DENSITY AND TOPOLOGICAL RIGIDITY FOR HOLOMORPHIC FOLIATIONS

Preliminary version

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Abstract

Let us denote by $\mathcal{X}(n)$ the space of degree $n \in \mathbb{N}$ foliations of the complex projective plane $\mathbb{C}P(2)$ which leave invariantthe line at infinity. We prove that for each $n \geq 2$, there exists an open dense subset $\mathcal{R}ig(n) \subset \mathcal{X}(n)$ such that any analytic deformation $\{\mathcal{F}_t\}_{t \in \mathbb{D}}$ of $\mathcal{F}_0 = \mathcal{F} \in \mathcal{R}ig(n)$ which is topological trivial in $\mathbb{C}^2$ must be analytical trivial in $\mathbb{C}P(2)$ for $t \approx 0$. We stress the fact that a priori, our deformations are allowed to move the line L_∞ which is $\mathcal{F}_0$-invariant by hypothesis. We also state results for deformations which are topological trivial in $\mathbb{C}^2$ not necessarily in $\mathbb{C}P(2)$. Finally we obtain a link between the analytic classification of the unfolding and the one of its germs at the singularities $p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$.

Introduction

A holomorphic foliation by curves on $\mathbb{C}P(2)$ is given in an affine space $\mathbb{C}^2 \hookrightarrow \mathbb{C}P(2)$ by a polynomial vector field $X = (P, Q) \in \mathfrak{X}(\mathbb{C}^2)$ with $\gcd(P, Q) = 1$. We fix the line at infinity $L_\infty = \mathbb{C}P(2) \backslash \mathbb{C}^2$ and denote by $\mathcal{X}(n)$ the space of foliations of degree $n \in \mathbb{N}$ which leave invariant L_∞. Let us denote by $\mathcal{F}(n)$ the space of degree n foliations on $\mathbb{C}P(2)$ as introduced in [Li]. We are interested in the two following questions:

(1) Under which conditions topologically trivial deformations of a foliation $\mathcal{F} \in \mathcal{F}(n)$ are analytically trivial?

(2) When does a foliation $\mathcal{F} \in \mathcal{F}(n)$ has dense leaves?

Part I
TOPOLOGICAL RIGIDITY

We refer to [Il] and [LiSaSc] for the notions of analytic deformation of a foliation, topological and analytical equivalence and topological rigidity. We only recall the following less unusual definition:

Definition. Let $\mathcal{C} \subset \mathcal{F}(n)$ be a class of foliations. A foliation $\mathcal{F}_0 \in \mathcal{C}$ is topologically rigid in the class if any topologically trivial deformation $\{\mathcal{F}_t\}_{t \in \mathbb{D}}$ of $\mathcal{F}_0$ with $\mathcal{F}_t \in \mathcal{C}$ is analytically trivial.

We also say that $\mathcal{F}_0 \in \mathcal{C}$ is $\mathcal{U}$-topological rigid in the class $\mathcal{C}$, where $\mathcal{U} \subset \mathbb{C}P(2)$ is an open subset, if any analytic deformation $\{\mathcal{F}_t\}_{t \in \mathbb{D}}$ of $\mathcal{F}_0$ with $\mathcal{F}_t \in \mathcal{C}$, $\forall t$; which is topologically trivial in $\mathcal{U}$, is in fact analytically trivial in $\mathbb{C}P(2)$.

A remarkable result of Y.Ilyashenko states topological rigidity for a residual set of foliations on $\mathcal{X}(n)$ if $n \geq 2$.

More precisely we have:

Theorem 0.1 [Il] *For any $n \geq 2$ there exists a residual subset $\mathcal{I}(n) \subset \mathcal{X}(n)$ whose foliations are topologically rigid in the class $\mathcal{X}(n)$.*

This result has been later improved by A. Lins Neto, P. Sad and B. Scardua as follows:

Theorem 0.2 [LiSaSc] *For each $n \geq 2$, $\mathcal{X}(n)$ contains an open dense subset $\mathcal{R} \subset \mathcal{X}(n)$ whose foliations are topologically rigid in the class $\mathcal{X}(n)$*

We stress the fact that in both theorems above we consider deformations $\{\mathcal{F}_t\}_{t \in \mathbb{D}}$ in the class $\mathcal{X}(n)$, that is, $\mathcal{F}_t$ leaves invariant $L_\infty, \forall t \in \mathbb{D}$; and we assume topological triviality in $\mathbb{C}P(2)$. We relax slightly this last hypothesis by requiring topological triviality for the set of separatrices through the singularities at L_∞:

Theorem 0.3 [LiSaSc]*For any $n \geq 2$, $\mathcal{X}(n)$ contains an open dense subset $\mathcal{SR}\mathrm{ig}(n)$ whose foliations are s-rigid in the class $\mathcal{X}(n)$.*

According to [LiSaSc] a foliation $\mathcal{F}_0 \in \mathcal{X}(n)$ is s-rigid if for any deformation $\{\mathcal{F}_t\}_{t \in \mathbb{D}} \subset \mathcal{X}(n)$ of $\mathcal{F}_0$ with the s-triviality property that is: If $S_t \subset \mathbb{C}^2$ denotes the set of separatrices of $\mathcal{F}_t$ which are transverse to L_∞ then there exists a continuous family of maps $\phi_t : S_0 \to \mathbb{C}^2$ such that ϕ_0 is the inclusion map and ϕ_t is a continuous injection map from S_0 to $\mathbb{C}^2$ with $\phi_t(S_0) = S_t$; then $\{\mathcal{F}_t\}$ is analytically trivial.

Remark. Topological triviality in $\mathbb{C}^2$ implies s-triviality.

Let us change now our point of view.

A deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of a foliation $\mathcal{F}_0$ on a manifold M, is an *unfolding* if there exists an analytic foliation $\tilde{\mathcal{F}}$ on $M \times \mathbb{D}$ with the property that: $\tilde{\mathcal{F}}|_{M\times\{t\}} \equiv \mathcal{F}_t$, $\forall t \in \mathbb{D}$. In other words, *an unfolding is a deformation which embeds into an analytic foliation.* The trivial unfolding of $\mathcal{F}$ is given by the $\mathcal{F}_t := \mathcal{F}$, $\forall t \in \mathbb{D}$ and $\tilde{\mathcal{F}}$ is the product foliation $\mathcal{F} \times \mathbb{D}$ in $M \times \mathbb{D}$.

Two unfoldings $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ and $\{\mathcal{F}_t^1\}_{t\in\mathbb{D}}$ of $\mathcal{F}$ are topologically equivalent respectively analytically equivalent if there exists a continuous respectively analytic map $\phi : M \times \mathbb{D} \to M$ such that each map $\phi_t : M \to M$, $\phi_t(p) = \phi(p,t)$, is a topological respectively analytical equivalence between $\mathcal{F}_t$ and $\mathcal{F}_t^1$.

Definition. An unfolding $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of a foliation $\mathcal{F}_0$ on M is said to be topologically rigid in the class $\mathcal{C} \subset \mathcal{F}(n)$ if any analytic unfolding $\{\mathcal{F}_t^1\}_{t\in\mathbb{D}}$ of $\mathcal{F}$ ($\mathcal{F}_t^1 \in \mathcal{C}$, $\forall t$), which is topologically equivalent to $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$, is necessarily analytically equivalent.

These notions rewrite theorems 0.1 and 0.2 as follows:

Theorem 0.4 [Il]*For any $n \geq 2$ there exists a residual subset $\mathcal{I}(n) \subset \mathcal{X}(n)$ whose foliations are topologically rigid trivial unfolding in the class $\mathcal{X}(n)$.*

Theorem 0.5 [LiSaSc]*For each $n \geq 2$, $\mathcal{X}(n)$ contains an open dense subset $\mathcal{R}ig(n) \subset \mathcal{X}(n)$ whose foliations are topologically rigid trivial unfolding in the class $\mathcal{X}(n)$.*

We are now in conditions of stating our main results concerning topological rigidity. We stress the fact that a priori, our deformations are allowed to move the line L_∞ (Theorems A, B and C) which is $\mathcal{F}_0$-invariant by hypothesis. We also state results for deformations which are topological trivial in $\mathbb{C}^2$ not necessarily in $\mathbb{C}P(2)$. Finally, we may relax the hypothesis of hyperbolicities for $\mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$ by allowing quasi-hyperbolic singularities(defined below) and obtaining this way a link between the analytical classification of the unfolding and the one of its germs at the singularities $p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$.

Our main results are the following:

Theorem A. *Given $n \geq 2$ there exists an open dense subset $\mathcal{R}ig(n) \subset \mathcal{X}(n)$ such that any foliation in $\mathcal{R}\mathrm{ig}(n)$ is $\mathbb{C}^2$-topological rigid: any deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of $\mathcal{F} = \mathcal{F}_0$ which is topologically trivial in $\mathbb{C}^2$ must be analytically trivial in $\mathbb{C}P(2)$ for $t \approx 0$.*

Theorem B. *Let $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ be topological trivial (in $\mathbb{C}^2$) analytic deformation of a foliation $\mathcal{F}_0$ on $\mathbb{C}^2$ such that:*

(i) $\mathcal{F}_0$ leave L_∞ invariant

(ii) $\forall p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$, p is a quasi-hyperbolic singularity

(iii) $\mathcal{F}_0$ has degree $n \geq 2$ and exhibits at least two simple singularities in L_∞.

Then we have two possibilities:

(a) $\mathcal{F}$ is a Darboux (logarithmic) foliation.

(b) $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ is an unfolding.

In this last case the unfolding is analytically trivial if and only if given a singularity $p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$ the germ of the unfolding $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ at p is analytically trivial for $t \approx 0$.

Theorem C. *Let $\mathcal{F}_0$ be a foliation on $\mathbb{C}P(2)$ with the following properties:*

(i) $\mathcal{F}_0$ leaves L_∞ invariant.

(ii) $\forall p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$, p is a quasi-hyperbolic singularity

(iii) $\mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$ has at least two simple singularities.

Given two topologically equivalent unfolding $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ and $\{\mathcal{F}_t^1\}_{t\in\mathbb{D}}$ of $\mathcal{F}_0$ we have that they are analytically equivalent if and only if the germs of unfoldings are analytically equivalent at the singular points $p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$.

1 PRELIMINARIES

We denote by $\mathcal{F}(n)$ the space of degree n foliations on $\mathbb{C}P(2)$. Then $\mathcal{F}(n)$ has a natural structure of projective manifold and we consider the following subsets:

$\mathcal{S}(n) := \{\mathcal{F} \in \mathcal{F}(n)|$ the singularities of $\mathcal{F}$ are non-degenerated$\}$

$\mathcal{T}(n) := \{\mathcal{F} \in \mathcal{S}(n)|$ any characteristic number λ of $\mathcal{F}$ satisfies $\lambda \in \mathbb{C}\backslash\mathbb{Q}_+\} = \{\mathcal{F} \in \mathcal{S}(n)|\mathcal{F}$ has simple singularities$\}$

$\mathcal{A}(n) := \mathcal{T}(n) \bigcap \mathcal{X}(n)$

$\mathcal{H}(n) := \{\mathcal{F} \in \mathcal{A}(n)|$all singularities of $\mathcal{F}$ in L_∞ are hyperbolic $\}$

Proposition 1.1 [Li][LiSaSc] *$X(n)$ is an analytic subvariety of $\mathcal{F}(n)$ and also if $n \geq 2$ then:*

(i)$\mathcal{T}(n)$ contains an open dense subset of $\mathcal{F}(n)$

(ii)$\mathcal{H}(n)$ contains an open dense subset $\mathcal{M}_1(n)$ such that if $\mathcal{F} \in \mathcal{M}_1(n)$, $n \geq 2$ then:

(a)L_∞ is the only algebraic solution of $\mathcal{F}$

(b)The holonomy group of the leaf $L_\infty\backslash\mathrm{Sing}(\mathcal{F})$ is nonsolvable.

(iii)$\mathcal{T}(n) \subset \mathcal{H}(n) \subset \mathcal{X}(n)$ are open subsets.

Lemma 1.2 *Let $\mathcal{F} \in \mathcal{M}_1(n)$, $n \geq 2$; then each leaf $L \neq L_\infty$ is dense in $\mathbb{C}P(2)$.*

Proof. First we notice that L must accumulate L_∞. Since L is non-algebraic leaf it must accumulate some regular point $p \in L_\infty\backslash\mathrm{Sing}(\mathcal{F})$. Choose a small transverse disk $\Sigma \pitchfork L_\infty$ with $\Sigma \subset V$, V is a flow-box neighborhood of p. We consider the holonomy group $\mathrm{Hol}(\mathcal{F}, L_\infty, \Sigma)$. Then L accumulates the origin $p \in \Sigma$ and since by [Na] (see also theorem 2.1) G has dense pseudo-orbits in a neighborhood the origin, it follows that L is dense in a neighborhood of p in Σ. Any other leaf L' of $\mathcal{F}$, $L' \neq L_\infty$ must have the same property. Using the continuous dependence of the solutions with respect to the initial conditions we may conclude that L accumulates any point $q \in L'$, $\forall L' \neq L_\infty$. Thus L is dense in $\mathbb{C}^2$ and since L_∞ is $\mathcal{F}$-invariant, L is dense in $\mathbb{C}P(2)$.

Proposition 1.3 *Let $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$, $\mathcal{F}_0 = \mathcal{F} \in \mathcal{M}_1(n)$ is an unfolding then it is analytically equivalent to the trivial unfolding of $\mathcal{F}$ for $t \approx 0$.*

Proof. Denote by $\tilde{\mathcal{F}}$ the foliation on $\mathbb{C}P(2) \times \mathbb{D}$ such that

$\forall t \in \mathbb{D}\ \tilde{\mathcal{F}}|_{\mathbb{C}P(2)\times\{t\}} = \mathcal{F}_t$,

$\pi : \mathbb{C}^3\backslash\{0\} \to \mathbb{C}P(2)$ the canonical projection and

$\Pi : (\mathbb{C}^3\backslash\{0\}) \times \mathbb{D} \to \mathbb{C}P(2) \times \mathbb{D}$ the map

$\Pi(p,t) := (\pi(p), t)$.

Denote by $\mathcal{F}^* := \Pi^*(\tilde{\mathcal{F}})$, pull-back foliation on $(\mathbb{C}^3\backslash\{0\}) \times \mathbb{D}$. Then $\mathcal{F}^*$ extends to a foliation on $\mathbb{C}^3 \times \mathbb{D}$ by a Hartogs type argument.

Claim. We may choose an integrable holomorphic 1-form Ω which defines $\mathcal{F}^*$ on $\mathbb{C}^3 \times \mathbb{D}$ such that

$$\Omega \quad = \quad A(x,t)dt + \sum_{i=1}^{3} B_j(x,t)dx_j,$$

where B_j is a homogeneous polynomial of degree $n+1$ in x, A is a homogeneous polynomial of degree $n+2$ in x, $\sum_{i=1}^3 x_j B_j(x,t) \equiv 0$ and $\Omega_t := \sum_{i=1}^3 B_j(x,t)dx_j$ defines $\pi^*(\mathcal{F}_t)$ on $\mathbb{C}^3$.

Proof of the claim. First we remark that by triviality of Dolbeault and Cech cohomology groups of $\mathbb{C}^3 \times \mathbb{D}$, $\mathcal{F}^*$ is given by an integrable holomorphic 1-form, say, ω in $\mathbb{C}^3 \times \mathbb{D}$.

The restriction $\omega_t := \omega|_{\mathbb{C}^3\times\{t\}}$ defines $\mathcal{F}_t^* := \pi^*(\mathcal{F}_t)$ in $\mathbb{C}^3$. Thus we may write $\omega = \alpha(x,t)dt + \sum_{k=1}^3 \beta^k(x,t)dx_k = \alpha(x,t)dt + \omega_t(x)$

Since the radial vector field R is tangent to the leaves of $\mathcal{F}^*$ we have $\omega \circ R = 0$ so that $\omega_t \circ R = 0$, i.e. $\sum_{k=1}^3 x_k\beta^k(x,t) = 0$. Now we use the Taylor expansion in variable $x = (x_1, x_2, x_3)$ of ω around a point $(0,t)$ so that $\omega = \sum_{j=\nu}^{+\infty} \omega_j$ where $\omega_j(x,t) := \alpha_j(x,t)dt + \sum_{k=1}^3 \beta_j^k(x,t)dx_k = \alpha_j(x,t)dt + \omega_j^t$ and α_j, β_j^k are holomorphic in (x,t), polynomial of degree j in x, $\omega_\nu \equiv 0$. Now the main argument is the following:

Lemma 1.4 *$\Omega = \alpha_{\nu+1}dt + \omega_\nu^t$ defines $\mathcal{F}^*$ in $\mathbb{C}^3 \times \mathbb{D}$.*

Proof. Indeed, $\omega \wedge d\omega = 0 \Rightarrow i_R(\omega \wedge d\omega) = i_R(\omega).d\omega - \omega \wedge i_R(d\omega) = 0$

$\omega \wedge i_R(d\omega) = 0$ (since $i_R(\omega) = 0$) $\Rightarrow i_R(d\omega) = f\omega$ for some holomorphic function f (Divisor lemma of Saito). Therefor the Lie derivative of ω with respect to R is

$$L_R(\omega) \quad = \quad i_R(d\omega) + d(i_R(\omega)) = f\omega. \tag{1}$$

On the other hand since $\omega = \sum_{j=\nu}^{+\infty} \omega_j = \sum_{j=\nu}^{+\infty} (\alpha_j(x,t)dt + \omega_j^t)$ we obtain

$$\begin{aligned}
L_R(\omega) &= \sum_{j=\nu}^{+\infty} L_R(\alpha_j(x,t)dt + \omega_j^t) \\
&= \sum_{j=\nu}^{+\infty} \frac{d}{dz}[\alpha_j(e^z x,t)dt + \sum_{k=1}^{3} \beta_j^k(e^z x,t)e^z dx_k]|_{z=0} \\
& \quad \text{(The flow of} \quad R \quad \text{is} \quad R_z(x,t) = (e^z x,t)) \\
&= \sum_{j=\nu}^{+\infty} [j\alpha_j(x,t)dt + (j+1)\omega_j^t]. \qquad (2)
\end{aligned}$$

Now we write the Taylor expansion also for f in the variable x. $f(x,t) = \sum_{j=0}^{+\infty} f_j(x,t)$ where $f_j(x,t)$ is holomorphic in (x,t) homogeneous polynomial of degree j in x. We obtain from (1) and (2)

$$\begin{aligned}
\sum_{j=\nu}^{+\infty} j\alpha_j dt + (j+1)\omega_j^t &= (\sum_{k=0}^{+\infty} f_k)(\sum_{l=\nu}^{+\infty} \omega_l) \\
&= \sum_{j\geq\nu}(\sum_{l+k=j} f_k\omega_l) j\alpha_j dt + (j+1)\omega_j^t \\
&= \sum_{l+k=j} f_k\omega_l \\
&= \sum_{l+k=j} (f_k\alpha_l dt + f_k\omega_l^t) \quad l \geq \nu \quad \text{and} \quad \forall j \geq \nu
\end{aligned}$$

Then

$$j\alpha_j \quad = \quad \sum_{l+k=j} (f_k\alpha_l) \tag{3}$$

$$(j+1)\omega_j^t \quad = \quad \sum_{l+k=j} (f_k\omega_l^t) \quad \forall j \geq \nu \quad \text{and} \quad l \geq \nu \tag{4}$$

In particular (3) and (4) imply $f_0\alpha_\nu = \nu\alpha_\nu$ and $f_0\omega_\nu^t = (\nu+1)\omega_\nu^t$ then $f_0 = \nu+1, \alpha_\nu = 0$.

An induction argument shows that:
$j \geq \nu \Rightarrow (\alpha_{j+1}dt + \omega_j^t) \wedge \Omega = 0$, $(\Omega := \alpha_{\nu+1}dt + \omega_\nu^t)$

Finally since the degree of the foliation $\mathcal{F} = \mathcal{F}_0$ is n we have $\nu = n+1$. This proves the lemma (2.4).

Lemma 1.5 *There exists a complete holomorphic vector field X on $\mathbb{C}^3 \times \mathbb{D}_\epsilon$, $\mathbb{D}_\epsilon \subset \mathbb{D}$ small subdisk, such that $X(x,t) = 1\frac{\partial}{\partial t} + \sum_{j=1}^{3} F_j(x,t)\frac{\partial}{\partial x_j}$, $\Omega \circ X = 0$ and $F_j(x,t)$ is linear on x.*

Proof. We may present $\Omega = A(x,t)dt + \sum_{j=1}^{3} B_j(x,t)dx_j = A(x,t)dt + \omega_t$ where $i_R(\omega_t) = 0$, B_j is a homogeneous polynomial of degree $n+1$ in x, A is a homogeneous polynomial of degree $n+2$ in x.
Claim. $\forall t \in \mathbb{D}_\epsilon$ ($\epsilon \geq 0$ small enough) we have $\mathrm{Sing}(\mathcal{F}_t) \subset \{A(.,t) = 0\}$.
Proof of the claim. Since $\Omega \wedge d\Omega = 0$ we have the coefficients of $dt \wedge dx_i \wedge dx_j$ equal to zero, that is:

$$A(\frac{\partial B_j}{\partial x_i} - \frac{\partial B_i}{\partial x_j}) + B_j\frac{\partial B_j}{\partial t} - B_i\frac{\partial B_j}{\partial t} + B_i\frac{\partial A}{\partial x_j} - B_j\frac{\partial A}{\partial x_i} \quad = \quad 0 \tag{5}$$

Now given $p_0 \in \mathrm{Sing}(\mathcal{F}_{t_0})$, ($t_0 \approx 0$, so that $\mathcal{F}_{t_0} \in \mathcal{M}_1(n)$) we have from (5) that $(B_i(p_0,t_0) = B_i(p_0,t_0) = 0) : A(p_0,t_0)(\frac{\partial B_j}{\partial x_i}(p_0,t_0) - \frac{\partial B_i}{\partial x_j}(p_0,t_0))$. Since $\mathcal{F}_{t_0} \in T(n)$ we have $\frac{\partial B_j}{\partial x_i}(p_0,t_0) \neq \frac{\partial B_i}{\partial x_j}(p_0,t_0)(i \neq j)$ and $A(p_0,t_0) = 0$.

Using now Noether's lemma for foliations we conclude that there exist $F_j(x,t)$ holomorphic in (x,t), homogeneous polynomial of degree $1 = (n+2)-(n+1)$ in x, such that $A(x,t) = \sum_{j=1}^{3} F_j(x,t)B_j(x,t)$. Now we define $X(x,t) := 1\frac{\partial}{\partial t} + \sum_{j=1}^{3} F_j(x,t)\frac{\partial}{\partial x_j}$ so that $\Omega \circ X = A - \sum_{j=1}^{3} F_jB_j = 0$.

In addition X is complete because each F_j is of degree one in x. The flow of X writes $X_z(x,t) = (\Psi_z(x,t), t+z)$. Clearly the $\Psi_z : \mathbb{C}^3\backslash\{0\} \longrightarrow \mathbb{C}^3\backslash\{0\}$ defines an analytic equivalence between $\mathcal{F}$ and $\mathcal{F}_z$. The proposition 2.3 is now proved.

Another important remark is the following:

Proposition 1.6 *Let $\mathcal{F}$, $\mathcal{G}$ be foliations with hyperbolic singularities on $\mathbb{C}P(2)$. Assume that L_∞ is the only algebraic leaf of $\mathcal{F}$ and that $\mathcal{F}|_{\mathbb{C}^2}$ and $\mathcal{G}|_{\mathbb{C}^2}$ are topologically equivalent. Then L_∞ is also $\mathcal{G}$-invariant.*

Proof. Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a topological equivalence between $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{C}^2$. We notice that given a singularity $p \in \mathrm{Sing}(\mathcal{F}) \cap L_\infty$, there exist local coordinates $(x,y) \in U$, $x(p) = y(p) = 0$, $L_\infty \cap U = \{y = 0\}$ such that $\mathcal{F}|_U : xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and $U \cap \mathrm{Sing}(\mathcal{F}) = \{p\}$. Let $U^* = U\backslash(L_\infty \cap U)$, $V^* = \phi(U^*) \subset \mathbb{C}^2$, $\Gamma := (x = 0)$, $\Gamma^* := \Gamma \cap U^* = \Gamma\backslash\{p\}$.$\Gamma$ is the local separatrix of $\mathcal{F}$ at p, transverse to L_∞. We put $\Gamma_1^* = \phi(\Gamma^*) \subset V^*$. We remark that Γ_1^* is contained in a leaf of $\mathcal{G}$ and it is closed in V^*. On the other hand if we take any local leaf L of $\mathcal{F}|_{U^*}$, $L \neq \Gamma$; then by the hyperbolicity of $p \in \mathrm{Sing}(\mathcal{F})$ we have that L accumulates Γ. Thus the image $L_1 = \phi(L)$ is a leaf of $\mathcal{G}|_{V^*}$ that accumulates $\Gamma_1^* \neq L_1$.

Assume by contradiction that L_∞ is not $\mathcal{G}$-invariant. The curve $\Gamma_1^* \subset \mathbb{C}^2$ accumulates L_∞. By the Flow Box Theorem, a point of accumulation

$q \in L_\infty \cap \bar{\Gamma}_1^*$ which is not a singularity of $\mathcal{G}$, must be a point near to which the closure (in $\mathbb{C}P(2)$) $\bar{\Gamma}_1^*$ is analytic.

Thus if there are no singularities of $\mathcal{G}$ in $\bar{\Gamma}_1^* \cap L_\infty$ then $\bar{\Gamma}_1^*$ is an algebraic $\mathcal{G}$-invariant curve in $\mathbb{C}P(2)$. This implies that if L_0 is the leaf of $\mathcal{F}$ on $\mathbb{C}P(2)$ that contains Γ^* then $\bar{L}_0$ is an algebraic invariant curve and $\mathcal{F}$-invariant. Since $\bar{L}_0 \neq L_\infty$ we have a contradiction to our hypothesis. Therefor Γ_1^* must accumulate to some singularity r of $\mathcal{G}$ in L_∞. Once again by the local behavior of the leaves close to Γ_1^* and due to the fact that r is hyperbolic, it follows that $\bar{\Gamma}_1^*$ is locally a separatrix of $\mathcal{G}$ at r. Since L_∞ is not $\mathcal{G}$-invariant, we have two local separatrices Λ_1, Λ_2 for $\mathcal{G}$ at r with $\Lambda_j \not\subseteq L_\infty$, $j = 1, 2$. Thus $\bar{\Gamma}_1^*$ is locally contained in $\Lambda_1 \cup \Lambda_2$ and in particular $\bar{\Gamma}_1^*$ is analytic around r. Since (as we have seen) $\bar{\Gamma}_1^*$ is also analytic around the points $q \in \mathrm{Sing}(\mathcal{G})$, it follows that $\bar{\Gamma}_1^*$ is analytic in $\mathbb{C}P(2)$ and once again it is an algebraic curve. Again we conclude that Γ is contained in an algebraic leaf of $\mathcal{F}$, other than L_∞. Contradiction!

The proof given above also shows us:

Proposition 1.7 *Let $\mathcal{F}$, $\mathcal{G}$ be foliations on $\mathbb{C}P(2)$ both leaving invariant the line L_∞. Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a topological invariant equivalence for $\mathcal{F}|_{\mathbb{C}^2}$ and $\mathcal{F}|_{\mathbb{C}^2}$. Then ϕ takes the separatrix set $\mathrm{S}_\mathcal{F}$ onto the separatix set $\mathrm{S}_\mathcal{G}$.*

Here $\mathrm{S}_\mathcal{F}$ and $\mathrm{S}_\mathcal{G}$ are respectively the set of separatrices of $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{C}^2$ that are transverse to L_∞ at some singular point $p \in \mathrm{Sing}(\mathcal{F})$.

Corollary 1.8 *Let $\mathcal{F}_0 \in \mathcal{H}(n)$, $n \geq 2$. Then any $\mathbb{C}^2$-topologically trivial deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of $\mathcal{F}_0$, is a deformation in the class $\mathcal{H}(n)$ and it is also s-trivial if we consider* $\mathrm{t} \approx 0$.

Proof. First we recall that $\mathcal{H}(n)$ is open in $\mathcal{X}(n)$. Thus it remains to use proposition 1.6 to conclude that $\mathcal{F}_t \in \mathcal{H}(n)$, $\forall t \approx 0$ and then we use proposition 1.7 to conclude that $\{\mathcal{F}_t\}_{t\approx 0}$ is s-trivial.

2 FIXED POINTS AND ONE-PARAMETER PSEUDOGROUP

$\mathrm{Diff}(\mathbb{C}, 0)$ denotes the group of germs of complex diffeomorphisms fixing $0 \in \mathbb{C}$, $f(z) = \lambda z + \sum_{n\geq 2} a_n z^n$; $\lambda \neq 0$.

Let $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a finitely generated subgroup with a set of generators $g_1, \cdots, g_r \in G$ defined in a compact disk $\bar{\mathbb{D}}_\epsilon$.

Theorem 2.1 [BeLiLo], [Na] *Suppose G is nonsolvable. Then:*

(i) The basin of attraction of (the pseudo-orbits of) G is an open neighborhood of the origin Ω.($0 \in \Omega$)

(ii) Either G has dense pseudo-orbits in some neighborhood $0 \in V \subset \Omega$ or there exists an invariant germ of analytic curve Γ (equivalent to $\mathrm{Im} z^k = 0$ for some $k \in \mathbb{N}$) where G has dense pseudo-orbits and such that G has also dense pseudo-orbits in each component of $V \backslash \Gamma$.

(iii) G is topologically rigid: Given another nonsolvable subgroup $G' \subset \mathrm{Diff}(\mathbb{C}, 0)$ and a topological conjugation $\phi : \Omega \to \Omega'$ between G and G', then ϕ is holomorphic in a neighborhood of 0.

(iv) There exists a neighborhood $0 \in W \subset V \subset \Omega$ where G has a dense set of hyperbolic fixed points.

Remark. In case G is nonsolvable contains some $f \in G$ with $f'(0)^n \neq 1$, $\forall n \in \mathbb{Z} \backslash \{0\}$ (i.e., $f'(0) = e^{2\pi i \lambda}$, $\lambda \notin \mathbb{Q}$) we have the following from (iii).

(iii)$'$ (Dense Orbits Property): There exists a neighborhood $0 \in V \subset \Omega$ where the pseudo-orbits of G are dense.

HOLOMORPHIC DEFORMATIONS IN $\mathrm{Diff}(\mathbb{C}, 0)$

Let $g \in \mathrm{Diff}(\mathbb{C}, 0)$ defined in some open neighborhood $0 \in \Omega$. A *holomorphic (one-parameter) deformation* of g is a map $G : \mathbb{D}_\epsilon \to \mathrm{Diff}(\mathbb{C}, 0)$, ($\epsilon > 0$) which verifies the four properties:

(1) $G(0) = g$ as germs

(2) The Taylor expansion coefficients of $G(t)$ depend holomorphically on t

(3) The radii of convergence of $G(t)$ and $G(t)^{-1}$ are both uniformly minorated by some constant $R \geq 0$ ($\forall t \in \mathbb{D}_\epsilon$)

(4) The modules of the linear coefficient of $G(t)$ is uniformly minorated by some constant $C \geq 0$. In particular $|(G(t)^{-1})'(0)|$ is uniformly majorated by $0 < t < \infty$.

Given a finitely generated pseudo-group $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ with a set of generators $g_1, \cdots g_r \in G$; a holomorphic (one parameter) deformation of G is given by holomorphic deformation of g_j, $j = 1, \cdots, r$. We may restrict ourselves to the following situation:

G_t is an one-parameter analytic deformation of G with $t \in \mathbb{D}$, $G_0 = G$. We have $g_{1,t} \cdots g_{r,t}$ as a set of generators for G_t, all of them defined in a disk $\bar{\mathbb{D}}_\delta$ (uniformly on t). We will consider dynamical and analytical properties of such deformations. The results we state below have their proofs reduced to the following case which is studied in [Wi3].

$g_{1,t}(z) = g_1(z) + tz^{D+1}$ where $D \in \mathbb{N}$ is fixed,

$g_{2,t}(z) = g_2(z), \cdots, g_{r,t}(z) = g_r(z)$.

For such deformations we have:

Theorem 2.2 [Wi3] *Given a hyperbolic fixed point $p \approx 0$ for a word $f = f_n \circ f_{n-1} \circ \cdots \circ f_1$ in G, we consider the corresponding word $f_t = f = f_{n,t} \circ \cdots \circ f_{1,t}$ in G_t. Then f_t has a hyperbolic fixed point $p(t)$ given by the implicit differential equation with initial conditions:*

$$\frac{dp(t)}{p(t)^{D+1}dt} = \frac{f_t'(p(t))}{f_t'(p(t))-1} f_{1,t}'(p(t)), \quad p(0) = p.$$

In particular $p(t)$ depends analytically on t as well as its multiplicator ${f_t}'(p(t))$.
This holds for $|t| < \epsilon$ if $\epsilon > 0$ is small enough.

We also have:

Theorem 2.3 [Wi2] *Let f and g be two non-commuting complex diffeomorphisms defined in some neighborhood of the origin $0 \in \mathbb{C}$, fixed by f and g. Assume that $f'(0) = e^{2\pi i \lambda}$, $g'(0) = e^{2\pi i \mu}$ with $\lambda, \mu \in \mathbb{C} \backslash \mathbb{R}$, $\mathrm{Re}\lambda, \mathrm{Re}\mu \notin \mathbb{Q}$. Then there exist some bound $K > 0$ and some radius $r_0 > 0$ such that if $r \in (0, r_0)$ and $|t| \leq Kr$ then the orbits of the pseudo-group generated by g and $f_t(z) = t + f(z-t)$ are dense in $\bar{\mathbb{D}}_r$.*

Corollary 2.4 [Wi1] *Let f and g be as above. Any holomorphic deformation of the subgroup $< f, g > \subset \mathrm{Diff}(\mathbb{C}, 0)$ preserves locally at the origin the dense orbits property.*

3 PROOF OF THEOREM A

We use the terminology of [LiSaSc]and some of the original ideas of [Il]. We give the main steps, for remaining details are found in [LiSaSc]. Let therefore $\{\mathcal{F}_t\}_{t \in \mathbb{D}}$ be a $\mathbb{C}^2$-topological trivial deformation of $\mathcal{F}_0 \in \mathcal{H}(n)$, $n \geq 2$. As we have proved in corollary (2.8) there exists $\epsilon > 0$ such that $\{\mathcal{F}_t\}_{t \in \mathbb{D}_\epsilon}$ is a s-trivial deformation of $\mathcal{F}_0$ in the class $\mathcal{H}(n)$. Now we consider the continuous foliation $\tilde{\mathcal{F}}$ on $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$ defined as follows:

(i)$\mathrm{Sing}(\tilde{\mathcal{F}}) = \bigcup_{|t|<\epsilon} \mathrm{Sing}(\mathcal{F}_t) \times \{t\}$

(ii)The leaves of $\mathcal{F}_t$ are the intersections of the leaves of $\tilde{\mathcal{F}}$ with $\mathbb{C}P(2) \times \{t\}$, $\forall |t| < \epsilon$.

Because of the topological triviality $\tilde{\mathcal{F}}$ is a continuous foliation on $\mathbb{C}^2 \times \mathbb{D}_\epsilon$. This foliation extends to a continuous foliation on $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$ by adding the leaf with singularities $L_\infty \times \mathbb{D}_\epsilon$. In order to prove that $\tilde{\mathcal{F}}$ is holomorphic we begin by proving that it has holomorphic leaves and then it is transversely holomorphic. This is basically done by the following lemma:

Lemma 3.1 *Let $p_1, \cdots, p_{n+1} \in L_\infty$ be the singularity of $\mathcal{F}_0$ in L_∞. Then*

(i) There exist analytic functions $p_j(t)$, $t \in \mathbb{D}_\epsilon$ such that $\{p_1(t), \cdots, p_{n+1}(t)\} =$ $\mathrm{Sing}(\mathcal{F}_t) \cap L_\infty$, $p_j(0) = p_j$, $j = 1, \cdots, n+1$.

Fix $q \in L_\infty \backslash \mathrm{Sing}(\mathcal{F}_0)$ and take small simple loops $\alpha_j \in \pi_1(L_\infty \backslash \mathrm{Sing}(\mathcal{F}_0), q)$ and a small transverse disk $\Sigma \pitchfork L_\infty$. Then for $\epsilon > 0$ small we have:

(ii) The holonomy group $G_t := \text{Hol}(\mathcal{F}_t, L_\infty, \Sigma_i) \subset \text{Diff}(\Sigma, q)$ is generated by the holonomy maps $f_{j,t}$ associated to the loops α_j (α_j is also simple loop around $p_j(t)$).

In particular we obtain

(iii)$\{G_t\}_{t\in\mathbb{D}_\epsilon}$ is an one-parameter holomorphic deformation of $G_0 = \text{Hol}(\mathcal{F}_0, L_\infty, \Sigma)$.

(iv)The group G_t is nonsolvable with the Density Orbits Property, a dense set $\eta_t \subset \Sigma \times \{t\}$ of hyperbolic fixed points around the origin (q,t). Moreover, given any $p_0 \in \eta_0$, $p_0 = f_0(p_0)$, there exists an analytic curve $p_t \in \eta_t$ such that $p(0) = p_0$, $f_t(p_t) = p_t$ where $f_t \in G_t$ is the corresponding deformation of f_0.

Using above lemma we prove that $\tilde{\mathcal{F}}$ is holomorphic close to $L_\infty \times \mathbb{D}_\epsilon$: Given a point $p_0 \in \eta_0$ and $f_0 \in G_0$ as above, the curve $p(t)$ and $f_t \in G_t$ given by (iv) above we have $\{p(t), |t| < \epsilon\} \subset \tilde{L}_{p_0} \cap (\Sigma \times \mathbb{D}_\epsilon)$ where $\tilde{L}_{p_0}$ is the $\tilde{\mathcal{F}}$-leaf through p_0. On the other hand $\tilde{L}_{p_0}$ is already holomorphic along the cuts $\tilde{L}_{p_0} \cap (\mathbb{C}P(2) \times \{t\})$ for $L_{{p'}_0,t}$ for $p_0 = ({p'}_0, 0)$. This implies that $\tilde{L}_{p_0}$ is analytic.

Since the curves $\{p(t), |t| < \epsilon\}$ with $p_0 \in \eta_0$ are analytic and locally dense around $\{q\} \times \mathbb{D}_\epsilon \subset \Sigma \times \mathbb{D}_\epsilon$ it follows that any leaf $\tilde{L}$ of $\tilde{\mathcal{F}}$ is a uniform limit of holomorphic leaves $\tilde{L}_{p_0}$ and it is therefore holomorphic. Thus $\tilde{\mathcal{F}}$ has holomorphic leave. We proceed to prove that it is transversely holomorphic. This is in fact a consequence of topological rigidity theorem [Na] for nonsolvable groups of $\text{Diff}(\mathbb{C}, 0)$.

Fix transverse section $\Sigma \pitchfork L_\infty$ as above. We may assume that $\Sigma \subset V$ where V is a flow-box neighborhood for $\mathcal{F}_0$ with $q \in V$. The homeomorphisms $\phi_t : \mathbb{C}^2 \to \mathbb{C}^2$ take the separatrices S_0 of $\mathcal{F}_0$ onto the set of separatrices S_t of $\mathcal{F}_t$. Now we use the following proposition:

Proposition 3.2 *Given $\mathcal{F} \in \mathcal{H}(n)$, $n \geq 2$, the set of separatrices $S_\mathcal{F}$ of $\mathcal{F}$ is dense in $\mathbb{C}P(2)$ and it accumulates densely a neighborhood of the origin for any transverse disk $\Sigma \pitchfork L_\infty$, $q \notin \text{Sing}\mathcal{F}$.*

Proof. Indeed, given a separatrix $\Gamma \subset S_\mathcal{F}$ the leaf $L \supset \Gamma$ is nonalgebraic for $\mathcal{F} \in \mathcal{H}(n)$. This implies that $L \backslash \Gamma$ accumulates L_∞ and therefor any transverse disk Σ as above is cut by L. Now it remains to use the density of the pseudo-orbits of $\text{Hol}(\mathcal{F}, L_\infty)$ stated in theorem 2.1.

Returning to our argumentation we fix any $p \in \Sigma$, separatrix $(p_0 \in)\Gamma_0 \subset \mathcal{S}_0$ of $\mathcal{F}_0$ and denote by $P(\Gamma_0, p)$ the local plaque of $\mathcal{F}_0|_V$ that is contained in $\Gamma_0 \cap V$ and contains the fixed point p. Put $\Gamma_t = \phi_t(\Gamma_0)$ and consider the map $t \mapsto p(t) := P(\Gamma_t, p)$. clearly we may write $p(t) = \phi_t(P(\Gamma_0, p)) \cap \Sigma \times \{t\}$ by choosing Σ and $|t|$ small enough. This map $t \mapsto p(t)$ is holomorphic as a consequence of proposition below:

Proposition 3.3 *Given any singularity $p_j^0 \in \mathrm{Sing}\mathcal{F}$ there exists a connected neighborhood $(p_j^0 \in)\mathcal{U}_j$, a neighborhood $\mathcal{U} \ni \mathcal{F}_0$ in $\mathcal{S}(n)$ and a holomorphic map $\psi_j : \mathcal{U} \to \mathcal{U}_j$ such that $\forall \mathcal{F} \in \mathcal{U}$, $\psi_j(\mathcal{F}) = \mathrm{Sing}\mathcal{F} \cap \mathcal{U}_j$, $\psi_j(\mathcal{F}_0) = p_j^0$. In particular, if $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ is a deformation of $\mathcal{F}_0 \in \mathcal{H}(n)$, $n \geq 2$; then given $\Gamma_0 \in S_0 = S_{\mathcal{F}}$, $\Sigma \pitchfork L_\infty$, V and $p \in \Gamma_0 \cap \Sigma$ as above, there exist analytic curves $p_j(t)$ and $p(t)$ such that: $p_j(t) = \mathrm{Sing}\mathcal{F}_t \cap \mathcal{U}_j$, $p_j(0) = p_j^0$, $p(t) = P(\Gamma_t, p(t))$, $p(0) = p$ and $p(t) \in \Gamma_t \cap \Sigma$.*

Roughly speaking, the proposition says that both the singularities and the separatrices of a foliation with nondegenerate singularities, move analytically under analytic deformations of the foliation.

Finally we define $h_t(p) := p(t)$ obtaining this way an injective map in a dense subset of Σ ($\mathcal{F}_0$ has dense separatrices in (Σ, q)), so that by the λ-lemma for complex mapping we may extend h_t to a map that $h_t : \Sigma \to \Sigma$. Moreover, it is clear that if $f_{j,t}$ is a holonomy map as above then we have

$$h_t(f_{j,0}(p)) \quad = \quad f_{j,t}(h_t(p))$$

Because f_0 and f_t fix the separatrices. Therefore, by density we have $h_t \circ f_{j,0} = f_{j,t} \circ h_t$, $\forall j \in \{1, \cdots, n+1\}$ and the mapping h_t conjugates the holonomy groups $G_t = \mathrm{Hol}(\mathcal{F}_t, L_\infty, \Sigma)$ and G_0. By the topological rigidity theorem h_t is holomorphic which implies that $\tilde{\mathcal{F}}$ is transversely holomorphic close to $L_\infty \times \mathbb{D}_\epsilon$ [Na]. The density of $\mathcal{S}_t$, $\forall t$ assures that $\tilde{\mathcal{F}}$ is in fact holomorphic in $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$.

Summarizing the discussion above we have:

Proposition 3.4 *Let $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ be a $\mathbb{C}^2$-topologically trivial deformation of $\mathcal{F}_0 \in \mathcal{H}(n)$, $n \geq 2$. Then there exists $\epsilon > 0$ such that $\{\mathcal{F}_t\}_{t\in\mathbb{D}_\epsilon}$ is an unfolding of $\mathcal{F}_0$ in $\mathbb{C}P(2)$.*

Proof of Theorem A. The proof is a consequence of propositions 1.3 and 3.4 above.

4 GENERALIZATIONS

foliations on other projective spaces. This is the goal of this section.Before going further into generalizations we state a kind of Noether's lemma for foliations.

Lemma 4.1 *Let $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ be a holomorphic unfolding of a foliation $\mathcal{F}_0$ of degree n on $\mathbb{C}P(2)$. Assume that for each singularity $p \in \mathrm{Sing}\mathcal{F}_0 \cap L_\infty$ the germ of unfolding at p is analytically trivial. Then there exists $\epsilon > 0$ such that $\{\mathcal{F}_t\}_{|t|<\epsilon}$ is analytically trivial.*

Proof. Denote by

$\pi : \mathbb{C}^3\backslash\{0\} \to \mathbb{C}P(2)$ the canonical projection and by

$\Pi : (\mathbb{C}^3\backslash\{0\}) \times \mathbb{D} \to \mathbb{C}P(2) \times \mathbb{D}$ the map $\Pi(p,t) := (\pi(p), t)$.

Choose a holomorphic integrable 1-form Ω which defines $\tilde{\mathcal{F}}^*$ extension of $\Pi^*(\mathcal{F})$ to $\mathbb{C}^3 \times \mathbb{D}$, so that we may choose

$$\Omega \quad = \quad A(x,t)dt + \sum_{i=1}^{3} B_j(x,t)dx_j,$$

where A, B_j are holomorphic in $(x,t) \in \mathbb{C}^3 \times \mathbb{D}$, homogeneous polynomial in x of degree $n+2$, $n+1$; $\sum_{i=1}^{3} x_j B_j = 0$. The foliation $\pi^*(\mathcal{F}_t)$ extends to $\mathbb{C}^3$ and this extension $\mathcal{F}_t^*$ is given by $\Omega_t = 0$ for $\Omega_t := \sum_{i=1}^{3} B_j dx_j$.

Claim. Given point $q \in \mathbb{C}^3 \times \mathbb{D}_\epsilon$, $q \notin \{0\} \times \mathbb{D}$, there exist a neighborhood $U(q)$ of q in $\mathbb{C}^3 \times \mathbb{D}_\epsilon$ and local holomorphic vector field $X_q \in \mathfrak{X}(U(q))$ such that $A = \Omega \circ X_q$ in $U(q)$, for ϵ small enough.

Proof of the claim. If $q = (x_1, t_1)$ with $x_1 \notin \operatorname{Sing}(\mathcal{F}_0)$ then $x_1 \notin \operatorname{Sing}(\mathcal{F}_t)$ for $|t|$ small enough and in particular $x_1 \notin \operatorname{Sing}(\mathcal{F}_{t_1})$. Thus the existence of $X_q \in \mathfrak{X}(U(q))$ is assured in this case. On the other hand if $x_1 \notin \operatorname{Sing}(\mathcal{F}_0)$ then we still have the existence of $X_q \in \mathfrak{X}(U(q))$ because of the local analytical triviality hypothesis for the unfolding at x_1.

Using the claim we obtain an open cover $\{U_\alpha\}_{\alpha\in\mathbb{Q}}$ of $M := \mathbb{C}^3\backslash\{0\} \times \mathbb{D}$ with U_α connected and $X_\alpha \in \mathfrak{X}(U_\alpha)$ such that $A = \Omega \circ X_\alpha$ in U_α, $\forall \alpha \in \mathbb{Q}$. Let $U_\alpha \cap U_\beta \neq \varnothing$ then we put $X_{\alpha\beta} := (X_\alpha - X_\beta)|_{U_\alpha \cap U_\beta}$ to obtain $X_{\alpha\beta} \in \mathfrak{X}(U_\alpha \cap U_\beta)$ such that $\Omega \circ X_{\alpha\beta} = 0$. Take now the rotational vector field

$$\begin{aligned} Y &= rot(B_1, B_2, B_3) \\ &= \left(\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3}\right)\frac{\partial}{\partial x_1} + \left(\frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1}\right)\frac{\partial}{\partial x_2} + \left(\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2}\right)\frac{\partial}{\partial x_3}. \end{aligned}$$

$Y \in \mathfrak{X}(\mathbb{C}^3 \times \mathbb{D})$ and for each $t \in \mathbb{D}$ we have $i_Y(\mathrm{Vol}) = d\Omega_t$ where $\mathrm{Vol} = dx_1 \wedge dx_2 \wedge dx_3$ is the volume element of $\mathbb{C}^3$ in the x-coordinates. Fixed now $q = (x_1, t_1) \notin \operatorname{Sing}(\Omega_{t_1})$ then the leaf of $\mathcal{F}_{t_1}^*$ through q is spanned by $Y(q)$ the radial vector field $R(q)$, as a consequence of the remark above: actually, we have $i_R i_Y(\mathrm{Vol}) = i_R(d\Omega_t) = (n+1)\Omega_t$.

Given thus $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing$, since $\Omega_t(X_{\alpha\beta})$ we have that $X_{\alpha\beta}$ is tangent to $\mathcal{F}_t^*$ outside the points $(x,t) \in \operatorname{Sing}(\Omega_t)$ so that we can write $X_{\alpha\beta} = g_{\alpha\beta}R + h_{\alpha\beta}Y$ for some holomorphic functions $g_{\alpha\beta}$, $h_{\alpha\beta} \in \mathcal{O}(U_{\alpha\beta}\backslash\operatorname{Sing}(\Omega_t))$. Since $\operatorname{Sing}(\Omega_t)$ is an analytic set of codimension ≥ 2, Hartogs extension theorem [Si] implies that $g_{\alpha\beta}$, $h_{\alpha\beta}$ extend holomorphically to $U_{\alpha\beta}$. Now if $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$ then $0 = X_{\alpha\beta} + X_{\beta\gamma} + X_{\gamma\alpha} = (g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha})R + (h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha})Y$ and since R and Y are linearly independent outside $\operatorname{Sing}(\Omega_t)$ we obtain: $g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = 0$, $h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha} = 0$.

Thus $(g_{\alpha\beta})$, $(h_{\alpha\beta})$ are additive cocycles in M and by Cartan's theorem (for $\mathbb{C}^{n+1}\backslash\{0\}$, $n \geq 2$) these cocycles are trivial, that is, $\exists g_\alpha, h_\alpha \in \mathcal{O}(U_\alpha)$

such that if $U_\alpha \cap U_\beta \neq \phi$ then $g_{\alpha\beta} = g_\alpha - g_\beta h_{\alpha\beta} = h_\alpha - h_\beta$ in $U_\alpha \cap U_\beta$. This gives $X_\alpha - X_\beta = X_{\alpha\beta} = g_{\alpha\beta}R + h_{\alpha\beta}Y(g_\alpha R + h_\alpha Y) - (g_\beta R + h_\beta Y)$ in $U_\alpha \cap U_\beta \neq \phi$. Thus, in $U_\alpha \cap U_\beta \neq \phi$ we obtain $X_\alpha - g_\alpha R - h_\alpha Y = X_\beta - g_\beta R - h_\beta Y$ and this gives a global vector field $\tilde{X} \in \mathfrak{X}(M)$ such that $\tilde{X}|_{U_\alpha} := X_\alpha - g_\alpha R - h_\alpha Y$. This vector field extends holomorphically to $\mathbb{C}^3 \times \mathbb{D}$ and we have $(\Omega_t \circ \tilde{X})|_{U_\alpha} = \Omega_t \circ X_\alpha - g_\alpha \Omega_t \circ R - h_\alpha \Omega_t \circ Y = A$ so that $\Omega_t \circ \tilde{X} = A$.

It remains to prove that we may choose $\tilde{X}$ polynomial in the variable x. Indeed, we write $\tilde{X} = \sum_{k=0}^{\infty} \tilde{X}_k$ for the Taylor expansion of $\tilde{X}$ around the origin, in the variable x.

Then $\tilde{X}_k$ is holomorphic in (x,t) and homogeneous polynomial of degree k in the variable x. We have $A = \Omega_t \circ \tilde{X} = \sum_{k=0}^{+\infty} \Omega_t(\tilde{X}_k)$ and since it is polynomial homogeneous of degree $n+2$ in x it follows that $k \neq 1 \Rightarrow \Omega_t(\tilde{X}_k) = 0$ and $\Omega_t(\tilde{X}_1) = A$. Since $\tilde{X}_1$ is linear, as before in ... the flow of $\tilde{X}_1$ gives an analytic trivialization for $\{\mathcal{F}_t\}_{t \in \mathbb{D}_\epsilon}$.

QUASI-HYPERBOLIC FOLIATIONS

Now we recall some of the features coming from [MaSa2]. A germ of holomorphic foliation at $0 \in \mathbb{C}^2$, is quasi-hyperbolic if after its reduction of singularities process[Se] we obtain an exceptional divisor that is a finite union of invariant projective lines meeting transversely at double points and a foliation with saddle-type singularities: $xdy - \lambda y dx = 0$, $\lambda \in (\mathbb{C} \backslash \mathbb{R}) \cup \mathbb{R}_-$.

In [MaSa2] we also find the notion of generic quasi-hyperbolic germs of foliation with some dynamical restrictions on the structure of the foliation after the reduction process.

The outstanding result is:

Theorem 4.2 *A topological trivial deformation of a generic quasi-hyperbolic germ of foliation is an (equisingular) unfolding.*

Using the concept of singular holonomy[CaSc] we may strength this result as follows:

Theorem 4.3 [MaSa2] *Let $\{\mathcal{F}_t\}_{t \in \mathbb{D}}$ be a topologically trivial analytic deformation of a germ of quasi-hyperbolic foliation $\mathcal{F}_0$ at $0 \in \mathbb{C}^2$. We have the following possibilities:*

(i) $\mathcal{F}_0$ admits a Liouvillian first integral and all its projective holonomy groups are solvable,

(ii) $\{\mathcal{F}_t\}_{|t|<\epsilon}$ is an (equisingular) unfolding.

Proof. First we recall that $\{\mathcal{F}_t\}_{t \in \mathbb{D}}$ is equireducible as a consequence of [MaSa2]. Assume that all the projective singular holonomy groups of $\mathcal{F}_0$

are solvable. In this case according to [CaSc] $\mathcal{F}_0$ has a Liouvillian first integral.(here we use strongly the fact that $\mathcal{F}_0$ is quasi-hyperbolic) We may therefore consider the case where some component of the exceptional divisor has non solvable singular holonomy group. This implies topological rigidity and abundance of hyperbolic fixed points as well as the Dense Orbits Property for this group as well as for all the projective singular holonomy groups, which are the main ingredients in the proof of (0.26)[MaSa1], so that a slight adaptation of their original proof gives us that $\{\mathcal{F}_t\}_{|t|<\epsilon}$ is an unfolding for $|t| < \epsilon$.

Proof of Theorem B. First we remark that by the topological triviality on $\mathbb{C}P(2)$ we may assume that L_∞ is an algebraic leaf for $\mathcal{F}_t$ and that $\phi_t(L_\infty) = L_\infty$, $\forall t \in \mathbb{D}$. In fact, we take $S_t = \phi_t(L_\infty) \subset \mathbb{C}P(2)$. Then S_t is compact $\mathcal{F}_t$-invariant and of dimension one, so that S_t is an algebraic leaf of $\mathcal{F}_t$. By a well-known theorem of Zariski S_t is smooth. Since the self-intersection number is a topological invariant we conclude that S_t has self-intersection number one and by Bezout's theorem S_t has degree one, that is, S_t is a straight line in $\mathbb{C}P(2)$.

The problem here is that S_t may do not depend analytically on t. That is where we use the hypothesis that there exist at least two simple singularities $p_1, p_2 \in \mathrm{Sing}\mathcal{F}_0 \cap L_\infty$. Since p_j is simple there exists an analytic curve $p_j(t) \in \mathrm{Sing}\mathcal{F}_t$ such that $p_j(t)$ is simple singularity of $\mathcal{F}_t$ and $p_j(t) = \phi_{t(p_j)}$, $p_j(0) = p_j$. since the line S_t contains $p_1(t) \neq p_2(t)$ it follows that S_t depends analytically on t and there exists a unique automorphism $T_t : \mathbb{C}P(2) \to \mathbb{C}P(2)$ such that $T_t(S_t) = S_0 = L_\infty$; $T_t(p_j(t)) = p_j$, $j = 1, 2$. Thus $\psi_t = T_t \circ \phi_t : \mathbb{C}P(2) \to \mathbb{C}P(2)$ gives a topological trivialization for the deformation $\{\mathcal{F}_t^1\}_{t\in\mathbb{D}}$ of $\mathcal{F}_0$, where $\mathcal{F}_t^1 := T_t(\mathcal{F}_t)$, and L_∞ is an algebraic leaf of $\mathcal{F}_t^1$, $\forall t \in \mathbb{D}$. Thus we may assume that L_∞ is $\mathcal{F}_t$-invariant, $\forall t \in \mathbb{D}$. Now we proceed after performing the reduction of singularities for $\mathcal{F}_0|_{L_\infty}$ we consider the exceptional divisor $D = \cup_{j=1}^r D_j$, $D_0 \cong L_\infty$, $D_j \cong \mathbb{C}P(1)$, $\forall j \in \{1, \cdots, r\}$ and observe that if the singular holonomy groups of the components D_j are all solvable then according to [CaSc](using the fact that the singularities $p \in \mathrm{Sing}(\mathcal{F}_0) \cap \mathrm{L}_\infty$ are quasi-hyperbolic) we get that $\mathcal{F}$ is a Darboux (logarithmic) foliation. We assume therefor that some singular holonomy groups is nonsolvable, then it follows that by definition of singular holonomy group and due to the fact that the divisor D is invariant and connected and has saddle-singularities at the corners, we can conclude that all components of D has nonsolvable singular holonomy groups. This implies, that each germ of $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ at a singular point $p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$ is an unfolding(these germs are evidently topologically trivial). Using now arguments similar to the ones in proof of (0.27) we conclude that $\{\mathcal{F}_t\}_{t\in\mathbb{D}_\epsilon}$ is an unfolding for $\epsilon > 0$ small enough.

If we assume that for any singularity $p \in \mathrm{Sing}(\mathcal{F}_0) \cap L_\infty$ the germs of unfolding is analytically trivial, then as consequence of (0.25) we conclude that $\{\mathcal{F}_t\}_{t\in\mathbb{D}_\epsilon}$ is analytically trivial for $\epsilon > 0$ small enough. Theorem B is now proved

Remark. Above theorem is still true if one replace condition (iii) by the following

(iii)'$\phi_t(L_\infty) = L_\infty$, $\forall t \in \mathbb{D}$.

Part II
DENSITY

The problem of density of the leaves of generic foliations $\mathcal{F}$ on $\mathbb{C}P(2)$ has been considered by several authors,e.g. [Mj][Il] and [Sh1]. It is also related to the problem of the existence of foliations on $\mathbb{C}P(2)$ having an exceptional minimal set[CaLiSa].

It is now well-known that given any integer $n \geq 2$ there exists an open dense subset $M_1(n)$ of degree n foliations leaving invariant the line at infinity whose foliations have dense leaves on $\mathbb{C}P(2)$ except the line at infinity. we demand basically whether it is possible to perform deformations of an element $\mathcal{F}_0 \in M_1(n)$ in such a way that the perturbed foliation has all leaves dense in $\mathbb{C}P(2)$. Towards this we have the following result:

Theorem 4.4 ***(Density theorem).*** *Let $\mathcal{F}_0 \in M_1(n)$ be given with $n \geq 2$. Then any analytic deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of $\mathcal{F}_0$ in $\mathbb{C}P(2)$ has the property that if t is close enough to zero then $\mathcal{F}_t$ has dense leaves in $\mathbb{C}^2$. In case L_∞ is not invariant for $\mathcal{F}_t$ then the leaves are dense in $\mathbb{C}P(2)$.*

As a corollary of this result we obtain:

Theorem 4.5 *Given $\mathcal{F}_0 \in M_1(n)$, $n \geq 2$, there exist analytic deformations $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of $\mathcal{F}_0$ in $\mathbb{C}P(2)$ whose foliations have dense leaves in $\mathbb{C}P(2)$.*

5 PRELIMINARIES

Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$. The following result is a well-known consequence of the Maximum Principle for holomorphic foliations:

Proposition 5.1 *Let L be a leaf of $\mathcal{F}$. Then L accumulates L_∞.*

Proof. Indeed, take a point $p_0 \in L \cap \mathbb{C}^2$ and let $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ be an isolated singularity polynomial vector field defining $\mathcal{F}_0$ in $\mathbb{C}^2$; $p_0 = (x_0, y_0)$, $X(p_0) \neq 0$. Denote by $\phi(z,(x_0,y_0))$ the local solution of X that satisfies $\phi(0,(x_0,y_0)) = (x_0,y_0)$. Assume by contradiction that $\bar{L} \cap L_\infty = \varnothing$, then $\bar{L}$ is a compact set on $\mathbb{C}^2$ and therefor $\phi(z,(x_0,y_0)) \in \bar{L}, \forall z$. This implies (by a canonical ODE argument) that $\phi(z,(x_0,y_0))$ is defined for all $z \in \mathbb{C}$. Now Liouville's theorem is applied in order to prove that $\phi(z,(x_0,y_0))$ is constant which gives $X(p_0) = 0$, contradiction.

This first remark allows us to restrict in a certain sense, our argumentation to when occurs in a neighborhood of L_∞. This will be done by studying the deformations of the holonomy group of $\mathcal{F}_0$ associated to this algebraic leaf. Let us thus consider the setting we will work with:

$\mathcal{F}_0 \in M_1(n)$, $n \geq 2$. Denote by $\{p_1, \ldots, p_{n+1}\} \subset L_\infty$ the singularities of $\mathcal{F}_0$ in L_∞. Fix a point $O \in L_\infty$, $O \neq p_j$, $\forall j$ and take cycles $\gamma_1, \ldots, \gamma_n \in \pi_1(L_\infty \backslash \{p_1, \ldots, p_{n+1}\})$ which are generators of this homotopy group with no relations. Let Σ be a transverse section to $\mathcal{F}_0$ at O, $\Sigma \pitchfork L_\infty$, $\Sigma \approx \mathbb{D}$.

Given a polynomial vector field $X = (P, Q)$ with g.c.d$(P, Q) = 1$, which defines $\mathcal{F}_0$ in $\mathbb{C}^2$ and given any holomorphic 1-parameter deformation $\{\mathcal{F}_t\}_{t \in \mathbb{D}}$ of $\mathcal{F}_0$ in $\mathbb{C}P(2)$. We may find a 1-parameter holomorphic family $\{P_t, Q_t\} \subset \mathbb{C}[x, y]$, $\forall t \in \mathbb{D}$ with g.c.d$(P_t, Q_t) = 1$, $P_0 = P$, $Q_0 = Q$ such that $\mathcal{F}_t$ is given by $X_t = (P_t, Q_t)$ in $\mathbb{C}^2$. If $\epsilon > 0$ is small enough then for $|t| < \epsilon$, the foliation $\mathcal{F}_t$ has leaves transverse to Σ. Let us change projective coordinates in $\mathbb{C}P(2)$ so that $L_\infty \cap \mathbb{C}^2$ becomes the x-axis $(y = 0)$, O is the origin $(0, 0)$. Since $(y = 0)$ is now $\mathcal{F}_0$ invariant we may take Σ as small disk $\mathbb{D} \subset (x = 0)$. The singularities p_j write $p_j = (a_j, 0)$, $j = 1, \cdots, n$ and $p_{n+1} = (x = \infty, y = 0)$ and the cycles γ_j write $\gamma_j(s) = (x_j(s), 0)$, $s \in [0, 1]$, $x_j(0) = x_j(1) = 0$, $x_j(s) \neq a_j, \forall s, \forall i, j = 1, \cdots, n$.

Given any point $(x_0, y_0) \in \mathbb{C}^2$ such that $X(x_0, y_0) \neq 0$ and $P(x_0, y_0) \neq 0$ we consider the solution $\phi(x, (x_0, y_0), t)$ of the Cauchy problem

$$\frac{dy}{dx} = \frac{Q_t(x, y)}{P_t(x, y)} \quad y(x_0) = y_0. \tag{6}$$

Note that since L_∞ is $\mathcal{F}_0$-invariant we may arbitrarily choose small neighborhood U_j of p_j in $\mathbb{C}P(2)$ such that $\exists$ neighborhood V of L_∞ in $\mathbb{C}P(2)$ and $\epsilon > 0$ with: $|t| < \epsilon$, $(x, y) \in V \backslash \bigcup_{j=1}^{n+1} U_j \Rightarrow X_t(x, y) \neq 0$ and $P_t(x, y) \neq 0$. In what follows, we will restrict ourselves to $|t| < \epsilon$ and $(x, y) \in V \cap \mathbb{C}^2$. Denote by $\mathcal{Y}_j(s, y_0, t)$, $s \in [0, 1]$ the analytic continuation of $\phi(x, (0, y_0), t)$ along the curve γ_j, with the initial value $\mathcal{Y}_j(0, y_0, t) = (0, y_0) \in \Sigma$. We denote by ${f_j}^t = \mathcal{Y}_j(1, y_0, t)$ the final value of this analytic continuation so that ${f_j}^t(y_0) \in \Sigma$ for $y_0 \approx 0$ enough and t ≈ 0 enough. Indeed ${f_j}^0(0) = 0$ because L_∞ is $\mathcal{F}_0$-invariant. Thus we obtain that $\{{f_1}^0, \ldots, {f_n}^0\} \subset \mathrm{Diff}(\Sigma, \mathrm{O})$ generate the holonomy group $\mathrm{Hol}(\mathcal{F}_0, L_\infty)$. Let H_t be the pseudogroup of local map $\Sigma \to \Sigma$ generated by the mappings ${f_1}^t, \ldots, {f_n}^t$. Then H_t is an one-parameter pseudogroup holomorphic deformation of $\mathrm{H}_0 = \mathrm{Hol}(\mathcal{F}_0, L_\infty)$.

Since $\mathcal{F}_0 \in M_1(n)$ the holonomy group is nonsolvable so that we have as a consequence of [Na]:

Proposition 5.2 *For $\epsilon > 0$ small enough the pseudo-orbits of H_t, $|t| < \epsilon$ are dense in an open neighborhood of the origin in Σ.*

Remark. Any leaf which dose not contain a separatrix must accumulate both separatrices of a hyperbolic singularity.

Now we state several lemmas which are consequence of the analytic dependence of the solutions of a one-parameter family of ODEs in the parameter and initial condition, or also of lemma [??] in part I and above remark.

The important fact is that L_∞ is $\mathcal{F}_0$-invariant and that $\mathcal{F}_t$ has hyperbolic singularities for $|t| < \epsilon$. Let $\Sigma' \subset \Sigma$ be a compact subdisk.

Lemma 5.3 *There exists a compact set K such that:*

(i)$\forall |t| < \epsilon$, $K \cap \mathrm{Sing}\mathcal{F}_t = \varnothing$

(ii)$\forall |t| < \epsilon$, L^t is a leaf of $\mathcal{F}_t$ not containing any separatrix of singularity $p_j(t) \Rightarrow L^t \cap K \neq \varnothing$

(iii) $\forall p \in K$ the $\mathcal{F}_0$-leaf, L^0_p cuts Σ'

Lemma 5.4 *We may choose Σ' in such a way that there exists a neighborhood W of Σ' with $\mathcal{F}_t$ trivial in a neighborhood of W, $\forall |t| < \epsilon$ so that if $q \in W$ then the $\mathcal{F}_t$ leaf, L^t_q cuts Σ*

Lemma 5.5 *$\exists$ $0 < \epsilon_1 < \epsilon$ such that if $|t| < \epsilon_1$ and $L^t \cap K \neq \varnothing$ then $L^t \cap W \neq \varnothing$*

The existence of W as in lemma 5.4 is an easy consequence of the Flow Box theorem for one parameter families of foliation and of the fact that given a vertical transverse disk Σ' in a flow box neighborhood W then any leaf of the trivialized (horizontal) foliation must cut this disk.

The existence of compact set K satisfying the conditions of lemma 5.3 may obtain as follows:

Take initially the U_j as small bidisks, $U_j = \mathbb{D}(p_j, \alpha) \times \mathbb{D}(0, 2\delta)$ a compact tubular neighborhood $A = L_\infty \times \mathbb{D}_\delta$ of L_∞ and put $B = A \backslash (\bigcup_{j=1}^{n+1} U_j \cap A)$

Notice that $L_\infty \backslash \bigcup_{j=1}^{\infty} (U_j \cap L_\infty)$ is compact and $\mathcal{F}_0$-invariant so that (Since U_j may be chosen arbitrarily small if $|t| < \epsilon$ and ϵ is small enough) we may choose K arbitrarily small and such that

$\forall p \in K$, $L^0_p \cap \Sigma' \neq \varnothing$

$U_j \subset A$, $\forall j$, U_j is a bidisk. Now we recall that according to proposition [??] the singularity p_j as well as its separatrices move analytically in t. In particular if we take the vertical fibration and recall that L_∞ is invariant and transverse to the fibration for $\mathcal{F}_0$ then we will have for small $|t|$ a smooth separatrices for $\mathcal{F}_t$ at $p_j(t)$ that is still transverse to the vertical fibration in a neighborhood of p_0. Now, take $V_j = \mathbb{D}(p_j, \alpha) \times \mathbb{D}(0, \delta_1)$ for $0 < \delta_1 << \delta$. Then it follows that for $|t| < \epsilon_1 << \epsilon$ we have a separatrix of $\mathcal{F}_t$ through $p_j(t)$, which is transverse to the vertical fibration and meeting the boundary ∂V_j at $\mathbb{S}(p_j, \alpha) \times \mathbb{D}(0, \delta_1)$ transversely so that the corresponding leaf enters the compact set A. Now, according to lemma [] given any leaf L^t of $\mathcal{F}_t$ we know that L^t must intersect the compact neighborhood of L_∞ given by $L_\infty \times \mathbb{D}(0, \delta_1)$. In particular we may conclude that if $|t| < \epsilon_1$ then we have two possibilities:

(i)Either L^t intersects K or some V_j and therefor $L^t \cap K = \varnothing$

(ii)Either L^t dose not intersect K and $L^t \cap V_j$ is always contained in some separatrix of $\mathcal{F}_t|_{V_j}$ which correspond to the deformation of the separatrix of $\mathcal{F}_0$ at p_j transverse to L_∞.

In this last case L^t cuts L_∞ transversely and it is transverse to the horizontal fibration $(x = cte)$ so that we conclude that L^t is analytic in a neighborhood of L_∞. This implies that L^t is analytic in $\mathbb{C}P(2)$ and therefor it is an algebraic and $\mathcal{F}_t$-invariant curve.

6 PROOF OF DENSITY THEOREM

We are now able to prove theorem 4.4:
Given an one-parameter holomorphic deformation $\mathcal{F}_{t\in\mathbb{D}}$ of a foliation $\mathcal{F}_0 \in \mathcal{M}_1(n)$, $n \geq 2$ on $\mathbb{C}P(2)$ we may consider $\Sigma \pitchfork L_\infty$ at $O \in L_\infty$, H_t and $|t| < \epsilon_1$ as above. Then $|t| < \epsilon_1$ implies any non-algebraic $\mathcal{F}_t$-leaf L^t intersects any immersed compact subdisk $\Sigma_1 \subset \Sigma$ so that in particular using the fact that H_t has dense pseudo-orbits in a fixed neighborhood of the origin in Σ we conclude that all the non-algebraic leaves of $\mathcal{F}_t$ are dense in $\mathbb{C}P(2)$. This proves theorem 4.4.

In order to prove theorem 4.5., it is enough to use theorem 4.4. and the following result of Lins Neto[Li]:
For every $n \geq 2$ the set of foliations of degree n and without algebraic solutions in $\mathbb{C}P(2)$, contains an open dense subset.

References

[BeLiLo] Belliart M., Liousse I. and Loray F.: Sur l'existence de points fixes attractifs pour les sous-groupes de $Aut(\mathbb{C}, 0)$. C. R. Acad. Sci.,Paris, Ser. I, Math. 324(1997) 443-446

[CaLiSa] Camacho C., Lins Neto A. and Sad P.: Minimal sets of foliations on complex projective spaces. Inst. Hautes Éudes Sci. Publ. Math. No. 68(1989), 187–203

[CaSc] Camacho C. and Scrádua, B.: Holomorphic foliations with Liouvillian first integrals. Ergodic Theory Dynam. Systems 21 (2001), no. 3, 717–756.

[CeSa] Cerveau D. and Sad p.: Problémes de modules pour les formes différentielles singulières dans le plan complexe. Comment. Math. Helv. 61 (1986), no. 2, 222–253.

[Il] Ilyashenko Y.: Topology of phase portraits of analytic differential equations on a complex projective plane. Trudy Sem. Petrovsk. No. 4 (1978), 83–136

[Li] Lins Neto, Alcides Algebraic solutions of polynomial differential equations and foliations in dimension two. Holomorphic dynamics (Mexico, 1986), 192–232, Lecture Notes in Math., 1345, Springer, Berlin, 1988

[LiSaSc] Lins Neto A., Sad P. and Scrádua, B.: On topological rigidity of projective foliations. Bull. Soc. Math. France 126 (1998), no. 3, 381–406.

[MaSaSu] Mañé,, R.; Sad, P.; Sullivan, D.: On the dynamics of rational maps. Ann. Sci. Écõle Norm. Sup. (4) 16 (1983), no. 2, 193–217.

[MaSa1] Mattei J.F. and Salem E.: Classification topologique locale des feuilletages singuliers en dimention 2, In prepration.

[MaSa2] Mattei J.F. and Salem E.: Complete systems of topological and analytical invariants for a generic foliation of $(\mathbb{C}^2, 0)$, Math. Res. Lett. 4 (1) (1997), 131-141

[Mj] Mjuller, B. The density of the solutions of a certain equation in $\mathbb{C}P(2)$. Mat. Sb. (N.S.) 98(140) (1975), no. 3(11), 363–377

[Na] Nakai, I.: Separatrices for nonsolvable dynamics on $(\mathbb{C}, 0)$. Ann. Inst. Fourier (Grenoble) 44 (1994), no. 2, 569–599.

[Sc1] Scrádua, B.: Integration of complex differential equations. J. Dynam. Control Systems 5 (1999), no. 1, 1–50

[Sh1] Shcherbakov, A.: Density of the orbit of a pseudogroup of conformal mappings and generalization of the Khudai -Verenov theorem. Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1982, no. 4, 10–15, 84

[Si] Siu Y,: Techniques of extension of analytic objects. Lecture Notes in Pure and Applied Mathematics, Vol. 8. Marcel Dekker, Inc., New York, 1974.

[Wi1] Wirtz, B.: A property of global density for generic pseudo-groups of holomorphic diffeomorphisms of $\mathbb{C}$ with linearizable generators. Preprint 1999.

[Wi2] Wirtz, B.: Persistence de densite locale D'orbites de pseudo-groups de diffeomorphisms holomorphes par perturbation de points fixes. Preprint, 1999.

[Wi3] Wirtz, B.: Fixed points of one parameter-family of pseudo-groups of $\mathrm{Diff}(\mathbb{C}, 0)$ 1998