\theta_{p(\tilde{g})}(x)\in T_i\right\}.$$

Then the *fundamental germ at infinity* (at $(x,\hat{x})$) is

$$[\![\overset{\infty}{\pi}]\!]_1(L_{\mathfrak{B}};\, x,\hat{x}) \;=\; [\![H]\!]$$

8.5. **The Reeb Foliation.** In this section, we consider a lamination which is not weakly-minimal: the Reeb foliation F_{Reeb} of the solid 3-torus $\mathbf{T}^3$.

Let $\mathbb{R}_+$ = the non-negative reals, $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ and write

$$(\mathbb{C}\times\mathbb{R}_+)^*=\mathbb{C}\times\mathbb{R}_+\setminus\{(0,0)\}.$$

Choose $\lambda\in\mathbb{C}$ and $\mu\in\mathbb{R}_+$ so that

$$|\lambda|,\ \mu>1 \quad\text{and}\quad \lambda\neq\mu,$$

and define an equivalence relation on $(\mathbb{C}\times\mathbb{R}_+)^*$ by

$$(z,\,t)\;\sim\;(\lambda z,\,\mu t).$$

The Reeb foliation of $\mathbf{T}^3$ may be defined

$$F_{\mathsf{Reeb}}=(\mathbb{C}\times\mathbb{R}_+)^*/\sim.$$

Let $P:(\mathbb{C}\times\mathbb{R}_+)^*\to F_{\mathsf{Reeb}}$ denote the projection map.

Note 70. The leaves of F_{Reeb} are of two types:

(1) $L_{[t]}=P(\mathbb{C}\times\{t\})$, $t>0$. Such a leaf is isomorphic to $\mathbb{C}$.
(2) $L_0=P(\mathbb{C}^*\times\{0\})$. This leaf is called the boundary leaf: it may be identified with the quotient $\mathbb{C}^*/<\lambda>$, which is in turn isomorphic to $\mathbb{T}^2$. L_0 is the minimal set of F_{Reeb}.

Note 71. The fiber tranversals of F_{Reeb} are of two types:

(1) $T_{[z]}=P(\{z\}\times\mathbb{R}_+)$, $z>0$. Such a transversal is homeomorphic to $\mathbb{R}_+$, and intersects every leaf of F_{Reeb}.
(2) $T_0=P(\{0\}\times(0,\infty))$. Such a transversal is homeomorphic to S^1, and intersects every leaf except L_0.

There is an action of $\mathbb{Z}$ on $F_{\mathsf{Reeb}}\setminus L_0$ induced by the map

$$(z,\,t)\mapsto(\lambda^n z,\,t).$$

For $x\in F_{\mathsf{Reeb}}\setminus L_0$, we write this action $x\mapsto n\cdot x$.

Note 72. For every $t > 0$ and $z \neq 0$,

$$n \cdot L_{[t]} = L_{[t]} \quad \text{and} \quad n \cdot T_{[z]} = T_{[z]}.$$

F_{Reeb} is a quotient of the form $(D \setminus \{\mathsf{point}\})/\mathbb{Z}$, where D is a deck of cards and $\mathbb{Z}$ acts properly discontinuously in a diagonal fashion. We will therefore view F_{Reeb} as suspension-like, and define $[\![\overset{\infty}{\pi}]\!]_1(F_{\mathsf{Reeb}};\, x,\hat{x})$ accordingly.

Let us consider the points $x = P((1,1)) \in L_{[1]}$ and $\hat{x} = P((1,0)) \in L_0$. Note that $x, \hat{x} \in T_{[1]}$. Let U_i be a neighborhood basis about $\hat{x}$ in $T_{[1]}$. Define a nested set in $\mathbb{Z}$ by

$$H_i \;=\; \{n \mid\; n \cdot x \in U_i\}.$$

The *fundamental germ at infinity* (at $(x,\hat{x})$) of the Reeb foliation is

$$[\![\overset{\infty}{\pi}]\!]_1(F_{\mathsf{Reeb}};\, x,\hat{x}) \;=\; [\![H]\!].$$

Theorem 19. $[\![\overset{\infty}{\pi}]\!]_1(F_{\mathsf{Reeb}};\, x,\hat{x}) \;=\; {}^*\mathbb{Z}.$

Proof. A sequence $\{n_\alpha\}$ converges with respect to the nested set H if and only if it is infinite. Thus the pre-germ $[\![H]\!]^{\mathrm{pre}}$ is identifiable with ${}^*\mathbb{Z}_\infty := {}^*\mathbb{Z} \setminus {}^*\mathbb{Z}_{\mathrm{fin}}$, the infinite nonstandard integers. Then

$$[\![H]\!] \;=\; {}^*\mathbb{Z}_\infty - {}^*\mathbb{Z}_\infty \;=\; {}^*\mathbb{Z}.$$

□

Note 73. Intuitively, $[\![\overset{\infty}{\pi}]\!]_1(F_{\mathsf{Reeb}};\, x,\hat{x})$ records the approximation by the dense leaf of the circumferential cycle $c \subset L_0$ through $\hat{x}$. This is borne out by the fact that ${}^*\pi_1(c,\hat{x}) \cong {}^*\mathbb{Z}$ as well.

Note 74. The Reeb foliation $F_{\mathsf{Reeb}}(S^3)$ of the 3-sphere is obtained from two copies F, F' of F_{Reeb} by identifying the toral boundaries L_0, L_0' so that the circumference of L_0 is identified with the waist of L_0'. It follows that $[\![\overset{\infty}{\pi}]\!]_1(F';\, x',\hat{x})$ records the approximation by a dense leaf $L' \subset F'$ of the waist cycle of the toral leaf of $F_{\mathsf{Reeb}}(S^3)$. Thus, all of the topology of the toral leaf of $F_{\mathsf{Reeb}}(S^3)$ is predicted by the two germs at infinity calculated from within and without the toral leaf.

9. The Higher Order Fundamental Germ

When attempting to relate fundamental germs $[\![\pi]\!]_1(L,x)$ and $[\![\pi]\!]_1(L,y)$, where x and y lie on distinct leaves, one faces the problem of converting a sequence class $[\![u]\!] \in [\![\pi]\!]_1(L,x)$ to one in $[\![\pi]\!]_1(L,y)$. Unfortunately, unless L admits enough leaf-shuffling symmetries, this is not possible. In this section, we replace equivalence classes of convergent sequences by certain classes of convergent sets to obtain the higher order fundamental germ $\{\!\!\{\pi\}\!\!\}_1$. It has the advantage of always allowing us to transform elements calculated with respect to one leaf to those calculated with respect to another.

9.1. Boolean Tri-algebras and Power Set Germs.

Let G be a group. The power set

$$2^G$$

is a Boolean algebra with respect to the operations $\cup$ and $\cap$. In addition, it has a third operation $\boxdot$ defined

$$A \boxdot B = \{ab \mid a \in A \text{ and } b \in B\}.$$

The operation $\boxdot$ is distributive over both $\cup$ and $\cap$: thus 2^G has three mutually distributive operations. The unit elements of these operations are respectively $1_\cup = \emptyset$, $1_\cap = G$ and $1_\boxdot = \{1\}$. (We define $A \boxdot \emptyset = \emptyset \boxdot A = \emptyset$.)

Definition 9. A Boolean algebra possessing a third operation which distributes over $\cup$ and $\cap$ is called a *Boolean tri-algebra*.

Note 75. The map
$$G \hookrightarrow 2^G, \quad g \mapsto \{g\}$$
is an isomorphic embedding of G as a subgroup with respect to the operation $\boxdot$.

The *non-standard power set* of G is defined as the ultrapower
$$^{\circledast}G \ := \ ^{*}(2^G).$$

Note 76. The operations $\cup$, $\cap$ and $\boxdot$ are defined on $^{\circledast}G$ in the obvious way, making $^{\circledast}G$ a Boolean tri-algebra.

Note 77. There is a natural inclusion of Boolean tri-algebras
$$^{\circledast}G \hookrightarrow 2^{^{*}G} \ :$$
if $^{*}A$ is represented by the sequence $\{A_i\}$, we define
$$^{*}A \longmapsto \prod_{\mathfrak{U}} A_i \ \in \ 2^{^{*}G}.$$
Note that this map is not surjective.

Given $G = \{G_i\}$ a nested set about 1 in G, let $^{\circledcirc}G$ to be the set of sequences $\{A_\alpha\}$, $A_\alpha \in 2^G$, which converge with respect to G: that is, for every i, $A_\alpha \subset G_i$ for α sufficiently large. The direct limit
$$\{\!\{G\}\!\} \ := \ \varinjlim \left({}^{\circledcirc}G \boxdot {}^{\circledcirc}G^{-1} \right)$$
is called the *power set germ*.

$\{\!\{G\}\!\}$ is a subring of $^{\circledast}G$ with respect to $\cup$ and $\cap$, but in general the operation $\boxdot$ is only partially defined. In this event, we call this structure a *Boolean tri-algebroid*.

Note 78. There is an isomorphic inclusion
$$[\![G]\!] \hookrightarrow \{\!\{G\}\!\}, \qquad [\![g_i]\!] \mapsto \{\!\{\{g_i\}\}\!\}.$$
In addition, there is a map
$$\{\!\{G\}\!\} \longrightarrow 2^{[\![G]\!]}$$
defined by the association
$$\{\!\{A_\alpha\}\!\} \longmapsto \prod_{\mathfrak{U}} A_\alpha.$$

Theorem 20. *If $G = \{H_i\}$ is a nested set of subgroups of G, $\{\!\{G\}\!\}$ is a Boolean tri-algebra.*

Proof. The product $A \boxdot B \subset H_i$ whenever $A, B \subset H_i$. □

Theorem 21. *Let $\mathfrak{G}$ ($\rho : G \to \mathfrak{G}$) be a (representation into a) topological group and let $G = \{U_i\}$ ($G = \{\rho^{-1}(U_i)\}$) be (the pull-back of) a neighborhood basis about 1. Then $\{\!\{\mathfrak{G}\}\!\}$ ($\{\!\{G\}\!\}$) is a Boolean tri-algebra.*

Proof. If $A_\alpha, B_\alpha \to 1$, the same is true of the product $A_\alpha \boxdot B_\alpha$. □

For example, if $G = \mathbb{Z}^p$, $\mathfrak{G} = \mathbb{T}^q$ and ρ is defined as in *Germ Example* 6, §2.2, then $^{\circledast}\mathbb{Z}_{\mathbf{R}} := \{\!\{\mathbb{Z}\}\!\}$ is a Boolean tri-algebra.

9.2. **Full Subalgebroids and The Higher Order Fundamental Germ.** Let be a weakly-minimal, algebraic lamination, $x \in L$. The fundamental germ $[\![\pi]\!]_1(L,x)$ was defined in terms of the action of a group G on some space T : a model fiber F (in the case of a suspension), a quasi-fiber $F^{\natural}$ (in the case of a quasi-suspension), a local section or transversal T (in the case of either a double coset or a locally free Lie group action). We would like to distinguish elements of 2^G that correspond to open sets of T.

We then say that $A \in 2^G$ is *full* if the following condition is satisfied, according to case:

- $L = \widetilde{B} \times_{\rho} F$ is a suspension, $G = \pi_1 B$, $\mathsf{T} = F$. There exists an open set $U \subset F$ such that
$$A \;=\; \{g \in \pi_1 B \mid\; g(x) \in U\}.$$
- $L = H \backslash L_0$ is a quasi-suspension, G = the group generated by the actions of $\pi_1 B_0$ and H on the dense leaf $L_0 \subset L_0$, $\mathsf{T} = F'$ a section over the quasi-fiber $F^{\natural}$. There exists an open set $U \subset F'$ such that
$$A \;=\; \{h\gamma \mid\; h \in H,\, \gamma \in \pi_1 B_0 \text{ and } \overline{\gamma} \cdot x \in U\}.$$
- $L = F_{\mathfrak{H},\Gamma}$ is a double coset foliation associated to the triple $(\mathfrak{G},\mathfrak{H},\Gamma)$, x covered by minimal $g \in \mathfrak{G}$, $G = \widetilde{\mathfrak{H}}$, $\mathsf{T} = T$ a local section about 1. There exists an open set $U \subset T$ such that
$$A \;=\; \{\tilde{h} \in \widetilde{\mathfrak{H}} \mid\; \sigma_g(p(\tilde{h})) \in \Gamma \cdot U\}.$$
- $L = L_{\mathfrak{B}}$ is a lamination occurring as a locally free action of a Lie group $\mathfrak{B}$, $G = \widetilde{\mathfrak{B}}$, $\mathsf{T} = T$ a transversal through x. There exists an open set $U \subset T$ such that
$$A \;=\; \{\tilde{g} \in \widetilde{\mathfrak{B}} \mid\; \theta_{(p(\tilde{g}))}(x) \in U\}.$$

Definition 10. Let L, x, G be as above. The *higher order fundamental germ* of L (at x) is the sub Boolean tri-algebroid
$$\boldsymbol{\{}\pi\boldsymbol{\}}_1(L,x) \;\subset\; \{\!\{G\}\!\}$$
formed from elements $\{A_{\alpha}\} \in {}^{\circledcirc}G$ in which A_{α} is full for all α, or $\{\!\{A_{\alpha}\}\!\} = \emptyset$.

Note 79. We have
$$\boldsymbol{\{}\pi\boldsymbol{\}}_1(L,x) \cap [\![\pi]\!]_1(L,x) = \emptyset$$
as long as the topology of T is such that points are never open.

The *higher order fundamental germ at infinity*
$$\boldsymbol{\{}\overset{\infty}{\pi}\boldsymbol{\}}_1(L;\, x,\hat{x})$$
may be defined in the same fasion: we leave the details to the reader.

10. Dependence on Data

We fixed three pieces of data in order to define the fundamental germ of an algebraic lamination L: a point $x \in L$, a nested set $T = \{T_i\}$ of open sets in a space T and an ultrafilter $\mathfrak{U}$. In this section, we discuss what happens when the choice of data $(x, T, \mathfrak{U})$ is changed.

10.1. **Change of T.** We proceed case by case.

10.1.1. *(Quasi)-suspensions.*

Proposition 29. *Let (L, x) be a (quasi)-suspension. Then $[\![\pi]\!]_1(L, x)$ is independent of the choice of basis T in the (quasi)-fiber. If L is topologically tame, the complete groupoid structure is independent of T as well.*

Proof. Suppose L is a suspension. If we replace the basis T about x in the fiber F by T' we obtain compatible nested sets G and G' in $\pi_1 B$: in particular, ${}^\bullet G = {}^\bullet G'$, so the resulting germs are equal. In case L is topologically tame, denote by $[\![\overline{\pi}]\!]_1$ and $[\![\overline{\pi}]\!]_1'$ the germ completions corresponding to G and G'. Then the right coset topologies on either are compatible. If $T = T_0 \neq T' = T_0'$, then $[\![\overline{\pi}]\!]_1 \neq [\![\overline{\pi}]\!]_1'$. However, by an argument similar to that employed in Theorem 16, the identity map yields a groupoid isomorphism with respect to the unit spaces $[\![\overline{\pi}]\!]_1, [\![\overline{\pi}]\!]_1'$.

The case of a quasi-suspension is similar, and is left to the reader. □

Note 80. Proposition 29 is false for the lock-step germs considered in §2.1, *Germ Example* 2: if G, G' are compatible nested sets of subgroups of G,

$$[G]_G \neq [G]_{G'}.$$

10.1.2. *Double Cosets and Locally Free Lie Group Actions.* Let $F_{\mathfrak{H},\Gamma}$ be a weakly-minimal double coset foliation, in which ${}^\star\widetilde{\mathfrak{H}}$ is a group. Consider two bases of local sections about 1, T and T'. Denote by $[\![\pi]\!]_1$ and $[\![\pi]\!]_1'$ the corresponding fundamental germs.

Proposition 30. *There is a canonical isomorphism*

$${}^\star\iota : [\![\pi]\!]_1(F_{\mathfrak{H},\Gamma}, \bar{g}) \longrightarrow [\![\pi]\!]_1'(F_{\mathfrak{H},\Gamma}, \bar{g}).$$

If $F_{\mathfrak{H},\Gamma}$ is topologically tame, ${}^\star\iota$ is an isomorphism of complete groupoid structures.

Proof. By *Note* 25, $[\![\pi]\!]_1(F_{\mathfrak{H},\Gamma}, \bar{g})$ and $[\![\pi]\!]_1'(F_{\mathfrak{H},\Gamma}, \bar{g})$ map injectively into ${}^\star\widetilde{\mathfrak{H}}$ with identical image. This defines ${}^\star\iota$. In case $F_{\mathfrak{H},\Gamma}$ is topologically tame, then the completions $[\![\overline{\pi}]\!]_1$ and $[\![\overline{\pi}]\!]_1$ also map injectively into ${}^\star\widetilde{\mathfrak{H}}$ with compatible image. Again, using an argument similar to that of Theorem 16, ${}^\star\iota$ yields a groupoid isomorphism. □

The case of locally free Lie group actions is entirely analogous to that of double cosets, and is left to the reader.

10.2. **Change of Base Point I.** We consider the effect of replacing the pointed lamination (L, x) by (L, x'). Let L, L' be the leaves containing x, x'. In this section, we limit ourselves to the case $L = L'$.

As before, we proceed case by case.

10.2.1. *(Quasi)-suspensions.*

Theorem 22. *Let L be a weakly-minimal (quasi)-suspension. If x, x' lie on the same leaf L, then*

$$[\![\pi]\!]_1(L, x) \;=\; [\![\pi]\!]_1(L, x').$$

If L is topologically tame, the complete groupoid structures are identical as well.

Proof. Suppose L is a suspension. The nested set used to define the germ depends only on the action of the deck group D_L on L and not on x. In particular, if L is topologically tame, the germ completions are identical, as are the attendant complete groupoid structures. The case of a quasi-suspension is left to the reader. □

10.2.2. *Double Cosets and Locally Free Lie Group Actions.* Let $F_{\mathfrak{H},\Gamma}$ be a weakly minimal double coset foliation, in which ${}^{\star}\widetilde{\mathfrak{H}}$ is a group.

Theorem 23. *Let $g, g' \in \mathfrak{G}$ be minimal such that $\bar{g}$ and $\bar{g}'$ lie on the same leaf. Then there exists an isomorphism*

$$[\![\iota]\!] : [\![\pi]\!]_1(F_{\mathfrak{H},\Gamma}, \bar{g}) \longrightarrow [\![\pi]\!]_1(F_{\mathfrak{H},\Gamma}, \bar{g}').$$

If $F_{\mathfrak{H},\Gamma}$ is topologically tame, $[\![\iota]\!]$ gives an isomorphism with respect to the complete groupoid structures as well.

Proof. If $\bar{g}$ and $\bar{g}'$ lie on the same leaf, then there exists $h \in \mathfrak{H}$ and $\gamma \in \Gamma$ such that

$$g \;=\; \gamma g' h_0.$$

By Proposition 21, it suffices to assume that $\gamma = 1$. Then

$$\sigma_g(p(\tilde{h})) \;\in\; \Gamma \cdot T_i$$

implies that

$$\sigma_{g'}(\sigma_{h_0}(p(\tilde{h}))) \;\in\; \Gamma \cdot T_i.$$

Let $\tilde{h}_0 \in \widetilde{\mathfrak{H}}$ be such that $p(\tilde{h}_0) = h_0$. Then the association

$$\tilde{h} \mapsto \sigma_{\tilde{h}_0}(\tilde{h})$$

defines the desired isomorphism. □

The analogue of Theorem 23 for locally free Lie group actions is very similar: the statement and its proof are left to the reader.

10.3. **Change of Base Point II.** In this section, we consider a base point change $x \mapsto x'$, where $L \neq L'$. Here, it is not even possible, in general, to assert the existence of a map of fundamental germs. On the level of higher order germs, however, we may at least define a set-theoretic map.

We begin with a positive case.

10.3.1. *Homogeneous (Quasi)-Suspensions.* Let $\mathfrak{G}$ be a topological group. A suspension

$$L = \widetilde{B} \times_{\rho} \mathfrak{G}$$

is called homogeneous if $\rho : \pi_1 B \to \mathsf{Homeo}(\mathfrak{G})$ is the right-multiplication representation induced by a representation $\theta : \pi_1 B \to \mathfrak{G}$. More specifically,

$$\rho_\gamma(g) = g \cdot (\theta(\gamma))^{-1}.$$

Note 81. Toral foliations and inverse limit solenoids are homogeneous suspensions.

Proposition 31. *A weakly-minimal homogeneous suspension is minimal. The fundamental germ $[\![\pi]\!]_1(L, x)$ is a group and is topologically tame.*

Proof. ρ has a dense orbit if and only if $\theta(\pi_1 B)$ is dense. But this means all orbits are dense. The nested set induced by ρ is the same as that induced by θ, so: 1) $[\![\pi]\!]_1(L, x)$ is a group by Proposition 11 and 2) L is topologically tame by the results of §5.3. □

Note 82. All of the leaves of a homogeneous suspension are homeomorphic to $\widetilde{B}/\mathsf{Ker}(\theta)$. In particular, there is a global action of $\pi_1 B$ on L, induced by the map of $\widetilde{B} \times \mathfrak{G}$ defined

$$(\tilde{x},\, g) \mapsto (\gamma \cdot \tilde{x},\, g).$$

A quasi-suspension $L^{\natural}$ is called homogeneous if it is formed from a homogeneous suspension.

Theorem 24. *Let $\mathcal{L}$ be a minimal, homogeneous (quasi)-suspension, and consider $x, x' \in \mathcal{L}$, possibly on different leaves. Then*

$$[\![\pi]\!]_1(\mathcal{L}, x) \;=\; [\![\pi]\!]_1(\mathcal{L}, x').$$

Proof. First, we assume that $\mathcal{L}$ is a suspension. We may assume that x, x' lie in the same fiber $F \approx \mathfrak{G}$. Let $h \in \mathfrak{G}$. The left-multiplication map on $\widetilde{B} \times \mathfrak{G}$, defined

$$(\tilde{x},\, g) \mapsto (\tilde{x},\, hg),$$

descends to a homeomorphism m_h of $\mathcal{L}$. In particular, there exists $h \in \mathfrak{G}$ such that $m_h(x) = x'$. The map m_h takes a basis about x to one about x': moreover, for $\gamma \in \pi_1 B$,

$$m_h(\gamma \cdot x) \;= \gamma \cdot m_h(x) \;= \gamma \cdot x'.$$

It follows that the germs are equal.

If $\mathcal{L}^\natural$ is a homogeneous quasi-suspension, then by the previous paragraph, we obtain the same nested set using either x or x'. □

10.3.2. *The Higher Order Map.*

Theorem 25. *Let $\mathcal{L}$ be a weakly-minimal, algebraic lamination and consider $x, x' \in \mathcal{L}$, possibly on different leaves. Then there exists a bijection*

$$\{\iota\} : \{\overset{\infty}{\pi}\}_1(\mathcal{L};\, x, x') \;\longleftrightarrow\; \{\pi\}_1(\mathcal{L}, x'). \tag{17}$$

Proof. Without loss of generality, we may assume that x and x' lie in a transversal T, with respect to which we define the relevant fundamental germs. Let $\{U_\alpha\} = \{A_\alpha \cdot B_\alpha^{-1}\}$ define an element $\{U\} \in \{\overset{\infty}{\pi}\}_1(\mathcal{L};\, x, x')$. Since each A_α, B_α is full, there exists a corresponding pair of sequences $\{V_\alpha\}$, $\{W_\alpha\}$ of open sets in T converging to x'. These sequences in turn uniquely define a pair of sequences $\{A'_\alpha\}$, $\{B'_\alpha\}$ which yield an element $\{U'\} \in \{\pi\}_1(\mathcal{L}, x')$. The correspondence $\{U\} \leftrightarrow \{U'\}$ yields the desired bijection. □

Theorem 26. *Let $\mathcal{L}$ be a minimal, homogeneous (quasi)-suspension. Then the map $\{\iota\}$ is an isomorphism.*

Proof. We assume again that x, x' lie in the same fiber $F \approx \mathfrak{G}$. Identify x with 1, x' with $\hat{g}$; for $\gamma \in \pi_1 B$, we simply write γ for $\theta(\gamma) \in \mathfrak{G}$. Let $\{A_\alpha \cdot B_\alpha^{-1}\}$ represent an element of $\{\overset{\infty}{\pi}\}_1(\mathcal{L};\, x, x')$, where $\{A_\alpha\}$, $\{B_\alpha\}$ are full sequences of subsets of $\pi_1 B$ converging to $\hat{g}$ in $\mathfrak{G}$. Denote by $\{V_\alpha\}$, $\{W_\alpha\}$ the corresponding open sets in $\mathfrak{G}$ that converge to $\hat{g}$. The map $\{\iota\}$ associates to $\{A_\alpha \cdot B_\alpha^{-1}\}$ a class represented by a sequence $\{A'_\alpha \cdot (B'_\alpha)^{-1}\}$, where for example

$$A'_\alpha \;=\; \{h \in \pi_1 B \mid\; h\hat{g} \in U_\alpha\}.$$

But then

$$A'_\alpha \cdot (B'_\alpha)^{-1} \;=\; A'_\alpha \hat{g} \cdot (B'_\alpha \hat{g})^{-1} \;\in\; V_\alpha \cdot W_\alpha^{-1},$$

so that $\{A'_\alpha \cdot (B'_\alpha)^{-1}\}$ also defines an element of $\{\!\{\overset{\infty}{\pi}\}\!\}_1(\mathcal{L};\, x, x')$. But this element is uniquely determined by the sequence $\{V_\alpha \cdot W_\alpha^{-1}\}$: that is, $A'_\alpha \cdot (B'_\alpha)^{-1} = A_\alpha \cdot B_\alpha^{-1}$. We leave the argument for quasi-suspensions to the reader. □

10.4. **Change of Ultrafilter.** Let G be a group of cardinality at most that of the continuum. The following is an immediate consequence of the proof of Theorem 1:

Theorem 27. *Let $\mathfrak{U}$, $\mathfrak{U}'$ be nonprincipal ultrafilters on $\mathbb{N}$, $\mathcal{G}$ a nested set about 1 in G. Denote by $[\![\mathcal{G}]\!]_{\mathfrak{U}}$ and $[\![\mathcal{G}]\!]_{\mathfrak{U}'}$ the corresponding germs. Then assuming the continuum hypothesis, there is an isomorphism*

$$[\![\mathcal{G}]\!]_{\mathfrak{U}} \;\cong\; [\![\mathcal{G}]\!]_{\mathfrak{U}'}.$$

If $\mathcal{G}$ is topologically tame, then the isomorphism is with respect to the complete groupoid structure.

11. Functoriality

A lamination map $F : \mathcal{L} \to \mathcal{L}'$ is a map such that

1. For every leaf $L \subset \mathcal{L}$, the image $F(L)$ is contained in a leaf of $\mathcal{L}$.
2. For every $x \in \mathcal{L}$, there exist transversals $T \ni x$, $T' \ni F(x)$, such that $F(T) \subset T'$.

Note 83. Let $\mathcal{F}$ be a foliation, M the underlying manifold (viewed as a lamination with one leaf). Then the canonical inclusion $\imath : \mathcal{F} \to M$ is not a lamination map (since the transversals of M are points).

11.1. **The Map of Fundamental Germs Induced by a Lamination Map.** Let

$$F : (\mathcal{L}, x) \longrightarrow (\mathcal{L}', x')$$

be a lamination map. Let $\mathcal{T} = \{T_i\}$ be the basis of transversals used to define $[\![\pi_1]\!](\mathcal{L}, x)$, $\mathcal{T}' = \{T_i'\}$ the basis of transversals used to define $[\![\pi_1]\!](\mathcal{L}', x')$. Let $\widetilde{F} : \widetilde{L} \to \widetilde{L}'$ be the lift of the restriction $F|_L$ to the universal covers. Denote by $\widetilde{\mathcal{T}} \subset \widetilde{L}$, $\widetilde{\mathcal{T}}' \subset \widetilde{L}'$ the pre-images of $\mathcal{T} \cap L$, $\mathcal{T}' \cap L'$.

In view of the results found in §10.1, we may assume without loss of generality that $\widetilde{F}(\widetilde{\mathcal{T}}) \subset \widetilde{\mathcal{T}}'$. It follows that we obtain a map

$$[\![\widetilde{\mathcal{T}}]\!] \longrightarrow [\![\widetilde{\mathcal{T}}']\!]$$

which induces, in particular, a map on fundamental germs

$$[\![F]\!] : [\![\pi_1]\!](\mathcal{L}, x) \longrightarrow [\![\pi_1]\!](\mathcal{L}', x').$$

Note 84. In exactly the same way, one may define a map $[\![\overset{\infty}{F}]\!]$ on the germ at infinity. If F is an open map along transversals, we may also define maps $\{F\}$ and $\{\overset{\infty}{F}\}$ of higher order germs.

Proposition 32. *Let $\mathcal{L} = \widetilde{B} \times F$ be a suspension. Then the map $[\![\xi]\!]$ induced by the projection $\xi : (\mathcal{L}, x) \to (B, x')$ is an injective groupoid homomorphism.*

Proof. The nested set of transversals $\mathcal{T}$ used to define $[\![\pi_1]\!](\mathcal{L}, x)$ is contained in $F_x = \xi^{-1}(x')$. In particular,

$$\widetilde{\mathcal{T}} \;\subset\; \pi_1 B \cdot \tilde{x},$$

where $\tilde{x} \in \widetilde{B}$ is the base point of $\widetilde{B}$. It follows immediately that $[\![\xi]\!]$ is the inclusion

$$[\![\pi_1]\!](\mathcal{L}, x) \;\subset\; {}^*\pi_1(B, x').$$

Since the product in $[\![\pi_1]\!](\mathcal{L}, x)$ is induced by multiplication in ${}^*\pi_1(B, x')$, $[\![\xi]\!]$ is a groupoid homomorphism. □

Unfortunately, we cannot assert in general that the map $[\![F]\!]$ induced by a lamination map F defines a groupoid homomorphism [9]. In the next section, we discuss a class of lamination maps which is sufficiently well-behaved so as to allow us to say more.

11.2. **Trained Lamination Maps.** Let $\mathcal{F}$ be a foliation, M the underlying space of $\mathcal{F}$, and $\imath : \mathcal{F} \to M$ the inclusion. Although $\imath$ is not a lamination map, we may nevertheless define a map on germs $[\![\imath]\!]$. The impetus comes from the following

Note 85. An element

$$[\![u]\!] = [\![gh^{-1}]\!] \in [\![\pi_1]\!](\mathcal{F}, x)$$

may be regarded as a class of sequences of homotopy classes of curves

$$\{\gamma_{u_\alpha}\} = \{\gamma_{g_\alpha} \circ \gamma_{h_\alpha}^{-1}\}$$

lying within the leaf containing x , where $\gamma_{g_\alpha}(0) = x = \gamma_{h_\alpha}(0)$, $\gamma_{u_\alpha}(0) = h_\alpha x = \gamma_{h_\alpha}(1)$ and $\gamma_{u_\alpha}(1) = g_\alpha x = \gamma_{g_\alpha}(1)$.

Given $[\![u]\!] \in [\![\pi]\!]_1(\mathcal{F}, x)$, let $\{u_\alpha\}$ be a representative. For α large, there is an open disc $O \subset M$ about x such that

$$\gamma_{u_\alpha}(0),\ \gamma_{u_\alpha}(1) \ \in\ O.$$

By connecting $\gamma_{u_\alpha}(0)$, $\gamma_{u_\alpha}(1)$ to x by paths contained in O, we obtain a map

$$[\![\imath]\!] : [\![\pi_1]\!](\mathcal{F}, x) \longrightarrow {}^*\pi_1(M, x)$$

which depends neither on O nor the choice of connecting paths.

More generally, given $\mathcal{L}$ a lamination and $\imath : \mathcal{L} \to X$ a map into a path-connected space, we may define a map $[\![\imath]\!] : [\![\pi_1]\!](\mathcal{L}, x) \to {}^*\pi_1(X, x)$.

Note 86. We may also define in the same way maps $[\![\overset{\infty}{\imath}]\!]$, $\{\imath\}$ and $\{\overset{\infty}{\imath}\}$ (the last two maps take values in ${}^{\circledast}\pi_1(X, x)$).

Definition 11. Let $\mathcal{L}$ be a weakly minimal lamination, X a path connected space. A map $\imath : (\mathcal{L}, x) \to (X, x)$ is called a *fidelity* if $[\![\imath]\!]$ is an injective homomorphism. We say that $\mathcal{L}$ is *faithful* if it has a fidelity.

Note 87. By Proposition 32, the projection $\xi : \mathcal{L} \to B$ of a suspension onto its base is a fidelity.

Proposition 33. *Let $\mathcal{F}_V$ be the foliation of $\mathbb{T}^{p+q}$ induced by the p-plane $V \subset \mathbb{R}^{p+q}$. Then the inclusion $\imath : \mathcal{F}_V \to \mathbb{T}^{p+q}$ is a fidelity.*

Proof. Recall (see §4.4) that for some $q \times p$ matrix $\mathbf{R}$, $[\![\pi]\!]_1 \mathcal{F}_V = {}^*\mathbb{Z}^p_{\mathbf{R}}$. Then for ${}^*\mathbf{n} \in {}^*\mathbb{Z}^p_{\mathbf{R}}$, the map $[\![\imath]\!]$ is

$$[\![\imath]\!]({}^*\mathbf{n}) = ({}^*\mathbf{n}, {}^*\mathbf{n}^\perp) \ \in\ {}^*\mathbb{Z}^{p+q} \ =\ {}^*\pi_1 \mathbb{T}^{p+q},$$

where ${}^*\mathbf{n}^\perp$ is the dual to ${}^*\mathbf{n}$. $[\![\imath]\!]$ is then clearly an injective homomorphism. □

The problem of the existence of fidelities for algebraic laminations seems interesting but difficult. We conjecture:

Conjecture 6. *Every weakly-minimal algebraic lamination $\mathcal{L}$ has a fidelity. If $\mathcal{L} = \mathcal{F}$ is a foliation with underlying manifold M, then the inclusion $\imath : \mathcal{F} \to M$ is a fidelity.*

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[9] Essentially, the problem is that we have restricted the nested set with which we construct $[\![\pi]\!]$ to live in a very particular group of transformations of $\widetilde{\mathcal{L}}$. In the presence of Riemannian geometry on $\widetilde{\mathcal{L}}$ with certain homogeneity properties, one can define a geometric fundamental germ which in this sense is more flexible: allowing us to assert that a much wider family of maps induce homomorphisms [Ge1], [Ge2].

Definition 12. A lamination map $F : \mathcal{L} \to \mathcal{L}'$ is *trained* if $\mathcal{L}$ and $\mathcal{L}'$ are faithful, and there exist fidelities $\imath : \mathcal{L} \to X$, $\imath' : \mathcal{L}' \to X'$ and a map $G : X \to X'$ such that

$$ {}^{*}G \circ [\![\imath]\!] = [\![\imath']\!] \circ [\![F]\!]. \tag{18}$$

The triple $(\imath, \imath', G)$ is called a *training* for F.

Theorem 28. *Let $F : (\mathcal{L}, x) \to (\mathcal{L}', x')$ be a trained lamination map. Then the induced map $[\![F]\!]$ is a groupoid homomorphism.*

Proof. Denote the product in $[\![\pi_1]\!](\mathcal{L}, x)$, $[\![\pi_1]\!](\mathcal{L}', x')$ by $\otimes$. Let $(\imath, \imath', G)$ be a training for F. Then by (18),

$$[\![\imath']\!][\![F]\!]\big([\![u]\!] \otimes [\![v]\!]\big) \;=\; [\![\imath']\!]\big([\![F]\!][\![u]\!] \otimes [\![F]\!][\![v]\!]\big).$$

Since $[\![\imath']\!]$ is injective,

$$[\![F]\!]\big([\![u]\!] \otimes [\![v]\!]\big) \;=\; [\![F]\!][\![u]\!] \otimes [\![F]\!][\![v]\!].$$

□

Corollary 4. *Let $F : (\mathcal{F}, x) \to (\mathcal{F}', x')$ be a map of foliations. Suppose that the inclusions into the underlying manifolds $\imath : \mathcal{F} \to M$, $\imath' : \mathcal{F}' \to M'$ are fidelities. Then $[\![F]\!]$ is a groupoid homomorphism.*

Proof. Take $G : M \to M'$ to be F, viewed as a map on underlying manifolds. Then $(\imath, \imath', G)$ is a training. □

In particular,

Corollary 5. *Any map $F : \mathcal{F}_V \to \mathcal{F}_{V'}$ of toral foliations induces a homomorphism $[\![F]\!]$ of fundamental germs.*

12. Germ Covering Spaces

We end our study with a brief discussion of covering space theory for $[\![\pi]\!]_1$. The themes we treat are universal covering, the correspondence between subgroupoids and lamination coverings maps and deck groupoids. Throughout this section, we assume that $\mathcal{L}$ is a weakly-minimal algebraic lamination with topologically tame fundamental germ.

12.1. **The Germ Universal Cover.** Let $x \in \mathcal{L}$ be contained in a dense leaf L. Let $p : \widetilde{L} \to L$ be the universal cover and denote by $[\![\overline{\pi}]\!]_1\mathcal{L}$ the germ completion associated to $[\![\pi]\!]_1(\mathcal{L}, x)$, equipped with the enlargement topology ${}^{*}\tau$. Given $[\![\overline{g}]\!] \in [\![\overline{\pi}]\!]_1\mathcal{L}$ represented by $\{g_\alpha\}$ and $\tilde{z} \in \widetilde{L}$, $[\![\overline{z}]\!] = [\![\overline{g}]\!] \cdot \tilde{z}$ is the element of ${}^{*}\widetilde{L}$ defined by the sequence $\{g_\alpha \cdot \tilde{z}\}$. We define the *germ universal cover* to be

$$[\![\overline{\mathcal{L}}]\!] \;=\; \left\{ [\![\overline{g}]\!] \cdot \tilde{z} \;\middle|\; [\![\overline{g}]\!] \in [\![\overline{\pi}]\!]_1\mathcal{L},\; \tilde{z} \in \widetilde{L} \right\}.$$

The trivial lamination

$$\widetilde{L} \times [\![\overline{\pi}]\!]_1\mathcal{L}$$

projects onto $[\![\overline{\mathcal{L}}]\!]$ by the map

$$(\tilde{z}, [\![\overline{g}]\!]) \mapsto [\![\overline{g}]\!] \cdot \tilde{z}.$$

We give $\widetilde{L} \times [\![\overline{\pi}]\!]_1\mathcal{L}$ the product topology and $[\![\overline{\mathcal{L}}]\!]$ the quotient topology with respect to the projection.

Note that $[\![\overline{\mathcal{L}}]\!]$ may be decomposed as a disjoint union of leaves: the leaf through $[\![\overline{z}]\!] = [\![\overline{g}]\!] \cdot \tilde{z}$ is the set

$$\widetilde{L}_{[\![\overline{z}]\!]} \;:=\; \left\{ [\![\overline{g}]\!] \cdot \tilde{w} \;\middle|\; \tilde{w} \in \widetilde{L} \right\}.$$

The map $\widetilde{L} \to \widetilde{L}_{[\![\overline{z}]\!]}$, defined $\tilde{w} \mapsto [\![\overline{g}]\!] \cdot \tilde{w}$, is a continuous bijection.

Let $L = \widetilde{B} \times_{\rho} F$ be a suspension, so that $\widetilde{L} \approx \widetilde{B}$. Note that there is a representation $[\![\rho]\!] : \pi_1 B \to \mathsf{Homeo}([\![\overline{\pi}]\!]_1 L)$ defined

$$[\![\overline{g}]\!] \mapsto [\![\overline{g}]\!] \cdot \gamma = [\![\overline{g} \cdot \gamma^{-1}]\!].$$

Theorem 29. *If L is a suspension, then $[\![\overline{L}]\!]$ is equal to the homogeneous suspension*

$$\widetilde{L} \times_{[\![\rho]\!]} [\![\overline{\pi}]\!]_1 L.$$

Proof. $[\![\overline{g}]\!]_1 \cdot \tilde{z}$ and $[\![\overline{g}]\!]_2 \cdot \tilde{w}$ define the same element of $[\![\overline{L}]\!]$ if and only if

$$\tilde{z} = [\![\overline{g}]\!]_1^{-1} [\![\overline{g}]\!]_2 \cdot \tilde{w} = \gamma \cdot \tilde{w}$$

for some $\gamma \in \pi_1 B$. □

In particular, we see that $[\![\overline{L}]\!]$ is itself a lamination.

Note 88. If L is the toral foliation F_V, we have

$$[\![\overline{F_V}]\!] = \mathbb{R}^p \times_{[\![\rho]\!]} {}^*\mathbb{Z}^q.$$

If L is the algebraic universal cover $\widehat{M}$ of a manifold M, we have

$$[\![\overline{\widehat{M}}]\!] = \widetilde{M} \times_{[\![\rho]\!]} {}^*\pi_1 M.$$

We define a projection

$$[\![\overline{p}]\!] : [\![\overline{L}]\!] \longrightarrow L$$

as follows. Given $[\![\overline{z}]\!] \in [\![\overline{L}]\!]$ represented by $\{\tilde{z}_\alpha\} \subset \widetilde{L}$, we write $[\![\overline{p}]\!][\![\overline{z}]\!] = \hat{z}$ if $z_\alpha = p(\tilde{z}_\alpha)$ converges (transversally) to $\hat{z}$ in L. Note that $[\![\overline{p}]\!]$ is a continuous map.

On the other hand, there is a natural left groupoid action of $[\![\pi]\!]_1(L,x)$ on $[\![\overline{L}]\!]$, defined

$$[\![u]\!] \cdot ([\![\overline{g}]\!] \cdot \tilde{z}) = ([\![u]\!] \cdot [\![\overline{g}]\!]) \cdot \tilde{z}$$

whenever $[\![\overline{g}]\!] \in \mathsf{Dom}[\![u]\!]$. Let

$$\mathsf{Dom}_{[\![\overline{L}]\!]}[\![u]\!] = \{[\![\overline{g}]\!] \cdot \tilde{z} \mid [\![\overline{g}]\!] \in \mathsf{Dom}[\![u]\!]\}.$$

Then $\mathsf{Dom}_{[\![\overline{L}]\!]}[\![u]\!]$ is a sublamination of $[\![\overline{L}]\!]$, and $[\![\pi]\!]_1(L,x)$ is a groupoid with respect to the unit space $[\![\overline{L}]\!]$.

Proposition 34. *The quotient*

$$[\![\pi]\!]_1 L \backslash [\![\overline{L}]\!]$$

is homeomorphic to L.

Proof. The equivalence relation enacted by the action of $[\![\pi]\!]_1 L$ identifies precisely those points of $[\![\overline{L}]\!]$ which map to the same point $z \in L$ by $[\![\overline{p}]\!]$. It remains to show that the quotient topology is that of L. For $D \subset \widetilde{L}$ a sufficiently small open and $G_i g$ a coset, the product

$$D \times {}^*G_i g \subset \widetilde{L} \times [\![\overline{\pi}]\!]_1 L$$

maps homeomorphically onto an open $[\![\overline{O}]\!]$ of $[\![\overline{L}]\!]$; conversely, a basis of $[\![\overline{L}]\!]$ consists of opens of this form. The image $O = [\![\overline{p}]\!]\left([\![\overline{O}]\!]\right)$ of such an open is a flow box of L. Thus $[\![\overline{p}]\!]$ is an open map; hence the map $[\![\pi]\!]_1 L \backslash [\![\overline{L}]\!] \to L$ is a homeomorphism. □

The utility of $[\![\overline{L}]\!]$ stems from the map lifting property.

Proposition 35. *Let $F : \mathcal{L} \to \mathcal{L}'$ be a lamination map. Then F induces an injective map*

$$[\![\overline{F}]\!] : [\![\overline{\mathcal{L}}]\!] \longrightarrow [\![\overline{\mathcal{L}'}]\!].$$

Proof. Let $F(L) \subset L'$, and denote by $p' : \widetilde{L}' \to L'$ the universal cover. We use L' to define all relevant germs. Let $[\![\overline{z}]\!] \in [\![\overline{\mathcal{L}}]\!]$ be represented by the sequence $\{\tilde{z}_\alpha\}$, and

$$\tilde{z}'_\alpha \ = \ \widetilde{F}(z_\alpha).$$

Then $\{p(\tilde{z}'_\alpha)\}$ converges transversally to a point $\hat{z}' \in \mathcal{L}'$. Thus it is asymptotic to a unique sequence of the form $\{g'_\alpha w'\}$, where $\{g'_\alpha\}$ defines an element $[\![\overline{g}]\!] \in [\![\overline{\pi}]\!]_1 L'$ and $\tilde{w}' \in \widetilde{L}'$. The map $[\![\overline{F}]\!]$ defined $[\![\overline{z}]\!] \mapsto [\![\overline{g}]\!]' \cdot \tilde{w}'$ is an injective, continuous map taking leaves to leaves. □

12.2. **Covering Maps and Subgroupoids.** A surjective lamination map $P : \mathcal{L} \to \mathcal{L}'$ is called a laminated covering if $P|_L$ is a covering map for every leaf $L \subset \mathcal{L}$.

Note 89. A lamination map which is a covering map (in the classical sense) is a laminated covering.

Note 90. The map $\mathcal{L} \to \mathcal{L}^\natural$ from a suspension to a quasi-suspension is a laminated covering.

Note 91. Let $\mathcal{L} = \widetilde{B} \times_\rho F$ be a suspension. Then the projection $\xi : (\mathcal{L}, x) \to (B, x')$ is a laminated covering map. It is not however a covering map.

Proposition 36. *Let $P : \mathcal{L} \to \mathcal{L}'$ be a laminated covering map. Then the induced map*

$$[\![P]\!] : [\![\pi]\!]_1(\mathcal{L}, x) \longrightarrow [\![\pi]\!]_1(\mathcal{L}', x')$$

is injective.

Proof. Let L, L' be dense leaves in $\mathcal{L}$, $\mathcal{L}'$ containing x, x'. Then the lift of the restriction $P|_L$,

$$\widetilde{P} : \widetilde{L} \longrightarrow \widetilde{L}',$$

is a homeomorphism. It follows that the induced map $[\![P]\!]$ is injective. □

Here, again, we cannot say if P is a homomorphism. If it is, then the image

$$\mathsf{C} = [\![P]\!]\big([\![\pi]\!]_1(\mathcal{L}, x)\big)$$

is a subgroupoid of $[\![\pi]\!]_1(\mathcal{L}', x')$. In this section, we characterize the subgroupoids of the fundamental germ which give rise to laminated covering maps.

Abbreviate $[\![\overline{\pi}]\!]_1 = [\![\overline{\pi}]\!]_1 \mathcal{L}$. Let $\mathsf{C} \subset [\![\pi_1]\!](\mathcal{L}, x)$ be a subgroupoid. Thus, for all $[\![u]\!] \in \mathsf{C}$,

$$\mathsf{Dom}^{\mathsf{C}}[\![u]\!] \subset \mathsf{Dom}[\![u]\!] \quad \text{and} \quad [\![u]\!]^{-1} \in \mathsf{C}.$$

The unit space of C is then

$$[\![\overline{\pi}]\!]_1^{\mathsf{C}} \ := \ \bigcup \mathsf{Dom}^{\mathsf{C}}[\![u]\!],$$

where $[\![u]\!]$ ranges over C and is not equal to $[\![1]\!]$. Note that since $\mathsf{Ran}^{\mathsf{C}}[\![u]\!] = \mathsf{Dom}^{\mathsf{C}}[\![u]\!]^{-1}$, we have

$$\mathsf{Ran}^{\mathsf{C}}[\![g]\!] \ \subset \ [\![\overline{\pi}]\!]_1^{\mathsf{C}}$$

for all $[\![u]\!] \in \mathsf{C}$. The enlargement and internal topologies ${}^*\tau$ and $[\tau]$ on $[\![\overline{\pi}]\!]_1^{\mathsf{C}}$ are induced via the inclusion in $[\![\overline{\pi}]\!]_1$. Let $[\![\tau]\!]^{\mathsf{C}}$ be a topology on $[\![\overline{\pi}]\!]_1^{\mathsf{C}}$ in which ${}^*\tau \subset [\![\tau]\!]^{\mathsf{C}} \subset [\![\tau]\!]$.

Define the relation

$$[\![\overline{g}]\!] \ \sim_{\mathsf{C}} \ [\![\overline{g}]\!]'$$

if and only if

(1) There exists an infinitesimal coset $[H]^{\mathsf{pre}} \subset [\![\overline{\pi}]\!]_1$ with $[\![\overline{g}]\!]$, $[\![\overline{g}]\!]' \in [H]^{\mathsf{pre}}$.
(2) There exists $[\![u]\!] \in \mathsf{C}$ such that

$$[\![u]\!] \cdot [\![\overline{g}]\!] \;=\; [\![\overline{g}]\!]'.$$

Clearly $\sim_{\mathsf{C}}$ is reflexive and symmetric.

Note 92. If we denote by R and R^{C} the subsets of $[\![\overline{\pi}]\!]_1 \times [\![\overline{\pi}]\!]_1$ and $[\![\overline{\pi}]\!]_1^{\mathsf{C}} \times [\![\overline{\pi}]\!]_1^{\mathsf{C}}$ defined by $\sim$ and $\sim_{\mathsf{C}}$, then $\mathsf{R}_{\mathsf{C}} \subset \mathsf{R}$.

$[\![\overline{L}]\!]^{\mathsf{C}}$ is defined by replacing $[\![\overline{\pi}]\!]_1$ by $[\![\overline{\pi}]\!]_1^{\mathsf{C}}$ in the definition of $[\![\overline{L}]\!]$. C acts on the left of $[\![\overline{L}]\!]^{\mathsf{C}}$.

We say that C is a *regular subgroupoid* of $[\![\pi]\!]_1(L,x)$ if

(1) $\sim_{\mathsf{C}}$ is transitive (*i.e* is an equivalence relation).
(2) R_{C} is saturated in R: *i.e.* for every $[\![\overline{g}]\!] \in [\![\overline{\pi}]\!]_1$,

$$\mathsf{R}\big([\![\overline{z}]\!], \cdot\big) \bigcap [\![\overline{\pi}]\!]_1^{\mathsf{C}} \;\neq\; \emptyset.$$

(3) The quotient

$$\mathsf{T}^{\mathsf{C}} \;:=\; [\![\overline{\pi}]\!]_1^{\mathsf{C}} / \sim_{\mathsf{C}}$$

with respect to the quotient $[\![\tau]\!]^{\mathsf{C}}$-topology is a Haussdorf space.
(4) For every $[\![\overline{z}]\!] \in [\![\overline{L}]\!]^{\mathsf{C}}$,

$$\mathsf{Stab}_{\mathsf{C}}(\widetilde{L}_{[\![\overline{z}]\!]}) \;=\; \left\{ [\![u]\!] \in \mathsf{C} \;\middle|\; [\![u]\!] \cdot \widetilde{L}_{[\![\overline{z}]\!]} \;=\; \widetilde{L}_{[\![\overline{z}]\!]} \right\}$$

is a group.

We use the notation

$$\mathsf{C} \;\lessdot\; [\![\pi]\!]_1(L,x)$$

to indicate that C is a regular subgroupoid. The complete groupoid structure of C is defined with respect to $\sim_{\mathsf{C}}$ as in §7.1.

By property (1) above, the quotient

$$L^{\mathsf{C}} \;:=\; \mathsf{C} \setminus [\![\overline{L}]\!]^{\mathsf{C}}$$

is well-defined as a set.

Theorem 30. *If* $\mathsf{C} \lessdot [\![\pi]\!]_1(L,x)$, *then* L^{C} *is a laminated covering of* L.

Proof. C preserves leaves by definition of its action on $[\![\overline{L}]\!]^{\mathsf{C}}$. Let $\mathsf{Stab}(\widetilde{L}_{[\![\overline{z}]\!]})$ be the stabilizer of $\widetilde{L}_{[\![\overline{z}]\!]}$ with respect to the action of $[\![\pi]\!]_1(L,x)$. We have

$$\mathsf{Stab}_{\mathsf{C}}(\widetilde{L}_{[\![\overline{z}]\!]}) \;\subset\; \mathsf{Stab}(\widetilde{L}_{[\![\overline{z}]\!]}), \tag{19}$$

hence $\mathsf{Stab}_{\mathsf{C}}(\widetilde{L}_{[\![\overline{z}]\!]})$ must act properly discontinuously. Thus the image of each leaf of $[\![\overline{L}]\!]^{\mathsf{C}}$ in L^{C} is a manifold. In addition, the image of a transversal is homeomorphic to the Haussdorf space T^{C}. In particular, L^{C} is a lamination.

Since R_{C} is a saturated subset of R, there is a well-defined surjective map $P: L^{\mathsf{C}} \to L$. P is lamination map because ${}^*\tau \subset [\![\tau]\!]^{\mathsf{C}}$ and because transversals are mapped to transversals. By (19), every restriction $P|_{L^{\mathsf{C}}}$, where L^{C} is a leaf of L^{C}, is a covering map. □

Let us restrict ourselves to the case where $[\![\pi]\!]_1(L,x)$ is a group, and consider subgroups $\mathsf{C} < [\![\pi]\!]_1(L,x)$ acting on the unit space $[\![\overline{\pi}]\!]_1$. Let X be the set of all topologies $[\![\tau]\!]$ on $[\![\overline{\pi}]\!]_1$, ordered with respect to inclusion, which have the property that $\mathsf{C} \notin [\![\tau]\!]$. Then X is closed under chains, so we may find a maximal element $[\![\tau]\!]^{\mathsf{C}}$, called the *regular topology*.

Lemma 5. *The quotient* $C\backslash[\![\overline{\pi}]\!]_1$ *is Hausdorff with respect to the quotient* $[\![\tau]\!]^C$*-topology.*

Proof. Since $[\![\tau]\!]^C$ is maximal,

$$C = \bigcap_{C\subset U\in[\![\tau]\!]^C} U.$$

The lemma follows immediately. ☐

Theorem 31. $C \lessdot [\![\pi]\!]_1(L,x)$.

Proof. By Lemma 5, The relation $\sim_C$ is transitive since C is a group, and is trivially saturated with repsect to $\sim$. T^C is Haussdorf by Lemma 5. Since C is a group, every stabilizer $\mathsf{Stab}_C(\widetilde{L}_{[\![\overline{z}]\!]})$ is a group. ☐

Two laminated coverings $P_i : L_i \to L$, $i = 1,2$, are isomorphic if there exists a lamination homeomorphism $F : L_1 \to L_2$ such that $P_1 = P_2 \circ F$.

Proposition 37. *Let* $[\![\pi]\!]_1(L,x)$ *be a group,* C *a subgroup. Then the isomorphism class of the laminated covering* $L^C \to L$ *depends only on the conjugacy class of* C *in* $[\![\pi]\!]_1(L,x)$.

Proof. Suppose that $C' = [\![u]\!]\cdot C\cdot[\![u]\!]^{-1}$. The homeomorphism $[\![\overline{L}]\!] \to [\![\overline{L}]\!]$, defined

$$[\![\overline{g}]\!]\cdot\tilde{z} \longmapsto ([\![u]\!]\cdot[\![\overline{g}]\!])\cdot\tilde{z},$$

descends to the desired homeomorphism $L^C \to L^{C'}$. ☐

12.3. **Deck Germs.** In this section, we continue to assume that $[\![\pi]\!]_1(L,x)$ is a group. Let $C \lhd [\![\pi]\!]_1(L,x)$ be a normal subgroup, $[\![\tau]\!]^C$ the regular topology and $P^C : L^C \to L$ the associated covering.

Theorem 32. *The quotient group*

$$\mathsf{Deck}(P^C) \;:=\; [\![\pi]\!]_1(L,x)/C$$

acts by homeomorphisms on L^C*, with quotient* L.

Proof. Every element of L^C is a class $C\cdot[\![\overline{z}]\!]$, for $[\![\overline{z}]\!]\in[\![\overline{\pi}]\!]_1^C$. The action of $\mathsf{Deck}(P^C)$ on such a class is well-defined, and the quotient is L. ☐

Deck groups give among other things, a natural way to construct groups of homeomorphisms of laminations. For example, in the case of foliation of $\mathbb{T}^2$ by irrational lines, the suspension map $F_r \to S^1$ yields a deck group

$${}^*\mathbb{Z}/{}^*\mathbb{Z}_r$$

which contains elements of all finite orders, as well as elements of infinite order.

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