# SOME OPEN PROBLEMS IN HIGHER DIMENSIONAL COMPLEX ANALYSIS AND COMPLEX DYNAMICS

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by

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# 1 Introduction

We present here a collection of problems in complex analysis and complex dynamics in several variables. The list contains some old questions which are well known and some new ones. There is no pretention to be exhaustive and the leading line was just that one of us was interested recently in these problems. The questions are of various nature, some would lead to a breakthrough and require new ideas, others are reasonably easy. We thank Bedford, Berndtsson, Burns, Henkin, Ohsawa, Pinchuk, who have proposed some of the questions.

# 2 Complex Dynamics Problem List

We divide the questions into discrete and continuous ones. The discrete questions are further divided in two sets depending on whether the stable or chaotic features are most dominant.

## 2.1 Discrete Dynamics

### 2.1.1 Fatou sets, Stable sets

**Question 2.1** *Let $\mathcal{F}$ be the family of holomorphic endomorphisms of $\mathrm{P}^k$ of degree $d$ with infinitely many sinks. Can $\mathcal{F}$ have interior or have positive measure? (See ([Ga]) and ([Bu]).)*

**Question 2.2** *Can maps on $\mathrm{P}^k, k \geq 2$ have wandering Fatou components?*

**Question 2.3** *Can Fatou-Bieberbach domains for Hénon maps have $\mathcal{C}^\infty$ boundary? Recall that a Hénon map in $\mathbb{C}^2$ is a biholomorphism $f$ of the following form:*

$$f(z,w) = (p(z) + aw, bz),\ ab \neq 0,$$

*with $p$ a polynomial of degree $d \geq 2$. Assuming that $0$ is an attracting fixed point it is known that $\Omega := \{q; f^n(q) \to p\}$ is biholomorphic to $\mathbb{C}^2$, but $\overline{\Omega} \neq \mathbb{C}^2$. The question is about the smoothness of $\partial\Omega$.*
*It is known that there are domains $U \subset \mathbb{C}^2, \overline{U} \neq \mathbb{C}^2$, $U$ biholomorphic to $\mathbb{C}^2$ and $U$ has smooth boundary ([St]).*

**Question 2.4** *Let $M^k$ be a complex manifold. Assume that $M = \cup\phi_n(B)$, $\phi_n : B \to M$, $1-1$ and $\phi_{n+1}(B) \supset \phi_n(B)$, $B$ is the unit ball. Assume that the Kobayashi metric $K_M \equiv 0$. See ([FS11])) Works as well for stable manifolds of hyperbolic maps. For those stable manifolds $W^s, Kob \equiv 0$. Is $M = \mathbb{C}^k$? (False when $k \geq 3$. See ([FS11]). Open when $k = 2$.*
*Example: Stable manifolds, Eric Bedford:*
*Let $F : \mathbb{C}^n \to \mathbb{C}^n$ a biholomorphism, $K$ a hyperbolic saddle. Assume that the stable dimension of $K$ is $k$. For $p \in K$ is $W^s(p)$ biholomorphic to $\mathbb{C}^k$? This is true for periodic saddle points. See ([JV]) for results on generic stable manifolds.*

**Question 2.5** *(M. Herman) Symplectic question: $H = (f(z) - w, z)$, $f$ entire. Find $f$ such that $f$ has a Siegel domain at 0, $f(0) = f'(0) = 0$. Equivalently is it possible to conjugate $H$ to a linear map in a neighborhood of 0?*

**Question 2.6** *Can a Siegel domain for a Hénon map or, more generally, a biholomorphism of $\mathbb{C}^2$ have strongly pseudoconvex boundary? In one variable, R. Perez-Marco has constructed Siegel discs with smooth boundary ([PM]).*

**Question 2.7** *Classify recurrent Fatou components for an endomorphism of $\mathbb{P}^k$. ([FS6]). Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an endomorphism. Assume that $U$ is a Fatou component such that $f(U) = U$. Assume that $(f^n)(z) \to \partial U$. Does there exist a parabolic fix point on $\partial U$? See ([JL]).*

**Question 2.8** *Classify those Reinhardt domains which are biholomorphic to Siegel domains, associated to an automorphism of $\mathbb{C}^k$ or to an endomorphism of $\mathbb{P}^k$.*

**Question 2.9** *Suppose that the boundary of a Fatou component of a Hénon map has a smooth piece. What can be said about the whole boundary? In one variable the whole boundary must be smooth.*

**Question 2.10** *(The kicked rotor; Infinite dimensional complex dynamics) The kicked rotor describes a particle on a circular orbit subject to a periodic kick. In quantum mechanics it describes an elementary particle in an atom subject to a periodically pulsed electric field.*
*Classical description: If angular momentum is $p$, mass is $m$, $K$ is the kick*

*strength, $T$ is the time between kicks, then $R(\theta,p) = (\theta + \frac{pT}{m}, p + K\sin(\theta + \frac{pT}{m}))$ describes the change in coordinates from just after one kick till just after the next kick. (This is also called the Standard Map). Prove ([Gr]) that if $k = \frac{Km}{T} > 1$ then almost all orbits are unbounded.*
*Quantum description: If $\psi = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$ is an $L^2$ function on $[0,2\pi]$ describing the state of the particle after one kick, then after the next kick, the state is $R(\psi) = e^{-i\frac{K\cos\theta}{\hbar}} \sum_n e^{-\frac{\hbar i n^2 T}{2m}} c_n e^{in\theta}$. Show that the dynamics localizes. More precisely, if $R^k(\psi) = \sum_n c_n^k e^{in\theta}$ then for some $C = C(\psi)$ we have $\sum_{|n|\leq C} |c_n^k|^2 > 1/2$ for all $k$. It is necessary to avoid resonances (IS]). Computer experiments support this statement ([CCIF]). See ([F2]), ([W]) for rigorous results for a simplified model.)*

**Question 2.11** *Define chaos in the setting of infinite dimensional complex dynamics. See ([F2]).*

**Question 2.12** *Suppose that $f$ is an endomorphism of $\mathbb{P}^k$ or an automorphism of $\mathbb{C}^k$ with an attracting basin $\Omega$ for an attracting fixed point $p$, $f(p) = p$. Let $\delta > 0$. For $q \in \Omega$, $dist(q,\partial\Omega) > \delta$, $dist(q,p) > \delta$, i.e. $q \in \Omega^*_\delta$. Find $N(\delta)$ so that no orbit $\{q, f(q), \ldots, f^N(q)\} \subset \Omega^*_\delta$. Example $(z^2 + \epsilon w, z)$ with $p = 0$. The question here is to get explicit estimates of $N(\delta)$ in terms of $f, p$ and $\delta$.*

### 2.1.2 Julia sets, Currents

**Question 2.13** *Let $f$ be a holomorphic selfmap of a complex manifold $M$. Let $\Omega(f)$ be the nonwandering set. Recall that a point is nonwandering if for every neighborhood $U$ of $p$, there is an $n > 1$ such that $f^n(U) \cap U \neq \emptyset$. The question is to describe $\Omega(f)$.*
*Define a Siegel set as an analytic set $X$ in an open set $V \subset M$ such that there is a subsequence $n_i \to \infty$ with $f^{n_i}_{|X} \to Id_{|X}$. Is $\Omega(f)$ the closure of the periodic points together with Siegel analytic sets? See ([FS3],[FS6],[BS2]).*

**Question 2.14** *Let $f$ be a holomorphic selfmap of a complex manifold $M$. Is the closure of the repelling periodic orbits open in $\Omega(f)$? Can there be a counterexample for entire maps in $\mathbb{C}^n$? Could there be a sequence of oscillating orbits converging to a repelling point? Can a sequence of saddles converge to a repelling point for a holomorphic self-map on $\mathbb{C}^2$? (This cannot* *happen for biholomorphisms because for the inverse map the repelling point becomes attracting.) We say that the orbit of a point p is oscillating if some subsequence is uniformly bounded and some other subsequence converges to infinity. For construction of oscillating domains see ([FS5]). On dynamics of transcendental maps see ([FS7],[FS9]).*

**Question 2.15** *Let $f$ be a polynomial map on $\mathbb{C}^k$ of topological degree $d_t$, which extends to a holomorphic map on $\mathbb{P}^k$. Let $\omega$ be the Kähler form on $\mathbb{P}^k$. It is known that*

$$\frac{(f^n)^*\omega^k}{d_t^{nk}} \to \mu$$

*where $\mu$ is a mixing probability measure ([FS1]). The measure $\mu$ is the unique measure of maximal entropy ([BD2]) and repelling periodic points are dense in $S_\mu$, the support of $\mu$ ([BD1]). So $S_\mu \subset \Omega(f)$. Is the support of $\mu$ open in $\Omega(f)$? It is the case when $S_\mu$ is hyperbolic.*
*Recall that the Hausdorff dimension of a probability measure $\nu$ is the minimal Hausdorff dimension of a Borel set with $\nu(B) = 1$.*
*For a polynomial map in $\mathbb{C}$, $\mu$ coincides with the harmonic measure with respect to $\infty$ of the compact $K_P := \{z; P^n(z)$ is bounded$\}$. It follows from the work of Makarov ([Ma]), Bishop-Jones ([BJ]), T. Wolff ([W]) that the Hausdorff dimension of $\mu$ is 1.*
*What is the Hausdorff dimension of $\mu$ for an endomorphism of $\mathbb{P}^k$? A first case is to study small perturbations $(z^2 + \epsilon w, w^2 + \delta z)$. When is $\mu$ absolutely continuous with respect to Lebesgue measure.*

**Question 2.16** *The Julia set $\mathcal{J}$ of an endomorphism on $\mathbb{P}^k$ has a nonwandering set $\mathcal{J}_0$. What can be said about the Hausdorff dimension of $\mathcal{J}, \mathcal{J}_0, \mathcal{J} \setminus \mathcal{J}_0$? Describe $\mathcal{J}_0 \setminus Supp(\mu)$.*

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**Question 2.17** *Given an endomorphism of $\mathbb{P}^k$, is the current $T$ extremal? This is true for maps of the form $[P_1 : \cdots : P_k : t^d]$. Berndtsson-Sibony (unpublished).*

**Question 2.18** *Find families of endomorphisms of $\mathbb{P}^2$ of critically finite maps. (Each critical component is preperiodic.) ([FS12], [J2])*

**Question 2.19** *For $f$ an endomorphism of $\mathbb{P}^k$, when is $Supp(\mu) = \mathbb{P}^k$? For how many maps? Say, find the Hausdorff dimension of this set of maps. When is $\mu$ absolutely continuous with respect to Lebesgue measure.*

**Question 2.20** *Let $f$ be a holomorphic endomorphism of $\mathbb{P}^k, k > 1$. A closed set $A$ is attracting if there is an open set $U(A)$ such that $f(U(A)) \subset\subset U(A), \cap f^n(U) = A$. $A$ is an attractor if it is attracting and has a dense orbit. (Sometimes one assumes instead that it is chain transitive, i.e. $\delta-$pseudo-orbits are dense.) The attractor is non-trivial if it is not a periodic orbit or the whole space.*
*Show that non trivial attractors are robust in $\mathbb{P}^2$, i.e. for the endomorphisms $(f_c)$ on $\mathbb{P}^2$, find a set of positive measure in the $c-$space with a nontrivial attractor.*
*Give estimates for the Hausdorff dimension of $A$. For examples of non trivial attractors on $\mathbb{P}^2$ see ([J]), [FS8]). There are some estimates on Hausdorff dimension of attractors in ([FS8]).*

**Question 2.21** *Does there exist a Hénon map $f = (g, h), f^n = (g_n, h_n)$ with the following property: There is a $p = (x, y) \in S_\mu$ such that for all $0 < \epsilon << 1$ and every $q \in S_\mu$, there is an integer $n$ so that $\|g_n(p) - g_n(q)\| < \epsilon$? This question arises naturally in the study of attractors: A collision-attractor absorbs all points whose orbits are closer than some radius for some iterate. If one considers $x$ as a space variable, and $y$ as a momentum variable, one measures distance using the first variable only. If one replaces the inequality by $\|f^n(p) - f^n(q)\| < \epsilon$, there is no such $p$, (see [BF]).*

**Question 2.22** *Let $f(z, w) = (e^{i\theta}z, e^{-i\theta}w)+$ higher order terms be a generalized Hénon map. Give diophantine conditions on $\theta$ for which $f$ has two Siegel discs, $D_1, D_2 \subset J^+ \cap J^-$. Recall that generalized Hénon maps are finite compositions of Hénon maps. They are the dynamically interesting polynomial biholomorphisms of $\mathbb{C}^2$.*

**Question 2.23** *Consider $\mathcal{S}_{\mathbb{R}}$, the class of holomorphic symplectomorphisms of $\mathbb{C}^{2k}$ preserving $\mathbb{R}^{2k}$. $f : \mathbb{C}^{2k} \to \mathbb{C}^{2k}, f^*\omega = \omega$, where $\omega = \sum_{j=1}^k dz_j \wedge dw_j$ and $f(\mathbb{R}^{2k}) = \mathbb{R}^{2k}$, $z_j = x_j + ix'_j, w_j = y_j + iy'_j$ with Whitney fine topology, $\mathcal{S}_{\mathbb{R}}$ is a Baire space. For $f \in \mathcal{S}_{\mathbb{R}}$, let $K_f^{\mathbb{R}} := \{(x, y) \in \mathbb{R}^{2k}; \{f^n(x, y)\}_n$ is bounded$\}$. Prove that the set $\mathcal{S}'_{\mathbb{R}}$ of $f \in \mathcal{S}_{\mathbb{R}}$ such that $K_f^{\mathbb{R}}$ is of empty interior in $\mathbb{R}^{2k}$ is a $G_\delta$ dense set. ([FS7],[FS9])*

**Question 2.24** *Let* $f(z,w) = (e^{i\theta}z, e^{i\psi}w)+$ *higher order terms. Study the dynamics near the origin. Hakim-Abate-Weickert have studied the case where* $f$ *is tangent to the identity. ([Ha],[Ab],[We].)*

**Question 2.25** *(D. Burns) Suppose that* $f : X_{\mathbb{C}} \to X_{\mathbb{C}}$ *is an endomorphism on a projective variety over* $\mathbb{C}$*. Assume that* $f$ *has "large" dynamics. For example* $f$ *has positive entropy or a "large" non wandering set. Consider now* $X_{(k)}$ *as a variety over a number field* $k$ *such that* $[k : \mathbb{Q}] < \infty$*. Does largeness of the dynamics imply largeness of the set of rational points in* $X_{(k)}$*? For example, are rational points Zariski dense in* $X_{\mathbb{C}}$*? (J. B. Bost). The existence of mappings with large dynamics should imply arithmetic properties of* $X_{(k)}$*.*

**Question 2.26** *Let* $K$ *be a hyperbolic solenoid with stable dimension* $2$*. How to get a stable current? Are the stable leaves biholomorphic to* $\mathbb{C}^2$*?*

For endomorphisms on $\mathbb{P}^k$ see ([FS4],[FS2],FS8],[FS11],[F1],[S],[BD1],[BD2],[Ga],[Bu])

## 2.2 Continuous Dynamics

**Question 2.27** *Is there a compact set* $K \subset \mathbb{P}^2$ *which is laminated by smooth holomorphic curves, except a compact curve? There are such compacts in* $\mathbb{P}^3$*. See survey by Ghys ([Gh]). There is no closed* $(1,1)$ *current directed by the lamination in* $\mathbb{P}^2$*. (i.e. if locally the current is of the form* $\int d\mu_\theta[V_\theta]$*, then the lamination is a compact curve.) See ([HM]). The origin of the question seems to be related to a Poincaré-Bendixson Theorem for holomorphic foliations on* $\mathbb{P}^2$*. ([CLS]) More precisely, let* $\mathcal{F}$ *be a holomorphic foliation on* $\mathbb{P}^2$*. the singularity set of* $\mathcal{F}$*, Sing* $\mathcal{F}$ *is never empty. The question is whether the closure of any leaf* $L$ *intersects Sing* $\mathcal{F}$*. If not* $\overline{L}$ *will be laminated by smooth holomorphic curves ([CLS]).*

**Question 2.28** *Let* $P_N$ *consist of the holomorphic polynomials of degree at most* $N$ *in* $\mathbb{C}^{2k}$*,* $k \geq 2, N \geq 2$*. Show that for almost every* $P_N$ *and almost every point* $z \in \mathbb{C}^{2k}$ *the orbit of the Hamiltonian vectorfield* $X_{P_N}$ *is unbounded. Recall that when* $h$ *is an entire map in* $\mathbb{C}^{2k}$*,*

$$X_h := \left(-\frac{\partial h}{\partial w_1}, \ldots, -\frac{\partial h}{\partial w_k}, \frac{\partial h}{\partial z_1}, \ldots, \frac{\partial h}{\partial z_k}\right).$$

*([FS7],[FS9]) contain results on Hamiltonian vector fields and dynamical symplectomorphisms. (When $k = 1$, see ([Du]).)*

# 3 Several Complex Variable Problem List

We divide the questions into three sets, depending whether they fit most naturally together with $\overline{\partial}$, the Levi Problem or Holomorphic Mappings

## 3.1 $\overline{\partial}$

**Question 3.1** *(Henkin) Let $X$ be a normal analytic set of pure dimension $p$ in the unit ball $B \subset \mathbb{C}^n$. Let $f \in \mathcal{C}^\infty_{(\ell,1)}(X \cap B)$, i.e. $f$ is the restriction to $X$ of a smooth form in $B$. Assume $\overline{\partial}\left(f_{|Reg(X)}\right) = 0$. Does there exist $u \in \mathcal{C}^\infty_{(\ell,0)}(Reg\ X \cap B)$ such that $\overline{\partial}u = Id^*_{Reg(X)} f$ on $Reg(X) \cap B$? See ([HP], [M], [AG1], [AG2])*

**Question 3.2** *Let $\Omega$ be a weakly pseudoconvex domain with smooth boundary in $\mathbb{P}^n, n \geq 2$. Let $f \in \mathcal{C}^\infty_{(0,1)}(\overline{\Omega})$ be a $\overline{\partial}-$ closed $(0,1)$ form. Does there exist a smooth solution $u \in \mathcal{C}^\infty(\overline{\Omega})$ to the equation $\overline{\partial}u = f$?*

**Question 3.3** *(Lempert-Henkin) Let $D_N = \{z \in \mathbb{C}^N; \sum_{j=1}^N |z_j| < 1\}$. Prove (or find a counterexample) that for $f \in \mathcal{C}_{(0,1)}(\overline{D}_N)$ with $\overline{\partial}f = 0$ there is $u \in \mathcal{C}(\overline{D}_N)$ such that $\overline{\partial}u = f$ and*

$$\|u\|_{\mathcal{C}(\overline{D}_N)} \leq C\|f\|_{\mathcal{C}(\overline{D}_N)}.$$

*The key point here is that $C$ should be independent of $N$. One can ask the same question for $D^2_N = \{z \in \mathbb{C}^N; \sum_{j=1}^N |z_j|^2 < 1\}$. It seems likely that there is a counterexample in this case. (([L]), Lempert.)*

**Question 3.4** *(Berndtsson) Let $\phi$ be a bounded strictly plurisubharmonic function in the unit ball $\mathbb{B} \subset \mathbb{C}^n$. Let $\Omega = i\partial\overline{\partial}\phi$, the Kähler form associated to $\phi$. Let $f$ be a $(0,1)$ form on $B$ such that $|f|^2_\Omega + |\partial f|_\Omega \leq C$. The subscript means that the norm for $f$ and $\partial f$ are measured with respect to $\Omega$. Does the equation $\overline{\partial}u = f$ have a bounded solution in $B$? The question is related to the Corona theorem in several variables. Ref. ([Be])*

**Question 3.5** *Let $X$ be a closed analytic set of pure dimension $p$ in $\mathbb{C}^{n+p}$ with singularities. Let $Reg(X) = X \setminus Sing(X)$ denote the set of smooth points of $X$. Give $Reg(X)$ the metric induced by the imbedding $Reg(X) \hookrightarrow \mathbb{C}^{n+p}$.*
*(a) Let $f \in L^2(Reg(X)), 1 \leq q \leq p$, $\overline{\partial} f = 0$ in the weak sense in $L^2_{0,q+1}(Reg(X))$. Find the obstructions to solving $\overline{\partial} u = f$ in the weak sense in $L^2_{0,q}(Reg(X))$. Recall that $\overline{\partial} u = f$ in the weak sense in $L^2_{0,q}(Reg(X))$ if and only if $u \in L^2_{0,q-1}(Reg(X))$ and for all $\psi \in \mathcal{C}^\infty_{p,p-q}(Reg(X))$, compactly supported in $Reg(X)$ we have*

$$\int_{Reg(X)} u \wedge \overline{\partial}\psi = (-1)^q \int_{Reg(X)} f \wedge \psi.$$

*(b) Let $h \in L^2_{(\alpha,\beta)}(Reg(X))$. We say that $h$ belongs to the domain of $\overline{\partial}$ with Dirichlet boundary conditions - $h \in Dom(\overline{\partial}_D)$ - if and only if there exist $h_n \in \mathcal{C}^\infty_{\alpha,\beta}(Reg(X))$, compactly supported in $Reg(X)$ and $g \in L^2_{\alpha,\beta+1}(Reg(X))$ such that $h_n \to h$ in $L^2_{\alpha,\beta}(Reg(X))$ and $\overline{\partial} h_n \to g$ in $L^2_{\alpha,\beta+1}(Reg(X))$. In that case we write $\overline{\partial}_D h =: g$. Let $f \in L^2_{(p,q)}(Reg(X)), 1 \leq q \leq p$ such that $\overline{\partial}_D f = 0$. Find the obstructions to solving $\overline{\partial}_D u = f$, (i.e. obstructions to obtaining $u_n \in \mathcal{C}^\infty_{(p,q-1)}( Reg(X))$, compactly supported in $Reg(X)$ such that $u_n \to u$ in $L^2_{(p,q-1)}(Reg(X))$ and $\overline{\partial} u_n \to f$ in $L^2_{(p,q)}(Reg(X))$.*
*In ([PS]), (a) and (b) are solved when $X$ is a projective surface with isolated singularity. They also computed obstructions to solving $\overline{\partial}$ weakly for $\overline{\partial}$ closed $(p,q)$ forms when $X$ is a projective variety of dimension $p$, as well as obstructions to solving $\overline{\partial}_D$ for $(0,q)$, $\overline{\partial}_D-$ closed forms again when $X$ is a projective variety of dimension $p$.*
*See ([BeSi]) for an alternative point of view: Solving $\overline{\partial}$ on positive currents (especially positive, closed currents of bidegree (1,1)).*
*See ([DFV]) for obstructions to solving $\overline{\partial}$ weakly for $(0,1)$ forms near 2-dimensional isolated singularities in $\mathbb{C}^n$, $n \geq 3$.*

**Question 3.6** *a) Let $T$ be a positive closed current of bidegree$(2,2)$ in$\mathbb{C}^3$.*
*Is $T$ a limit of $c_j[V_j]$, dim $V_j = 1$, analytic and $c_j > 0$.*
*b) Can $\overline{V_j} \to$ Supp $T$? or even in the Hausdorff metric? The case when $T$ is a bidegree $(1,1)$ current(i.e. i.e. dim $V_j$=2) is classical. For the recent refinements see Demailly ([D1], [D2]), Duval-Sibony ([DS1]) , Guedj ([Gu]) for currents in complex manifolds.*

## 3.2 Levi problem

**Question 3.7** *Let $X \subset \mathbb{C}^n, 0 \in X$ be a complex analytic set with an isolated normal singularity, $K$ a compact subset of $X \setminus (0), 0 \in \hat{K}$. Suppose that $L$ is compact in $X$, containing $K$ in it's relative interior. Does $\hat{L}$ contain a neighborhood of $0$ in $X$? This is equivalent to a Runge problem: If $(\Omega_t)_{t\in(-1,1)}$ is a continuously increasing family of Stein open subsets of $X$, $0 \in X \setminus \Omega_t, t \leq 0$, $0 \in \Omega_t, t > 0$. Is $\Omega_{t_1}$ Runge in $\Omega_{t_2}$ if $t_1 \leq 0 < t_2$?*

**Question 3.8** *Suppose that $X$ is Stein, $\Omega \subset\subset X$. Suppose that for all $p \in \partial\Omega$ there is an open neighborhood $U(p)$ so that $U \cap \Omega$ is Stein. Is $\Omega$ Stein? ([FN], [Si])*

**Question 3.9** *Let $\Omega \subset\subset M$ be a compact complex manifold, $\partial\Omega$ connected. Suppose that $f$ is holomorphic in a neighborhood of $\partial\Omega$. Under which natural conditions does $f$ extend to $\Omega$ or its complement? Exclude $M = \Delta * N$, $N$ compact. It is true if $\Omega$ is Stein. Same question when $f$ is CR on $\partial\Omega$.*

**Question 3.10** *Does there exist a $\mathcal{C}^\infty$ Levi flat hypersurface in $\mathrm{P}^2$? Same question for topological hypersurfaces, i.e. locally graphs which are locally union of disjoint holomorphic discs. Describe all $\mathcal{C}^\infty$ Levi flat hypersurfaces in a complex torus. This is related to Question 2.6. See ([Oh]), $\mathcal{C}^\infty$ in $\mathrm{P}^n, n \geq 3$. Done by ([Siu2]) Also Lins-Neto ([LN]) in the real analytic case.*

**Question 3.11** *(Henkin) (i) Construct a strictly pseudoconvex domain $M$ with smooth boundary in a complex 2-dimensional surface in $\mathrm{P}^3$ which intersects every compact Riemann surface in $\mathrm{P}^3$. B. Fabre ([Fab1], p. 88) has constructed an $M \subset \mathrm{P}^3$ with Levi-flat boundary and which intersects any compact Riemann surface.*
*(ii) Let $V$ be an algebraic hypersurface in $\mathbf{P}^3$. Let $M$ be a strictly pseudoconvex domain in V. Is there a Riemann surface $S$ in $\mathrm{P}^3$ with $S \cap V \subset V \setminus M$?*

**Question 3.12** *(Gromov) Let $\mathcal{A}_p$ be the space of analytic sets of pure dimension $p$ in $\mathbf{P}^n$. When $p = 1$, ([Bo] F. Bogomolov) has constructed a sequence of Riemann syrfaces $(V_j)$ in $\mathcal{A}_1$ such that $\sup_j diam(V_j) = \infty$. Here the diameter in $V_j$ is computed with respect to the Hermitian metric induced from $\mathrm{P}^n$. Does a similar phenomenon occur for subvarieties in $\mathcal{A}_p$ for $1 < p < n$?*

**Question 3.13** *Let $M$ be a totally real compact manifold in $\mathbb{C}^n$, $dim(M)_{\mathbb{R}} = n$. Does it bound a smooth Riemann surface, smooth up to the boundary? For $\epsilon > 0$, there is a Riemann surface with boundary in $M(\epsilon)$ ([Vi]). Here $M(\epsilon) = \{z; dist(z, w) < \epsilon\}$. Gromov showed the existence of discs when $M$ is a Lagrangian, for a Kähler form. There are examples without discs. Ref. Gromov ([Gr]), Alexander ([Al]), Duval-Sibony ([DS1], [DS2]), Viterbo ([Vi]).*

**Question 3.14** *Let $M$ be a smooth projective manifold, $\Omega$ a proper open subset, locally Stein at the boundary. When is $\Omega$ holomorphically convex? See ([N]). (Grauerts counterexample is a Levi flat domain on a complex torus, $(\{(z, w); a < \Re z < b\}$ with irrational slope ).) All holomorphic functions are constant, so we assume that $\partial\Omega$ has 1 strongly pseudoconvex boundary point. See ([Si]). What are the Leviflat domains in a complex torus?*

**Question 3.15** *(L. Nirenberg) Let $D \subset \mathbb{C}^n, n \geq 2$ be a domain with smooth boundary. Let $\gamma$ be a smooth curve in $\partial D$ transverse to $T_p^{\mathbb{C}}(\partial D)$ for every $p$ in the image of $\gamma$. Prove that if $f \in \mathcal{A}^\infty(\overline{D})$ and $f^{(k)}_{|\gamma} = 0 \; \forall k \in \mathbb{Z}^n_+$ then $f \equiv 0$. When $\gamma$ is real analytic, the result and some consequences are in ([N]).*

## 3.3 Holomorphic Mappings

**Question 3.16** *Can one embed all Stein Riemann surfaces as closed complex submanifolds of $\mathbb{C}^2$? (See [GS]))*

**Question 3.17** *Let $\Omega$ be a smoothly bounded pseudoconvex domain with non-compact automorphism group $G$. Does $G$ contain $\mathbb{R}$?*
*Method: Pinchuk scaling technique. Scaling limit should be biholomorphic to the original domain. When $\Omega$ is convex, see ([Fr]).*

**Question 3.18** *(Pinchuk) Let $\Sigma$ be a real analytic hypersurface, $\partial\Omega$ a real analytic strictly pseudoconvex, nonspherical hypersurface or simply connected. Suppose $f$ is a holomorphic map defined on $\Sigma$ in a neighborhood of a strictly pseudoconvex point, and to $\partial\Omega$. Can $f$ be extended along any curve?*
*Some special cases are considered in ([Pi], [Sh], [BP]).*

**Question 3.19** *Let $\Sigma$ be a real analytic hypersurface of finite type. Let $X = \sum a_j \frac{\partial}{\partial z_j}$, ($a_j$ holomorphic) be a holomorphic vector field, $\Re X$ tangent to $\Sigma$. Classify these $X$ and use them to classify $\Sigma$. (Convex domains with compact automorphism groups ([BP]))*

**Question 3.20** *Let $f : \Omega \to V$ be biholomorphic, $\Omega, V$ open sets in $\mathbb{C}^n$ with $\mathcal{C}^\omega$ boundary. Does $f$ extend continuously to the boundary (or $\mathcal{C}^\omega$)? How about $\mathcal{C}^\infty$ domains? Do there exist pseudoconvex $\Omega, V$, biholomorphism, not extending smoothly to the boundary? Case to analyze: $\{(\Re w + |z_1|^2 - |z_2|^2)^2 + \Re\eta < 0.\}$ (All known methods fail.)*

**Question 3.21** *Jacobian Conjecture: Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping. Assume that the Jacobian of $f$ does not vanish. Then $f$ is invertible. ([BCW], [V], [O].)*

**Question 3.22** *Let $\Omega$ be a domain with real analytic boundary, $p \in \partial\Omega$, $p$ not in the envelope of holomorphy, $p$ of finite type. Does there exist an $\epsilon > 0$ so that the infinitesimal Kobayashi metric satisfies the estimate*

$$F(q, X) \geq \frac{|X|}{|p-q|^\epsilon}?$$

*This will be useful to prove that proper, holomorphic maps $\Omega \to \Omega'$ extend contiuously to $\overline{\Omega}$.*

**Question 3.23** *Kobayashi hyperbolicity problem. Let $X \subset \mathbb{P}^n$ be a generic hypersurface of degree $\geq 2n+1$, is $\mathbb{P}^n \setminus X$ Kobayashi hyperbolic? More precisely, is any holomorphic map $\mathbb{C} \to \mathbb{P}^n \setminus X$ constant? A lot of work has been done in this direction. We only mention Siu-Yeung ([SY]) and Demailly-El Goul ([DG]), Noguchi, Winkelman, Katsutoshi ([NWK]).*

**Question 3.24** *(Pinchuk) Let $H \subset B^n$, $H$ a $\mathcal{C}^\infty$ hypersurface of finite type. Let $X_n$ be a sequence of closed subvarieties of $B^n$, $0 \in X_n \cap H$. Can the cluster set of $X_n$ be a contained in $H$? When $H$ is pseudoconvex, then every point has a plurisubharmonic barrier ([S2]) and hence the cluster set is not contained in $H$.*

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