RANDOM CONFORMAL DYNAMICAL SYSTEMS

BERTRAND DEROIN & VICTOR KLEPTSYN

1. INTRODUCTION

Given a probability measure on the group of homeomorphisms of a manifold, one can study the asymptotic behaviour of large composition of elements chosen randomly with respect to this measure. From the 50's the questions of the behaviour of the random walk on the group, composition of random matrices, equidistribution of the orbits, Lyapunov exponents (if the action is by C^1 diffeomorphisms) etc. has been largely studied and understood. We are unable to review here all the history of these problems and we refer the interested reader to an excellent survey of Furman [F]. Nevertheless, we would like to mention here some important works in the development of this theory: Kakutani [Kak], Furstenberg [Fu1], Arnold-Krylov [A-K], Furstenberg-Kesten [F-K], Guivar'ch [Gui].

More recently was developed the idea that the maps could be taken from a pseudo-group, rather than a group. This has been introduced in the paper of Garnett [Ga] for the pseudo-group of a foliation, and then studied by Ghys [Gh2, Gh3], Kaimanovich [Kai1], Ledrappier [Led] and Candel [Can]. Following the lines of the "Sullivan's dictionary", one can extend these ideas to other pseudo-groups, like for instance the one generated by an endomorphism or a correspondence.

In this work we study random compositions of the elements of a pseudo-group acting conformally on a manifold. Our results concern actions of a group by conformal transformations, conformal correspondences or transversely conformal foliations (for one dimensional dynamical systems, we suppose that the maps are of class C^1); most of them were known in the case of a group acting conformally on a compact manifold, when the conditional probabilities do not depend on the point (see [F]).

Observe that the existence of a measure which is invariant by every element of the pseudo-group is rather rare. In the symmetric case

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(i.e. when the probabilities are symmetric) we prove the following dichotomy. On a minimal subset, either is supported a probability measure which is preserved by all the elements of the pseudo-group, or the system has the property of exponential contraction: for any point and almost every random composition of elements of the pseudogroup there exists a neighborhood of the point which is contracted exponentially.

We deduce some results about the equidistribution of the orbits of the system along random compositions. In the case where the system have the property of exponential contraction (even if the system is not symmetric), we prove that the orbit of a point by almost every random compositions is distributed with respect to a unique measure. We also give examples of non symmetric systems for which the exponential contraction property and the equidistribution property are not satisfied.

1.1. Presentation of the results for a foliation. We begin by a survey on Garnett's theory [Ga] (see also [Can]).

Let \mathcal{F} be a foliation of a compact manifold M, whose leaves are of class C^{∞} , and g a Riemannian metric on the leaves of \mathcal{F} . In [Ga], Garnett studies the diffusion process along the leaves of \mathcal{F} . Namely, the metric g induces the Laplace-Beltrami operator along the leaves, which we denote Δ ; given a continuous function $f_0 : M \to \mathbb{R}$, one studies the heat equation along the leaves of the foliation

$$\frac{\partial f}{\partial t} = \Delta f$$

with initial condition $f(\cdot, t) = f_0$. As it is well-known [Cha], because the leaves are complete and of bounded geometry, the solution to the heat equation is unique, defined for all positive time, and is expressed by convolution of the initial condition with the heat kernel p(x, y; t). A fundamental Lemma due to Garnett (see also Candel [Can]), asserts that the functions $f(\cdot, t)$ on M are continuous, and that the diffusion semi-group of operators D^t defined for all $t \ge 0$ by

$$D^t f(0, \cdot) = f(t, \cdot)$$

acts continuously on $C^0(M)$.

Associated to this diffusion semi-group, Garnett considers the Brownian motion along the leaves of the foliation: this is a Markovian process with continuous time, whose trajectories stays every time in the same leaf, and whose transition probability distributions are volume forms whose leafwise density is the heat kernel. It is known that this

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process can be realized as a process with continuous trajectories. For any point x in M, let Γ_x be the set of continuous paths parametrized by $[0, \infty)$, starting at x, and whose image is contained in the leaf \mathcal{F}_x passing through the point x. The space Γ_x is equipped with the uniform topology on compact subsets; there is a probability Borel measure induced by the Brownian motion process, expressing the probability that a trajectory occurs. This probability measure is called the *Wiener mea*sure and denoted W_x .

We recall the definition of the holonomy pseudo-group. Because the manifold M is compact, there is a finite cover of M by foliated box $B_i \times T_i$, in which the foliation \mathcal{F} is the horizontal fibration. The change of coordinates from $B_i \times T_i$ to $B_j \times T_j$ are of the form

$$(x_i, t_i) \mapsto (x_j = x_j(x_i, t_i), t_j(t_i)).$$

The maps $t_j(t_i)$ generates a pseudo-group on the union $T = \bigcup_i T_i$. A measure on T which is invariant by the holonomy pseudo-group is called a *transversely invariant measure*. These measures have been introduced by Schwartzman for flows [Scm], by Plante and Ruelle-Sullivan for foliations [Pl, R-S] and by Sullivan for other kind of dynamical systems [Su1].

Now, consider a continuous path γ contained in a leaf, parametrized by a closed interval. It crosses successively the foliation boxes $B_{i_1} \times T_{i_1}, \ldots, B_{i_k} \times T_{i_k}$. The composition of the associated change of transverse coordinates is by definition the holonomy map h_{γ} corresponding to γ . The following result describes the asymptotic behaviour of the holonomy maps $h_{\gamma|_{[0,t]}}$ when t goes to infinity, for a generic Brownian path along the leaf passing throw a point x, when the foliation is transversely conformal.

Theorem 1.1 (Main Theorem). Let \mathcal{F} be a transversely conformal foliation of class C^1 of a compact manifold. Then either there exists a transversely invariant measure. Or \mathcal{F} has a finite number of minimal sets $\mathcal{M}_1, \ldots, \mathcal{M}_k$ equipped with probability measures μ_1, \ldots, μ_k , and there exists a real $\alpha > 0$ such that:

 Contraction. For every point x in M and almost every leafwise Brownian path γ starting at x, there is a neighborhood T_γ of x in T and a constant C_γ > 0, such that for every t > 0, the holonomy map h_{γ|[0,t]} is defined on T_γ and

$$h_{\gamma|_{[0,t]}}(T_{\gamma})| \le C_{\gamma} \exp(-\alpha t).$$

 Distribution. For every point x in M and almost every leafwise Brownian path γ starting at x, the path γ tends to one of the \mathcal{M}_j and is distributed with respect to μ_j , in the sense that

$$\lim_{t \to \infty} \frac{1}{t} \gamma_* \operatorname{leb}_{[0,t]} = \mu_j,$$

where $leb_{[0,t]}$ is the Lebesgue measure on the interval [0,t].

- Attraction. The probability $p_j(x)$ that a leafwise Brownian path starting at a point x of M tends to \mathcal{M}_j is a continuous leafwise harmonic function.
- **Diffusion.** When t goes to infinity, the diffusions $D^t f$ of a continuous function $f : M \to \mathbb{R}$ converge uniformly to the function $\sum_j c_j p_j$, where $c_j = \int f d\mu_j$. In particular, the functions p_j form a base in the space of continuous leafwise harmonic functions.

The existence of a transversely invariant measure for a transversely conformal foliation is a very strong condition. An ergodic component of such a measure is either supported on a compact leaf, or it is diffuse. All the examples we know of a transversely conformal foliation having a diffuse transversely invariant measure have a transverse metric which is transversely invariant. In the case of codimension one foliation of class C^2 , this is an easy consequence of Sacksteder Theorem. For higher transversely conformal foliations this has been conjectured by Ghys [Gh1], and for codimension 3 and higher with an additional restriction of minimality this conjecture was proven by Tarquini [Ta].

The distribution part of the theorem was proved by Garnett [Ga] in the case of the stable foliation of the geodesic flow on the unitary tangent bundle of a surface of constant negative curvature, using the contraction property and the similarity, which is straightforward in this case. This was also extended to the case of a manifold of negative variable curvature of any dimension by Ledrappier [Led].

The diffusion part of the Theorem 1.1 gives example of foliations of a compact manifold that have non trivial continuous leafwise harmonic function. Such example were constructed in [F-G].

1.2. **Organization of the proof.** In [Ga], Garnett studies the ergodic properties of the leafwise diffusion semi-group and of the leafwise Brownian motion. She introduces the notion of *harmonic measure*, which is a probability measure invariant by the diffusion semi-group. By the Kakutani fixed point Theorem, such a measure exists. The relation with the Brownian motion goes as follows. Consider the space Γ of all the continuous paths contained in a leaf of \mathcal{F} . There is a semi-group $\{\sigma_t\}_{t>0}$ of transformations of Γ defined for every $t, s \geq 0$ and every $\gamma \in \Gamma$ by

$$\sigma_t(\gamma)(s) = \gamma(t+s).$$

If μ is a measure on M, then one consider the probability measure $\overline{\mu}$ on Γ which is defined by

$$\overline{\mu}(B) = \int_M W_x(B \cap \Gamma_x) d\mu(x),$$

for every Borel subset B contained in Γ (recall that for every $x \in M$, Γ_x is the set of pathes starting at x and contained on the leaf through x, and W_x is the Wiener measure on Γ_x). It is straightforward to see that if μ is harmonic, then $\overline{\mu}$ is invariant by $\{\sigma_t\}_{t>0}$, and reciprocally.

If a harmonic measure can not be written as a convex sum of different harmonic measures it is called *ergodic*. The Random Ergodic Theorem, due to Kakutani ([Kak], see also [F]) states that if μ is an ergodic harmonic measure, then $\overline{\mu}$ is an ergodic invariant measure of $\{\sigma_t\}_{t\geq 0}$. This implies in particular that for μ -almost every point x in M, W_x almost every path $\gamma \in \Gamma_x$ is distributed with respect to μ . Thus, the ergodic properties of harmonic measures on the foliation \mathcal{F} can be studied via the classical ergodic theory of one dimensional semi-groups of transformations.

We study the Lyapunov exponent of a harmonic measure for a transversely conformal foliation \mathcal{F} of class C^1 (see [Can, De]). Let $|\cdot|$ be a transverse metric. Then if γ is a continuous path of Γ_x starting at a point x, consider the Lyapunov exponent

$$\lambda(\gamma) := \lim_{t \to \infty} \frac{1}{t} \log |Dh_{\gamma|_{[0,t]}}|,$$

when it is defined. Recall that $h_{\gamma|_{[0,t]}}$ is the holonomy map from a transversal $T_{\gamma(0)}$ passing through $x = \gamma(0)$ to a transversal $T_{\gamma(t)}$ passing through $\gamma(t)$. By the Birkhoff Ergodic Theorem and the Random Ergodic Theorem, if μ is an ergodic harmonic measure, then for μ -almost every point x, and W_x -almost every Brownian path γ starting at x, the Lyapunov exponent of γ converges to a number depending only on μ ; we call it the Lyapunov exponent of the measure μ and denote it by $\lambda(\mu)$.

We begin by the study of the case where the Lyapunov exponent is negative.

Theorem A. Let \mathcal{F} be a transversely conformal foliation of class C^1 of a compact manifold, and suppose that on a minimal subset \mathcal{M} is supported a harmonic ergodic measure μ with negative Lyapunov exponent. Then the following properties are satisfied:

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• Contraction. Let α , $0 < \alpha < |\lambda(\mu)|$ be chosen. Then for any $x \in \mathcal{M}$, and almost every Brownian path $\gamma \in \Gamma_x$, there exist a transversal T_{γ} at x and a constant $C_{\gamma} > 0$ such that for every t > 0, the holonomy map $h_{\gamma|_{[0,t]}}$ is defined on T_{γ} , and

 $|h_{\gamma|_{[0,t]}}(T_x)| \le C_{\gamma} \exp(-\alpha t).$

- Unique ergodicity. For any point $x \in \mathcal{M}$, almost every Brownian path starting at x is distributed with respect to μ . Thus μ is the unique harmonic measure on \mathcal{M} .
- **Diffusion.** The diffusions $D^t f$ of a continuous function $f : \mathcal{M} \to \mathbb{R}$ converge uniformly to the constant function $\int_{\mathcal{M}} f d\mu$.
- Attraction. Suppose $\mathcal{M} \neq M$, and let $p_{\mathcal{M}}(x)$ be the probability that a Brownian path starting at x tends to \mathcal{M} , is distributed with respect to μ , and contracts exponentially a transversal at x. Then $p_{\mathcal{M}}$ is lower semi-continuous and leafwise harmonic. In particular, $p_{\mathcal{M}}$ is bounded from below be a positive constant in some neighborhood of \mathcal{M} .

Theorem A is proved in section 2. The idea is the following. A lemma of contraction (Lemma 2.2) implies, together with the fact that the Lyapunov exponent is negative, that for μ -almost every point x, almost every Brownian path starting at x contracts a transverse ball exponentially. We prove that if \overline{x} is a point which is close to such a point x, then there is a similarity between the Brownian motion on the leaf $L_{\overline{x}}$ and on L_x . This comes from the fact that in a lot of "direction", the leaves approach each other exponentially. All the properties annouced in the Theorem A are deduced from this property of similarity.

Remark 1.2. Theorem A is also true if the foliation is singular, but the minimal set does not contain any singularity. In particular, our result applies for singular holomorphic foliations on complex compact surfaces. For instance, we prove that if \mathcal{M} is a minimal subset of a holomorphic foliation of the complex projective plane, then on \mathcal{M} is supported a unique harmonic measure and the Lyapunov exponent is negative. This has been recently proved by Fornaess and Sibony for a lamination by holomorphic curves of class C^1 contained in the complex projective plane [F-S].

In section 3, we prove that there is a dichotomy between the case of negative Lyapunov exponent and the case where there exists a transversely invariant measure:

Theorem B. Let \mathcal{F} be a transversely conformal foliation of a compact manifold. Then on a minimal subset, either there exists a transversely

invariant measure, or the harmonic measure is unique and the Lyapunov exponent is negative.

In the case of a group of diffeomorphisms, the first result of this kind was proved by Furstenberg [Fu1]: if G is an irreducible subgroup of projective transformations of $\mathbb{R}P^n$, equipped with a probability measure of finite first moment, then it has the contraction property. For a group of diffeomorphisms of class C^1 of an arbitrary compact manifold, the dichotomy "there exists a measure which is invariant by all the elements of the group or there is a stationary measure with negative sum of the Lyapunov exponents" were proved by Baxendale [Ba].

From Theorem B and a Theorem of Candel [Can], we obtain the following:

Corollary 1.3. A minimal subset of a transversely conformal foliation of a compact manifold (if the codimension is one, the foliation is supposed of class C^1) carries an invariant measure, or there is a loop contained in a leaf with hyperbolic holonomy.

Our main theorem is a consequence of Theorem A and B. At the end of section 3, we prove the main theorem.

In [Can], Candel extends Garnett's theory to the case of a non symmetric Laplace operator on the leaves of a foliation. For this kind of processes, Theorem A is still valid: negative Lyapunov exponent implies unique ergodicity, contraction etc. However, in the non-symmetric case Lyapunov exponent can be positive, the dynamics not uniquely ergodic, even if the foliation is minimal. Such an example is presented in Section 3.4.

The last part is a tentative to prove unique-ergodicity for a Laplace operator whose drift vector field preserves the Riemannian volume. We prove this when the foliation together with the Laplace operator are *similar*. A foliation equipped with a Laplace operator on the leaves is similar if there exists a transverse foliation which leaves the Laplace operator invariant.

Theorem C. Let (\mathcal{F}, Δ) be a similar codimension one foliation of a compact manifold, which is transversely continuous. Suppose that the operator Δ is obtained by drifting the Laplacian of a Riemannian metric by a vector field that preserves the volume. Then on every minimal subset is supported a unique harmonic measure. Moreover, if Δ is symmetric, every ergodic harmonic measure is supported on a minimal.

The idea of the proof of Theorem C is, as in Theorem A, based on the fact that the leaves through close points stay close in a lot of "directions". This idea is due to Thurston ([Th], see also [C-D, Fe]). Because we already know that the Brownian motion on different leaves are similar, such a property implies unique-ergodicity. This property is proved by constructing a harmonic transverse distance and to use the Martingale Theorem. This is done in section 4.

1.3. Other examples of dynamical systems. Following the lines of the "Sullivan's dictionary", our results are still valid for discrete conformal dynamical systems. The idea is that, instead of considering a foliation by smooth manifolds, one can consider foliations by *graphs*.

Let M be a compact manifold together with a conformal structure of class C^1 , and Γ be a finitely generated pseudo-group of conformal transformations of M. Then for any x in M, we denote by O(x) the orbit of x under the action of the elements of Γ . Given a symmetric system of generators of Γ , we consider a distance on every orbit O(x): the distance between x and a point y in O(x) is the minimal number of elements of the system of generators that is necessary to map x to y.

Consider a family $\{\mu_x\}_{x \in M}$ of probability measures on M whose support is the orbit of x. We ask that for any element γ of the pseudogroup G, the function

$$x \in \operatorname{dom}(\gamma) \mapsto \mu_x(\gamma(x)) \in (0,1)$$

is Hölder. The *diffusion* operator acts on the space of continuous functions by the formula

$$Df(x) = \int_{O_x} f d\mu_x$$

for any $f \in C^0(M)$ and any $x \in M$. An invariant measure by the diffusion semi-group always exists, and is called a *stationary* measure.

Associated to this diffusion process on the leaves, we consider the Markov process induced by the measures μ_x on the orbits of Γ , which is the discrete analog of the Brownian motion on the leaves of a foliation. We define the Lyapunov exponent when the following finiteness hypothesis holds:

$$\forall x \in M, \quad \forall \alpha > 0, \quad \int_{O_x} \exp(\alpha d(x, y)) d\mu_x(y) < \infty.$$

Our results extends to the context of a pseudo-group with the following analogies:

• the leaves together with the Brownian motion process is replaced by the orbit of a point together with the Markovian process with transition probabilities $p(x, y) = \sum_{\gamma(x)=y} \mu_x(\gamma(x))$. • the symmetry condition in the discrete case means that there exists a Hölder function $v: M \to \mathbb{R}$ such that for any points $x, y \in M$:

$$p(x, y) \exp(v(x)) = p(y, x) \exp(v(y)).$$

The important examples are the pseudo-group generated by the action of a group on \mathbf{S}^n by conformal transformations, or the action of an affine conformal correspondence on a torus \mathbf{T}^n .

2. Negative Lyapunov exponent

In this section, we are going to prove Theorem A for a codimension one foliation of class C^1 . In this case, we can find a transversal foliation of dimension one and of class C^1 , and this will simplify the proof. Such a foliation does not necessarily exist in the case of higher codimension, but at the appendix (Section 5.3) we explain how to adapt the proof. Finally, we may suppose \mathcal{F} to be transversely orientable (and we do so from this moment), as it is always true up to a 2-folded cover, and passing a finite cover does not change our results.

Let \mathcal{F} be a codimension one foliation of class C^1 of a compact manifold M, and \mathcal{G} a transverse foliation of class C^1 . Consider some minimal subset $\mathcal{M} \subset M$. Let us suppose that for some ergodic harmonic measure μ supported on \mathcal{M} , we have $\lambda(\mu) < 0$.

2.1. Contraction. The goal of this paragraph is to prove that for generic Brownian paths, the holonomy contracts a transverse interval exponentially, which is true infinitesimally:

Proposition 2.1. Suppose that there exists an ergodic harmonic measure μ on M, such that $\lambda(\mu) < 0$. Let $\alpha > 0, \alpha < |\lambda(\mu)|$. Then for μ -almost every $x \in M$ and for W_x -almost every $\gamma \in \Gamma_x$, there exists a transversal neighborhood I and a constant C > 0, such that

$$\forall t > 0 \quad \left| h_{\gamma|_{[0,t]}}(I) \right| < C |I| e^{-\alpha t}.$$

In particular, all the holonomy maps $h_{\gamma|_{[0,t]}}$ are defined in the same transversal neighborhood I and, as $t \to \infty$, this neighborhood is exponentially contracted.

To this end, we have to connect the derivatives of holonomy maps in one point x and the diameter of the images of the transversal neighborhood I. The following Contraction Lemma is in the "folklore". For the completeness of the text, we present here both its statement and proof. This lemma extends the Distortion Lemmas, used in C^2 case by Schwartz [Sch], Denjoy [Den] and Sacksteder [Sa], in $C^{1+\tau}$ by Sullivan [Su2] and Hurder [Hur].

Lemma 2.2 (Contraction Lemma). Let $x_0, x_1, \dots \in \mathbb{R}$ be points in \mathbb{R} , $I_j \subset \mathbb{R}$, $I_j = U_{\varepsilon}(x_j)$ be their ε -neighborhoods. Let

$$h_j: I_j \to \mathbb{R}, \quad h_j(x_j) = x_{j+1}, \quad j = 0, 1, 2, \dots$$

be C^1 -diffeomorphisms onto their image. Let $\overline{f}_j(y) = h_j(y+x_j) - x_{j+1}$ be diffeomorphisms of $U_{\varepsilon}(0)$, and suppose, that these diffeomorphisms are bounded in the C^1 topology. Denote

$$F_n = h_n \circ \cdots \circ h_1 : I_1 \to \mathbb{R}, \quad n \in \mathbb{N}$$

and suppose that

(2.1)
$$\limsup_{n \to \infty} \frac{1}{n} \log F'_n(x_0) = \lambda < 0,$$

and let $\alpha > 0, \alpha < |\lambda|$. Then there exist such an $\varepsilon_1 > 0$ and a constant C, that for any interval $J \subset I_0$, such that $x_0 \in J, |J| < \varepsilon_1$ all the compositions F_n are defined on J and we have a bound

$$\forall n \in \mathbb{N} \quad |F_n(J)| \le |J| \cdot Ce^{-\alpha n}.$$

Proof. Let us choose β , $\alpha < \beta < |\lambda|$. Then, the condition (2.1) implies that the supremum

$$C = \sup_{n \in \mathbb{N}} e^{\beta n} F'_n(x_0)$$

is finite. Then, for every n we have $F'_n(x_0) \leq Ce^{\beta n}$.

Note, that due to pre-compactness property the logatithms of the derivatives $\log h'_n$ are uniformly continuous. Thus, there exists $\varepsilon_0 > 0$, such that for every n and for every $y, z \in I_n$, $|y - z| < \varepsilon_0$ we have $h'_n(y)/h'_n(z) < e^{\beta - \alpha}$.

Now, let $\varepsilon_1 = \min(\varepsilon_0, \varepsilon)/C$, and let J be an interval of length less than ε , containing x_0 . We are going to prove the following statement: for every n, the length of $|F_n(J)| < Ce^{-\alpha n}|J|$. In fact, by the mean value theorem $|F_n(J)| = F'_n(y)|J|$ for some point $y \in J$. Now,

(2.2)
$$\frac{F'_n(y)}{F'_n(x)} = \prod_{j=1}^n \frac{h'_j(y_{j-1})}{h'_j(x_{j-1})} < (e^{\beta - \alpha})^n = e^{(\beta - \alpha)n}.$$

Here $y_j = F_j(y)$, and the inequality $\frac{h'_j(y_{j-1})}{h'_j(x_{j-1})} < e^{\beta - \alpha}$ is satisfied due to the recurrence hypothesis and the choice of ε_0 :

$$|x_{j-1} - y_{j-1}| \le |F_{j-1}(J)| \le Ce^{-\alpha n}|J| \le C\varepsilon_1 < \varepsilon_0.$$

Now, from (2.2) we have:

$$F'_n(y) \le e^{(\beta - \alpha)n} F'_n(x) \le e^{(\beta - \alpha)n} \cdot C e^{\beta n} = C e^{-\alpha n}.$$

Hence,

$$|F_n(J)| = F'_n(y) \cdot |J| < Ce^{-\alpha n}|J|.$$

This proves the recurrence step, and thus the entire lemma.

Now we are going to apply the Contraction Lemma to the Brownian motion on the leaves. In order to do that, we would like to decompose

the holonomy map in time t as a composition of some number n of maps, forming a Diff¹-pre-compact set, with the quotient n/t being bounded from above and from below.

Suppose $\delta > 0$ be given. Then to any trajectory $\gamma \in \Gamma_x$, we associate a sequence of points $(x_n) = (\tilde{\gamma}(n\delta))$ on the universal cover $\tilde{\mathcal{F}}_x$, where the path $\tilde{\gamma}$ is the covering path for the path γ , and a sequence of numbers $k_n = [\operatorname{dist}(x_{n-1}, x_n)] + 1$ (here [z] denotes the integer part of z). Let us divide a segment of shortest geodesic line, joining x_{n-1} and x_n , in k_n equal parts; let us denote the vertices of this partition by $y_n^0 = x_{n-1}, y_n^1, \ldots, y_n^{k_n-1}, y_n^{k_n} = x_n$. Then, the holonomy map $h_{x_0x_n}$ between x_0 and x_n can be written as

$$h_{x_0x_n} = h_{x_{n-1}x_n} \circ h_{x_{n-2}x_{n-1}} \cdots \circ h_{x_0x_1}$$

and thus as

$$(2.3) \quad h_{x_0x_n} = (h_{y_n^{k_n-1}y_n^{k_n}} \circ \dots \circ h_{y_n^0y_n^1}) \circ \dots \circ (h_{y_1^{k_n-1}y_1^{k_n}} \circ \dots \circ h_{y_1^0y_1^1}).$$

The total number of the maps in the right hand side of (2.3) is $K_n = k_1 + \cdots + k_n$. The following lemma is some form of the Large Numbers Law. Though it is rather clear that such statement should take place, its rigorous proof is rather long, and we have put it in Section 5.

Lemma 2.3. There exists a constant c > 0, such that for any $x \in M$ for W_x -almost every path $\gamma \in \Gamma_x$ we have $K_n/n < c$ for all n sufficiently big.

Thus, the discretization of the Brownian motion is "quasi-preserving" the time: the number of terms in the right-hand side of the representation (2.3) is comparable with the time passed.

Proof of Proposition 2.1. Suppose that a point $x \in M$ is such that for almost every path $\gamma \in \Gamma_x$ we have

(2.4)
$$\lim_{t \to \infty} \frac{1}{t} \log h'_{\gamma|_{[0,t]}}(x) = \lambda(\mu) < 0.$$

Let us show that for this point the conclusion of Proposition 2.1 holds. This will prove the Proposition — as μ -almost every point x satisfies (2.4).

Recall that $\alpha < |\lambda|$. As it follows from (2.4), for W_x -almost every path $\gamma \in \Gamma_x$ there exists a constant $C_0 > 0$, such that for every t > 0

(2.5)
$$h'_{\gamma|_{[0,t]}}(x) < C_0 e^{-\alpha t}.$$

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Let us consider a discretization (x_n, k_n) of such path γ . As it follows from Proposition 2.3, for almost every path γ we have also

(2.6)
$$\exists N : \forall n > N \quad \frac{K_n}{n} < c.$$

Suppose that for γ both (2.5) and (2.6) are satisfied. Choosing some c' > c, we may suppose that for all $n \in \mathbb{N}$ we have $K_n < c'n$. As it follows from (2.3), for every $n \in \mathbb{N}$ the holonomy map $h_{x_0x_n}$ can be written as a composition of $K_n < c'n$ maps $h_{y_l^j y_l^{j+1}}$, each one being a holonomy between two points at the distance at most 1. The set of holonomy maps along paths of length at most 1 is pre-compact (it is a continuous image of a compact set), and the derivative of such composition at the point x is less then $C_0 e^{-\alpha n} < C_0 e^{-\alpha K_n/c'}$, thus, we still have an exponential decrease of derivatives at x with respect to the number of maps. The application of the Lemma 2.2 concludes the proof.

2.2. Similarity of the Brownian motions on different leaves. Suppose that the point x is typical in the sense of Proposition 2.1, and that almost every Brownian path starting at x is distributed with respect to μ . Then there exist a transversal interval I, and constants C_0 , $\alpha > 0$, such that the set

$$E_x = \left\{ \gamma \in \Gamma_x \mid \forall t \quad \left| h_{\gamma|_{[0,t]}}(I) \right| < C_0 \exp(-\alpha t) |I| \right\}$$

has positive Wiener measure: $W_x(E_x) > 0$ (in fact, this probability can be made arbitrary close to 1 by choosing sufficiently small I and sufficiently big C_0). Let us fix such I and C_0 .

If \bar{x} is a point close to x, consider the set $E_{\bar{x}}$ of Brownian paths $\bar{\gamma} \in \Gamma_{\bar{x}}$ which are exponentially asymptotic to a path γ of $\Gamma_{\bar{x}}$:

$$\forall t \ge 0, \qquad d(\gamma(t), \bar{\gamma}(t)) < C_0 \exp(-\alpha t) |I|.$$

For instance, a path of $E_{\bar{x}}$ can be constructed from a path of E_x by following the foliation \mathcal{G} .

Lemma 2.4 (Similarity of the Brownian motions). There exists a neighborhood U of x such that for any $\bar{x} \in U$, the Wiener measure of $E_{\bar{x}}$ is positive. Moreover, it can be made arbitrarily close to 1 by choosing sufficiently small I, large C_0 and small U. Finally, $W_{\bar{x}}$ -almost every path $\gamma \in E_{\bar{x}}$ is distributed with respect to μ .

We are going to outline the proof of this Lemma. However, a formal realization of the ideas faces some difficulties and becomes very technical. Thus, we have postponed it until Appendix (section 5).

First, we observe that the lemma is simple to prove when the foliation is similar, meaning that the foliation \mathcal{G} preserves the Laplace operator. In this case, the \mathcal{G} -along holonomy preserves the metric and thus translates the Brownian motion on an initial leaf to the Brownian motion of the range leaf. In particular, the probabilities $W_x(E_x)$ and $W_{\bar{x}}(E_{\bar{x}})$ coincide if the points x and \bar{x} are in the same \mathcal{G} -leaf. On the other hand, if we consider Brownian motions starting at a point y and at some point $\bar{y} \in \mathcal{F}_y$ close enough to y, the distribution of their values at the moment δ will be absolutely continuous with respect to each other, and the density will be close to 1. Thus, the probabilities of any tail-type properties (in particular, of belonging to E_y and $E_{\bar{y}}$, being distributed with respect to μ , etc.) are close enough. The $W_x(E_x) > 0$ property implies the same for the points close enough on the same leaf of \mathcal{F} and for the points close enough on the same leaf of \mathcal{G} . Thus, this property is satisfied in some neighborhood of x (as \mathcal{F} and \mathcal{G} are transversal). This proves the lemma for similar foliations.

In the general case, the \mathcal{F} -leafwise implication is still valid. Unfortunately, it is much more difficult to prove the \mathcal{G} -along implication, when the Riemannian metric is not invariant by the foliation \mathcal{G} . Let us suppose that $\bar{x} \in \mathcal{G}_x$. Denote by $\Phi_{x,\bar{x}} : E_x \to E_{\bar{x}}$ the holonomy map along the transversal foliation \mathcal{G} . Except for similar foliations, $\Phi_{x,\bar{x}}$ does not translate the Wiener measure W_x on E_x to a measure which is absolutely continuous with respect to the Wiener measure $W_{\bar{x}}$ on $E_{\bar{x}}$. This effect comes from small movements — typical path of the Brownian motion, being considered on arbitrary small interval of time, allows to reconstruct the Riemannian metrics (on its support). Thus, we are going to pass from the Brownian paths to their discretization as it was already done in the previous paragraph.

Let $\delta > 0$ be given. Let us define the *discretization map* $F^{\delta} : E_x \to (\widetilde{\mathcal{F}}_x)^{\infty}$ as

(2.7)
$$F(\gamma) = \{\widetilde{\gamma}(n\delta)\}_{n=0}^{\infty}.$$

Denote by E_x^{δ} the image of E_x under F^{δ} , and by W_x^{δ} measure on E_x^{δ} that is the image of the Wiener measure on Γ_x , restricted on E_x , under F^{δ} . We claim that if $\bar{x} \in \mathcal{G}_x$ is sufficiently close to x, then the induced map

$$\Phi_{x,\bar{x}}^{\delta} : (E_x^{\delta}, W_x^{\delta}) \to (E_{\bar{x}}^{\delta}, W_{\bar{x}}^{\delta})$$

is absolutely continuous and its Radon-Nykodym derivative is uniformly close to 1 on a set of a large measure. This comes from the fact that for a finite number of steps of discretization the density can be written explicitly: the density is the product of quotients of heat kernels on these two leaves. The step of discretization is constant and equals δ , and the leaves approach each other along the trajectories of E_x exponentially. Thus, it is natural to expect that such product would converge. Moreover, the closer initial points x and \bar{x} are, the closer to 1 will be the product. These facts imply that the probabilities $W_x(E_x)$ and $W_{\bar{x}}(E_{\bar{x}})$ are sufficiently close and thus imply the Lemma; their rigorous proof can be found in the appendix (Section 5).

To simplify the notations, we note $\sigma = \sigma_{\delta} : \Gamma \to \Gamma$:

$$\forall t \ge 0 \quad \sigma(\gamma)(t) = \gamma(t+\delta).$$

Corollary 2.5 (Similarity of Brownian motions). For any $y \in \mathcal{M}$ for W_y -almost every path $\gamma \in \Gamma_y$ there exists n, such that $\gamma(n\delta) \in U$, $\sigma^n(\gamma) \in E_{\gamma(n\delta)}$, and γ is distributed with respect to μ .

Proof. Let us consider the map $D : \Gamma \to \Gamma$, defined in the following "algorithmic" way:

$$D = D_1 \circ D_2,$$

where $D_2(\gamma) = \sigma^n(\gamma), n = \inf\{j \mid \gamma(j\delta) \in U\},$
$$D_1(\gamma) = \begin{cases} \text{STOP,} & \text{if } \gamma \in E_{\gamma(0)}, \\ \sigma^n(\gamma), & n = \inf\{j \mid |h_{\gamma|_{[0,j\delta]}}(I)| \ge Ce^{-\alpha \cdot j\delta}|I|\}. \end{cases}$$

Note, that $D_2(\gamma)$ is defined on W_z -almost every $\gamma \in \Gamma_z$ for every $z \in \mathcal{M}$ due to the minimality of \mathcal{M} and due to the Markovian property of the Brownian motion: on every step of discretization, the probability of hitting U is positive and bounded from below; thus the probability of never hitting U is 0.

Now, note that for every z the probability of D returning "STOP" on $\gamma \in \Gamma_z$ is bounded from below due to Lemma 2.4. Also, due to the Markovian property for every z, z' the conditional distribution of $D(\gamma), \gamma \in \Gamma_z$ with respect to the condition $D(\gamma)(0) = z'$, coincides with $W_{z'}$. Thus, the probability of D not stopping on γ in k iterations tends to zero exponentially. In particular, W_z -almost surely some iteration $D^k(\gamma)$ stops, which means, that $D_2(D^{k-1}(\gamma)) \in E_{\bar{z}}$, where $\bar{z} = D_2(D^{k-1}(\gamma))(0)$. Together with the definitions of D_1 and D_2 , this proves the first part of the corollary.

Finally, recall that for any n the conditional distribution of $\sigma^n(\gamma)$ with respect to the condition $\gamma(n\delta) = z$ coincides with W_z . Lemma 2.4 states, that for every $z \in U$ almost every path $\gamma \in E_z$ is distributed with respect to μ , hence, the constructed path $\sigma^n(\gamma)$ almost surely is distributed with respect to μ . A finite shift can not change the asymptotic distribution of a path, and so the corollary is proven.

Before going into the applications of the preceeding results, we would like to make a disgression that may clarify the meaning of "similarity of the Brownian motion on the leaves". Recall that every Riemannian manifold N of bounded geometry has a boundary associated to the Brownian motion process on it: this is the Poisson boundary P(N). We recall the following facts, that characterize the Poisson boundary (see [Kai1]). For every x of N there is a canonical projection $\pi_x : \Gamma_x \to P(N)$. The family of probability measures $\nu_x = (\pi_x)_* W_x$ depends harmonically on N, in the sense that for any bounded measurable function f on P(N), the function

$$P(f)(x) = \int_{P(N)} f d\nu_a$$

is a bounded harmonic function on N. Moreover, every bounded harmonic function on N is obtained in this way.

There is a following question, which (even if the answer is negative), in our opinion, clarifies the proof of Lemma 2.4, giving the good general idea.

Question 2.6. Given two leaves \mathcal{F}_x and $\mathcal{F}_{\bar{x}}$, does the argument of the proof of Lemma 2.4 allow to identify large parts of the Poisson boundaries of their universal covers \mathcal{F}_x and $\mathcal{F}_{\bar{x}}$, corresponding to the couple of "directions" in which the leaves are converging exponentially to each other?

In the case of a foliation by hyperbolic surfaces of a 3-manifold, Thurston has constructed the "circle at infinity" (see [C-D, Fe]). A leaf of the universal cover of such a foliation is isometric to the hyperbolic plane, and its boundary (as a Gromov hyperbolic space) is a topological circle. Thurston has proved that there is a natural topological identification of the circle at infinity of the leaves of the foliation on the universal cover.

Question 2.7. The boundary (as a Gromov hyperbolic space) of the hyperbolic plane is also the Poisson boundary. Is it true that the topological identifications of the boundaries of Thusrton's theorem preserve the structure of the Poisson boundary as well?

2.3. Proof of Theorem A.

2.3.1. Contraction property. Let $x \in \mathcal{M}$. Then by Corollary 2.5, for almost every Brownian path γ starting at x, there exists n such that $\gamma(n\delta)$ belongs to U, and $\sigma^n \gamma \in E_{\gamma(n\delta)}$. Thus starting from the time $n\delta$, the path γ contracts a transverse interval exponentially, with exponent α . The contraction property is proved.

2.3.2. Unique ergodicity. Let $x \in \mathcal{M}$. Recall first that almost every Brownian path starting at x is distributed with respect to μ , as it is claimed by Corollary 2.5.

Let us prove that μ is the only harmonic measure supported on \mathcal{M} . Let μ' be an ergodic harmonic measure. Then for μ' -almost every point y, W_y -almost every Brownian path $\gamma \in W_y$ is distributed with respect to μ' . Fix one of these points y. By the preceeding argument, W_y -almost every Brownian path γ is also distributed with respect to μ . Thus $\mu' = \mu$ and μ is the unique harmonic measure supported on \mathcal{M} .

2.3.3. Attraction. Lower semicontinuity of the function $p_{\mathcal{M}}$ is immediately implied by the proof of Lemma 2.4. Now, $p_{\mathcal{M}}|_{\mathcal{M}} = 1$ due to Corollary 2.5. Thus, $p_{\mathcal{M}}$ is bounded from below by a positive constant in some neighborhood of \mathcal{M} .

Now, due to the Markovian property of the Brownian motion, for any initial point x the process $p_{\mathcal{M}}(\gamma(t))$ is a martingale. Due to Ito formula, $p_{\mathcal{M}}$ is leafwise harmonic.

The Theorem A is proven unless for the diffusion part, which is absolutely analogous to the proof of the diffusion part of the Main Theorem, see paragraph 3.2.

Remark 2.8. In fact, for any point $y \in M$ almost all the paths $\gamma \in \Gamma_y$, tending to \mathcal{M} (if such paths exist), are distributed with respect to \mathcal{M} . In particular,

$$p_{\mathcal{M}}(x) = W_x(\{\gamma \in \Gamma_x \mid \gamma \to \mathcal{M}\})$$

Let us prove this. For a trajectory starting at sufficiently small distance from \mathcal{M} , with the probability close to 1 this trajectory will be distributed with respect to μ . Decompose trajectories starting at y and tending to \mathcal{M} in two parts: finite part before arriving close to \mathcal{M} and infinite afterwards. The Markovian property implies, that the probability of the trajectory being distributed with respect to μ is arbitrary close to total probability of tending to \mathcal{M} . Thus, as the distance of decomposition can be chosen arbitrary small, almost every trajectory, tending to \mathcal{M} , is distributed with respect to μ .

Denote by $Attr(\mathcal{M})$ the basin of attraction of \mathcal{M} : this is the union of the leaves whose closure contains \mathcal{M} . Now, it is rather clear that the function $p_{\mathcal{M}}$ is positive exactly in the points of $Attr(\mathcal{M})$.

Corollary 2.9. Any harmonic ergodic measure different from μ has a support disjoint from $Attr(\mathcal{M})$.

Proof. A trajectory starting at $y \in Attr(\mathcal{M})$ tends to \mathcal{M} and is distributed with respect to μ with positive probability. If there exists another harmonic ergodic measure μ' with the support not disjoint from $Attr(\mathcal{M})$, for μ' -almost every point $z \in Attr(\mathcal{M})$ almost all trajectories, starting at z, would be distributed with respect to μ' , not with respect to μ . But there are no such points, which gives us the desired contradiction.

2.4. Examples: holomorphic foliations on complex surfaces. Let \mathcal{F} be a singular holomorphic foliation of a compact complex surface S, and let g be a hermitian metric on $T\mathcal{F}$. We note Δ the Laplacian of g along the leaves of \mathcal{F} . With the use of harmonic measures, one can extend certain notions that we have for compact holomorphic curves; for instance the Euler characteristic. The following definition is due to Candel (see [Gh4]).

Definition 2.10. Let $E \to S$ be a holomorphic line bundle over S, and μ a harmonic measure. The *Chern-Candel class* of E against μ is

$$c_1(E,\mu) := \frac{1}{2\pi} \int_S \operatorname{curvature}(|\cdot|) d\mu,$$

where $|\cdot|$ is a hermitian metric on E of class C^2 . Recall that the curvature of a hermitian metric is the function such that

$$\operatorname{curvature}(|\cdot|) := -2\Delta \log |s|,$$

where s is a *local* non vanishing holomorphic section of E. Because the curvature of two different smooth hermitian metrics on E differs by the Laplacian along \mathcal{F} of a smooth function, the Chern-Candel class of E does not depend on the choice of the hermitian metric. The *Euler* characteristic of T is the first Chern class of the tangent bundle of \mathcal{F} .

The following lemma expresses the Lyapunov exponent in algebraic terms.

Lemma 2.11. Let \mathcal{M} be a closed minimal subset, which does not contain singularities of \mathcal{F} . Let g be a hermitian metric on $T\mathcal{F}$ and μ a harmonic measure supported on \mathcal{M} . Then

$$\lambda(\mu) = -\pi c_1(N_{\mathcal{F}}, \mu).$$

Proof. Let $|\cdot|$ be a smooth conformal transverse metric of the foliation. Let us compute the curvature of the normal bundle $N_{\mathcal{F}}$. Let (z, t) some local coordinates where the foliation is defined by dt = 0. The section $\frac{\partial}{\partial t}$ induces a non vanishing holomorphic section of the normal bundle. Thus, the curvature of $N_{\mathcal{F}}$ is

$$\operatorname{curvature}(|\cdot|) = -2\Delta \log |\frac{\partial}{\partial t}|.$$

One gets

$$c_1(N_{\mathcal{F}}, T_{g,\mu}) = -\frac{1}{\pi} \int \Delta \log |\frac{\partial}{\partial t}| d\mu$$

and the lemma is proved by applying Lemma 3.1.

Theorem 2.12. Let \mathcal{F} be a singular holomorphic foliation of a complex surface, and \mathcal{M} be an exceptional minimal subset of \mathcal{F} . Suppose that the normal bundle of \mathcal{F} has a metric of positive curvature along \mathcal{M} . Then on \mathcal{M} is supported a unique harmonic measure of negative Lyapunov exponent.

Proof. Let μ be an ergodic harmonic measure supported on \mathcal{M} . The normal bundle of \mathcal{F} has a metric of positive curvature. Thus, by Lemma 2.11, the Lyapunov exponent of μ is negative. The theorem is a corollary of Theorem A.

Recently Fornaess-Sibony proved that on a compact lamination by holomorphic curves of class C^1 of $\mathbb{C}P^2$, there is a unique harmonic measure [F-S]. We recover this result here for an exceptional minimal set of a singular holomorphic foliation of $\mathbb{C}P^2$.

Corollary 2.13. An exceptional minimal subset of a singular holomorphic foliation of $\mathbb{C}P^2$ carries a unique harmonic measure of negative Lyapunov exponent.

Proof. The normal bundle of a singular holomorphic foliation of $\mathbb{C}P^2$ is O(d+2) where d > 0 is the degree. This bundle has a metric of positive curvature everywhere.

3. Symmetric case

A Laplace operator is called symmetric if it is the Laplace-Beltrami operator of a Riemannian metric. In this section we consider a foliation equipped with a Riemannian metric on the leaves, and the Laplace-Beltrami operator Δ along the leaves.

3.1. The dichotomy: proof of Theorem B. In this paragraph we prove Theorem B: on a minimal subset of a transversely conformal foliation is supported a unique harmonic measure with negative Lyapunov exponent, or there is a transversely invariant measure.

If there exists a harmonic measure for which the Lyapunov exponent is negative, then Theorem A shows that this is the unique harmonic measure. Thus we will prove that if the Lyapunov of *any* harmonic measure is non negative, then there exists a transversely invariant measure.

We use an integral formula which expresses the Lyapunov exponent, and which has been founded in [Can, De]. Let $|\cdot|$ be a transverse conformal metric. In a foliation box we consider a transverse vector field u which is invariant by the holonomy, and define $\varphi := |u|$. Note that φ is well-defined up to the multiplication by a leafwise constant function. Thus the function $f = \Delta \log \varphi$ is a well-defined continuous function on M.

Lemma 3.1. For any ergodic harmonic measure μ on M, $\lambda(\mu) = \int_M f d\mu$.

Proof. For every $t \ge 0$, let $L_t : \Gamma \to \mathbb{R}$ be the functional defined by:

$$L_t(\gamma) = \log |Dh_{\gamma|_{[0,t]}}|.$$

The family $\{L_t\}_{t\geq 0}$ is a cocycle with respect to the shift semi-group, in the sense that for any $s, t \geq 0$:

$$L_{t+s} = L_t + L_s \circ \sigma_t.$$

Thus the integrals $\int_{\Gamma} L_t d\overline{\mu}$ depends linearly of t. Because by definition the limit

$$\lambda(\gamma) = \lim_{t \to \infty} \frac{L_t(\gamma)}{t}$$

exists for $\overline{\mu}$ -almost every path and is equal to the Lyapunov exponent of μ , we have for every $t \ge 0$:

$$\int_{\Gamma} L_t d\overline{\mu} = t\lambda(\mu).$$

Now, let x, y be two points in the universal cover \widetilde{L} of a leaf L. Define the cocycle

$$c(x,y) := \log h'_{x,y}$$

where $h_{x,y}$ is the holonomy between x and y. Let

$$\lambda_t(x) := \int_{\widetilde{L}} p(x, y; t) c(x, y) \operatorname{vol}_g(y).$$

The work of Garnett ([Ga, p. 288, Fact 1]) shows that λ_t is a continuous function on M. Moreover we have the formula

(3.1)
$$t\lambda(\mu) = \int_{\Gamma} L_t d\overline{\mu} = \int_M \lambda_t(x) d\mu(x).$$

Observe that φ is a function defined up to multiplication by a constant on \widetilde{L} and that by definition $\varphi(y)/\varphi(x) = h'_{x,y}$. Thus we get $c(x,y) = \log \varphi(y) - \log \varphi(x)$ and taking the derivative at t = 0:

(3.2)
$$\frac{d\lambda_t(x)}{dt}\Big|_{t=0} = \Delta \log \varphi(x).$$

Differentiating (3.1) at t = 0 and substituting (3.2), we obtain the desired result.

We will also use another formula which expresses the conservation of mass of the diffusion semi-group. Let v be the volume form on M, induced by the leafwise volume form vol_q and the transverse metric.

Lemma 3.2. $\int_{M} (\Delta \varphi + |\nabla \varphi|^2) v = 0.$

Proof. Let v_t be the transverse volume form induced by the transverse metric. Then, in a foliation box $B \times T$, there is a volume form θ on T such that

$$v_t = \exp(q\varphi)\theta_t$$

where q is the codimension of \mathcal{F} . This is by definition of φ .

Consider a partition of unity: $1 = \sum_{i} f_i$, where the support of each function f_i is contained in a foliation box $B_i \times T_i$. By Green's formula:

$$\int_{M} \Delta f_{i} v = \int_{B_{i} \times T_{i}} (\Delta f_{i}) \exp(\varphi_{i}) \operatorname{vol}_{g} \wedge \theta_{i} =$$

$$= \int_{B_{i} \times T_{i}} f_{i}(\Delta \exp(\varphi_{i})) \operatorname{vol}_{g} \wedge \theta_{i} = \int_{B_{i} \times T_{i}} f_{i}(\Delta \varphi + |\nabla \varphi|^{2}) \exp(\varphi) \operatorname{vol}_{g} \wedge \theta_{i} =$$

$$= \int_{M} f_{i}(\Delta \varphi + |\nabla \varphi|^{2}) v.$$

By summing these equalities and using the fact that $\Delta 1 = 0$, we obtain

$$0 = \int_{M} \Delta 1 v = \sum_{i} \int_{M} \Delta f_{i} v =$$
$$= \sum_{i} \int_{M} f_{i} (\Delta \varphi + |\nabla \varphi|^{2}) v = \int_{M} (\Delta \varphi + |\nabla \varphi|^{2}) v.$$

Lemma 3.2 is proved.

3.1.1. Proof of Theorem B when $\mathcal{M} = M$. By the integral definition of a harmonic measure (see [Ga, Lemma B, p. 294]) and Lemma 3.1, the fact that the Lyapunov exponent of every harmonic measure is non negative is trivially satisfied if the function f is bounded from below by the laplacian along the leaves of a smooth function h:

$$f \ge \Delta h$$

Let us study this case as a relevant example. Let v_t be the transverse volume form induced by the transverse metric $|\cdot|$, and v'_t be defined by $v'_t = \exp(-qh) v_t$, where q is the codimension of \mathcal{F} . We claim that v'_t is transversely invariant, or, what is the same, the measure $\mu = \operatorname{vol}_g \wedge v'_t$ is totally invariant. By Lemma 3.2, we have

$$\int_{M} (\Delta \varphi' + |\nabla \varphi'|^2) d\mu = 0.$$

Because φ' is subharmonic, this implies that φ' is locally constant along the leaves. Thus μ is a totally invariant measure.

Remark 3.3. A more geometric proof of this goes as follows: we consider the leafwise gradient of the function $\log \varphi'$, which is well-defined everywhere. Due to sub-harmonicity of φ' it dilates leafwise volume. By definition it also dilates the transverse volume. Thus, the total volume must increase everywhere. The only way is that the function φ' must be leafwise constant, meaning that the measure μ is totally invariant.

Unfortunately, there exists a diffeomorphism of the circle, which is minimal, and whose invariant transverse measure is singular [K-H]. Therefore, we can not hope to solve the functional inequality

 $f \ge \Delta_{\mathcal{F}} h$

in general. However, it is still possible to solve this inequality "approximatively". The following result is due to Ghys [Gh2, Gh3], and the proof is based on the use of the Hahn-Banach theorem; this idea goes back to the famous paper [Su1] by Sullivan on the foliation cycles.

Lemma 3.4. Let $f: M \to \mathbb{R}$ be a continuous function such that

$$\int_M f \, d\mu \ge 0$$

for every harmonic measure μ . Then there exists a sequence of smooth functions $\psi_n : M \to \mathbb{R}$ such that uniformly,

$$\liminf_{n \to \infty} f - \Delta_{\mathcal{F}} \psi_n \ge 0.$$

Proof. In the Banach space $C^0(M)$, consider the closed subspace E of uniform limit of leafwise Laplacian of smooth functions, the cone C of everywhere positive functions. Let $F = C^0(M)/E$ and $\overline{C} \subset F$ be the closure of the image of the cone C under the natural projection $C^0(M) \to F$. The conclusion of Lemma 3.4 is equivalent to the fact that the image of f in F is in \overline{C} . Suppose it is not the case. Then, by Hahn-Banach separation theorem, there exists a linear functional which is non-negative on \overline{C} and negative on the image of f. Such a linear functional is by definition a harmonic measure (due to the integral definition of harmonicity), if it is conveniently normalized, and the corresponding Lyapunov exponent is negative. It gives us the desired a contradiction. The lemma is proved.

To prove Theorem A in the case where $\mathcal{M} = M$, we consider the family of volume forms

$$\mu_n = \exp(-q\psi_n) \operatorname{vol}_g \wedge v_t$$

where the functions ψ_n are given by Lemma 3.4. After normalizing them to probability measures and taking a subsequence, they converge to a probability measure μ on M.

Lemma 3.5. The measure μ is transversely invariant.

Proof. It is more convenient to consider the family of transverse forms $v_{t,n} = \exp(-q\psi_n)v_t$ as currents, i.e. as operators on the space of *p*-forms along the leaves. Define:

$$C_n(\omega) = \int_M \omega \wedge v_{t,n},$$

for every p-form ω , where $p = \dim(\mathcal{F})$. The family of currents $\{C_n\}$ is bounded, thus after taking a subsequence they converge to a current C. By construction, by choosing well the subsequence of currents

converging to C, we have

$$\mu = \operatorname{vol}_q \wedge C.$$

We prove that C is a *closed* current, or equivalently that C_n is asymptotically closed as n goes to infinity. Thus by Sullivan's Theorem ([Su1, Theorem I.12, p. 235]) the measure μ is totally invariant.

Consider a (p-1)-form α . Using a partition of unity, we can write α as a finite sum of (p-1)-forms whose support is contained in a foliation box. Thus we will suppose that the support of α is contained in a foliation box $B \times T$. Let θ be a volume form on T; write

$$v_{t,n} = \exp(\varphi_n)\theta_t$$

then

(3.3)
$$\mu_n = v_{t,n} \wedge \operatorname{vol}_g = \exp(\varphi_n)\theta \wedge \operatorname{vol}_g.$$

We have

$$\int_{M} d\alpha \wedge v_{t,n} = \int_{M} d\alpha \wedge \exp(\varphi_{n})\theta =$$
$$\int_{M} \alpha \wedge d(\exp(\varphi_{n})\theta) = \int_{M} (\alpha \wedge d\varphi_{n}) \wedge v_{t,n}$$

Thus, by Schwarz inequality, (3.4)

$$\left| \int_{M} d\alpha \wedge v_{t,n} \right| \le c |\alpha|_{\infty} \int_{M} |\nabla \varphi_{n}| \mu_{n} \le c |\alpha|_{\infty} \left(\int_{M} |\nabla \varphi_{n}|^{2} \mu_{n} \right)^{1/2},$$

where c is a constant, and ∇ denotes the leafwise gradient. Observe that φ_n is well-defined up to addition of a leafwise constant function, so that $\nabla \varphi_n$ is well-defined.

By Lemma 3.2:

$$\int_{M} |\nabla \varphi_n|^2 \mu_n = -\int_{M} \Delta \varphi_n \mu_n,$$

and by Lemma 3.4, the Laplacians $\Delta \varphi_n$ verify uniformly

$$\liminf_{n \to \infty} \Delta \varphi_n \ge 0.$$

Thus the integrals

$$\int_M |\nabla \varphi_n|^2 \mu_n$$

tend to 0 when n goes to infinity. By formula 3.4, we conclude that C is closed. Hence Lemma 3.5 is proved.

The proof of this lemma concludes the proof of Theorem B when $\mathcal{M} = M$.

Remark 3.6. In general, for a codimension q foliation of class C^1 , Oseledets' Theorem states the existence of q Lyapunov exponents $\lambda_1(\mu), \ldots, \lambda_q(\mu)$ associated to any harmonic ergodic measure μ . In fact, the method we have used in this section proves that for any symmetric Laplace foliation of class C^1 : either there exists a harmonic measure μ for which $\lambda_1(\mu) + \ldots + \lambda_q(\mu)$ is negative, or there exists a transversely invariant measure. This result is analogue to the one of Baxendale [Ba].

However, let us mention some differences between the context of groups and the one of foliations. The theorem of Baxendale does not require the dynamics to be symmetric. It is interesting to note that Baxendale's Theorem does not work for non symmetric Laplace foliations: there exists an example of a minimal foliation (drifted geodesic flow, see paragraph 3.4), for which every harmonic measure has positive Lyapunov exponent. The fact that the probability does not depend on the point in the Theorem of Baxendale should be interpreted in the foliation context by the concept of similarity. There is also an example of a similar Laplace lamination which does not verify Baxendale's Theorem (in this setting the Lyapunov exponent should be defined by using the transverse 2-adic structure).

3.1.2. Proof of Theorem B in the exceptional minimal set case. First, let us notice that \mathcal{F} can not support a harmonic measure with positive Lyapunov exponent. A general idea implying this is the following. Consider first the case of a similar foliation. Then, a harmonic measure on \mathcal{M} induces a harmonic transverse measure (see Definition 4.3), which is harmonic. Due to the Ito Formula, the transversal measure of the image of a small transverse ball under the holonomy map $h_{\gamma|_{[0,t]}}$ associated to the Brownian path $\gamma|_{[0,t]}$ is a martingale.

Hence, its expectation at any Markovian moment equals the measure of the initial transverse ball. On the other hand, a small transverse ball around a typical point (in the sense of convergence of the Lyapunov exponent) is exponentially expanded by a typical Brownian path.

Thus, the martingale takes (at some big moments of time) large values with a large probability. Thus, the expectation is large. This contradicts the fact that the initial transverse ball can be chosen arbitrarily small. Rigorous proof of this statement is presented in Section 5, Lemma 5.4. Thus, all the Lyapunov exponents are equal to 0.

The fact that the Laplace operator is symmetric was used when we stated that conditional measures are harmonic: if the Laplace operator is non-symmetric, these measures are harmonic in the sense of the adjoint operator Δ^* , not Δ .

Now, let us continue the proof using the fact that all the Lyapunov exponents vanish. Choose some $\varepsilon > 0$. Then, the arguments of Lemma 3.4 imply that there exists a function ψ_{ε}^0 , such that $-\varepsilon < f - \Delta \psi_{\varepsilon}^0 < \varepsilon$. By continuity, the same inequality holds in some neighborhood U^{ε} of \mathcal{M} , that we suppose to be contained in the ε neighborhood of \mathcal{M} .

Lemma 3.7. If $\varepsilon > 0$ is small enough, there exists a function ψ_{ε} , such that $\Delta \psi_{\varepsilon} \geq -\varepsilon$ and such that at least $(1 - \varepsilon)$ -part of the measure $\mu_{\varepsilon} = e^{-\psi_{\varepsilon}^{0} + \psi_{\varepsilon}} \operatorname{vol}_{g} \wedge v_{t}$ is concentrated in U^{ε} .

Once such functions are constructed for any ε , the proof will be finished in the same way as the proof in a minimal case. Namely, suppose that such functions are constructed. Let us find any weak limit μ of a subsequence of a family $\frac{1}{\mu_{\varepsilon}(M)}\mu_{\varepsilon}$ as $\varepsilon \to 0$. Note that μ is supported on \mathcal{M} : this comes from the fact that $\mu_{\varepsilon}(U^{\varepsilon}) \geq 1-\varepsilon$ and that $\bigcap_{\varepsilon} U^{\varepsilon} = \mathcal{M}$. Also, note that μ is a weak limit for the same subsequence of the (non-normalized) measures $\mu_{\varepsilon}|_{U^{\varepsilon}}$. But these restricted measures can be written (locally) as $e^{\varphi'} \operatorname{vol}_g \wedge \theta$, where θ is a transverse measure, and $\Delta \varphi' = q \Delta \varphi - q \Delta \psi_{\varepsilon}^0 + \Delta \psi_{\varepsilon} \geq -2\varepsilon$. Passing to the limit and making estimates as in Lemma 3.5, we obtain that μ is totally invariant. Thus the proof of the Theorem will be complete after the proof of Lemma 3.7.

Proof of Lemma 3.7. Recall that $\varepsilon > 0, \psi_{\varepsilon}^{0}$ and U^{ε} are already chosen. We choose another neighborhood V of $\mathcal{M}, V \subset U^{\varepsilon}$. We are going to find a function $\psi = \psi_{\varepsilon}$ verifying Lemma 3.7.

First, let us suppose that in U^{ε} there is no other minimal set, and thus that every leaf passing through a point in $U^{\varepsilon} \setminus \mathcal{M}$ intersects ∂U^{ε} . We are going to look for the function ψ as a solution of the Poisson equation

$$\Delta \psi(x) = -\varepsilon, \quad x \in U^{\varepsilon} \setminus V; \quad \psi|_{\partial(U^{\varepsilon} \setminus V)} = 0,$$

extended by 0 to the complementary of $U^{\varepsilon} \setminus V$. The solution of this problem always exists and can be found in the following way:

$$\psi(x) = \varepsilon \mathbb{E}T(\gamma),$$

where $T(\gamma)$ denotes the first intersection moment of a Brownian path γ with the boundary of $U^{\varepsilon} \setminus V$:

$$T(\gamma) = \min\{t : \gamma(t) \in \partial(U^{\varepsilon} \setminus V)\},\$$

and \mathbb{E} is the expectation of a function on the probability space (Γ_x, W_x) . Note that $\Delta \psi$ equals $-\varepsilon$ in $U^{\varepsilon} \setminus V$, and 0 in V and in the complementary of U^{ε} . Moreover, on $\partial (U^{\varepsilon} \setminus V)$, $\Delta \psi$ is a positive distribution. Thus, one has $\Delta \psi \geq -\varepsilon$. Now, let us show that for an appropriate choice of V the major part of μ_{ε} is concentrated in U^{ε} , with the precise estimates of the Lemma. To do this, it suffices to check that as V tends to \mathcal{M} , the part of the measure μ_{ε} , concentrated in U^{ε} , tends to 1. Note that the (non-normalized!) measure μ_{ε} of $M \setminus U^{\varepsilon}$ does not change, so that we have to prove that the measure of U^{ε} tends to infinity. By the monotone convergence theorem, it is equivalent to the fact that if we let $\bar{\psi} = \lim_{V \to \mathcal{M}} \psi$ (maybe, $\bar{\psi}$ equals infinity at some points), the function $e^{\bar{\psi}}$ will be non-integrable in U^{ε} . Note that the function $\bar{\psi}$ can be written as:

$$\bar{\psi}(x) = \varepsilon \mathbb{E} T_0(\gamma),$$

where

$$T_0(\gamma) = \min\{t : \gamma(t) \in \partial U^{\varepsilon}\}$$

(if such an intersection does not occur, we define $T_0(\gamma) = \infty$). Thus, we have to estimate the mean $\mathbb{E}T_0(\gamma)$.

Note that in a neighborhood U^{ε} we have $f - \Delta h_{\varepsilon} < \varepsilon$. Thus, for a distance \tilde{d} induced by a transversal metric $e^{-h_{\varepsilon}} |\cdot|$, we have $\Delta \log \tilde{d}(\cdot, \mathcal{M}) < \varepsilon$ in U^{ε} .

Now, let us consider a random process

$$\xi_0(t,\gamma) = \log \tilde{d}(\gamma(t), \mathcal{M}) - \varepsilon t,$$

and let us stop it at the moment $T_0(\gamma)$:

$$\xi(t,\gamma) = \xi_0(\min(t,T_0(\gamma)),\gamma).$$

Then, the Ito formula implies that $\xi(t, \gamma)$ is a supermartingale:

$$\begin{aligned} \frac{\partial}{\partial s}\Big|_{s=t+0} \mathbb{E}(\xi(s,\gamma)|\gamma|_{[0,t]}) &= \\ &= \begin{cases} (\Delta \log \tilde{d}(\cdot,\mathcal{M}))(\gamma(t)) - \varepsilon, & \gamma|_{[0,t]} \subset U^{\varepsilon} \\ 0, & T_0(\gamma) \ge t. \end{cases} \le 0 \end{aligned}$$

Note also, that the function $\log d(\cdot, \mathcal{M})$ is Lipschitz on the leaves, thus, the conditional second moments

$$\mathbb{E}(r_n^2(\gamma)|\gamma|_{[0,t]})$$

of the increasements

$$r_n(\gamma) = \xi(n+1,\gamma) - \xi(t,\gamma)$$

are bounded uniformly on n and $\gamma|_{[0,n]}$. Thus, due to the theory of martingales, for every Markovian moment τ with finite expectation the expectation of ξ at this moment does not exceed its initial value.

Let us now use this process to estimate from below the expectation $\mathbb{E}T_0(\gamma)$. Either this expectation is infinite (in which any lower bound is satisfied automatically). Or it is finite, and in this case the expectation of a value of a supermartinagle ξ in a Markovian moment $T_0(\gamma)$ does not exceed its initial value, that is

$$\mathbb{E}\left[\log \tilde{d}(\gamma(T_0(\gamma)), \mathcal{M}) - \varepsilon T_0(\gamma)\right] \le \log \tilde{d}(x, \mathcal{M}).$$

The expectation in the left side can be rewritten as

$$-\varepsilon \mathbb{E}T_0(\gamma) + \mathbb{E}\log \tilde{d}(\gamma(T_0(\gamma)), \mathcal{M}) = -\varepsilon \mathbb{E}T_0(\gamma) + O(1),$$

for at the moment of exiting U^{ε} the distance to \mathcal{M} is separated from 0. So, we have

$$\bar{\psi} = \mathbb{E}T_0(\gamma) \ge -\frac{1}{\varepsilon} \log \tilde{d}(x, \mathcal{M}) + C_0.$$

This implies that

$$e^{\bar{\psi}(x)} \ge \frac{C}{(\tilde{d}(x,\mathcal{M}))^{1/\varepsilon}}.$$

Thus, if ε is less than $1/(\operatorname{codim} \mathcal{F})$, the function $e^{\overline{\psi}}$ is non-integrable and the effect of concentration takes place. This completes the proof under the hypothesis that \mathcal{M} is the unique minimal subset of U^{ε} .

To conclude the proof, we remark that if U^{ε} contains another minimal set, we can replace \mathcal{M} by the closure of the union of all the leaves, entirely contained in U^{ε} , and repeat the previous arguments.

The proof of Theorem B is completed in all the cases.

3.2. **Proof of the Main Theorem.** We suppose that the foliation \mathcal{F} is transversely conformal and does not have a transversely invariant measure. By Theorem B, on any minimal set is supported a unique harmonic measure with negative Lyapunov exponent. Because of the attraction property, any minimal set has a neighborhood which does not contain any other minimal set. Thus, there is a finite number of minimal sets $\mathcal{M}_1, \ldots, \mathcal{M}_k$. Denote by μ_1, \ldots, μ_k their unique harmonic measure, and $\lambda_1, \ldots, \lambda_k$ the corresponding Lyapunov exponents. Note that every point $x \in M$ belongs to the basin of attraction of at least one of these sets; the reason is that the set

$$M \setminus (\bigcup_{j=1}^k Attr(\mathcal{M}_j))$$

is closed, consists only of entire leaves and does not contain any minimal subset.

Let $\alpha > 0$ be a real number such that $\alpha < |\lambda_j|$ for every j. For every point $x \in M$ we consider the probability

$$p_j(x) = W_x(\{\gamma \in \Gamma_x \mid \gamma(t) \xrightarrow[t \to \infty]{} \mathcal{M}_j\}),$$

that a Brownian path starting at x tends to \mathcal{M}_i . Note that almost every Brownian path tending to \mathcal{M}_i (if such path exists) is distributed with respect to μ_i , and contracts a transverse ball at x exponentially with exponent $-\alpha$ (see Remark 2.8). We claim that the sum of these probabilities is equal to 1; in other words, W_x -almost every trajectory tends to one of the minimal sets, with the distribution and transverse contraction properties. We show this in the following way: for arbitrary small neighborhoods U_1, \ldots, U_k of $\mathcal{M}_1, \ldots, \mathcal{M}_k$ respectively, the complementary $R = M \setminus \bigcup_j U_j$ is a closed set without any minimal subset, thus containing no entire leaf. Hence, for any point x of Rthere exists a leafwise path leading to one of the neighborhoods U_i ; moreover, by compactness of M, the length of such a path is bounded uniformly on R. Thus, for a point $x \in R$, the probability that it lies in one of the U_i at time 1 is bounded from below by a positive uniform constant. Hence, for any point $x \in M$, almost every trajectory starting in x meets one of the neighborhoods U_i .

To complete the proof, let us show that $\sum_j p_j(x) > 1 - \varepsilon$ for any $\varepsilon > 0$. To do this, let us choose U_j so close to \mathcal{M}_j that for every point in U_j the probability of attracting to \mathcal{M}_j with the distribution and the transverse contraction properties is at least $1 - \varepsilon$; it is possible due to Lemma 2.4. Now, let us use the Markovian property: for any $x \in \mathcal{M}$, almost every trajectory $\gamma \in \Gamma_x$ meets one of the U_j , and for a starting point in U_j the probability of attracting to the corresponding \mathcal{M}_j is at least $1 - \varepsilon$. Thus, the probability of attracting to one of the \mathcal{M}_j is at least $1 - \varepsilon$. As $\varepsilon > 0$ was chosen arbitrary, we have proven that almost every trajectory tends to one of the U_j .

Recall that the functions p_j are leafwise harmonic and lower semicontinuous. Because their sum is equal to the constant function 1 the functions p_j are continuous.

Thus, we have proved the Contraction, Distribution and Attraction parts of the main theorem. We end the proof of Theorem 1.1 by proving the statement about the asymptotic behaviour of the diffusion. First, we shall prove a weaker form. Namely, we prove that the *time-averages* of diffusions tends to the same limit:

$$\frac{1}{T} \int_0^T D^t f \, dt \stackrel{x \in M}{\Rightarrow} \psi(x),$$

where $\psi(x) = \sum_{j} p_j \int f d\mu_j$. In the case where the foliation is minimal, it is implied by unique ergodicity (see analogous arguments in [Fu2]). Namely, the value of the time-average of the diffusions at a point $x \in M$ can be rewritten as an integral:

$$\frac{1}{T} \int_0^T (D^t f)(x) dt = \frac{1}{T} \int_0^T \int_M D^t f \, d\delta_x \, dt = \\ = \frac{1}{T} \int_0^T \int_M f \, d(D^t_* \delta_x) \, dt = \int_M f \, dm_{x,T},$$

where

$$m_{x,T} = \frac{1}{T} \int_0^T (D^t_* \delta_x) \, dt$$

is the time-average of the diffusions of the measure δ_x . Note that due to a classical argument in ergodic theory, a weak limit of a sequence m_{x_n,t_n} with $t_n \to \infty$ is harmonic. As there exists a unique harmonic measure μ , the time averages $m_{x,t}$ converge to μ uniformly in x as t tends to infinity. Thus, the integrals of f with respect to these measures also converge uniformly to $\int_M f d\mu$, which implies the desired statement.

In the case of an exceptional minimal set we notice that the timeaverages of the diffusions can be rewritten as

(3.5)
$$\frac{1}{T}\int_0^T (D^t f)(x) dt = \int_{\Gamma_x} \left(\frac{1}{T}\int_0^T f(\gamma(t)) dt\right) dW_x(\gamma).$$

We know that W_x -almost all trajectories tend to one of the \mathcal{M}_j 's and are distributed with respect to the corresponding harmonic measure μ_j . The probability that a point x tends to \mathcal{M}_j is equal to $p_j(x)$. Hence, the right hand side of (3.5) is equal to

$$\sum_{j=1}^{k} p_j(x) \int_{\mathcal{M}_j} f \, d\mu_j.$$

Moreover, a uniform argument on $(1 - \varepsilon)$ -measure of trajectories (similarity of Brownian motions) implies that this convergence is uniform in $x \in \mathcal{M}$.

Thus, in every case we have shown that the time-average of the diffusions converge to the right-hand side, which we denote ψ .

Now, let us finish the proof using the arguments analogous to these of Kaimanovich [Kai1]. Namely, notice that due to the diffusion of the Brownian motion, there exists $\varepsilon, \varepsilon' > 0$ such that for any point x, and any point $y \in U_{\varepsilon}(x)$, the densities p(x, y, 1) and p(x, y, 2) are bounded from below by ε' . Thus one has for every x and every bounded function f:

$$|D^1 f(x) - D^2 f(x)| \le 2(1 - \varepsilon' \operatorname{vol}(U_{\varepsilon})(x)) |f|_{\infty},$$

where $|\cdot|_{\infty}$ is the uniform norm. Thus, because the leaves are of bounded geometry, we have

$$||D^1 - D^2||_{\infty} < 2,$$

where $|| \cdot ||_{\infty}$ is the norm of operators acting on L^{∞} . The "zero-two law" [Li] implies that $||D^n - D^{n+1}|| \to 0$ as $n \to \infty$. In particular, the time-averages of the diffusions converge if and only if the diffusions converge themselves to the same limit [Kai2]. Hence, the diffusions converge to the limit we have described.

Proof of Corollary 1.3. Let \mathcal{F} be a transversely conformal foliation of class C^1 of a compact manifold, and \mathcal{M} a minimal set of \mathcal{F} . By Theorem B, either \mathcal{M} supports a transversely invariant measure, or a harmonic measure of negative Lyapunov exponent. In this case, Candel has proved that there exists a loop contained in a leaf of \mathcal{M} with hyperbolic holonomy (see [Can, Theorem 8.18]).

3.3. Examples: codimension one foliations of class C^2 . In the case of codimension one foliations of class C^2 without compact leaf, the following result completes the Main Theorem:

Proposition 3.8. Let \mathcal{F} be a codimension one foliation of class C^2 of a compact manifold, without compact leaves. Then if \mathcal{F} has a totally invariant measure μ , this measure is the unique harmonic measure. Then, for every point x, almost every Brownian path starting at x is distributed with respect to μ , and the diffusions of a continuous function $f: M \to \mathbb{R}$ tend uniformly to the constant function $\int f d\mu$.

Proof. By Sacksteder Theorem, the foliation \mathcal{F} is minimal. We first prove that the measure μ is the unique totally invariant measure. By minimality of \mathcal{F} and Haefliger's argument [Hae], there exists a transverse circle C cutting every leaf. The transversely invariant measure (corresponding to μ) induces a measure θ on C, invariant by all the holonomy maps. This measure gives us a map $h: C \to \mathbb{R}/l\mathbb{Z} = C'$, where $l = \theta(C)$. This map semi-conjugates the pseudo-group induced by \mathcal{F} on C to a finitely generated group of rotations of C', which we note G. Because \mathcal{F} is minimal, and the holonomy pseudo-group is finitely generated, at least one of the rotations of G is irrational. Then, the Lebesgue measure is the unique probability measure invariant by G, and thus μ is the unique totally invariant measure on \mathcal{F} up to multiplication by a constant.

To conclude the case when there exists a totally invariant measure μ , it suffices to show that every harmonic measure is in fact a totally invariant measure. Observe that the group G is a group of rotations, so that the orbits of its action on C' have a polynomial growth. Hence, the same is true for the action of the holonomy group on C. Thus, every leaf grows polynomially. Kaimanovich ([Kai1, Corollary of Theorem 4]) proved that if for a harmonic measure, almost every leaf (with respect to this measure) has subexponential growth, then this measure is totally invariant. In our case, all the leaves have polynomial growth, hence every harmonic measure is in fact totally invariant.

The distribution and diffusion property is implied by the fact that the harmonic measure is unique. $\hfill \Box$

We end the paragraph by constructing a foliation by surfaces of a 3-dimensional compact manifold, with two exceptional minimal sets. The example is constructed in the following way. Let \mathcal{F} be an oriented codimension one foliation by oriented surfaces with an exceptional minimal set \mathcal{M} and suppose that there exists a transverse loop c which does not cut \mathcal{M} . Then a neighborhood of c in M is diffeomorphic to a solid torus $D^2 \times \mathbb{S}^1$, the foliation \mathcal{F} being the horizontal fibration by two dimensional balls D^2 . Now consider two copies N_1 and N_2 of the exterior of $D^2 \times \mathbb{S}^1$ in M. These two manifolds are foliated, and have a boundary component $\partial B \times \mathbb{S}^1$ transverse to the foliation \mathcal{F} . The foliation $\mathcal F$ induces the horizontal foliation by circles on it. Observe also that N_1 and N_2 have an exceptional minimal set in their interior. Thus, by gluing N_1 and N_2 along their boundary by a diffeomorphism which preserves the foliation and reverses the orientation, we construct a foliation by surfaces of a closed manifold with two exceptional minimal sets.

Now we are given an example of such a situation. We consider a surface Σ of bounded topology and constant negative curvature, with a cusp of infinite volume. The cusp determines an interval I in the boundary of the universal cover of Σ which has two remarcable properties. The first is that it is invariant by the action of the geodesic γ on $\tilde{\Sigma}$. The second is that it is a component of the exterior of the limit set of $\pi_1(\Sigma)$. Now consider a compact surface S of sufficiently large genus so that there exists a surjective morphism $\rho : \pi_1(S) \to \pi_1(\Sigma)$. We get an action of the fundamental group of S on the boundary of $\tilde{\Sigma}$, which leaves the limit set of $\pi_1(\Sigma)$ invariant, and for which there

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exists an element leaving I invariant, and which acts as a translation on it. Let (M, \mathcal{F}) be the supension of ρ : this is the foliation induced by a flat circle bundle over S whose holonomy is smoothly conjugated to the representation ρ (see 4.1.1). Then the saturated subset of the limit set of $\pi_1(\Sigma)$ is an exceptional minimal subset \mathcal{M} of \mathcal{F} . Now, by construction, there is a leaf L which intersects twice the component Iof the exterior of the limit set of $\pi_1(\Sigma)$ in $\partial \widetilde{\Sigma}$. By the standard Haefliger's argument, we construct a transverse circle which does not cut \mathcal{M} . Thus applying the preceeding arguments we construct a foliation with two minimal sets.

3.4. A counter-example in the non symmetric case. In [Can], Candel extends Garnett's theory to the case of non symmetric Laplace operators on a foliation. In the case of non symmetric Laplace operators, the dichotomy "The Lyapunov exponent is positive or there exists a transversely invariant measure" does not hold anymore. In this paragraph we describe a nice counter-example in the non symmetric case.

Consider a compact Riemannian manifold (M, g) of dimension 3 on which there is an orthonormal frame (H^s, V, H^u) , for which the vector fields H^s, V, H^u verify the relations:

$$[V, H^s] = -H^s, \quad [V, H^u] = H^u, \quad [H^u, H^s] = V.$$

Such manifolds are quotient of the universal cover of $SL(2, \mathbf{R})$ by a cocompact lattice Γ . If Σ is the quotient of the upper-half plane \mathbf{H} by Γ , then M is naturally identified with the unitary tangent bundle of Σ , and under this identification V is the geodesic flow of the hyperbolic surface, H^s and H^u the horocycle flows. The vector fields V and H^s generate a foliation \mathcal{F}^s which is the *stable* foliation of the flow V.

Let g^s be the restriction of the metric g on \mathcal{F}^s . For any $\kappa \in \mathbb{R}$, consider the Laplace operator Δ_{κ} defined by

$$\Delta_{\kappa} = \Delta_{g^s} + \kappa V,$$

where Δ_{g^s} is the Laplacian of g^s along the leaves of \mathcal{F}^s . Garnett proved that for the symmetric case $\kappa = 0$, the Liouville measure vol_g is the unique harmonic measure (see [Ga, Proposition 5, p. 305]). Note that the Liouville measure on M is also invariant by V, so that it is a harmonic measure for all the Laplace operators Δ_{κ} .

Theorem 3.9. For any κ , the Lyapunov exponent of any harmonic measure μ of $(\mathcal{F}^s, \Delta_{\kappa})$ is $\lambda(\mu) = \kappa - 1$. When $\kappa < 1$ the Liouville measure is the unique harmonic measure. When $\kappa > 1$, there exists a harmonic measure supported on every cylinder leaf (thus the foliation is not uniquely ergodic).

Proof. First, we compute the Lyapunov exponent of a harmonic measure μ of $(\mathcal{F}^s, \Delta_{\kappa})$. To this end we use the formula of Lemma 3.1:

$$\lambda(\mu) = \int_M \Delta_\kappa \log \varphi d\mu$$

Consider the metric $|\cdot|$ on the normal bundle of \mathcal{F} which is induced by g. We are going to compute the function φ , which is defined up to multiplication by a constant. This function verifies the relations

$$V\varphi = \varphi, \quad H^s\varphi = 0.$$

In the leaves we have local coordinates z = x + iy with values in the upper half-plane **H**, such that

$$V = y \frac{\partial}{\partial y}, \quad H^s = y \frac{\partial}{\partial x}.$$

(These coordinates are well defined up to an affine transformation of the upper half-plane). In these coordinates, $\varphi = y$ up to a multiplicative constant. The metric g^s and the Laplacian Δ_{g^s} are expressed by

$$g^s = \frac{dx^2 + dy^2}{y^2}, \quad \Delta_{g^s} = y^2 (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$$

Thus, we have $\Delta_{\kappa}\varphi = \kappa - 1$ identically, and the formula $\lambda(\mu) = \kappa - 1$ follows. In particular, when $\kappa < 1$ the only harmonic measure is the Liouville measure, because of Theorem A.

Now, let us suppose that $\kappa > 1$. We shall prove that every cylinder leaf supports a harmonic measure. Observe that these leaves are those containing a periodic orbit of the vector field V. Let L be such leaf, γ_0 be the closed orbit of V in L. Then, its universal cover is the hyperbolic plane, for which we choose the upper half-plane model **H**. Observe that we have the canonical coordinates up to an affine transformation, constructed before. Without loss of generality, we may suppose that the geodesic in **H**, corresponding to γ_0 , is the vertical geodesic x = 0going upwards. Denote by A the length of γ_0 ; then the transformation of **H** corresponding to γ_0 as an element of $\pi_1(L)$, is $z \mapsto e^A z$. Thus, the leaf L is obtained from **H** by identifying z and $e^A z$.

Now, let us consider a typical Brownian trajectory γ in L and its lift to the universal cover $\tilde{\gamma}(t) = (x(t), y(t))$. Note that $\tilde{\gamma}$ satisfies the following stochastic differential equation:

$$\begin{cases} \dot{x} = \sqrt{2y} \, dW_t^1, \\ \dot{y} = cy + \sqrt{2y} \, dW_t^2, \end{cases}$$

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where W_t^1 and W_t^2 are two independent Wiener processes. Here the coefficient $\sqrt{2}$ comes from our definition of Brownian motion: as we have defined it using the heat kernel, its intensivity equals 2 (instead of its common value 1). Let us make a change of variables: let $u = \log y$, v = x/y. Then

$$\dot{u} = (\log y)^{\cdot} = (\log \cdot)'(y) \cdot \kappa y + \frac{1}{2}(\log \cdot)''(y) \cdot 2y^2 + (\log \cdot)'(y) \cdot \sqrt{2}y dW_t^1 = (-1+\kappa) + \sqrt{2}dW_t^1,$$

$$\dot{v} = \left(\frac{x}{y}\right)'_{y} \cdot \kappa y + \frac{1}{2} \left(\frac{x}{y}\right)''_{yy} \cdot 2y^{2} + \left(\frac{x}{y}\right)'_{x} \cdot \sqrt{2}y \, dW_{t}^{1} + \left(\frac{x}{y}\right)'_{y} \cdot \sqrt{2}y \, dW_{t}^{2} = \\ = -\kappa y \frac{x}{y^{2}} + \frac{2x}{y^{3}} y^{2} + \sqrt{2} \left(dW_{t}^{1} + \frac{x}{y} dW_{t}^{2}\right) = (2 - \kappa)v + \sqrt{2} \left(dW_{t}^{1} + v dW_{t}^{2}\right).$$

This implies that v satisfies a stochastic differential equation

 $\dot{v} = (2 - \kappa)v + \sqrt{2(1 + v^2)}dW_t.$

Let us now make another change of variable: we denote $\xi = f(v) = \log(v + \sqrt{1 + v^2})$. Then ξ satisfies the following stochastic differential equation:

$$\begin{split} \dot{\xi} &= f'(\xi)(2-\kappa)v + \frac{1}{2}f''(\xi)(\sqrt{2(1+v^2)})^2 + f'(\xi)\sqrt{2(1+v^2)}dW_t = \\ &= \frac{1}{\sqrt{1+v^2}}(2-\kappa)v - \frac{1}{4}\frac{2v}{\sqrt{1+v^2}}(2(1+v^2)) + \sqrt{2}\,dW_t = \\ &= \frac{v}{\sqrt{1+v^2}}(1-\kappa) + \sqrt{2}\,dW_t. \end{split}$$

For $\kappa > 1$ we notice that the Brownian component of this stochastic differential equation is constant, and the drift is towards 0 with the velocity separated from zero for large v. Thus, there exists a probability stationary measure for this process on the real line. By lifting this measure to the initial cylinder (by a product with the Lebesgue measure in $\log y$), we obtain a stationary measure on L. We have constructed a harmonic measure on L.

4. Similar foliations

A Laplace foliation (\mathcal{F}, Δ) is called *similar* if there exists a transverse continuous foliation \mathcal{G} of dimension $\operatorname{codim}(\mathcal{F})$ such that the operator Δ is invariant by \mathcal{G} ; thus \mathcal{G} preserves the metric g and the drift vector field V.

The main goal of this part is to prove the unique ergodicity property for a codimension 1 similar Laplace foliation whose drift vector field preserves the volume, and whose transverse structure is just supposed continuous. We begin by giving examples of these foliations.

4.1. Some examples.

4.1.1. Suspension. Let $(N, \overline{\Delta})$ be a compact manifold equipped with a Laplace operator, and $\rho : \pi_1(N) \to Homeo(F)$ be a representation of its fundamental group into the group of homeomorphisms of a compact manifold F. Let \widetilde{N} be the universal cover of N. The diagonal action of the discrete group $\pi_1(N)$ on the product $\widetilde{N} \times F$ is discontinuous and free. Moreover, it preserves the horizontal foliation and the vertical fibration. Thus, the quotient $N \ltimes_{\rho} F$ is equipped with a foliation \mathcal{F} (quotient of the horizontal foliation) and with a transverse fibration $F \to M \xrightarrow{\pi} N$ (quotient of the vertical fibration). Let Δ be the Laplace operator on the leaves of \mathcal{F} such that $(\pi_{\mathcal{F}})_*\Delta = \overline{\Delta}$. By construction the foliation (\mathcal{F}, Δ) is similar. Such foliations are called suspensions.

4.1.2. Linear Anosov diffeomorphism. Let $A : \mathbf{T}^n \to \mathbf{T}^n$ be a linear Anosov diffeomorphism of the torus $\mathbf{T}^n = \mathbf{R}^n / \mathbf{Z}^n$. Consider the quotient M of $(0, \infty) \times \mathbf{T}^n$ by the diffeomorphism $\widetilde{A}(t, x) = (2t, Ax)$: this is the fiber bundle over the circle whose fiber is \mathbf{T}^n and monodromy is given by A. Define the foliations \mathcal{F} and \mathcal{G} to be respectively the quotient of $(0, \infty) \times \mathcal{F}^u$ and of \mathcal{F}^s . We define a Laplace operator Δ on the leaves of \mathcal{F} of the form

$$\Delta = \Delta_t + t^2 \frac{\partial^2}{\partial t^2}$$

where $\{\Delta_t\}_{t>0}$ is a family of linear Laplace operators on \mathcal{F}^u depending smoothly on t, and verifying the relation

$$\Delta_{2t} = (A|_{\mathcal{F}^u})_* \Delta_t,$$

for every t > 0. The Laplace foliations (\mathcal{F}, g) are similar (the invariance of Δ by \mathcal{G} comes from the fact that the operators Δ_t are linear).

Remark 4.1. The similar foliations by surfaces of a compact 3-manifold are well known [Ca, Ep]: in this case, the only examples with interesting dynamics are the suspensions and the foliations induced by a linear Anosov diffeomorphism of a 2-torus. However, it may exist other examples in higher dimension.

4.2. Non divergence of the leaves. In this paragraph we prove that the leaves of a similar codimension 1 foliation whose drift vector field preserves the volume vol_g are not diverging in a set of directions of large measure. This has been observed by Thurston (see [C-D, Fe] for a topological proof).

Definition 4.2. Let \mathcal{F} be a similar foliation. A transverse harmonic measure on \mathcal{F} is a family $\{\nu_L\}$ of measures ν_L on every \mathcal{G} -leaf L, such that in a chart $B \times T$ in which \mathcal{F} and \mathcal{G} are respectively the horizontal and vertical foliations, the function

$$p \in B \mapsto \nu(\{p\} \times T) \in \mathbf{R}_+$$

is harmonic.

Lemma 4.3. On a codimension one similar Laplace foliation of a compact manifold whose drift vector field preserves the leafwise volume vol_g , and which is minimal, there exists a transverse harmonic measure.

Proof. The result follows from the existence of a harmonic measure for the adjoint operator Δ^* of Δ . Recall that Δ^* is defined on every \mathcal{F} -leaf L in such a way that for any smooth functions $u, v : L \to \mathbf{R}$ with compact support one has

$$\int_L u \Delta v d \mathrm{vol}_g = \int_L (\Delta^* u) v d \mathrm{vol}_g$$

An integration by parts shows that one has the following formula:

$$\Delta^* = \Delta_g - V + \operatorname{div}_{\operatorname{vol}_g} V,$$

where V is the drift vector fields of Δ (i.e. by definition $\Delta = \Delta_g + V$). If V preserves the volume vol_g , which means that the divergence of V vanishes identically, then the operator Δ^* is also a Laplace operator. In [Can], it is proved that for such operators there exists a harmonic measure.

Let μ be a harmonic measure on (\mathcal{F}, Δ^*) . Consider a foliation box $B \times T$ in which \mathcal{F} and \mathcal{G} are respectively the horizontal and vertical foliation. Let T' be an open subset of T. The image of the measure μ by the projection $B \times T' \to B$ is a Δ^* -harmonic measure on B, because the foliation \mathcal{F} is similar. Thus, there is a Δ -harmonic function $L_{T'}: B \to [0, \infty)$ such that for any continuous function $f \in C_c^0(B)$, one

has $\int_{B \times T'} f(b) d\mu(b,t) = \int L_{T'}(b) f(b) \operatorname{vol}_g(b)$. Because μ is a measure, if T_n are disjoint open subsets of T, one has the relations $\sum_n L_{T_n} = L_{\cup_n T_n}$; thus there exists a transverse measure on the leaves of \mathcal{G} such that $\nu(b \times T') := L_{T'}(b)$, for every Borel subset T' of T. This transverse measure is Δ -harmonic, by construction and the lemma is proved. \Box

Lemma 4.4. Let \mathcal{F} be a codimension one similar Laplace foliation of a compact manifold. Let [x, y] be an interval in a \mathcal{G} -orbit. There exists a uniquely defined map $I_{x,z} : \widetilde{L_x} \times [x, y] \to M$ which maps the horizontal Laplace foliation (given by the Laplace operator on $\widetilde{L_x}$) on (\mathcal{F}, Δ) , the vertical foliation on \mathcal{G} , and the interval $\{x\} \times [x, y]$ identically on [x, y].

Proof. Let $\gamma : [0,1] \to \widetilde{L_x}$ be a smooth path starting at x. The restriction of $I_{x,y}$ to $\gamma([0,1]) \times [x,y]$ is uniquely defined, if it exists. Let $0 \leq t \leq 1$ be the supremum of those t such that $I_{x,y}$ is defined on $\gamma([0,t]) \times [x,y]$. It is clear that t > 0, because locally we have foliation charts. Recall that the foliation \mathcal{G} preserves the metric g; thus, for any $z \in [x,y]$ the length of the curve $I_{x,y}(\gamma([0,s] \times z)$ equals the length of the curve $\gamma([0,s])$. This implies that it is possible to extend the map $I_{x,y}$ on the domain $\gamma([0,t]) \times [x,y]$, and because locally we have foliation charts, to a domain $\gamma([0,t_+)) \times [x,y]$, where $t_+ > t$. Thus t = 1 and the lemma is proved.

Recall that for any point $x \in M$, Γ_x is the set of continuous paths contained in the leaf through the point x, and the Laplace operator Δ induces a probability measure W_x on Γ_x . Here is the main result of the paragraph, where $J_{x,y}(p)$ is the point $I_{x,y}(p, 1)$.

Proposition 4.5. Let \mathcal{F} be a Laplace similar foliation of codimension 1 of a compact manifold, which is minimal, and whose drift vector fields preserves the volume vol_g. For every $\varepsilon > 0$, there exists a constant $\delta > 0$ such that if x and y are two points on the same \mathcal{G} -orbit with $d(x,y) \leq \delta$, then there exists a subset $E_{x,y} \subset \Gamma_x$ of W_x -measure 1/2 such that for any $\gamma \in E_{x,y}$, one has

$$\limsup_{t \to \infty} d(\gamma(t), J_{x,y}(\gamma(t)) \le \varepsilon.$$

Proof. Consider a transverse harmonic measure ν constructed in lemma 4.3. Because \mathcal{F} is minimal, the measure ν restricted to every leaf of \mathcal{G} has full support, and is diffuse. Thus, there exists $\delta' > 0$ such

that if [x, y] is an interval in a \mathcal{G} -leaf of ν -measure bounded by δ' , then the distance between x and y is bounded by ε . Moreover, there exists $\delta > 0$ such that if the distance between x and y is bounded by δ , then the ν -measure of the interval [x, y] is bounded by $\delta'/2$.

The function $p \in L_x \mapsto f(\gamma) = \nu(I_{x,y}(\{p\} \times [x, y])) \in (0, \infty)$ is harmonic. Thus by the martingale theorem, for w_x -almost every γ , the limit

$$\lim_{t \to \infty} f(\gamma(t))$$

exists, and its integral over Γ_x is $f(x) = \nu([x, y]) \leq \delta'/2$. In particular, there is a measurable subset $E_{x,y} \subset \Gamma_x$ of W_x -measure 1/2 such that for every $\gamma \in E_{x,y}$

$$\lim_{t \to \infty} f(\gamma(t)) \le \delta'.$$

For every γ of $E_{x,y}$ we have

$$\limsup_{t \to \infty} d(\gamma(t), I_{x,y}(\gamma)(t, 1)) \le \varepsilon.$$

The proposition is proved.

4.3. Application to unique ergodicity. We prove that there is only one harmonic measure on a codimension 1 similar foliation whose drift vector fields preserves the volume vol_g .

Theorem 4.6. Let (\mathcal{F}, Δ) be a similar Laplace foliation of codimension 1 of a compact manifold M, whose drift vector field preserves the volume vol_g . Then, on a minimal subset of \mathcal{F} is supported a unique harmonic measure.

Proof. Let \mathcal{M} be a minimal subset of \mathcal{F} . There are three possibilities:

- \mathcal{M} is a compact leaf.
- \mathcal{M} is transversely a Cantor set.
- $\mathcal{M} = M$.

In the first case, the only harmonic measure is the unique Δ -harmonic volume on the compact leaf. The second case can be reduced to the third one by collapsing the components of the leaf of \mathcal{G} outside \mathcal{M} . Thus, we suppose that $\mathcal{M} = M$, i.e. \mathcal{F} is minimal.

Lemma 4.7. Let \mathcal{F} be a similar minimal foliation of codimension 1 of a compact manifold M, and μ be an ergodic harmonic measure. Then for every $\alpha > 0$, and every continuous function $f : M \to \mathbf{R}$, the following

property holds. For every point $y \in M$, there exists a measurable subset $E_y \subset \Gamma_y$ of W_y -measure 1/2 such that for every $\gamma \in E_y$:

$$\int f d\mu - \alpha \leq \liminf_{n \to \infty} B_n(f, \gamma) \leq \limsup_{n \to \infty} B_n(f, \gamma) \leq \int f d\mu + \alpha,$$

where $B_n(f,\gamma) := \frac{1}{n} \sum_{1 \le k \le n} f(\gamma(k))$ are the Birkhoff sums of f along the path γ .

Proof. By the Birkhoff theorem for harmonic measures proved in [Ga], there is a measurable subset $X \subset M$ of full μ -measure, saturated by \mathcal{F} , so that for every $x \in X$ and w_x -almost every continuous path $\gamma \in \Gamma_x$, the Birkhoff sums $B_n(f, \gamma)$ converge to $\int f d\mu$.

Let $\varepsilon > 0$ such that if $d(x, y) \leq \varepsilon$ then $|f(x) - f(y)| \leq \alpha$. Let ybe any point of M. There exists a point $x \in X$ which is in the \mathcal{G} orbit of y and such that $d(x, y) \leq \delta$, the δ being given by Lemma 4.5. Let F_x denote the set of element $\gamma \in \Gamma_x$ for which the Birkhoff sums converge to $\int f d\mu$. Because x belongs to X this set is of full measure. Define $E_y := J_{x,y}(F_x \cap E_{x,y})$. The set E_y is of W_y -measure 1/2, because $F_x \cap E_{x,y}$ is of W_x -measure 1/2 and that $J_{x,y}$ sends W_x on W_y . Let $\gamma = J_{x,y}(\gamma') \in E_y$. Using Lemma 4.5, we have

$$\limsup_{t \to \infty} d(\gamma(t), \gamma'(t)) \le \varepsilon.$$

Thus one gets

$$\limsup_{n \to \infty} |B_n(f, \gamma) - B_n(f, \gamma')| \le \alpha,$$

and the lemma follows because the Birkhoff sums of γ' converge to $\int f d\mu$.

We are now able to finish the proof of the theorem. Let μ' be another ergodic measure, and f be a continuous function on M. We are going to prove that $\int f d\mu' = \int f d\mu$. Observe that by ergodicity of μ' there exists a point y on M such that for w_y -almost every $\gamma \in \Gamma_y$, the Birkhoff sums $B_n(f, \gamma)$ converge to $\int f d\mu'$. Denote by F_y the set of such paths γ which is of full w_y -measure. We apply Lemma 4.7 to the point y: for every $\gamma \in E_y$, the Birkhoff sums $B_n(f, \gamma)$ are tending to $\int f d\mu$ with an error of α . Because the measure of E_y is positive, it intersects F_y . Thus one gets $|\int f d\mu' - \int f d\mu| \leq \alpha$. Because α is arbitrary, there is only one ergodic harmonic measure, and thus only one harmonic measure. The theorem is proved. \Box **Proposition 4.8.** Let (\mathcal{F}, Δ_g) be a similar Laplace foliation of codimension 1 of a compact manifold M, where Δ_g is the Laplacian of a riemannian metric. Then every ergodic harmonic measure is supported on a minimal subset.

Proof. Let μ be an ergodic harmonic measure on (\mathcal{F}, Δ_g) , and \mathcal{M} be a minimal closed subset contained in the support of μ . We are going to prove that \mathcal{M} is exactly the support of μ . It is clear that one can suppose that μ does not charge any leaf.

Suppose that the foliation is oriented. Because the operator Δ_g is symmetric, the measure μ induces a transverse harmonic measure which is Δ_g -harmonic. From every point x of M, the positive \mathcal{G} -orbit of x intersects \mathcal{M} in a first time in a point y. Consider the function $f(x) = \nu([x, y))$. This is a continuous function, because μ does not charge any leaf, and it is harmonic on every leaf. By Garnett lemma [Ga], this function has to be constant on μ -almost every leaf, thus by continuity of f, f is constant on the support of μ . But on the minimal \mathcal{M} , f vanishes, so that the restriction of f to the support of μ is identically 0. Thus the support of μ has to be reduced to \mathcal{M} . The proposition is proved.

Example 4.9. The Theorem 4.6 seems to be false when the drift does not preserve the volume vol_g . We give an example of a similar Laplace *lamination* of a compact space which is minimal, transversely conformal, and not uniquely ergodic.

A lamination of a compact space X is an atlas of homeomorphisms from open sets of X to the product of an euclidian ball by a topological set, in such a way that the change of coordinates preserves the local fibration by balls, and the diffeomorphisms from a piece of ball to another one depends continuously of the transverse parameter in the smooth topology. The definition of a Laplace operator on a lamination is exactly the same as in the foliation case.

We are going to describe an example of a similar Laplace lamination of a compact space, which has been constructed by Sullivan [Su3]. Let **H** be the upper-half plane, whose hyperbolic metric is expressed by

$$g = \frac{dx^2 + dy^2}{y^2},$$

in the coordinates z = x + iy of **H**. Consider the unit vector fields that points on the direction ∞ of ∂ **H**. In the x, y coordinates, it is

expressed as

$$V = y \frac{\partial}{\partial y}$$

The direct isometries of (\mathbf{H}, g, V) are the maps of the form $z \mapsto az + b$ where a is a positive number, and b is a real number. For any real number κ , these transformations preserve the Laplace operator

$$\Delta_{\kappa} = \Delta_q + \kappa V.$$

Let $A(\mathbf{Z}[1/2])$ be the group of affine transformations $x \mapsto ax + b$, where *a* is a power of 2, and *b* is a dyadic integer of the form $p/2^n$, *p* and *n* being integers. The group $A(\mathbf{Z}[1/2])$ acts naturally on the product $\mathbf{H} \times \mathbf{Q}_2$ of the upper half plane by the field of 2-adic numbers, the action preserving the natural structure of the horizontal Laplace lamination ($\mathbf{H} \times \mathbf{Q}_2, \Delta_{\kappa}$). The action is discrete, without fixed point, and the quotient (X, Δ_{κ}) is similar Laplace lamination of a compact space.

When $\kappa > 1$ the Laplace laminations (X, Δ_{κ}) carry harmonic measures that charge any cylinder leaves. This can be seen by the same arguments as those given in 3.4. These examples are not codimension 1 foliations, but they share with codimension 1 foliation the property of being transversely conformal, which is the only property used in the proof of Theorem 4.6.

It seems to us that the hypothesis $div_{\text{vol}_g}V = 0$ is too strong. For instance, we conjecture that for any Laplace operator on the base of a suspension, the conclusion of 4.6 holds.

There are analog examples of similar Laplace laminations of a compact space, associated with tilings of the hyperbolic plane. They have been studied by Petite [Pe]; many of them are minimal but not uniquely ergodic even for a symmetric operator. The lack of unique ergodicity is due to the fact that they do not carry a transversely invariant "conformal" structure.

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5. Appendix: Technical proofs

5.1. **Proof of Proposition 2.3.** It is a well-known fact (see [C-L-Y, Ma]), that for a Brownian motion, the probability of making large steps decreases very fast:

(5.1)
$$\exists C_1, d_0: \forall p \in M, \forall d > d_0 \quad W_p(\operatorname{dist}(\widetilde{\gamma}(\delta), p) > d) \le e^{-C_1 d}.$$

Thus, we may choose a random variable ξ , such that $\xi \ge 0$, $E\xi < \infty$ and

(5.2)
$$\forall p \in M, \forall d > 0 \quad W_x(\operatorname{dist}(\widetilde{\gamma}(\delta), p) > d) \le P(\xi > d).$$

Let us choose for every $x \in \mathcal{M}$, a function $\chi_x : \Gamma_x \to \mathbb{R}$, such that χ_x depends only on $\gamma|_{[0,\delta]}$, has the same distribution as ξ and such that $\chi_x(\gamma) \geq k_1(\gamma)$ for every $\gamma \in \Gamma_x$. Recall, that $\sigma : \Gamma \to \Gamma$ stays for the map, erasing the first step of the δ -discretization: $\sigma(\gamma)(t) = \gamma(t + \delta)$. Then,

$$\forall j, \gamma \quad k_j(\gamma) = k_1(\sigma^{j-1}(\gamma))$$

Let us denote $\xi_j(\gamma) = \chi_{\gamma((j-1)\delta)}(\sigma^{j-1}(\gamma))$. Then,

$$\frac{k_1(\gamma) + \dots + k_n(\gamma)}{n} = \frac{k_1(\gamma) + k_1(\sigma(\gamma)) + \dots + k_1(\sigma^{n-1}\gamma)}{n} \le \frac{\chi_{\gamma(0)}(\gamma) + \chi_{\gamma(\delta)}(\sigma(\gamma)) + \dots + \chi_{\gamma((n-1)\delta)}(\sigma^{n-1}\gamma)}{n} = \frac{\xi_1(\gamma) + \dots + \xi_n(\gamma)}{n}$$

Note, that all the ξ_j are distributed identically with ξ . Moreover, from the Markovian property, the conditional distribution of the variable ξ_{n+1} with respect to every condition $\gamma|_{[0,n\delta]} = \overline{\gamma}$ coincides with the distribution of ξ . On the contrary, the variables ξ_1, \ldots, ξ_n are determined by such a condition. Thus, the variable ξ_{n+1} is independent from ξ_1, \ldots, ξ_n . As *n* is arbitrary, all the variables $\xi_1, \ldots, \xi_n, \ldots$ are independent and identically distributed.

Now, let us take $c = E\xi + 1$. We are going to show that for almost every $\gamma \in \Gamma$ we have $\limsup_{n \to \infty} K_n(\gamma)/n < c$. From the Large Numbers Law, we have:

$$\limsup_{n \to \infty} \frac{\xi_1(\gamma) + \dots + \xi_n(\gamma)}{n} = E\xi < c$$

 W_x -almost surely. But

$$\limsup_{n \to \infty} \frac{k_1(\gamma) + \dots + k_n(\gamma)}{n} \le \limsup_{n \to \infty} \frac{\xi_1(\gamma) + \dots + \xi_n(\gamma)}{n},$$

and thus

$$\limsup_{n \to \infty} \frac{k_1(\gamma) + \dots + k_n(\gamma)}{n} < c$$

 W_x -almost surely. This concludes the proof of the proposition.

5.2. **Proof of Lemma 2.4.** First, we are going to prove the lemma in the particular case of a codimension one foliation \mathcal{F} , using the transversal one-dimensional foliation \mathcal{G} , described in Section 2.

In order to prove the lemma, we shall study the behaviour of heat kernels for a time $t = \delta$ fixed on different but close enough leaves. We are going to use the following idea: major parts of the heat distribution measures on these two leaves are similar (i.e. the density of \mathcal{G} -holonomy image of one with respect to another is close to 1). Moreover, the infinite product of these densities converges along most Brownian paths, because along these paths the leaves approach exponentially. Thus, the measures $W_x^{\delta} = F_* W_x|_{E_x}$ and $(F \circ \Phi_{\bar{x},x})_* W_{\bar{x}}|_{E_{\bar{x}}}$ are absolutely continuous with respect to each other, and the density on the major part of trajectories is close to 1. In particular, the total measures $W_x(E_x)$ and $W_{\bar{x}}(E_{\bar{x}})$ are close to each other.

Proposition 5.1. Let $\bar{x} \in \mathcal{G}_x$, dist_{\mathcal{G}} $(x, \bar{x}) = \theta$. Consider the measures ν_x and $\nu_{\bar{x}}$, where the measure $\nu_z = p(z, \cdot; \delta) \operatorname{dvol}_g$ on \mathcal{F}_z gives the distribution of Brownian motion at the time δ . Also, let $\varepsilon_2 > 0$ and R be chosen. Then, there exists a set $S = S(R, \varepsilon_2, \theta, x) \subset B_R^{\mathcal{F}}(x)$, such that

- $\nu_x(S) > 1 \varepsilon_1$,
- $\frac{d\nu_x}{d(\Phi_{x,\bar{x}}^*\nu_{\bar{x}})}\Big|_S \in [1-\varepsilon_2, 1+\varepsilon_2],$

where $\varepsilon_1 = \frac{1}{\varepsilon_2} \left(G'_1 e^{G'_2 R} \theta + e^{-G_3 R^2} \right)$, and G'_1, G'_2, G_3 are geometric constants (depending only on the foliation \mathcal{F}).

Proof. First, let us equip the leaf \mathcal{F}_x with another Riemannian metric g', coinciding with $\Phi_* g|_{\mathcal{F}_x}$ inside $B_R^{\mathcal{F}}(x)$ and with $g|_{\mathcal{F}_x}$ outside $B_{2R}^{\mathcal{F}}(x)$; in the annulus left, the metrics g' is defined using cut-off function.

Note, that the distance (in C^{2d} -topology, where d is the dimension of the leaves of \mathcal{F}) between g and g' is at most $G_0 \theta e^{G_1 \cdot 2R}$, where G_1 is a constant, giving the maximum deviation of leaves of \mathcal{F} , and G_0 gives the maximum of derivative of g along \mathcal{G} .

Also, we recall, that the heat kernel can be constructed explicitly as a series of convolutions. This procedure is described in the books of Candel [C-C] and Chavel [Cha]. In a few words, a function L, which "almost satisfies" the heat equation, is explicitly constructed, and then

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the real heat kernel is obtained as a sum of L and a series of convolutions. These series are converging uniformly for every fixed moment of time t, and the dependence of the metrics is smooth (due to the explicit nature of the construction). Thus, the distance between the heat kernels for the metrics g and g' at the time $t = \delta$ can be bounded by the product of a constant G_2 (depending only on the geometry of foliation \mathcal{F} and of the moment δ) and of the distance between g and g'. This distance is at most $e^{2G_1R}\theta$; hence, the difference between these kernels is bounded by $G_2e^{2G_1R}\theta$.

Now, due to the upper bounds for the heat kernel [C-L-Y, Ma], the set of Brownian trajectories on the interval of time $[0, \delta]$, starting at x and exiting from the ball $B_R^{\mathcal{F}}(x)$ at some intermediate moment, has the measure at most $e^{-G_3R^2}$. We notice that these trajectories for the metric g' and for the metric $\Phi_{x,\bar{x}}^*g|_{\mathcal{F}_{\bar{x}}}$ are the same (they do not pass through the points where these metrics do not coincide). Thus, the parts of heat kernel at time δ , coming from these trajectories, are the same for these two metrics.

Let us define the set S in a following way: $z \in S$, if

- The density $p_{q'}(x, z; \delta)$ is at least $3G_2 e^{2G_1 R} \theta / \varepsilon_2$
- At least $1 \varepsilon_2/3$ of this density comes from the trajectories staying inside $B_R^{\mathcal{F}}$ (in particular, $z \in B_R^{\mathcal{F}}(x)$).

Then, the second condition for S (quotient of densities) is satisfied automatically: the maximum possible change of density at a point of Sis the sum of changes while passing from g to g' (at most $\varepsilon_2/3$ part) and from g' to $g|_{\mathcal{F}_{\bar{x}}}$ (at most $\varepsilon_2/3$ part due to common set of trajectories).

Now, let us estimate $\nu(\mathcal{F}_x \setminus S)$, thus verifying the first condition. The points of this complementary can be of two types: either points of $B_R^{\mathcal{F}}$ with too small value of density (we note this set X_1), or with too big part of this density coming from trajectories, exiting the ball $B_R^{\mathcal{F}}(x)$ (we note this set X_2).

The first part is estimated as

$$\nu(X_1) = \int_{B_R^{\mathcal{F}}(x)} p(x, z; \delta) d\operatorname{vol}_g \leq 3G_2 e^{2G_1 R} \theta / \varepsilon_2 \cdot \operatorname{vol}_g(B_R^{\mathcal{F}}(x)) \leq (3G_2 e^{2G_1 R} \theta / \varepsilon_2) \cdot e^{G_4 R},$$

where G_4 is the constant, bounding the growth of the leaves of \mathcal{F} .

The second part is estimated as follows: denote by $\rho(z)$ the part of the density $p(x, z; \delta)$, coming from the trajectories exiting from the ball

$$B_R^{\mathcal{F}}(x). \text{ Then,}$$

$$\nu(X_2) = \int_{\{z: \rho(z)/p(x,z;\delta) > \varepsilon_2/3\}} p(x,z;\delta) \, d\text{vol}_g(z) \leq \\ \leq \int_{\{z: \rho(z)/p(x,z;\delta) > \varepsilon_2/3\}} \frac{3}{\varepsilon_2} \cdot \rho(z) \, d\text{vol}_g(z) \leq \\ \leq \frac{3}{\varepsilon_2} \int_{\mathcal{F}_x} \rho(z) \, d\text{vol}_g(z) < \frac{3}{\varepsilon_2} \cdot e^{-G_3 R^2}$$

(the last inequality comes from the upper bound for the probability of all the set of trajectories, leaving the ball of radius R at some moment between 0 and δ).

Finally, we obtain

$$\nu(\mathcal{F}_x \setminus S) = \nu(X_1 \cup X_2) < \frac{3G_2 e^{(2G_1 + G_4)R}}{\varepsilon_2} \cdot \theta + \frac{3}{\varepsilon_2} \cdot e^{-G_3 R^2} = \frac{1}{\varepsilon_2} \left(G_1' e^{G_2' R} \theta + e^{-G_3 R^2} \right),$$

where $G'_1 = 3G_2, G'_2 = 2G_1 + G_4.$

The first condition on S is satisfied.

For any $R, \theta > 0$ let us denote

$$\Psi(R,\theta) = \sqrt{G_1' e^{G_2' R} \theta + e^{-G_3 R^2}}.$$

Also, let $r(\theta) = (\log \frac{1}{2\theta G'_1})/(2G'_2)$. Then $G'_1 e^{G'_2 r(\theta)} \theta = \frac{1}{2}\sqrt{\theta}$, and for all θ sufficiently small $e^{-G_3 r(\theta)^2} < \frac{1}{2}\sqrt{\theta}$. Thus, for all θ sufficiently small

$$\Psi(\theta) := \Psi(r(\theta), \theta) < \sqrt{\theta}.$$

Now, denote

$$S(\theta, x) = S(r(\theta), \Psi(\theta), \theta, x).$$

For this set, the conclusions of Proposition 5.1 is satisfied with $\varepsilon_1 = \varepsilon_2 < \sqrt{\theta}$. Let us choose a small transversal interval $J \subset I$, $J \ni x$, and consider the subset of E_x , defined as

(5.3)
$$E'_{x} = \{ \gamma \in E_{x} | \forall n \ge 0 \quad x_{n+1} \in S(\theta_{n}, x_{n}) \},$$

where $x_n = \gamma(n\delta)$ is the discretization sequence, corresponding to γ , and $\theta_n = |h_{\gamma|_{[0,n\delta]}}(J)|$ is the sequence of the corresponding transverse distances (exponentially decreasing due to the nature of E_x).

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Lemma 5.2. For $\bar{x} \in J$, the images of the measure $W_x|_{E'_x}$ under Fand of the measure $W_{\bar{x}}|_{E'_x}$ under $F \circ \Phi_{\bar{x},x}$ are absolutely continuous with respect to each other. The density can be made arbitrary close to 1 by choice of sufficiently small intervals I and J. Moreover, by such a choice, the differences $W_x(E_x) - W_x(E'_x)$ and $W_{\bar{x}}(E_{\bar{x}}) - W_{\bar{x}}(E'_{\bar{x}})$ can be made arbitrarily small.

Proof. Let us consider the projection map

$$\pi_n : (\widetilde{\mathcal{F}}_x)^{\infty} \to (\widetilde{\mathcal{F}}_x)^{n+1}, \quad \pi_n(\{x_j\}_{j=0}^{\infty}) = \{x_j\}_{j=0}^n$$

and its composition with F, which we denote $F_n: E'_x \to (\widetilde{\mathcal{F}}_x)^{n+1}$,

 $F_n(\gamma) = \{\widetilde{\gamma}(j\delta)\}_{j=0}^n.$

Let us denote

$$\mu_1 = F_* W_x|_{E'_x}, \quad \mu_2 = (F \circ \Phi_{\bar{x},x})_* W_{\bar{x}}|_{E'_{\bar{x}}},$$
$$\mu_1^n = (\pi_n)_* \mu_1 = (F_n)_* W_x|_{E'_x}, \quad \mu_2^n = (\pi_n)_* \mu_2 = (F_n \circ \Phi_{\bar{x},x})_* W_{\bar{x}}|_{E'_{\bar{x}}}.$$

Proposition 5.3. The measures μ_1^n and μ_2^n are absolutely continuous with respect to each other, and

$$\frac{d\mu_2^n}{d\mu_1^n}\Big|_{(x_j)} = \rho_n((x_j)) = \prod_{j=0}^{n-1} \frac{d\nu_{x_j}}{d(\Phi_{x_j,\bar{x}_j}^*\nu_{\bar{x}_j})},$$

To prove Lemma 5.2, it suffices to show that for μ_1 -almost every point $(x_j) \in (\widetilde{\mathcal{F}}_x)^{\mathbb{N}}$ the sequence $\rho_n((x_j))$ converges to some number between 0 and ∞ . It is equivalent to the convergence of the infinite product

(5.4)
$$\prod_{j=1}^{\infty} \frac{d\nu_{x_j}}{d(\Phi_{x_j,\bar{x}_j}^* \nu_{\bar{x}_j})} = \lim_{n \to \infty} \rho_n((x_j))$$

(again, for μ_1 -almost every $(x_j) \in (\widetilde{\mathcal{F}}_x)^{\mathbb{N}}$).

Now, for every (x_j) in the image $F(E'_x)$, the logarithm of the product (5.4) can be estimated as

(5.5)
$$\left| \log \prod_{j=1}^{\infty} \frac{d\nu_{x_j}}{d(\Phi_{x_j,\bar{x}_j}^* \nu_{\bar{x}_j})} \right| \le \sum_{j=0}^{\infty} \left| \log \frac{d\nu_{x_j}}{d(\Phi_{x_j,\bar{x}_j}^* \nu_{\bar{x}_j})} \right|$$

 $\le \sum_{j=0}^{\infty} \left| 2\sqrt{\theta_j} \right| \le \sum_{j=0}^{\infty} \left| 2\sqrt{Ce^{-\alpha j}|J|} \right| \le C_3\sqrt{|J|} \sum_{j=0}^{\infty} e^{-\alpha j/2} = C_4\sqrt{|J|}.$

Here we used the definition of E'_x to bound the density $\frac{d\nu_{x_j}}{d(\Phi^*_{x_j,\bar{x}_j}\nu_{\bar{x}_j})}$, and then again to estimate θ_n .

We have estimated the density; moreover, for J sufficiently small this density (due to (5.5)) can be made arbitrary close to 1.

Now, let us estimate the difference $W_z(E_z) - W_z(E'_z)$, where $z \in J$. Denote

$$E_{z,n} = \left\{ \gamma \in E_z | \forall j, 0 \le j < n \quad x_{n+1} \in S(\theta_n, x_n) \right\},$$
$$\widetilde{C}_{z,n} = F_n(E_{z,n}),$$

and let $\widetilde{\mu}_n$ be a measure on $(\widetilde{\mathcal{F}}_z)^{n+1}$, defined as the discretization image of $W_z|_{E_{z,n}}$. Also, consider projection maps $\widetilde{\pi}_n : \widetilde{C}_{z,n+1} \to \widetilde{C}_{z,n}$. Then, $(\widetilde{\pi}_n)_*\widetilde{\mu}_{n+1}$ is absolutely continuous with respect to $\widetilde{\mu}_n$, and the density is equal to

$$\widetilde{\rho}_n((x_j)) = \nu_{x_n}(S(\theta_n, x_n)) \ge 1 - 2\sqrt{\theta_n} \ge 1 - 2\sqrt{Ce^{-\alpha n}|J|}.$$

Thus,

$$W_{z}(E_{n,z} \setminus E_{n+1,z}) = \int_{\widetilde{C}_{n,z}} (1 - \widetilde{\rho}_{n})((x_{j})) d\widetilde{\mu}_{n}((x_{j})_{j=0}^{n}) \leq$$
$$\leq \int_{\widetilde{C}_{n,z}} 2\sqrt{Ce^{-\alpha n}|J|} d\widetilde{\mu}_{n}((x_{j})_{j=0}^{n}) \leq C_{3}e^{-\alpha n/2}\sqrt{|J|}.$$

Now, recall that $E'_{z} = \bigcap_{n=1}^{\infty} E_{z,n}$, hence,

(5.6)
$$W_z(E_z) - W_z(E'_z) = \sum_{n=0}^{\infty} W_z(E_{z,n} \setminus E_{z,n+1}) \le \le \sum_{n=0}^{\infty} C_3 e^{-\alpha n/2} \sqrt{|J|} \le C_4 \sqrt{|J|}.$$

The difference $W_z(E_z) - W_z(E'_z)$ tends to 0 as |J| tends to 0.

Proof of Lemma 2.4.

5.2.1. "Positive measure" part. First, let us prove the "positive measure" part. Namely, estimate the difference $|W_x(E_x) - W_{\bar{x}}(E_{\bar{x}})|$ if $\bar{x} \in J \subset I$:

$$|W_x(E_x) - W_{\bar{x}}(E_{\bar{x}})| \le |W_x(E_x) - W_x(E'_x)| + |W_x(E'_x) - W_{\bar{x}}(E'_{\bar{x}})| + |W_{\bar{x}}(E'_{\bar{x}}) - W_{\bar{x}}(E_{\bar{x}})|.$$

All the three differences can be estimated using Lemma 5.2. Thus, choosing any $p_1 < W_x(E_x)$, we can find a sufficiently small transversal interval J, such that for any $\bar{x} \in J$ we have $W_{\bar{x}}(E_{\bar{x}}) \ge p_1$.

Now, suppose that $y \in \widetilde{\mathcal{F}}_{\bar{x}}$. Then, the measures ν_y and $\nu_{\bar{x}}$ are absolutely continuous with respect to each other, and the density on the major (with respect to these measures) part of $\widetilde{\mathcal{F}}_{\bar{x}}$ is close to 1. Let us choose $\varepsilon_3 > 0$ and denote the set

$$\widetilde{S}(\bar{x}, y, \varepsilon_3) = \left\{ z \in \widetilde{\mathcal{F}}_{\bar{x}} \mid \frac{d\nu_{\bar{x}}}{d\nu_y}(z) \in \left[\frac{1}{1+\varepsilon_3}, 1+\varepsilon_3\right] \right\}.$$

Then,

$$\forall \varepsilon_3 > 0 \, \exists r > 0 : \, \forall y \in \widetilde{\mathcal{F}}_{\bar{x}}, d(y, \bar{x}) < r \quad \nu_{\bar{x}}(\tilde{S}) > 1 - \varepsilon_3, \nu_y(\tilde{S}) > 1 - \varepsilon_3.$$

Note that due to the Markovian property (the conditional distribution of $\sigma(\gamma)$ with respect to every condition $\sigma(\gamma)(0) = z'$ coincides with $W_{z'}$), we have

$$W_z(E_z) = \int_{\widetilde{\mathcal{F}}_z} W_{z'}(E_{z'}^1) \, d\nu_z(z'),$$

where

$$E_{z'}^{1} = \left\{ \gamma \mid \forall n \quad \left| h_{\gamma|_{[0,n\delta]}}(h_{z,z'}(I)) \right| < C_0 e^{-\alpha \cdot (n+1)\delta} |I| \right\}$$

(this is the definition of E_z , rewritten in terms of the shift $\sigma(\gamma)$). Thus, for $d(y, \bar{x}) < r$ we have

(5.7)
$$W_y(E_y) = W_y(E_y \cap \{x_1 \notin \widetilde{S}\}) + \int_{\widetilde{S}} W_z(E_z^1) \, d\nu_y(z),$$

(5.8)
$$W_{\bar{x}}(E_{\bar{x}}) = W_{\bar{x}}(E_{\bar{x}} \cap \{x_1 \notin \widetilde{S}\}) + \int_{\widetilde{S}} W_z(E_z^1) \, d\nu_{\bar{x}}(z) =$$

= $W_{\bar{x}}(E_{\bar{x}} \cap \{x_1 \notin \widetilde{S}\}) + \int_{\widetilde{S}} W_z(E_z^1) \frac{d\nu_{\bar{x}}}{d\nu_y}(z) \, d\nu_y(z).$

The first summands in the right hand sides of (5.7) and (5.8) are no greater than ε_3 , and the quotient of the second summands is bounded by $1 + \varepsilon_3$. Thus, for y and \bar{x} being sufficiently close to each other, the probabilities $W_{\bar{x}}(E_{\bar{x}})$ and $W_y(E_y)$ are also close to each other. In particular, for every $p_0 < p_1$ we can find r > 0, such that for U being the union of r-leafwise-neighborhoods of points of J, we have

$$\forall y \in U \quad W_y(E_y) \ge p_0.$$

It completes the proof of this part of the lemma.

5.2.2. Proof of the "distributions" part. Note that due to the arguments already used we may suppose that $y \in \mathcal{G}_x$: for y and y' on the same \mathcal{F} -leaf, the measures $\nu_{y'}$ and ν_y are absolutely continuous with respect to each other, and hence any tail-type property holds (or does not hold) simultaneously for typical trajectories in Γ_y and $\Gamma_{y'}$. Hence, if y does not belong to \mathcal{G}_x , we may replace it by y', which is an intersection point of \mathcal{F}_y and \mathcal{G}_x .

Recall that almost every trajectory $\gamma \in \Gamma_x$ (due to the choice of x) is distributed with respect to μ . Thus, for almost every path $\gamma \in \Gamma_x$ we have

(5.9)
$$\lim_{T \to \infty} \frac{1}{T} \gamma_* \operatorname{leb}_{[0,T]} = \mu.$$

We know that a trajectory of E_y approaches a trajectory of E_x . Unfortunately, we can not claim that the map giving the trajectory of E_x by a trajectory of E_y is absolutely continuous (or, what is the same, maps typical trajectories to typical ones): we have this statement only for discretizations of trajectories.

Thus, we have to prove the distributions property using discretizations behaviour. The following arguments are a technical realization of this idea.

Rewrite (5.9) in the terms of discretization. Let a continuous function φ on M be chosen. Then, for W_x -almost every $\gamma \in \Gamma_x$,

(5.10)
$$\left(\frac{1}{T}\gamma_* \operatorname{leb}_{[0,T]}\right)(\varphi) = \frac{1}{T}\int_0^T \varphi(\gamma(t)) \, dt.$$

It is clear that we can restrict to the moments of time of the form $T = n\delta$; for such T,

(5.11)
$$\frac{1}{n\delta} \int_0^{n\delta} \varphi(\gamma(t)) dt = \frac{1}{n\delta} \sum_{j=0}^{n-1} \int_{j\delta}^{(j+1)\delta} \varphi(\gamma(t)) dt =$$
$$= \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\delta} \int_{j\delta}^{(j+1)\delta} \varphi(\gamma(t)) dt.$$

Let $z \in M$ be some point, and let us rewrite (5.11) for a typical trajectory $\gamma \in \Gamma_z$. Namely, we divide this sum into discrete averaging

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and the rest term:

(5.12)
$$\frac{1}{n\delta} \int_0^{n\delta} \varphi(\gamma(t)) dt = \frac{1}{n} \sum_{j=0}^{n-1} \varphi(\gamma(j\delta)) + \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\delta} \int_{j\delta}^{(j+1)\delta} (\varphi(\gamma(t)) - \varphi(\gamma(j\delta))) dt.$$

We estimate the second term in the right hand side of (5.11), decomposing the sum in two parts, the one corresponding to j with $\operatorname{diam}(\gamma([j\delta, (j+1)\delta])) < r$ and the one with $\operatorname{diam}(\gamma([j\delta, (j+1)\delta])) \ge r$.

$$(5.13) \quad \left| \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\delta} \int_{j\delta}^{(j+1)\delta} (\varphi(\gamma(t)) - \varphi(\gamma(j\delta))) dt. \right| \leq \\ \leq \frac{1}{n} \sum_{j < n, \operatorname{diam}(\gamma([j\delta, (j+1)\delta])) < r} \frac{1}{\delta} \int_{j\delta''}^{(j+1)\delta''} |\varphi(\gamma(j\delta)) - \varphi(\gamma(t)) dt| + \\ + \frac{1}{n} \sum_{j < n, \operatorname{diam}(\gamma([j\delta, (j+1)\delta])) \ge r} \frac{1}{\delta} \int_{j\delta''}^{(j+1)\delta''} |\varphi(\gamma(j\delta)) - \varphi(\gamma(t)) dt| \leq \\ \leq \omega_{\varphi}(r) + 2 \frac{\#\{j : \operatorname{diam} \ge r\}}{n} \sup_{\mathcal{M}} |\varphi|,$$

where ω_{φ} is the modulus of continuity of the function φ .

Let us pass in (5.13) to the upper limit:

(5.14)
$$\lim_{n \to \infty} \sup_{n \to \infty} \left| \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\delta} \int_{n\delta}^{(j+1)\delta} (\varphi(\gamma(t)) - \varphi(\gamma(j\delta))) dt. \right| \leq \\ \leq \omega_{\varphi}(r) + 2 \sup_{\mathcal{M}} |\varphi| \cdot \limsup_{j \to \infty} \frac{\#\{j < n \mid \operatorname{diam}(\gamma[j\delta, (j+1)\delta]) \ge r\}}{n}.$$

For a function φ chosen, the first summand can be made arbitrarily small by a choice of r due to the continuity of φ . For every r > 0chosen, the second summand can be made arbitrarily small for almost all trajectories by a choice of sufficiently small $\delta > 0$ (which is uniform in $z \in M$) due to the same arguments as the ones used in the proof of Proposition 2.3.

Hence, for every $\varepsilon > 0$ we can find r and then δ small enough, such that for almost all $\gamma \in \Gamma_z$,

$$\limsup_{n \to \infty} \left| \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\delta} \int_{j\delta}^{(j+1)\delta} (\varphi(\gamma(t)) - \varphi(\gamma(j\delta))) \, dt \right| < \varepsilon.$$

The rest term in (5.12) is estimated, and taking it together with (5.10), for a discretization $(x_j) = F^{\delta}(\gamma)$ of a typical path $\gamma \in \Gamma_x$, we have

$$\limsup_{n \to \infty} \left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(x_j) - \int_M \varphi \, d\mu \right| < \varepsilon.$$

Now, for a $W_{\bar{x}}$ -typical path γ from E'_{y} let us estimate the difference

(5.15)
$$\limsup_{n \to \infty} \left| \frac{1}{n\delta} \int_0^{n\delta} \varphi(\gamma(t)) \, dt - \int_{\mathcal{M}} \varphi \, d\mu \right|.$$

We have:

$$\begin{split} \limsup_{n \to \infty} \left| \frac{1}{n\delta} \int_0^{n\delta} \varphi(\gamma(t)) \, dt - \int_{\mathcal{M}} \varphi \, d\mu \right| \leq \\ \leq \limsup_{n \to \infty} \left| \frac{1}{n\delta} \int_0^{n\delta} \varphi(\gamma(t)) \, dt - \frac{1}{n} \sum_{j=0}^{n-1} \varphi(y_j) \right| + \\ + \limsup_{n \to \infty} \frac{1}{n} \sum_{j=k}^{n-1} |\varphi(y_j) - \varphi(x_j)| + \\ + \limsup_{n \to \infty} \left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(x_j) - \int_{\mathcal{M}} \varphi \, d\mu \right|, \end{split}$$

where (y_n) is the discretization of the path $\gamma \in \Gamma_y$, and (x_n) is its $\Phi_{y,x}$ image. We know that the measures $F^{\delta}_* W_x|_{E'_x}$ and $(F^{\delta} \circ \Phi_{y,x})_* W_y|_{E'_y}$ are absolutely continuous; thus, the image of a typical sequence is a typical sequence. Hence, for a typical trajectory $\gamma \in E'_y$ the first and the last summands do not exceed ε . So, we have

(5.16)
$$\limsup_{n \to \infty} \left| \frac{1}{n\delta} \int_0^{n\delta} \varphi(\gamma(t)) \, dt - \int_{\mathcal{M}} \varphi \, d\mu \right| < 2\varepsilon$$

for a typical path $\gamma \in E'_{u}$, which implies, that

(5.17)
$$\limsup_{T \to \infty} \left| \frac{1}{T} \int_0^T \varphi(\gamma(t)) \, dt - \int_{\mathcal{M}} \varphi \, d\mu \right| < 2\varepsilon.$$

For a typical path from E'_y we have obtained an estimate on the difference between the integral of φ and its average along the path. Let us extend this statement to all the E_y . Namely, repeating the arguments used in the proof of Remark 2.8, we see that almost every path from E_y can be decomposed into a finite starting segment and a path from some E'_z for some z close to M. Then, the estimate (5.17) holds also for a W_y -typical path from E_y .

But the definition of E_y does not depend on δ , thus, choosing arbitrarily small δ and r, we have finally:

$$\limsup_{n \to \infty} \left| \frac{1}{T} \int_0^T \varphi(\gamma(t)) \, dt - \int_{\mathcal{M}} \varphi \, d\mu \right| = 0.$$

Hence,

$$\frac{1}{T} \int_0^T \varphi(\gamma(t)) \, dt \xrightarrow[T \to \infty]{} \int \varphi \, d\mu$$

for a typical path $\gamma \in E_y$.

Recall that by definition the measures μ_t weakly converge to μ if and only if for every continuous function φ we have

$$\int \varphi \, d\mu_t \to \int \varphi \, d\mu$$

Moreover, it suffices to check such convergence for a well-chosen countable family φ_k . We have already obtained, that for any function φ and for a typical path $\gamma \in E_y$

$$\left(\frac{1}{T}\gamma_* \mathrm{leb}_{[0,T]}\right)(\varphi) = \frac{1}{T} \int_0^T \varphi(\gamma(t)) \, dt \xrightarrow[T \to \infty]{} \int \varphi \, d\mu$$

A countable family of typically satisfied conditions still is a typically satisfied condition, and hence for W_y -almost every trajectory $\gamma \in E_y$

$$\lim_{t \to \infty} \frac{1}{t} \gamma_* \mathrm{leb}_{[0,t]} = \mu.$$

This completes the proof of the lemma.

5.3. Codimension higher than one. Here we present a construction which permits us to handle the case of transversely conformal foliation of codimension higher than one. Let \mathcal{F} be such a foliation. Equip Mwith a Riemannian metric and for any point $x \in M$, let a transversal \mathcal{G}_x be an image of the image of the exponential map for a small disk in $(T_x \mathcal{F})^{\perp}$ disk. We notice that these transversals depend smoothly on x, and in a small neighborhood of every point x for every point y there exists a unique point z in its small \mathcal{F} -leafwise neighborhood, such that $z \in \mathcal{G}_y$.

We remark that the family of transversals $\{\mathcal{G}_x\}_{x\in M}$ does not necessarily form a foliation, for the reason that we can not control intersections of transversals with starting points on different \mathcal{F} -leaves.

Now, for a point $x_0 \in \mathcal{M}$, let us consider the set

$$M = \bigcup_{y \in \widetilde{\mathcal{F}}_{x_0}} \{y\} \times \mathcal{G}_y.$$

Note that \overline{M} is a manifold with boundary, naturally inheriting from M its foliation structure (except for the fact that for \overline{M} some leaves intersect the boundary $\partial \overline{M}$) and Riemannian metric on the leaves. But, on \overline{M} we have a natural smooth transversal foliation \mathcal{G} , leaves of which are $\{y\} \times \mathcal{G}_y$. Now, we can apply the same arguments as these used in the codimension one case, to prove Lemma 2.4.

5.4. Non-positivity of Lyapunov exponents. This section is devoted to the following lemma:

Lemma 5.4. Let \mathcal{F} be a transversely conformal foliation, \mathcal{M} be a minimal set in \mathcal{F} , and μ be a harmonic ergodic measure supported on \mathcal{M} . Then, $\lambda(\mu) \leq 0$.

Proof. We will present the proof for the case of codimension one foliation, using the existence of a transversal foliation \mathcal{G} . Then, it is generalized to an arbitrary codimension case in the same way as in the proof of Lemma 2.4.

Assume the contrary: let $\lambda(\mu) > 0$. We take some $\alpha, \beta, \alpha < \beta < \lambda(\mu)$. Prove first that the measure μ does not charge any leaf. Assume the contrary: $\mu(L) > 0$ for some leaf L. Then, the density with respect to the volume $\frac{d\mu|_L}{d\text{vol}_g}$ is a harmonic function on the leaf L. Moreover, this function is positive, bounded (because of harmonicity and boundedness of geometry of L) and of integral 1. Extending this function by 0 to the complementary, we obtain a harmonic measurable positive leafwise integrable function. Garnett [Ga, Proposition 1, p. 295] have proved that such a function should be leafwise constant. Thus, it is equal to a positive constant on L and hence (as it is integrable) L is a compact leaf. But for a compact leaf the Lyapunov exponent equals 0, as the corresponding Dirac measure is transversely invariant.

Let us choose a point $x_0 \in \mathcal{M}$, typical in the sense of Lyapunov exponents: for W_{x_0} -almost every path $\gamma \in \Gamma_{x_0}$ the corresponding path has the Lyapunov exponent equal to $\lambda(\mu)$.

First, let us consider the simplest case: suppose, that the foliation \mathcal{G} preserves the metric g. Also, we suppose that any \mathcal{F} -along holonomy extends to some fixed neighborhood $U \subset \mathcal{G}_{x_0}$. Finally, we suppose that the measure μ does not charge any leaf, or equivalently, that measures ν have no points of positive measure.

In this case, the measure μ induces on every leaf \mathcal{G}_x a conditional measure ν_x , which is harmonic in the sense of measure-valued functions (see Section 4.2). Let us take some T > 0 and C > 1 and for every path γ , starting at x_0 , try to find a $\tau = \tau(\gamma)$, $T < \tau < 2T$, as a minimal

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value t_0 in this interval possessing the following property:

(5.18)
$$\forall t \in [0,\tau] \quad h'_{\gamma|_{[\tau,t]}}(\gamma(\tau)) < Ce^{-\beta(\tau-t)}.$$

Here, we use $h_{\gamma|_{[\tau,t]}}$ as a short notation for $h_{\gamma|_{[t,\tau]}}^{-1}$ ($t < \tau$, and thus the first notation is not absolutely clear). Note, that as the holonomy is taken in the inverse sense, from the moment τ to $t < \tau$, one can expect that the derivative is will be small.

Note that the non-existence of such τ means, that

$$\frac{h'_{\gamma|_{[0,2T]}}}{h'_{\gamma|_{[0,T]}}} \le e^{\beta T}(x_0),$$

so for W_{x_0} -almost every path γ and for every T, sufficiently big such τ exists. If in the interval [T, 2T] we can not find a moment satisfying (5.18), then we choose $\tau(\gamma) = 2T$. Finally, we remark that $\tau(\cdot)$ is a Markovian moment.

The transversal measures ν_x depend on x in a harmonic way, thus for a transversal interval $I \subset \mathcal{G}_{x_0}$ the measures of its holonomy images $\nu_{\gamma(t)}(h_{\gamma|_{[0,t]}}(I))$ form a martingale. The expectation of value of this martingale at the Markovian moment τ should be equal to its initial value:

(5.19)
$$\mathbb{E}\nu_{\gamma(\tau(\gamma))}(h_{\gamma|_{[0,\tau(\gamma)]}}(I)) = \nu_{x_0}(I).$$

Now, note that due to the definition of $\tau(I)$ for all the paths γ with $\tau(\gamma) < 2T$ a neighborhood V of $\gamma(\tau)$ is contracted exponentially by the holonomy $h_{\gamma|_{[\tau,0]}}$, the radius of V is bounded from below by means of C, α, β and the geometry of the foliation. Namely,

$$|h_{\gamma|_{[\tau,0]}}(U_{\varepsilon}(\gamma(\tau)))| \le e^{-\alpha\tau} \le e^{-\alpha T},$$

where $\varepsilon > 0$ does not depend on *T*. Passing from inverse to direct time we see that an exponentially small neighborhood of $x_0 = \gamma(0)$ is expanded:

$$h_{\gamma|_{[0,\tau]}}(U_{e^{-\alpha T}}(x_0)) \supset U_{\varepsilon}(\gamma(\tau)).$$

Now, let us estimate the left part of (5.19) for $I = U_{e^{-\alpha T}}(x_0)$:

$$(5.20) \quad \mathbb{E}\nu_{\gamma(\tau(\gamma))}(h_{\gamma|_{[0,\tau(\gamma)]}}(I)) \geq \\ \geq \int_{\{\gamma:\tau(\gamma)<2T\}} \nu_{\gamma(\tau(\gamma))}(h_{\gamma|_{[0,\tau(\gamma)]}}(I)) \, dW_{x_0} \geq \\ \geq \int_{\{\gamma:\tau(\gamma)<2T\}} \nu_{\gamma(\tau(\gamma))}(U_{\varepsilon}(\gamma(\tau))) \, dW_{x_0} \geq \\ \geq W_{x_0}\{\gamma:\tau(\gamma)<2T\} \cdot \inf_{y\in\mathcal{M}}(\nu_y(U_{\varepsilon}(y)))$$

As \mathcal{M} is compact and $\operatorname{supp} \mu = \mathcal{M}$, the infimum in the last term of (5.20) is positive. Thus, the last term stays separated from 0 as Ttends to infinity, so it is no less than some constant $c_0 > 0$. Hence (5.19) and (5.20) imply that

$$\nu_{x_0}(U_{e^{-\alpha T}}(x_0)) \ge W_{x_0}\{\gamma : \tau(\gamma) < 2T\} \cdot \inf_{y \in \mathcal{M}}(\nu_y(U_{\varepsilon}(y))) > c_0,$$

and thus the left term does not tend to 0 as T tends to infinity. This contradicts the fact that the measure ν_{x_0} can not have atoms. We have obtained the desired contradiction. So, this case is handled.

Let us now consider the case of generic Riemannian structure (not necessarily preserved by the transversal foliation). Note, that the harmonic measure μ still defines conditional measures on the transversals $\{\mathcal{G}_y\}$, which are its Fubini conditional measures with respect to vol_g on the leaves. We still suppose that the \mathcal{F} -along holonomy maps are defined on the entire transverse leaves $\{\mathcal{G}_y\}$. Also, we add the following (simplifying the explanation of this step) hypothesis: all the leaves of \mathcal{F} are simply connected.

For these conditional distributions, the harmonicity condition implies that for a transversal interval I at a point $x \in M$ and a function ρ , we have:

(5.21)

$$\int_{I} \rho(y) \, d\nu_x(y) = \mathbb{E} \int_{h_{\gamma|_{[0,t]}}(I)} \frac{p(h_{\gamma|_{[t,0]}}(z), z; t)}{p(x, \gamma(t); t)} \rho(h_{\gamma|_{[t,0]}}(z)) \, d\nu_{\gamma(t)}(z),$$

where the expectation is taken in the sense of W_x . To prove this formula, we take a smooth function f on M supported in the neighborhood of I, with integral on \mathcal{F}_y equal to $\rho(y)$. For every fixed t, as the support of f tends to I, the integral of f with respect to μ tends to the left hand side of (5.21), and the integral of $D^t f$ to the right hand side. The harmonicity of μ implies that these two integrals coincide, which proves the formula.

Now, let us repeat the arguments used in the similar case with the following modification: we consider only discrete moments of time $t = k\delta$, where sufficiently small $\delta > 0$ is fixed.

Applying (5.21) several times for the initial function $\rho = \mathbf{1}_I$, we obtain, that for a Markovian moment (taking discrete values) $t(\gamma) = k(\gamma)\delta$

(5.22)
$$\nu_x(I) = \mathbb{E} \int_{h_{\gamma|_{[0,t]}}(I)} \prod_{j=1}^k \frac{p(z_{j-1}, z_j; \delta)}{p(x_{j-1}, x_j; \delta)} \, d\nu_{\gamma(t)}(z),$$

where $x_j = \gamma(j\delta)$, $z_j = h_{\gamma|_{[k\delta,j\delta]}}(z)$, and the expectation is taken in the sense of the measure W_{x_0} .

Let us take, as in the previous case, for $T = K\delta$ sufficiently big, a Markovian moment $\tau(\gamma) = k(\gamma)\delta$ defined as the smallest value in the interval [T, 2T] such that for every $t = l\delta < \tau$

$$h_{\gamma|_{[\tau,t]}}'(\gamma(\tau)) < Ce^{-\beta(\tau-t)}$$

Once again, if such a moment does not exist, we take $\tau = 2T$.

For T sufficiently big, the probability of $\tau < 2T$ is close to 1. Now, let us take $I = U_{e^{-\alpha T}}(x_0)$. For most paths, as we know, the holonomy maps expand exponentially and thus $h_{\gamma|_{[0,\tau]}}(I) \supset U_{\varepsilon}(\gamma(\tau))$.

Note that for most of the paths starting at x_0 , and for a point z_0 in the holonomy preimage $h_{\gamma|_{[\tau,0]}}(U_{\varepsilon}(\gamma(\tau)))$, the product of the quotients of the heat kernels (due to estimates analogue to these of Lemma 2.4) is bounded from below by some constant $c_1 > 0$. Let us denote

$$N = \left\{ \gamma \in \Gamma_{x_0} : \tau(\gamma) < 2T, h_{\gamma|_{[0,\tau]}}(I) \supset U_{\varepsilon}(\gamma(\tau)), \\ \forall z_0 \in h_{\gamma|_{[\tau,0]}}(U_{\varepsilon}(\gamma(\tau))) \quad \prod_{j=1}^k \frac{p(z_{j-1}, z_j; \delta)}{p(x_{j-1}, x_j; \delta)} \ge c_1 \right\}$$

Thus, the right hand side of (5.22) can be estimated as

(5.23)
$$\mathbb{E} \int_{h_{\gamma|_{[0,t]}}(I)} \prod_{j=1}^{k} \frac{p(z_{j-1}, z_{j}; \delta)}{p(x_{j-1}, x_{j}; \delta)} d\nu_{\gamma(t)}(z) \geq \\ \geq \int_{N} \int_{h_{\gamma|_{[0,t]}}(I)} \prod_{j=1}^{k} \frac{p(z_{j-1}, z_{j}; \delta)}{p(x_{j-1}, x_{j}; \delta)} d\nu_{\gamma(t)}(z) dW_{x_{0}} \geq \\ \geq \int_{N} \int_{U_{\varepsilon}(\gamma(\tau))} c_{1} d\nu_{\gamma(t)}(z) dW_{x_{0}} = \\ = \int_{N} c_{1} \nu_{\gamma(t)} (U_{\varepsilon}(\gamma(\tau))) dW_{x_{0}} \geq W_{x_{0}}(N) \cdot c_{1} \cdot c_{0}$$

Once again we see that the measure $\nu_{x_0}(I)$ does not tend to 0 as I contracts to x_0 , which contradicts the fact that the measure ν can not have atoms.

The higher codimension case is handled in the same way by working in \overline{M} defined in Section 5.3. The measure μ defines a σ -finite measure on \overline{M} , which harmonic in the sense of integral definition for compactly supported test function. Considering leafwise Brownian motion in \overline{M} (with the possibility of exiting through the boundary) we see that this measure is superharmonic in the sense that $\mu \ge D_*^t \mu$, and the same is true for the conditional measures ν_x .

Note that as in (5.23) the only trajectories used for estimates are those who arrive in ε -neighborhood of the $\gamma(\tau)$; hence due to exponential contraction in the inverse time they stay closer and closer to the main leaf \mathcal{F}_{x_0} . In particular, they stay in \overline{M} for all the time in the interval $[0, \tau]$.

Adding all this together, we see that the same estimates work in this case. So, the general case is handled in the same way. \Box

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MAX PLANCK INSTITUT - LEIPZIG - GERMANY *E-mail address*: bderoin@mis.mpg.de

ÉCOLE NORMALE SUPÉRIEURE DE LYON - FRANCE *E-mail address*: Victor.Kleptsyn@umpa.ens-lyon.fr